# A Characterization of Euler's Ordinary Differential Operator

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**Abstract**

The necessary and sufficient condition that

$$u \mapsto G(x, u(x), u'(x), \dots, u^{(2k)}(x))$$

be Euler's differential operator $u \mapsto \sum_{j=0}^{k} (-l)^j \frac{d^j}{dx^j} \left( \frac{\partial L}{\partial u^{(j)}} \right)$ associated to a Lagrangian $L(x, u(x), u'(x), \dots, u^{(k)}(x))$ is that for all integers $p > 0$ one has

$$\sum_{j=0}^{\infty} (-1)^j \left( \begin{array}{c} p+j \\ p \end{array} \right) \frac{d^j}{dx^j} \left( \frac{\partial G}{\partial u^{(2p+j+1)}} \right) \equiv 0$$

This is not an original result as [1] and the references therein provide much more general solutions to the inverse euler problem. The present treatment employs somewhat different techniques and so the author felt it might still be useful to present it.

## 1 Introduction

Variational problems involving extremals of functionals of the form

$$J[u] = \int_a^b L(x, u'(x), \dots, u^{(k)}(x))\, dx$$

where the Lagrangian $L(x, y^0, \dots, y^k)$ is a function of $k+2$ variables, invariably lead to the so-called Euler's equation:

$$E(L)(x, u(x), u'(x), \dots, u^{(2k)}(x)) = \\ \sum_{j=0}^{k} (-1)^j \frac{d^j}{dx^j} \left( \frac{\partial L}{\partial y^j}(x, u'(x), \dots, u^{(k)}(x)) \right) = 0$$

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Well known analogous constructions hold for variational problems in several dependent and independent variables. In spite of the fact that Euler's equations probably define the most important class of differential equations of physical theories, little precise information is available about this class. We provide a necessary and sufficient condition that a differential operator

$$u \mapsto G(\cdot, u, u', \ldots, u^{(2k)})$$

be an Euler operator

$$u \mapsto E(L)(\cdot, u, u', \ldots, u^{(2k)})$$

for some L. This condition is that for all integers $p > 0$ and for all functions $u$ we have

$$\sum_{j=0}^{\infty} \left( \begin{array}{c} p+j \\ p \end{array} \right) \frac{d^j}{dx^j} \frac{\partial G}{\partial y^{2p+1+j}}(x, u(x), u'(x), \ldots, u^{(2k)}(x)) \equiv 0$$

## 2 Basic definitions and Conventions

**Definition 1** *We introduce the* jet bundles $J^k = \mathbb{R} \times \mathbb{R}^{k+1}$ *and* $J^\infty = \mathbb{R} \times \mathbb{R}^\infty$ *with generic points* $(x, y^0, y^1, \ldots, y^k)$ *and* $(x, y^0, y^1, \ldots, y^k, \ldots)$ *respectively and denote by* $\pi_k : J^\infty \to J^k$ *the canonical projection.*

**Convention 1** *By a* function *we shall always mean a* $\mathcal{C}^\infty$ *real valued function* $u$ *defined in some open set* $\mathcal{O} \in \mathbb{R}$.

**Definition 2** *Given a function* $u$ *we define* $\tau u : \mathcal{O} \to J^\infty$ *by*

$$\tau u(x) = (x, u(x), u'(x), \ldots, u^{(k)}(x), \ldots)$$

**Definition 3** *By a* jet function $F$ *we mean any map of the form*

$$\pi_k^{-1}(\mathcal{O}) \to J^k \stackrel{F_0}{\to} E$$

*where* $\mathcal{O} \subset J^k$ *is open and* $F_0$ *is* $\mathcal{C}^\infty$. *We write* $ord(F)$ *for the lowest integer* $k$ *for which such a factorization is possible and call it the* order *of* $F$.

**Definition 4** *Given a jet function* $F$, *the* differential operator associated *to* $F$ *is the operator* $u \mapsto F \circ \tau u$ *and the* differential equations associated *to* $F$ *is* $F \circ \tau u = O$.

Notice that a jet function is uniquely determined by its operator, whereas not in general by its equation.

We now define some differential operators that act on jet functions.

**Definition 5**

$$\mathbb{D} = \frac{\partial}{\partial x} + \sum_{j=0}^{\infty} y^{j+1} \frac{\partial}{\partial y^j}$$

We note that $\mathbb{D}$ can also be defined by its basic property

$$\frac{d}{dx}(F \circ \tau u) = (\mathbb{D}F) \circ \tau u$$

for any jet function $F$ and any function $u$.

**Definition 6**

$$F_{p,q} = \sum_{j=1}^{\infty} (-1)^j \left( \begin{array}{c} p+j \\ p \end{array} \right) \mathbb{D}^j \frac{\partial}{\partial y^{q+j}}$$

**Definition 7**

$$E = F_{0,0}$$

Note that $E(F)$ is the jet function determined by Euler's operator with $F$ as Lagrangian, and $E(F) \circ \tau u = 0$ is Euler's equation.

**Convention 2** *We deem the binomial coefficient* $\left( \begin{array}{c} r \\ s \end{array} \right)$ *to be zero unless* $r \geq 0$, $0 \leq s \leq r$. *We shall also deem* $\mathbb{D}^q$ *and* $\frac{\partial}{\partial y^q}$ *to be zero unless* $q \geq 0$.

# 3 Some technical lemmas

**Lemma 1**

$$\frac{\partial}{\partial y^r} \mathbb{D}^s = \sum_{j=0}^{\infty} \left( \begin{array}{c} s \\ j \end{array} \right) \mathbb{D}^{s-j} \frac{\partial}{\partial y^{r-j}}$$

The proof follows easily by induction on $s$ using definition 5.

**Lemma 2**

$$\frac{\partial}{\partial y^r} F_{p,q} = \sum_{j=0}^{\infty} (-1)^j \left( \begin{array}{c} p+j \\ p \end{array} \right) F_{p+j,q+j} \frac{\partial}{\partial y^{r-j}}$$

*Proof:* Using lemma 1 we have

$$\frac{\partial}{\partial y^r} F_{p,q} = \sum_{j=0}^{\infty} (-1)^j \left( \begin{array}{c} p+j \\ p \end{array} \right) \sum_{k=0}^{\infty} \left( \begin{array}{c} j \\ k \end{array} \right) \mathbb{D}^{j-k} \frac{\partial}{\partial y^{q+j}} \frac{\partial}{\partial y^{r-k}}$$

Setting $j = t + k$ and noting that

$$\left( \begin{array}{c} p+t+k \\ p \end{array} \right) \left( \begin{array}{c} t+k \\ k \end{array} \right) = \left( \begin{array}{c} p+t+k \\ p+k \end{array} \right) \left( \begin{array}{c} p+k \\ p \end{array} \right)$$

and that the double sum can be written as $\sum_{k=0}^{\infty} \sum_{t=0}^{\infty}$ we arrive at the result. Q.E.D

**Lemma 3**

$$F_{p,q}F_{r,s} = \sum_{a=0}^{\infty}(-1)^{a+p+1}\sum_{m=0}^{\infty}\frac{1}{2\pi i}\oint\frac{(1-z)^{m-p-1}}{z^{m-a-p}}\,dz\left(\begin{array}{c} r+m \\ m \end{array}\right)\mathbb{D}^{a}\frac{\partial}{\partial y^{q+a-m}}\frac{\partial}{\partial y^{s+m}}$$

*Proof:* Using lemma 1 we have

$$F_{p,q}F_{r,s} = \sum_{k=0}^{\infty}\sum_{m=0}^{\infty}\sum_{j=0}^{\infty}(-1)^{k+m}\left(\begin{array}{c} p+k \\ p \end{array}\right)\left(\begin{array}{c} r+m \\ m \end{array}\right)\left(\begin{array}{c} m \\ j \end{array}\right)\mathbb{D}^{k+m-j}\frac{\partial}{\partial y^{q+k-j}}\frac{\partial}{\partial y^{s+m}}$$

Setting $a = k + m - j$ we can rewrite the sum as

$$\sum_{a=0}^{\infty}\sum_{m=0}^{\infty}\sum_{j=0}^{\infty}(-1)^{a+j}\left(\begin{array}{c} a+j-m+p \\ p \end{array}\right)\left(\begin{array}{c} r+m \\ m \end{array}\right)\left(\begin{array}{c} m \\ j \end{array}\right)\mathbb{D}^{a}\frac{\partial}{\partial y^{q+a-m}}\frac{\partial}{\partial y^{s+m}}$$

Let us identify the sum

$$\sigma_{m,p} = \sum_{j=0}^{\infty}\left(\begin{array}{c} a+j-m+p \\ p \end{array}\right)\left(\begin{array}{c} m \\ j \end{array}\right)$$

Remembering the formulae:

$$\begin{aligned}\left(1-\frac{1}{z}\right)^{-p-1} &= \sum_{j=0}^{\infty}\left(\begin{array}{c} p+j \\ p \end{array}\right)z^{-j}\\ (1-z)^{m} &= \sum_{j=0}^{\infty}(-1)^{j}\left(\begin{array}{c} m \\ j \end{array}\right)z^{j}\end{aligned}$$

We see that the sum corresponds to the coefficients of $z^{m-a}$ in the $1/z$ expansion of $\left(1-\frac{1}{z}\right)^{-p-1}(1-z)^{m} = (-1)^{p+1}z^{p+1}(1-z)^{m-p-1}$ the validity of the said expansion holds for $|z| > 1$ and we have

$$\sigma_{m,p} = (-1)^{p+1}\frac{1}{2\pi i}\oint\frac{(1-z)^{m-p-1}}{z^{m-a-p}}\,dz$$

proving the lemma. Q.E.D

# 4 The main theorem

**Theorem 1** $\mathrm{Im}\,(E) = \bigcap_{p=0}^{\infty}\ker\left(F_{p,2p+1}\right)$

*Proof:* We first prove that $F_{p,2p+1}E = 0$ for all $p \geq 0$. Remembering that $E = F_{0,0}$ we have from lemma 3 that

$$F_{p,2p+1}E = \sum_{a=0}^{\infty}(-1)^{a+p+1}\sum_{m=0}^{\infty}\frac{1}{2\pi i}\left\{\oint\frac{(1-z)^{m-p-1}}{z^{m-a-p}}\,dz\right\}\mathbb{D}^a\frac{\partial}{\partial y^{2p+1+a-m}}\frac{\partial}{\partial y^m}$$

We note that by convention 2 the sum over $m$ extends only to $m = 2p+1+a$. The Cauchy integral gives contributions from poles at $z = 0$ and $z = 1$. These correspond to the following values of $m$:

pole at $z = 1$: $m = 0, 1, 2, \ldots, p$
pole at $z = 0$: $m = p+a+1, p+a+2, \ldots, 2p+a+1$

Let us compare the contribuition from the pole at $z = 1$ for $m = n$ and the pole at $z = 0$ for $m = 2p+a+1-n$. We note that the differential operator part

$$\mathbb{D}^a\frac{\partial}{\partial y^{2p+1+a-m}}\frac{\partial}{\partial y^m}$$

is the same for both poles. The transformation $w = 1 - z$ transforms

$$\oint\frac{(1-z)^{n-p-1}}{z^{n-a-p}}\,dz$$

into

$$-\oint\frac{(1-w)^{p+a-n}}{w^{p+1-n}}\,dw$$

which is just negative of the Cauchy integral corresponding to $n = 2p+1+a-n$. Thus the two contributions cancel precisely and $F_{p,2p+1}E = 0$.

Suppose now that $G$ is a jet function such that $F_{p,2p+1}G = 0$ for all $p \geq 0$. If $\mathrm{ord}\,(G) = 2k+1$ is odd then seeing that

$$F_{k,2k+1} = \frac{\partial}{\partial y^{2k+1}} - (k+1)\mathbb{D}\frac{\partial}{\partial y^{2k+2}} + \cdots$$

we have $\frac{\partial}{\partial y^{2k+1}}G = 0$ contradicting the definition of order. Thus $\mathrm{ord}\,(G) = 2k$ is even, $k \geq 0$. Consider the sequence of operators

$$\mathbb{D}^{2p}F_{p,2p+1} = \sum_{s=0}^{\infty}(-1)^s\begin{pmatrix} s-p \\ p \end{pmatrix}\mathbb{D}^s\frac{\partial}{\partial y^{s+1}}$$

We note that if $F$ is any linear combinations of these operators that $FG = 0$. Consider the $k \times k$ matrix of coefficients $(-1)^s\begin{pmatrix} s-p \\ p \end{pmatrix}$ for $k \leq s \leq 2k-1$ and $0 \leq p \leq k-1$. We show that this matrix is non-singular. It's enough to show the matrix $M_{p,s} = \begin{pmatrix} s-p \\ p \end{pmatrix}$ is non singular. Now in spite of the arbitrary nature of convention 2 we note that for the given range of indices we have $\begin{pmatrix} s-p \\ p \end{pmatrix} = Q_p(s)$ where $Q_p(x)$ is the polynomial given by.

$$Q_p(x) = \frac{(x-p)(x-p-1)\cdots(x-2p+1)}{1\cdot 2\cdots p}$$

If now some non-trivial linear combination of the rows of $M$ were zero then by the Lagrange interpolation formula the same linear combination would hold among the polynomials $Q_p$ which is impossible seeing that all are of different order. By using row operations we can therefore bring matrix $M$i to the identity matrix. This means that for $k \leq t \leq 2k-1$ we have a linear combination of the $\mathbb{D}^{2p}F_{p,2p+1}$ of the form

$$F = \sum_{s<k} a_s \mathbb{D}^s \frac{\partial}{\partial y^{s+l}} + \mathbb{D}^t \frac{\partial}{\partial y^{t+1}} + \sum_{s>2k-1} b_s \mathbb{D}^s \frac{\partial}{\partial y^{s+1}}$$

Now the third term doesn't contribute to $FG$ since $\operatorname{ord}(G) = 2k$. If we apply $\frac{\partial}{\partial y^{2k+t+1}}$ to $FG$ we see by lemma 1 that the first term gives zero contribution since for $r \leq s < k$ we have $2k+t+1-r > 2k+1$. Thus again by lemma 1 we conclude that

$$\frac{\partial}{\partial y^{2k+t+1}} FG = \frac{\partial}{\partial y^{t+1}} \frac{\partial}{\partial y^{2k}} G = 0$$

This means that $G = A + By^{2k}$ where $\operatorname{ord}(A) \leq 2k-1$ and $\operatorname{ord}(B) \leq k$. Let now

$$L_1(x, y^0, y^1, \ldots, y^k) = \int_a^{y^k} \int_a^{\eta} B(x, y^0, y^1, \ldots, y^{k-1}, \xi)\, d\xi\, d\eta$$

for some convenient $a$. Using lemma 2 we find

$$\frac{\partial}{\partial y^{2k}} EL_1 = \left(\frac{\partial}{\partial y^k}\right)^2 L_1 = B$$

and so $EL_1 = C + By^{2k}$ where $\operatorname{ord}(C) \leq 2k-1$. Let $G_1 = G - EL_1$. By the fact that $F_{p,2p+1}E = 0$ we see that $F_{p,2p+1}G_1 = 0$ and seeing that $\operatorname{ord}(G_1) \leq 2k-1$, the proof of the existence of a jet function $L$ such that $G = EL$ reduces by inductions to the case of $\operatorname{ord}(G) = 0$. In this case all of the equations $F_{p,2p+1}G = 0$ are trivially satisfied and defining $L(x, y^0) = \int_a^{y^0} G(x,\xi)\, d\xi$ for some convenient $a$ we see that $EL = G$. Q.E.D

We note that the proof also provides an algorithmic procedure, for calculating a Lagrangian $L$ from Euler's operator $G = EL$. The Lagrangian is not uniquely determined since $\ker(E) = \operatorname{Im}(\mathbb{D})$.

Little conceptual insight is furnished by the above proof. Necessary and sufficient conditions were first found for low order operators by trial and error providing linear combinations of the $F_{p,2p+l}G = 0$ conditions. The $F_{p,q}$ operators appear systematically as boundary terms in the first and second variation of the functional $J[u]$ but unfortunately we can offer no further insight. Analogous expressions for the case of several independent variables provide necessary but not sufficient conditions for a partial differential operator to be Euler's operator. We must also emphasize that we have not characterized Euler's equations, that is determine under what conditions given a jet function $F$ we can find a jet function $L$ such that $F^{-1}(0) = E(L)^{-1}(0)$, that is for which $F \circ \tau u = 0$ would hold.

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## References

[1] I. M. Anderson and T. Duchamp, "On the Existence of Global Variational Principles", *American Journ. Math.* **102** (1980) 781-868.