Cubics: lines, squares, and irrationality.

joint work with

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Theorem. Assume $\mathbb{L} \not\mid 0$.

Then... In particular, generic cubic fourfold is irrational.

$$\mathbb{L} := [\mathbb{A}^1] \in K_0(\mathrm{Var}_{/k})$$

$\mathbb{L} \not\mid 0$ means $\mathbb{L}$ is not a zero divisor

$$\mathbb{L} \cdot M = 0 \implies M = 0 \quad \forall M \in K_0(\mathrm{Vars})$$

(„Lefschetz element" (or Tate?))

k - field. $K_0(Vars/k)$ - Grothendieck ring of varieties/k.

Generators: $[X]$, X - variety/k

Relations: $[X] = [Y] + [Z]$

$Z \subset X$ - closed subvariety

$Y = X \setminus Z$ - open complement

Thm (Looij-a Bittner)
$\underline{\text{char } k = 0}$
enough for smooth projective

$$\begin{array}{ccc} E & \subset & W = Bl_Z X \\ \downarrow & & \downarrow \\ Z & \subset & X \end{array} \qquad X \setminus Z = Y = W \setminus E$$

$$E \simeq \mathbb{P}_Z(N_{X/Z}) \qquad [X] - [Z] = [W] - [E]$$

Ring $K_0(\mathrm{Vars}/k)$ <u>does</u> have zero divisors:

Poonen 2002 – generated by abelian varieties

Kollar 2005 – gen. by Conics (conic bundles)

„Motivic measure": any ring homomorphism
$\mu: K_0(Vars/k) \to R$

<u>Examples</u> • $k \subset \mathbb{F}_q$ $\quad \chi_q(X) = |X(\mathbb{F}_q)|$

• $k \subset \mathbb{C}$ $\quad \chi_{\mathbb{C}}(X) = \chi^{comp}_{top}(X(\mathbb{C}), \mathbb{R})$

• $k \subset \mathbb{R}$ $\quad \chi_{\mathbb{R}}(X) = \chi^{comp}_{top}(X(\mathbb{R}), \mathbb{Z}/2\mathbb{Z})$

After Bittner–Looijenga it is easy to construct/prove m.m. in char $k = 0$

Measures defined on smooth proj. X:

- $k \subset \mathbb{C}$ $\quad X \mapsto \sum h^{p,q}(X_{\mathbb{C}}) \cdot u^p \cdot v^q \in \mathbb{Z}[u,v]$
  $\qquad \qquad \qquad \qquad \qquad \qquad \qquad \qquad \parallel$
  Hodge-Deligne polynomial $\quad E_X(u,v)$
- $\cdots \quad X \to [HS(X)] \in K_0\left(\begin{matrix}\text{Hodge}\\ \text{Structures}\end{matrix}\right)$
- $X \mapsto [M(X)] \in K_0\langle \text{Chow Motives} \rangle$

char $k = 0$: OK for motives $/\mathbb{Z}$ (by B-L)
char $k > 0$: also OK with coeff in $\mathbb{Q}$ (Gillet-Soule)

- (BLL) $X \longmapsto [\text{dg-enh}(D^b_{coh} X)] \in K_0(\text{Cats})$
- (Larsen-Lunts) $\left(\sum h^{0,n} u^n\right) \in \mathbb{Z}\left[(1 + \mathbb{Z}[u], \cdot)\right]$
  values in group ring of mult. monoid.

Measures of $\mathbb{L}$:

$\chi_{\mathbb{C}}(\mathbb{L}) = 1$ $\chi_{\mathbb{R}}(\mathbb{L}) = -1$

$\chi_q(\mathbb{L}) = q$ $E_{\mathbb{L}}(u,v) = uv$

$\mu_{GS}(\mathbb{L}) =$ Lefschetz motive

$\mu_{BLL}(\mathbb{L}) = 1$ $\left(\mathcal{D}^b_{coh}(\mathbb{P}^1) = \langle \mathcal{O}, \mathcal{O}(1) \rangle \right)$

$\mu_{\mathbb{L}}(\mathbb{L}) = 0$!!! More generally

consider quotient-ring $K_0(Vars)/\mathbb{L}$

Theorem (Larsen-Lunts, 2003)

$$\begin{tikzcd} K_0(\mathrm{Vars}) \arrow[r] \arrow[dr] & \mathbb{Z}[SB] \\ & K_0(\mathrm{Vars})/\mathbb{L} \arrow[u, "\sim"'] \end{tikzcd}$$

isomorphism

SB – Monoid of Varieties modulo stably birational equivalence;

- $X \overset{\text{bir.}}{\dashleftarrow\dashrightarrow} X' \Rightarrow X \sim X'$ (Birational)
- $X \sim X \times \mathbb{P}^1$ (Stable)

This provides a link between Grothendieck ring $K_0(\mathrm{Vars})$ and birational geometry.

But, how to use it?

<u>Lemma</u> $X \dashrightarrow X'$ birational $\Rightarrow$

$[X] - [X'] = \mathbb{L} \cdot \mathcal{M}$,

where $\mathcal{M} = \sum \pm V_i$, $\dim V_i \leq \dim X - 2$

<u>Pf</u> Weak Factorization Thm.

<u>Cor.</u> X-rational $\Rightarrow \exists \mathcal{M}_X$ s.t. $[X] = [\mathbb{P}^d] + \mathbb{L} \cdot \mathcal{M}_X$

Beautiful Formula

For cubics use input from geometry:

$$[S^2X] = (1+\mathbb{L}^d)[X] + \mathbb{L}^2[F_X] - \mathbb{L}^d[X_{sing}]$$

$X \subset \mathbb{P}^{d+1}$ – reduced cubic hypersurface

F_X – variety of lines $\ell \subset X$

S^2X – symmetric square, $S^2X = X^2/\sigma$

variety of unordered pairs of points. $\sigma(x_1,x_2)=(x_2,x_1)$

X_{sing} – Singular Locus of X ($\emptyset$ for smooth)

$\mathbb{L} = [\mathbb{A}^1]$ – Lefschetz class

Beautiful Formula is Unconditional!

Derivation of irrationality from the Beautiful Formula.

Assume X is rational, by Lemma

$$[X] = [\mathbb{P}^d] + \mathbb{L} \cdot \mathcal{M}_X$$

Plug this into

$$[S^2 X] = (1+\mathbb{L}^d)[X] + \mathbb{L}^2[F]$$

to obtain

$$\mathbb{L}^2[F] = \mathbb{L}^2 \, S^2(\mathcal{M} + [\mathbb{P}^{d-2}]) - \mathbb{L}^d$$

Now use the Assumption, and divide by $\mathbb{L}^2$

$$[F] = S^2(\mathcal{M} + [\mathbb{P}^{d-2}]) - \mathbb{L}^{d-2}$$

Now take modulo $\mathbb{L}$

$$[F] = S^2(\mathcal{M} + [\mathbb{P}^{d-2}]) \pmod{\mathbb{L}}$$

$$\mathcal{M} + [\mathbb{P}^{d-2}] = \sum_i V_i - \sum_j W_j$$

$$S^2(\cdots) = \sum [\mathrm{Hilb}_2 V_i] + \sum [V_i V_{i'}] + \sum [W_j W_{j'}]$$
$$- \sum [\mathrm{Hilb}_2 W_j] - \sum [W_j W_{j'}] - \sum [V_i W_j]$$

Larsen-Lunts (Cor 2.6)

F_X is stably birational to either:

$\mathrm{Hilb}_2 V_i$ | $V_i V_{i'}$ | $W_j W_{j'}$

Corollary Let X be a smooth rational cubic fourfold over k, char $k=0$. Then F_X is birational to $\mathrm{Hilb}_2 S$ for a K3 surface S.

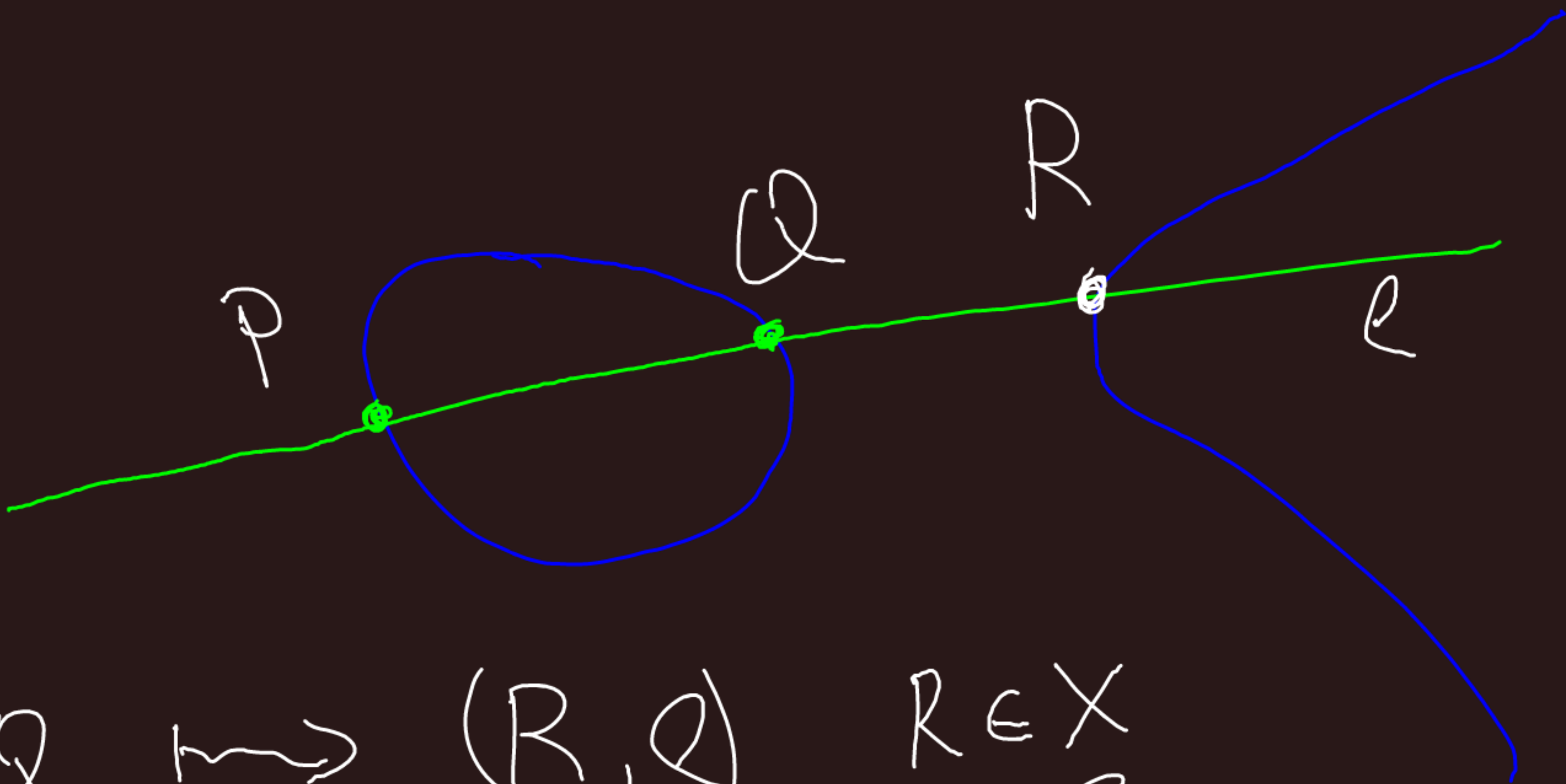
In particular, if $k=\mathbb{C}$ then there is an isomorphism of polarizable Hodge structures in sense of Hassett

Pf $F \not\leftrightarrow S_1 \times S_2$ because $1+t^2+t^4=(1+t+t^2)(1-t+t^2)$ non-positive coeff

$\Uparrow$ measure $\sum h^{0,n} t^n$

$F \leftrightarrow \mathrm{Hilb}_2 S$, S is K3 by class. of surf.

Proof of the Beautiful Formula.



$PQ \mapsto (R, Q)$ $\quad R \in X$, $R \in \ell$

Birational map between

$S^2 X$ and $\mathbb{P}_X(T_{\mathbb{P}}) \overset{\text{bir}}{\dashleftarrow\dashrightarrow} X \times \mathbb{P}^d$

The map is not everywhere
defined: points P, Q may coincide

- line $\ell = \overline{PQ}$ may lie inside X
- line $R \in \ell$ might be tangent to X

... in other point $P = Q$

Take this into account, to deduce formula

(Resolve the birational morphism)

Other Corollaries

- Cayley's 27 lines on surface
  Just Apply $\chi_{\mathbb{C}}$
  (Need also $\chi_{\mathbb{C}}(S^2 X) = \dfrac{\chi_{\mathbb{C}}(X)(\chi_{\mathbb{C}}(X)+1)}{2}$)
- Lines on real surface:
  27, 15, 7, 3 — Apply $\chi_{\mathbb{R}}$

Similar for surfaces with ADE singularities
(uniform formulas (proof)

- Relation between $X(\mathbb{F}_q)$, $X(\mathbb{F}_{q^2})$ and $F(\mathbb{F}_q)$. For $d=1$ — recursion

$$N_{q^2} = N_q(2+2q-N_q)$$

- Rational Hodge Structure of F_X in terms of $HS(X)$. (semi-simple category)

Apply μ_{Hdg}.

In particular, Hodge # of F.

Etc.

Thank you!

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