S. Galkin: CY3: Age of discovery?

Def Calabi-Yau 3-fold Y – a smooth proj., $\omega_Y \simeq \mathcal{O}_Y$, $h^{1,0}=0$ (either $\pi_1(Y)=0$ or at least $|H_1(Y)|<\infty$)

Thm (Bogomolov) $K_Y=0 \Rightarrow$ up to a finite étale cover $\hat{Y}\to Y$ we have $\hat{Y}\simeq$ Abelian $\times$ Hol. Symp! $\times$ CY

$\pi_1=0$, $H^i(Y,\mathcal{O}_Y)=\begin{cases}\mathbb{C}, & i=0,\dim Y\\ 0 & \text{otherwise}\end{cases}$

CY
- of type K $\hat{Y}=S\times E$ S – K3 surf, E – ell. curve
- of type A $\hat{Y}$ = Abelian 3-fold

Most interesting case $\pi_1(Y)=0$

Open question: Are there finitely or infinitely many families of CY 3-folds?

Hodge diamond

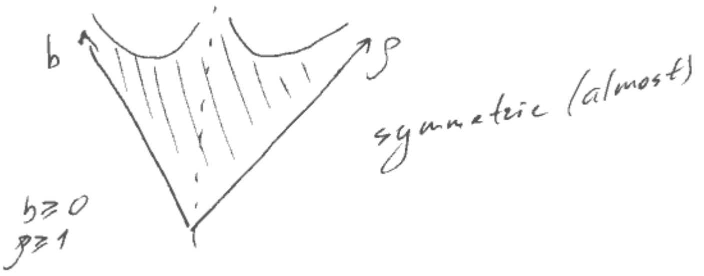
$$\begin{array}{ccccccc} & & & 1 & & & \\ & & 0 & & 0 & & \\ & 0 & & \rho & & 0 & \\ 1 & & b & & b & & 1 \\ & 0 & & \rho & & 0 & \\ & & 0 & & 0 & & \\ & & & 1 & & & \end{array}$$

2 non-trivial Hodge #
$\rho = h^{1,1}=h^{2,2}$ (Kähler modulus)
$b = h^{1,2}=h^{2,1}$ (complex modulus)

$T_Y\simeq\Omega_Y^2$ (because $\Omega_Y^3\simeq\mathcal{O}_Y$) $H^k(Y,T_Y)\simeq H^k(Y,\Omega_Y^2)=H^{2,k}$

$H^0(Y,T_Y)=0$; $H^1(Y,T_Y)\simeq\mathbb{C}^b$

Y has moduli space and it is b-dimensional



Other story – Fano 4-folds
Z; ω_Z^* – ample

Thm (Campana, KMM) $\exists$ only finitely many def. classes of smooth Fano 4-folds.

why? Z is rationally connected $\Rightarrow$ many rat. curves
$\Rightarrow$ bound the degree $(-K_Z)^n\le(2n)^n$ (if $\rho=1$)
bound $c_1^{n-2}c_2$

Conj. bound: $(-K_Z)^n\le(n+1)^n$
True for $n=4$ (Hwang)

Open question: how many families of Fano 4-folds?

$g=1$ — most interesting

$Y \in |-K_Z|$ if smooth it is CY 3-fold

Calabi-Yau Explorer

$b=0$ ⇒ Rigid CY3 Def. $/\bar{\mathbb{Q}}$ $\mathbb{Q} \subset K$, $[K:\mathbb{Q}]<\infty$

$H^3_{et}(Y, \mathbb{Q}_\ell)$ is 2-dim. 2 possibilities:

- $K \neq \mathbb{Q}$ — Galois action
- $K = \mathbb{Q}$

this represent. is modular

$L_3(Y)$ = Mellin transform (Modular form of weight 4)

$b=1$ — deform in 1-dim family

$g(\mathcal{M}_Y)=0$ coarse moduli space is a curve of genus 0

$$\begin{array}{l} U \\ \pi\downarrow \\ \mathcal{M}_Y \end{array}$$

fibres $\pi^{-1}(t) = \begin{cases} \text{smooth} \\ \text{singular} \end{cases}$

Sing. fibres $\pi_1(\mathbb{CP}^1 \setminus \{p_1 \dots p_n\}) \to Sp(4, \mathbb{Z})$ - monodr. repres.

image can be finite and infinite index subgr.

MUM point — when monodromy is max. unipotent: $(M-1)^4=0$, $(M-1)^3 \neq 0$

conifold point — monodromy M is quasi-reflection

$rk(M-1)=1$

$M(v) = v + \langle v,u \rangle u$ u - vanishing cycle

orbifold point - $M^a = 1$

Usually p is MUM point when $\pi^{-1}(p)$ looks like maximal simple normal crossing at some Q

$\pi(Q)=P$ (like $\{z_1 z_2 z_3 z_4 = 0\} \subset \mathbb{C}^4$)

Conifold point — usually some Morse point

π looks locally like this: $z \mapsto \sum_{i=1}^4 z_i^2$

$$\begin{array}{c} \hat{Y}_0 \leftarrow \text{crepant resolution, rigid CY 3-fold} \\ \downarrow \\ Y_t \xrightarrow[\text{degeneration}]{} Y_0 \end{array}$$

Can try to study these local systems or ODEs ③

"Mainz tables" – Calabi-Yau operators database.

($\simeq 50$ operators)

Z – Fano 4-fold, $r = 1$; $Y \subset Z$ CY3 with $\rho = 1$

$\rightsquigarrow$ construct $\pi : \mathcal{U} \longrightarrow \mathbb{CP}^1$ mirror dual family of CY 3 folds

It has at least 1 MUM point; T – conifold point

$$\mathcal{U} \supset (\mathbb{C}^*)^4 \qquad\qquad L \subset Z$$

$$\shortparallel \qquad\qquad \overset{\wr\wr}{(S^1)^4} \text{ Lagrangian}$$

$$\mathrm{Hom}(\pi_1(L), \mathbb{C}^*)$$

$\pi|_{(\mathbb{C}^*)^4}$ is LP with real positive coeffs.

Most famous CY 3folds $\cdot\ X_5 \subset \mathbb{CP}^4$

$\cdot$ complete intersections $\sum d_i = n+1$; $X_{d_1 \ldots d_{n-3}} \subset \mathbb{CP}^n$

<u>Orbifolding</u> Look for specific pencil of quintics (Dwork pencil)

$$\mu(x_1^5 + \ldots x_5^5) + \lambda x_1 \ldots x_5 = 0$$

$$(\mu : \lambda) \in \mathbb{CP}^1 \qquad (\mathbb{Z}/5)^5 \quad x_i \longmapsto \varepsilon^{a_i} x_i$$

$$\overset{\cup}{(\mathbb{Z}/5)^4} \longrightarrow \underset{G}{\underset{\shortparallel}{(\mathbb{Z}/5)^3}} \leftarrow \text{preserves the pencil}$$

$$Y_t^\vee = \widehat{(Y_t/G)} \quad \text{resolution}$$

Complete intersections in Grassmanians

$$Gr(2,7) \quad 10\text{-dim Fano, index } 7$$

$$\shortparallel$$

$$G = Gr(2, V) \subset \mathbb{P}(\Lambda^2 V) \cong \mathbb{P}^{20}$$

$G \cap \mathbb{P}(A) \to$ CY 3 fold $\rho = 1$; $\mathrm{Pic}\, Y = \mathbb{Z}H$,

codim $A = 7$ $\qquad H^3 = 42$

The mirror dual family (or at least ODE) has 5 singular points in the moduli space ($\simeq \mathbb{CP}^1$)

2 MUM, 3 conifolds $\qquad * \quad \cdot \quad \cdot \quad \cdot \quad *$

ODE games here predict

deg = 14 CY 3fold and Fano 4fold (1996)

Look at 2-dim quotients of V:

$$V \xrightarrow{\pi} \mathbb{C}^2 \qquad \Lambda^2\mathbb{C}^2 \simeq \mathbb{C} = \langle \omega_0 \rangle$$

$\pi^*\omega_0$ skew-symmetric matrix 7×7 rank 2

In $\mathbb{P}(\Lambda^2 V^*)$ look for matrices of rank 4

$$\mathcal{X}^{17} \subset \mathbb{P}^{20}$$

$GL(V)$ acts; $\operatorname{sing} \mathcal{X}^{17} \simeq Gr(2,V^*)$

Fano variety of index 14

$$\mathcal{X} \cap \mathbb{P}(A^\perp) \subset \mathbb{P}(A^\perp) \simeq \mathbb{P}^6$$

another CY 3-fold

$Gr(2,V)$ and $\mathcal{X}^{17}$ are classically proj. dual

Thm (Borisov–Caldararu 2003, Kuznetsov 2006) $D^b_{Coh}(G \cap \mathbb{P}(A)) \simeq D^b_{Coh}(\mathcal{X} \cap \mathbb{P}(A^\perp))$

1-st example of derived equiv., not birat. CY 3folds

Expectation: MUM points in Kähler moduli space ($\mathbb{CP}^1$) correspond to non-birational Fourier–Mukai partners.

More on Pfaffians $X \subset \mathbb{P}^N$ X – CY 3fold

If $\operatorname{codim} X \le 2 \Rightarrow X$ is a c.i.

$\operatorname{Codim} X = 3$

Thm (Walker) this variety is given as a generalized Pfaffian

$E/\mathbb{P}^N$ $E \xrightarrow{\varphi} E^\vee(u)$ φ is skew-symmetric

$\operatorname{rk} E = 2k+1$ $\operatorname{rk}\varphi \le 2k-2$

Hosono–Takagi example $\mathcal{X} = \mathbb{P}^4 \times \mathbb{P}^4 \overset{\text{Segre}}{\hookrightarrow} \mathbb{P}^{24} = \mathbb{P}(V \otimes V) \dashrightarrow \mathbb{P}(S^2 V)$

$\mathcal{X} \cap \mathbb{P}(A)$ CY3 $\rho = 2$

codim $A = 5$

$S^2\mathbb{P}^4$ $Y = S^2\mathbb{P}^4 \cap \mathbb{P}(A')$ – smooth CY 3fold

codim $A' = 5$

$\pi_1(Y) = \mathbb{Z}/2\mathbb{Z}$ $\rho(Y) = 1$ $H^3 = 35$

Kähler moduli space has 2 MUM's

prediction – should be another CY 3 fold with $H^3 = 10$

$S^2(\mathbb{P}^4)^* \leftarrow$ symmetric 5×5 matrices of rank 2

Reye congruence $\mathbb{P}^{14} \cong \mathbb{P}(S^2 V^*) \supset Y$

$\det\begin{pmatrix} a_{11} & a_{12} & \\ a_{12} & \ddots & \\ & & \end{pmatrix} = 0$ 13-dim quintic

$(rk \le 3) \subset Y^{13}$

$V \to \mathbb{C}^3$

$S^2(\mathbb{C}^3)^*$

$\underset{Gr(5,3)}{\mathbb{P}}(S^2 Q^*)$ 11-dim

$Y^{11} = (rk \le 3) \subset Y^{13}$

$Y^{13} \cap \mathbb{P}(A'^{\perp})$ sing quintic 3 fold

$\tilde{Y} \leftarrow$ quadric of $rk \le 4$ + choice of ruling sing in codim 5

$Y^{11} \subset \overset{\downarrow 2:1}{Y^{13}}$

$\tilde{Y} \cap \mathbb{P}(A'^{\perp})$ – smooth CY 3fold

„double quintic symmetroid"

Thm (Hosono–Takagi) $D^b_{Coh}(S^2(\mathbb{P}^4)^* \cap \mathbb{P}(A)) \cong D^b_{Coh}(\tilde{Y} \cap \mathbb{P}(A'^{\perp}))$ and one is a moduli space of curves on the other.

Moral: $\pi_1(Y)$ is not derived invariant,

$Br(Y)$ is not derived invar. $\leftarrow$ (Addington)

Pf: $H_1(Y) \oplus Br(Y)$ is derived invariant.

$Gr(2,5) \subset \mathbb{P}^9$

$X_1, X_2 \simeq Gr(2,5) \quad X_1 \cap X_2 \subset \mathbb{P}^9$

↑ CY 3fold $\rho = 1$

Kähler mod. sp. has 2 MUM's, Looks like self-dual, but particular points in mod. space may be different.

M. Kapustka noticed that some BPS # are squares

These are just ~~joints~~ joins

$X \subset \mathbb{P}(A) \qquad (C_{\mathbb{P}(B)} X) \cap (C_{\mathbb{P}(A)} Y) \subset \mathbb{P}(A \oplus B)$

$Y \subset \mathbb{P}(B)$

$E \xhookrightarrow{\ell_d} \mathbb{P}(\mathbb{C}^d) \qquad$ If CCE has smoothings

elliptic curve

<u>Funny example</u> $\quad Gr(2,4) \times Gr(2,4)$

$s \in \Gamma(G \times G, \mathcal{U}^* \boxtimes \mathcal{U}^*) \qquad (s=0)$ – Fano 4-fold $\rho = 1$ diagonal

Miura

$X_{12} \subset E_6/P \leftarrow$ Cayley plane $\mathbb{OP}^2$

↑ Fano index 9 $\quad$ Sing $X = \mathbb{P}^5$

$X_{12} \subset \mathbb{P}^{20}; \quad X_{12} \cap \mathbb{P}(A) \quad$ codim $A = 9$

KMS has 2 MUM's

prediction: $\exists$ CY 3 fold $\quad Z \subset \mathbb{P}^8$

Thm (–, Kuznetsov, Movshev)

Miura was right.

Explicit construction: V, $\dim V = 6$

$\{(A,u)\} = \Lambda^2 V^* \oplus V \qquad M_{12} = \{(A,u) \mid Au = 0, \ \mathrm{rk}\, A \le 2\}$

$M_{15} = \{Au = 0, \ \mathrm{rk}\, A \le 4\}$

Then $\dim \mathrm{Sing}\, M_{12} = 5$, $\dim \mathrm{Sing}\, M_{15} = 8 \qquad M_{12} \cap \mathbb{P}(A) \longleftrightarrow M_{15} \cap \mathbb{P}(A^\perp)$

smooth CY 3-folds $\rho = 1$