Quantum cohomology via small degenerations and low ramification

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Abstract. The talk will outline Golyshev's programme for classification of Fano varieties by studying their quantum D-modules and assosiated mirror-symmetric low ramified objects. We will provide some illustrations constructed from small toric degenerations approach. Whether time permits we will exhibit the recent calculations and conjectures on period-like limits for quantum differential equations.

Landau–Ginzburg models

Mirror duality at Hodge theoretic level states that
regularized quantum differential equation of n-dimensional smooth Fano variety X
coincides with Picard–Fuchs equation of some pencil w over $\mathbb{A}^1$
of $(n-1)$-dimensional Calabi–Yau varieties (the *Landau–Ginzburg model* mirror symmetric to X).

If X is a smooth toric Fano variety, an easy corollary from Givental's computation shows that the mirror symmetric to X Landau–Ginzburg model can be given by the Laurent polynomial w associated with X:

$$w_\Delta = \sum_{v \in Vertices(\Delta)} x^v$$

Obviously Newton polytope of w_Δ equals $\Delta = FanPolytope(X)$.

Euguchi-Hori-Xiong's mirror for Grassmannian $Gr(k, N)$ In 'Gravitational quantum cohomology' Euguchi, Hori and Xiong propose the following Laurent polynomial as Landau–Ginzburg model mirror-symmetric to Grassmanian $Gr(k, N)$:

$$w_{k,N} = X_{1,1} + \sum_{(i,j)=(1,1)}^{(k,N-k)} \frac{X_{i+1,j} + X_{i,j+1}}{X_{i,j}} + \frac{q}{X_{k,N-k}}$$

Here w is a Laurent polynomial of $\dim Gr(k, N) = k(N-k)$ variables i.e. an element of $\mathbb{C}[X_{i,j}^{\pm 1}]_{i=1,\ldots,k;j=1,\ldots,N-k}[q]$ and $X_{k+1,j} = X_{i,N-k+1} = 0$.

Toric degenerations explanation (Batyrev, Ciocan-Fontanine, Kim, van Straten 97)
By Sturmfels there is a degeneration of $Gr(k, N)$ to $T(k, N) = \mathbb{P}(Newton(w_{k,N}))$. This degeneration is small (singularities of $T(k, N)$ in codimension 3 are conifolds). If $T(k, N)$ would be smooth, then $w_{k,N}$ will be it's toric LG-model. And in the small degeneration case one expects the degenerated variety to be close enough to smooth so this still holds.

"Batyrev's ansatz"

Generalizing the previous example Batyrev states that if smooth Fano Y has a small degeneration $\pi : \mathcal{X} \to C$ to toric X (i.e. π is a flat projective morphism to a curve, X and Y are isomorphic to some fibers of π, X admits only Gorenstein terminal singularities, and the restriction map $Pic(\mathcal{X}) \to Pic(\mathcal{X}_t)$ is an isomorphism for all $t \in C$), then the Laurent polynomial $w_{\Delta(X)}$ is a Landau–Ginzburg model mirror symmetric to Y.

Consider

$$\Phi_w(t) = \frac{1}{(2\pi i)^{\dim X}} \int \frac{1}{1 - tf} \frac{dx}{x}$$

i.e. a constant (with respect to x) term of $\frac{1}{1-tf}$. $\Phi_w(t)$ is a solution of Picard–Fuchs equation for the pencil $\frac{1}{w} : (\mathbb{C}^*)^{\dim X} \to \mathbb{A}^1$.
Let $I_Y(t)$ be the I-series of Y.

Definition. w is a *weak Landau–Ginzburg* mirror-symmetric to Y if general fiber of w is Calabi–Yau and the following numerical relation holds:

$$\Phi_w(t) = I_Y(t).$$

By Danilov–Khovanskii the general (in the sense of Kushnirenko) Laurent polynomial w in a class of Laurent polynomials with fixed Newton polytope Δ has Calabi–Yau fibers $\iff$ 0 is unique lattice point inside Δ. Equivalently, the Fano toric variety with fan polytope Δ admits at most canonical singularities.

Examples

- Grassmannians (degeneration constructed by Strumfels)
- Partial flag manifolds (degenerations constructed by Goncuilea-Lakshimbai)
- (Batyrev) Hypersurfaces $X_d \subset \mathbb{P}^N$ for $2d \leqslant (N+1)$:

$$x_1 \cdot \dots \cdot x_d = x_{d+1} \cdot \dots \cdot x_{2d}$$

Questions

(1) Terminal surfaces are nonsingular. So all two-dimensional small toric degenerations are (iso)trivial. Is it possible to generalize for worse singularities?
(2) (Batyrev 97) Which of the threefolds admit small toric degenerations?
(3) Is Batyrev's ansatz correct and how to justify it?

Definition. *Deformation* is a flat proper morphism

$$\pi : \mathcal{X} \to \Delta,$$

where
Δ is a unit disc $\{|t| < 1\}$,
and $\mathcal{X}$ is an irreducible complex manifold.

All the deformations we consider are projective (π is a projective morphism over Δ).
Denote fibers of π by X_t, and let $i_{t\in\Delta}$ be the inclusion of a fiber $X_t \to \mathcal{X}$.

If all fibers $X_{t\neq 0}$ are nonsingular, then the deformation π is called a degeneration of $X_{t\neq 0}$ or a smoothing of X_0.
If at least one such morphism π exists, we say that varieties $X_{t\neq 0}$ are *smoothings* of X_0, and X_0 is a *degeneration of* $X_{t\neq 0}$.

For a coherent sheaf $\mathcal{F}$ on $\mathcal{X}$ over Δ and $t \in \Delta$ the symbol $\mathcal{F}_t$ stands for the restriction $i_t^*\mathcal{F}$ to the fiber over t.
In particular there is a well-defined restriction morphism on Picard groups

$$i_t^* : \mathrm{Pic}(\mathcal{X}) \to \mathrm{Pic}(\mathcal{X}_t).$$

Definition (Batyrev'97)**.** Degeneration (or a smoothing) π is *small*, if

(1) X_0 has at most Gorenstein terminal singularities
(2) and for all $t \in \Delta$ the restriction $i_t^* : \mathrm{Pic}(\mathcal{X}) \to \mathrm{Pic}(X_t)$ is an isomorphism.

All 3-dimensional terminal Gorenstein toric singularities are nodes i.e. ordinary double points analytically isomorphic to

$$(xy = zt) \subset \mathbb{A}^4 = \mathrm{Spec}\ \mathbb{C}[x, y, z, t].$$

Definition. The *index* of a (Gorenstein) Fano variety X is the highest $r > 0$, s.t. anticanonical divisor class $-K_X$ is an r-multiple of some integer Cartier divisor class H:

$$-K_X = rH.$$

Definition. Let $H \in \mathrm{Pic}(X)$ be a Cartier divisor an on n-dimensional variety X, and $D_1, \ldots, D_l$ be a base of lattice $H^{2k}(X, \mathbb{Z})/tors$. Define $d^k(X, H)$ as a discriminant of the quadratic form

$$(D_1, D_2) = (H^{n-2k} \cup D_1 \cup D_2)$$

on $H^{2k}(X, \mathbb{Z})/tors$.
For a Gorenstein threefold X denote by

$$d(X) = d^1(X, -K_X)$$

the anticanonical discriminant of X.

If X is a smooth variety and H is an ample divisor, then hard Lefschetz theorem states that $d^k(X, H)$ is nonzero.

Definition. Let X be a Fano threefold. Consider the numbers

(1)
$$\rho = \mathrm{rk}\,\mathrm{Pic}(X) = \dim H^2(X),$$
(2)
$$b = \frac{1}{2} \dim H^3(X),$$
(3)
$$\deg = (-K_X)^3,$$
(4) Fano index r
(5) and (anticanonical) discriminant d .

Numbers

$$\rho, r, \deg, b, d$$

form a set of *principal invariants* of smooth Fano 3-fold.

Theorem (3d)**.** *These and only these families of nontoric smooth Fano 3-folds Y do admit small degenerations to toric Fano threefolds:*

(1) 4 *families with* $\mathrm{Pic}(Y) = \mathbb{Z}$*:* Q, V_4, V_5,V_{22}*;*

(2) 16 *families with* $\mathrm{Pic}(Y) = \mathbb{Z}^2$*:* $V_{2.n}$*, where* $n = 12, 17$, 19, 20, 21, 22, 23, 24, 25, 26, 27, 28, 29, 30, 31, 32*;*

(3) 16 *families with* $\mathrm{Pic}(Y) = \mathbb{Z}^3$*:* $V_{3.n}$*, where* $n = 7$, 10, 11, 12, 13, 14, 15, 16, 17, 18, 19, 20, 21, 22, 23, 24*;*

(4) 8 *families with* $\mathrm{Pic}(Y) = \mathbb{Z}^4$*:* $V_{4.n}$*, where* $n = 1$, 2, 3, 4, 5, 6, 7, 8.

All these degenerations are listed below.

Here $V_{\rho,n}$ is a variety indexed $\rho.n$ in Mori-Mukai's table of varieties with Picard number ρ.

Remark. All these smooth 3-folds Y satisfy the following conditions

(1) Y is rational,
(2) $\deg(Y) \geqslant 20$,
(3) $\rho(Y) \leqslant 4$,
(4) $b(Y) \leqslant 3$,
(5) $b(Y) = 3$ only if Y is $V_{2.12}$,
(6) $b(Y) = 2$ only if Y is V_4 or $V_{2.19}$.

A sketch of the proof.

Consider a toric Fano 3-fold X with ordinary double points.

(1) There is only a finite number of such X.
All these threefolds X are explicitly classified.
(Kreuzer,Skarke 98 ; Kasprzyk 03 ; Nill 04)

(2) X admits a smoothing — a Fano threefold Y
(Friedman 82 ; Namikawa 97)

(3) Principal invariants of Y can be expressed via invariants of X.
(Clemens 83; Namikawa, Steenbrink 95 ; Jahnke, Radloff 06)

(4) Family of smooth Fano 3-folds Y is completely determined by it's principal invariants.
(Mori, Mukai 85)

(5) If some smooth Fano threefold Y admits a degeneration to a nodal toric Fano X,
then the pair (Y, X) comes from the steps above.

Y	ρ	deg	b	$[d]$	$(v,p,f)(X)$	$\#(X)$
V_{22}	1	22	0		(13,9,13)	1
V_4	1	32	2		(8,6,6)	1
V_5	1	40	0		(7,3,7)	1
Q	1	54	0		(5,1,5)	1

Y	ρ	deg	b	$[d]$	$(v,p,f)(X)$	$\#(X)$
$V_{2.12}$	2	20	3		(14,12,12)	1
$V_{2.17}$	2	24	1		(12,8,12)	1
$V_{2.19}$	2	26	2		(11,8,10)	1
$V_{2.20}$	2	26	0		(11,6,12)	2
$V_{2.21}$	2	28	0		(10,5,11)	2
$V_{2.21}$	2	28	0		(11,6,12)	1
$V_{2.23}$	2	30	1		(9,5,9)	1
$V_{2.22}$	2	30	0		(10,5,11)	1
$V_{2.22}$	2	30	0	$[-24]$	(9,4,10)	1
$V_{2.24}$	2	30	0	$[-21]$	(9,4,10)	1
$V_{2.25}$	2	32	1		(8,4,8)	1
$V_{2.25}$	2	32	1		(9,5,9)	1
$V_{2.26}$	2	34	0		(10,5,11)	1
$V_{2.26}$	2	34	0		(8,3,9)	1
$V_{2.26}$	2	34	0		(9,4,10)	1
$V_{2.27}$	2	38	0		(7,2,8)	1
$V_{2.27}$	2	38	0		(8,3,9)	2
$V_{2.28}$	2	40	1		(7,3,7)	1
$V_{2.29}$	2	40	0		(7,2,8)	1
$V_{2.29}$	2	40	0		(8,3,9)	1
$V_{2.30}$	2	46	0	$[-12]$	(6,1,7)	1
$V_{2.31}$	2	46	0	$[-13]$	(6,1,7)	1
$V_{2.31}$	2	46	0	$[-13]$	(7,2,8)	1
$V_{2.32}$	2	48	0		(6,1,7)	1
$V_{2.34}$	2	54	0		(6,1,7)	1

Y	ρ	deg	b	$[d]$	$(v,p,f)(X)$	$\#(X)$
$V_{3.7}$	3	24	1		(12,7,13)	1
$V_{3.10}$	3	26	0		(11,5,13)	1
$V_{3.11}$	3	28	1		(10,5,11)	1
$V_{3.12}$	3	28	0		(10,4,12)	1
$V_{3.12}$	3	28	0		(11,5,13)	1
$V_{3.13}$	3	30	0		(10,4,12)	2
$V_{3.13}$	3	30	0		(9,3,11)	1
$V_{3.14}$	3	32	1		(8,3,9)	1
$V_{3.15}$	3	32	0		(10,4,12)	1
$V_{3.15}$	3	32	0		(9,3,11)	3
$V_{3.16}$	3	34	0		(8,2,10)	1
$V_{3.16}$	3	34	0		(9,3,11)	1
$V_{3.17}$	3	36	0	[28]	(8,2,10)	2
$V_{3.17}$	3	36	0	[28]	(9,3,11)	1
$V_{3.18}$	3	36	0	[26]	(8,2,10)	1
$V_{3.18}$	3	36	0	[26]	(9,3,11)	1
$V_{3.19}$	3	38	0	[24]	(7,1,9)	1
$V_{3.19}$	3	38	0	[24]	(8,2,10)	1
$V_{3.20}$	3	38	0	[28]	(7,1,9)	1
$V_{3.20}$	3	38	0	[28]	(8,2,10)	1
$V_{3.20}$	3	38	0	[28]	(9,3,11)	1
$V_{3.21}$	3	38	0	[22]	(8,2,10)	1
$V_{3.22}$	3	40	0		(7,1,9)	1
$V_{3.23}$	3	42	0	[20]	(7,1,9)	1
$V_{3.23}$	3	42	0	[20]	(8,2,10)	1
$V_{3.24}$	3	42	0	[22]	(7,1,9)	1
$V_{3.24}$	3	42	0	[22]	(8,2,10)	1
$V_{3.25}$	3	44	0		(7,1,9)	1
$V_{3.26}$	3	46	0		(7,1,9)	1
$V_{3.28}$	3	48	0		(7,1,9)	1

Y	ρ	deg	b	$[d]$	$(v,p,f)(X)$	$\#(X)$
$V_{4.1}$	4	24	1		(12,6,14)	1
$V_{4.2}$	4	28	1		(10,4,12)	1
$V_{4.3}$	4	30	0		(10,3,13)	1
$V_{4.4}$	4	32	0	$[-40]$	(9,2,12)	1
$V_{4.5}$	4	32	0	$[-39]$	(9,2,12)	1
$V_{4.6}$	4	34	0		(10,3,13)	1
$V_{4.6}$	4	34	0		(9,2,12)	1
$V_{4.7}$	4	36	0		(8,1,11)	2
$V_{4.7}$	4	36	0		(9,2,12)	1
$V_{4.8}$	4	38	0		(8,1,11)	1
$V_{4.9}$	4	40	0		(8,1,11)	1

Quantum minimality and minimal ramification.

Definition. X is *minimal* if

$$H^{2k}(X, \mathbb{Z}) = \mathbb{Z}$$

or equivalently

$$\dim H^{even}(X, \mathbb{C}) = \dim X + 1.$$

Motivating example.

Assume for a moment:

- X is minimal, $H^{p,q}(X) = 0$ if $p \neq q$,
- $\mathcal{D}^b(X)$ admits a full exceptional collection E_i,
- singular points of w are ordinary.

Critical points of w correspond to the (Lagrangian) vanishing cycles L_i,
and by HMS L_i correspond to E_i.

$$\#(L_i) = \#(Sing(w)) = \#(E_i) = \operatorname{rk} K_0(X) = \dim X + 1$$
$$\implies \#(w(Sing(w)) \leqslant \dim X + 1$$

Let $can : \mathbb{C}[k] \to QH(X)$ send k to K_X, consider the subring $R = Im(can)$ of QH generated by K_X.

Definition. X is *quantum minimal* if

$$\dim R = \dim X + 1.$$

It is enough to ask for

$$\dim R \leqslant \dim X + 1.$$

In general the symbol of regularized quantum differential equation is the characteristic polynomial of quantum multiplication by canonical class. Quantum minimality is essentialy (up to aparent singularities) equivalent to the low ramification of Landau–Ginzburg.

Golyshev's low ramified local systems Let $\mathcal{L}$ be a local system over $\mathbb{P}^1$, and for $x \in \mathbb{P}^1$ let I_x be the group of local monodromy. Euler characteristic of $\mathcal{L}$ is summed up from local contributions:

$$\chi(\mathcal{L}) = \sum_{x \in \mathbb{P}^1} \dim(\mathcal{L}_x / \mathcal{L}_x^{I_x})$$

Definition. Local system $\mathcal{L}$ is called minimally (or low) ramified if

$$\chi(\mathcal{L}) = 2 \operatorname{rk} \mathcal{L}$$

In similar way one defines the low-ramified D-modules.

The main example of objects we are interested in are local systems $R^{\dim X - 1} w_* \mathbb{Z}$ (or $w_* \mathbb{Q}_l$) for LG-models w and quantum D-modules.

Proposition. *Constructed in that way local system mirror symmetric to minimal Fano varieties (with full exceptional collection) is low-ramified.*

There are $\dim X + 1$ conifold points over $\mathbb{A}^1$ that contribute 1's to $\chi(\mathcal{L})$ and the monodromy over ∞ is maximal unipotent, so contributes $\dim X - 1$ to $\chi(\mathcal{L})$.

Proposition. *The same is true for quantum minimal varieties.*

Hypothetically the only obstruction to being low-ramified is the existence of some "parasitic" algebraic cycles on X in the intermediate dimension.

Golyshev's programme
Search for pencils with minimal ramification.
In toric framework:

(1) Fix a polytope Δ
(2) Consider the space $L(\Delta) = Newton^{-1}(\Delta)$
(3) Stratify $L(\Delta)$ by the value of $\chi(\mathcal{L}_w) - 2 \operatorname{rk} \mathcal{L}_w$ or thiner by the gluings of singular points
(4) Aritnian up to isomorphisms strata is LG for quantum minimal (and other ot too bad) Fano varieties.

Evidences: HMS + Dubrovin's conjecture for minimal varieties, surfaces (theorem 2d),
Fano threefolds of first series (Przyjalkowski table).

Definition. A pair of a Fano variety X equipped with finite group action $G : X$ is *G-minimal* if

$$\dim H^{even}(X, \mathbb{C})^G = \dim X + 1,$$

i.e. even part of G-invariant cohomologies of X is generated by K_X.

Theorem (G-minimality implies quantum minimality)**.** *Let X be a Fano variety admitting some action of group G such that X is G-minimal. Then X is quantum minimal.*

Reason — quantum multiplication respects the group action:

Lemma. *Let X be a Fano variety with the action of the finite cyclic group G,*
χ_1, χ_2 — a pair of characters of G,
and

$$\gamma_i \in H^\bullet(X, \mathbb{C})^{\chi_i}, i = 1, 2$$

be a pair of G-eigenvector cohomology classes with characters χ_1, χ_2:

$$g\gamma_i = \chi_i(g)\gamma_i \text{ for } g \in G, i = 1, 2.$$

Then

$$\gamma_1 \star \gamma_2 \in H^\bullet(X, \mathbb{C})^{(\chi_1\chi_2)}[[q]].$$

Proof of the lemma. Gromov–Witten are well defined and are indeed invariant with respect to the isomorphims, so one has

$$\langle g^*\gamma_1, ..., g^*\gamma_n\rangle_\beta = \langle\gamma_1, \ldots, \gamma_n\rangle_{g_*\beta}$$

for any classes $\beta \in H_2(X)$ and $\gamma_i \in H^\bullet(X)$.

The canonical class K_X is G-invariant, so the action of G preserves anticanonical degrees of the curves in X: $(-K_X \cdot \beta) = (-K_X \cdot g_*\beta)$.

This implies the G-invariance of the correlators: $\langle g\gamma_1, ..., g\gamma_n\rangle_d = \langle\gamma_1, \ldots, \gamma_n\rangle_d$ for any n, d and $g \in G$.

Choose a basis of $H^\bullet(X, \mathbb{C})$ consisting of G-eigenvectors. Let $(\cdot, \cdot) = \langle\cdot, \cdot\rangle_0$ be the Poincare pairing. For a pair of eigenvectors α_1, α_2 with characters χ_1, χ_2 the pairing (α_1, α_2) is nonzero only if $\chi_1\chi_2 = 1$, so we need to show $(\gamma_1 \star \gamma_2, \gamma_3)$ is zero for any eigenvector γ_3 with any χ_3 different from $(\chi_1\chi_2)^{-1}$. By definition $(\gamma_1 \star \gamma_2, \gamma_3) = \sum_{d \geqslant 0} q^d \langle\gamma_1, \gamma_2, \gamma_3\rangle_d$, so the vanishing $(\gamma_1 \star \gamma_2, \gamma_3)$ is equivalent to the vanishing of all corelators $C_d = \langle\gamma_1, \gamma_2, \gamma_3\rangle_d$. But $C_d = \langle\gamma_1, \gamma_2, \gamma_3\rangle_d = \langle g\gamma_1, g\gamma_2, g\gamma_3\rangle = \langle\chi_1(g)\gamma_1, \chi_2(g)\gamma_2, \chi_3(g)\gamma_3\rangle = (\chi_1\chi_2\chi_3)(g) \langle\gamma_1, \gamma_2, \gamma_3\rangle_d = (\chi_1\chi_2\chi_3)(g)C_d$, so if $(\chi_1\chi_2\chi_3)(g) \neq 1$ for some g, then $C_d = 0$. □

Proof of the theorem. By lemma the subring R of quantum cohomology generated by $-K_X$ is contained inside $H^\bullet(X, \mathbb{C})[q]$. By the proposition of the theorem the dimension of $H^\bullet(X, \mathbb{C})(q)$ over $\mathbb{C}(q)$ is $\dim X + 1$. This implies dimension of $R \otimes \mathbb{C}(q)$ over $\mathbb{C}(q)$ is $\leqslant \dim X + 1$. □

Remark. There are two frameworks for quantum cohomology — symplectic and algebraic. One may notice neither of these definitions were used in the proof. Geometrical part is hidden behind the equality of correlators and the fact that correlators are invariant with respect to algebraic or symplectic isomorphisms.

Moreover, one can even apply the theorem in the case of non-geometric action of the Galois group (or mixed geometric and Galois action) on variety X and it's cohomologies (e.g. $H_{et}(X, \mathbb{Q}_l)$) if X is defined over $\mathbb{Q}$ (or over some number field). This is true since everything is defined over the base field of X: $M_{g,n}(X, \beta)$, evaluation map $ev : M_{g,n}(X, \beta) \to X^n$, ψ-classes and the virtual fundamental class.

For example in case of del Pezzo surfaces S_1 of degree 1 the whole Weyl group E_8 (automorphisms of $\mathrm{Pic}\, S_1$) may be realized as Galois group (Varilly-Alvarado, Zywina 08), but it is known the whole $E8$ may never be realized as a group of biholomorphic automorphisms.

Example. Let X be a del Pezzo surface different from blowup of a plane in 1 or 2 points. It is a classical result that such a surface admits some G-minimal action for some moduli (e.g. Dolgachev and Iskovskikh provide the description of all the possible minimal rational G-surfaces). Hence del Pezzo surfaces except of S_7 and S_8 are quantum minimal, and give rise to 6 Zagier's equations of type $D2$.

Theorem (2d)**.** *In the following table we list (du Val) weak Landau-Ginzburg models for quantum minimal del Pezzo surfaces of degree $\geqslant 3$. All LG-models constructed from different degenerations of the same del Pezzo surface are related by cluster type fiberwise birational symplectomorphisms. The associated local systems are low ramified.*

$12-d$.number	k	Laurent polynomial f	j-invariant
3.1	1	$x+y+x^{-1}y^{-1}$	j_3
4.1	1	$x+y+x^{-1}+y^{-1}$	j_4
4.3	1	$xy+2x+xy^{-1}+x^{-1}$	j_4
6.1	1	$3+xy+2y+x^{-1}y+3x^{-1}+3x^{-1}y^{-1}+x^{-1}y^{-2}$	j_6
6.2	1	$3+xy+2x+2y+xy^{-1}+yx^{-1}+x^{-1}$	j_6
6.3	1	$3+xy+2y+yx^{-1}+x+x^{-1}+y^{-1}$	j_6
6.4	1	$3+x+y+x^{-1}+y^{-1}+xy+x^{-1}y^{-1}$	j_6
7.1	1	$3+xy+2y+x^{-1}y+3x^{-1}+3x^{-1}y^{-1}+y^{-1}+x^{-1}y^{-2}$	j_7
7.2	1	$3+xy+2x+2y+xy^{-1}+yx^{-1}+x^{-1}+y^{-1}$	j_7
8.1	2	$y(x^{-2}+2+x^2)+y^{-1}$	j_8
8.1b	1	$y(x^{-2}+4x^{-1}+6+4x+x^2)+2(x^{-1}+3+x)+y^{-1}$	j_{8b}
8.2	2	$x^2y^{-1}+3x-2xy^{-1}+3y+y^{-1}+y^2x^{-1}+2yx^{-1}+x^{-1}$	j_8
8.3	2	$(x+x^{-1})(y+y^{-1})$	j_8
8.3b	1	$(x+2+x^{-1})(y+2+y^{-1})$	j_{8b}
9.1	3	$x^2y^{-1}+x^{-1}y^2+x^{-1}y^{-1}$	j_9
9.1b	1	$6+3(x+y+xy+x^{-1}+y^{-1}+x^{-1}y^{-1})+x^2y^{-1}+x^{-1}y^2+x^{-1}y^{-1}$	j_{9b}

$j_{8b}=\frac{t^2-16t+16)^3}{t(t-16)}$ $j_{9b}=\frac{t(t-24)^3}{t-27}$

Hori-Vafa models.

d	equation ($t=\frac{w-const}{t_0}$)	singularities
$d=4, S_4=X_{2,2}\subset\mathbb{P}^4$	$1-t\cdot xy(1-x)(1-y)=0$	$I_1^*I_4I_1$
$d=3, S_3=X_3\subset\mathbb{P}^3$	$1-t\cdot xy(1-x-y)=0$	$IV^*I_3I_1$
$d=2, S_2=X_4\subset\mathbb{P}(1112)$	$1-t\cdot xy^2(1-x-y)=0$	$III^*I_2I_1$
$d=1, S_1=X_6\subset\mathbb{P}(1123)$	$1-t\cdot x^2y^3(1-x-y)-t=0$	$II^*I_1I_1$

Fake minimal Fano threefolds

Let t be a coordinate on G_m and $D = t\frac{d}{dt}$.

Example (example of degree 28)**.** Consider the differential operator

$$L_{28} = D^3 - tD(D+1)(2D+1) - t^2(D+1)(59(D+1)^2+5) - \\ - 68t^3(2D+3)(D+2)(D+1) - 80t^4(D+3)(D+2)(D+1) \tag{0.1}$$

The solution of this equation in coordinate q is given by

$$\eta(q)\eta(q^2)\eta(q^7)\eta(q^{14}).$$

In coordinate t after the shift of first term to zero it is

$$I_{28}(t) = 1 + 8t^2 + 24t^3 + 240t^4 + 1440t^5 + 11960t^6 + 89040t^7 + \dots$$

Consider Laurent polynomials

$$f_{28}^{(1)} = x + y + z + \frac{1}{x} + \frac{1}{y} + \frac{1}{z} + \frac{y}{z} + \frac{z}{y} + xy + xz + xyz \tag{0.2}$$

$$f_{28}^{(2)} = x + y + z + \frac{1}{x} + \frac{1}{y} + \frac{1}{z} + xyz + \frac{1}{xyz} + xz + \frac{1}{yz} \tag{0.3}$$

$$f_{28}^{(3)} = x + y + z + \frac{1}{x} + \frac{1}{y} + \frac{1}{z} + xy + \frac{1}{xy} + xz + \frac{1}{yz} \tag{0.4}$$

Then $\Phi_{f_{28}^{(1)}}(t) = \Phi_{f_{28}^{(2)}}(t) = \Phi_{f_{28}^{(3)}}(t)$ and up to Givental's constant are equal to $I_{Y_{28}}(t)$.

$f_{28}^{(1)}$ is G-minimal with respect to the $\mathbb{Z}/2\mathbb{Z}$-action $(x,y,z) \to (x,z,y)$.

Here Y_{28} is smoothing of toric Fano varieties with fan polytopes given by Newton polytopes of f_{28}'s. Explicitly $Y_{28} = V_{2.21}$ is a blowup of a twisted quartic on a quadric.

Experimentally quantum minimality seems to be closely related with the existence of Kahler-Einstein metrics.

Example (example of degree 30)**.** Consider the differential operator

$$L_{30} = D^3 - tD(D+1)(2D+1) - t^2(D+1)(43(D+1)^2+5) - \\ - 78t^3(2D+3)(D+2)(D+1) - 216t^4(D+3)(D+2)(D+1) \tag{0.5}$$

The solution of this equation in coordinate q is given by

$$\eta(q)\eta(q^3)\eta(q^5)\eta(q^{15})$$

. In coordinate t after the shift of first term to zero it is

$$1 + 6t^2 + 24t^3 + 162t^4 + 1080t^5 + 7620t^6 + 55440t^7 + \ldots$$

Consider Laurent polynomials

$$f_{30}^{(1)} = x + y + z + \frac{1}{x} + \frac{1}{y} + \frac{1}{z} + \frac{x}{y} + \frac{y}{z} + \frac{z}{x} \tag{0.6}$$

$$f_{30}^{(2)} = x + y + z + \frac{1}{x} + \frac{1}{y} + \frac{1}{z} + xy + xz + \frac{1}{yz} + xyz \tag{0.7}$$

$$f_{30}^{(3)} = x + y + \frac{1}{x} + \frac{1}{y} + \frac{1}{z} + \frac{x}{y} + \frac{y}{x} + xy + xz + xyz \tag{0.8}$$

$f_{30}^{(1)}$ is G-minimal with respect to the $\mathbb{Z}/3\mathbb{Z}$-action $(x,y,z) \to (y,z,x)$.
Then $\Phi_{f_{30}^{(1)}}(t) = \Phi_{f_{30}^{(2)}}(t) = \Phi_{f_{30}^{(3)}}(t)$ and up to Givental's constant are equal to $I_{Y_{30}}(t)$.

Here Y_{30} is smoothing of toric Fano varieties with fan polytopes given by Newton polytopes of f_{30}'s. Explicitly $Y_{30} = V_{3.13}$ is a blowup of a a curve of bidegree $(2,2)$ on $W = \mathbb{P}(T_{\mathbb{P}^2})$.

Batyrev's ansatz + G-minimality of degeneration
$\Longrightarrow$ quantum minimality of smoothing
QH is Jacobian ring of w

Apery class

Let X be a Fano variety of index r: $-K_X = rH$, and q be a coordinate on the torus $\mathbb{Z} - K_X \otimes \mathbb{C}^* = G_m \in \mathrm{Pic}(X) \otimes \mathbb{C}^*$, and $D = q\frac{d}{dq}$ be an invariant vector field. Cohomologies $H^\bullet(X)$ are endowed with the structure of quantum multiplication and associativity of $\star$ implies that first Dubrovin's connection given by

$$D\phi = H \star \phi$$

is flat.

Constant terms of holomorphic solutions are coprimitive classes γ:

$$H \cup \gamma = 0$$

We say that solution is associated with Poincare-dual primitive class.

Givental: the solution

$$A = 1 + \sum_{n \geqslant 1} a^{(n)} q^n$$

associated with $1 \in H^0(X)$ is the I-series of X.

For coprimitive class γ consider the solution

$$A_\gamma = \sum_{n \geqslant 1} a_\gamma^{(n)} q^n = Pr_0(\gamma + \sum_{n \geqslant 1} A_\gamma^{(n)} q^n)$$

and the limit

$$Apery(\gamma) = \lim_{n \to \infty} \frac{a_\gamma^{(n)}}{a^{(n)}}$$

Apery is a linear map from coprimitive cohomologies to $\mathbb{C}$. A linear map on coprimitive cohomologies is dual to (nonhomogeneous) primitive cohomology class with coefficients in $\mathbb{C}$. We name it *Apery characteristic class* $A(X) \in H^{\leqslant \dim X}(X, \mathbb{C})$.

Consider the homogeneous ring $R = \mathbb{Q}[c_1, c_2, c_3, \dots], \deg c_i = i$ and a map $ev : R \to \mathbb{C}$ sending c_1 to Euler constant C [1], and c_i to $\zeta(i)$.

Conjecture 0.9. *Let X be any Fano variety and $\gamma \in H^\bullet(X)$ be some coprimitive with respect to $-K_X$ homogeneous cohomology class of codimension n. Consider two solutions of quantum D-module: A_0 associated with 1 and A_γ associated with γ. Then Apery number for A_γ (i.e. $\lim_{k\to\infty} \frac{a_\gamma^{(k)}}{a_0^{(k)}}$) is equal to $ev(f_\gamma)$ for some homogeneous polynomial $f_\gamma \in R^{(n)}$ of degree n.*

For toric varieties X the solutions of QDE are known to be pullbacks of hypergeometric functions, coefficients of hypergeometric functions are rational functions of Γ-values, and the Taylor expansion

$$\log \Gamma(1+x) = Cx + \sum_{k \geqslant 2} \frac{\zeta(k)}{k} x^k \tag{0.10}$$

suggests all Apery constants would probably be rational functions in C and $\zeta(k)$.

So toric degenerations (or hypergeometric pullback conjecture) $\Longrightarrow$ the conjecture.

[1] $C = \lim_{n\to\infty}(\sum_{k=1}^{n} \frac{1}{k}) - \ln n$

X	μ	p_2	p_4	p_6	p_8
$Gr(2,4)$	2	0			
$Gr(2,5)$	2	$\zeta(2)$			
$Gr(2,6)$	3	$2\zeta(2)$	0		
$Gr(2,7)$	3	$3\zeta(2)$	$\frac{27}{4}\zeta(4)$		
$Gr(2,8)$	4	$4\zeta(2)$	$16\zeta(4)$	0	
$Gr(2,9)$	4	$5\zeta(2)$	$\frac{111}{4}\zeta(4)$	$\frac{675}{16}\zeta(6)$	
$Gr(2,10)$	5	$6\zeta(2)$	$42\zeta(4)$	$108\zeta(6)$	0
$Gr(2,11)$	5	$7\zeta(2)$	$\frac{235}{4}\zeta(4)$	$\frac{3229}{16}\zeta(6)$	$\frac{18375}{64}\zeta(8)$
$Gr(2,12)$	6	$8\zeta(2)$	$78\zeta(4)$	$328\zeta(6)$	$768\zeta(8)$,
$Gr(2,13)$	6	$9\zeta(2)$	$\frac{399}{4}\zeta(4)$	$\frac{7855}{16}\zeta(6)$	$\frac{96111}{64}\zeta(8)$,
$Gr(2,14)$	7	$10\zeta(2)$	$124\zeta(4)$	$695\zeta(6)$	$\frac{7664}{3}\zeta(8)$,
$Gr(2,15)$	7	$11\zeta(2)$	$\frac{603}{4}\zeta(4)$	$\frac{15113}{16}\zeta(6)$	$\frac{768085}{192}\zeta(8)$,

The proof for the computation of p_2 (in slightly another $\mathbb{Q}$-basis) was given recently by Golyshev. Let us describe a transparent generalization of this method for the all primitive p_{2k} of $Gr(2, N)$. Quantum D-module for $Gr(r, N)$ is the r'th wedge power of quantum D-module for $\mathbb{P}^{N-1}$ (solutions of QDE for $Gr(r, N)$ are $r \times r$ wronskians of the fundamental matrix of solutions for $\mathbb{P}^{N-1}$). Let N be either $2n$ or $2n+1$. Consider the deformation of quantum differential equation for $\mathbb{P}^{N-1}$:

$$(D-u_1)(D+u_1)(D-u_2)(D+u_2)\cdot\dots\cdot(D-u_n)(D+u_n)\cdot D^{N-2n}-q$$

This equation has (at least) $2n$ formal solutions:

$$R_a = \sum_{k-a\in\mathbb{Z}_+} \frac{1}{\Gamma(k-u_1)\Gamma(k+u_1)\cdot\dots\cdot\Gamma(k-u_n)\Gamma(k+u_n\cdot\Gamma(k)^{N-2n}} q^k$$

for $a = u_1, -u_1, \dots, u_n, -u_n$. Let $S_i = R'_{u_i}R_{-u_i} - R'_{-u_i}R_{u_i}$ be the wronskians. Then $S_i = \sum_{k\geqslant 0} s_i^{(k)} q^k$ for $i = 1, \dots, n$ are n holomorphic solutions of the wedge square of the deformed equation . Using his explicit calculation for the monodromy of hypergeometric equation and Dubrovin's theory, Golyshev computes the monodromy of $\wedge^2()$ and demonstrates *the formula of sinuses*:

$$\lim_{k\to\infty} \frac{s_i^{(k)}}{s_j^{(k)}} = \frac{\sin(2\pi u_i)}{\sin(2\pi u_j)}$$

So in the base of $S_1, \dots, S_n$ Apery numbers are $\frac{\sin(2\pi u_i)}{\sin(2\pi u_1)}$. One then reconstructs the required Apery numbers by applying the inverse fundamental solutions matrix to this vector of sinuses, and limiting all u_i to 0.

$Gr(3,N)$

N	μ	$\frac{p_2}{\zeta(2)}$	$\frac{p_3}{\zeta(3)}$	$\frac{p_4}{\zeta(4)}$	p_5	$p_{\geqslant 6}$
6	3	0	-6			
7	4	1	-7	$-\frac{17}{4}$		$-\frac{49}{2}\zeta(3)^2 - \frac{945}{16}\zeta(6)$
8	5	2	-8	0	$-8\zeta(2)\zeta(3) - 4\zeta(5)$	$-32\zeta(3)^2 - 62\zeta(6)$
9	8	3	-9	$\frac{27}{4}$	$-\frac{27}{2}\zeta(2)\zeta(3) - \frac{9}{2}\zeta(5)$	$\pm(\frac{81}{2}\zeta(3)^2 + \frac{871}{16}\zeta(6)), \dots$
10	10	4	-10	16	$-20\zeta(2)\zeta(3) - 5\zeta(5)$	$\pm(50\zeta(3)^2 + 32\zeta(6)), \dots$
11	13	5	-11	$\frac{111}{4}$	$-\frac{55}{2}\zeta(2)\zeta(3) - \frac{11}{2}\zeta(5)$	$(-\frac{121}{2}\zeta(3)^2 + \frac{110}{16}\zeta(6)) \pm \frac{45}{16}\zeta(6), \dots$

$Gr(4,N)$

N	μ	p_2	p_3	p_4	p_4'	p_5	$p_{\geqslant 6}$
8	8	0	$-8\zeta(3)$	$-6\zeta(4)$	0	none	$32\zeta(3)^2 + 50\zeta(6)$ twice and 0_8
9	12	$\zeta(2)$	$-9\zeta(3)$	$\frac{21}{4}\zeta(4)$	$\zeta(4)$	$-\frac{9}{2}(\zeta(2)\zeta(3) + \zeta(5))$	$(\frac{81}{2}\zeta(3)^2 + \frac{117}{4}\zeta(6)) \pm \frac{159}{16}\zeta(6), \dots$
10	18	$2\zeta(2)$	$-10\zeta(3)$	$-2\zeta(4)$	$2\zeta(4)$	$-10\zeta(2)\zeta(3) - 5\zeta(5)$	$50\zeta(3)^2 + 31\zeta(6)$, $50\zeta(3)^2$, $0_6, \dots$
11	24	$3\zeta(2)$	$-11\zeta(3)$	$\frac{15}{4}\zeta(4)$	$3\zeta(4)$	$-\frac{33}{2}\zeta(2)\zeta(3) - \frac{11}{2}\zeta(5)$	$(\frac{121}{2}\zeta(3)^2 + \frac{35}{2}\zeta(6)) \pm \frac{197}{16}\zeta(6), \dots$

$Gr(5,N)$

N	μ	p_2	p_3	p_4	p_4'	p_5	p_5'
10	20	0	$-10\zeta(3)$	$-6\zeta(4)$	0	$10\zeta(5)$	$-10\zeta(5)$
11	32	$\zeta(2)$	$-11\zeta(3)$	$-\frac{21}{4}\zeta(4)$	$\zeta(4)$	$11(\zeta(5) - \zeta(2)\zeta(3))$	$-11\zeta(5)$

X	μ	Apery numbers
$B(3,2)$	2	$-2\zeta(2)$.
$B(4,2)$	3	$\zeta(2)$, $-\frac{41}{2}\zeta(4)$.
$B(4,3)$	3	$-4\zeta(2)$, $-4\zeta(3)$.
$B(4,4)$	2	$2\zeta(3)$.
$B(5,2)$	4	$3\zeta(2)$, $\frac{3}{2}\zeta(4) - \frac{1191}{8}\zeta(6)$.
$B(5,3)$	8	0_2, $-8\zeta(3)$, $-24\zeta(4)$, $20\zeta(5)$, $\frac{64}{3}\zeta(3)^2 + \frac{80}{3}\zeta(6)$, $32\zeta(3)\zeta(4) + \frac{232}{3}\zeta(7)$, $\frac{256}{21}\zeta(3)^3 + \frac{320}{7}\zeta(3)\zeta(6) - \frac{480}{7}\zeta(4)\zeta(5) - \frac{1000}{21}\zeta(9)$.
$B(5,4)$	8	$-6\zeta(2)$, $-6\zeta(3)$, $-45\zeta(4)$, $9\zeta(2)\zeta(3)+21\zeta(5)$, $15\zeta(3)^2 + \frac{1141}{24}\zeta(6)$, $56\zeta(2)\zeta(5) + 30\zeta(3)\zeta(4) + 52\zeta(7)$, $\frac{266}{5}\zeta(3)^3 - \frac{171}{5}\zeta(2)\zeta(7) - \frac{222}{5}\zeta(3)\zeta(6) - \frac{263}{5}\zeta(4)\zeta(5) + \frac{136}{5}\zeta(9)$.
$B(5,5)$	3	$4\zeta(3)$, $20\zeta(5)$.
$B(6,2)$	5	$5\zeta(2)$, $\frac{87}{4}\zeta(4)$, $-\frac{485}{8}\zeta(6)$, $-\frac{35073}{32}\zeta(8)$.
$B(6,3)$	12	$2\zeta(2)$, $-6\zeta(3)$, $-12\zeta(4)$, $-12\zeta(2)\zeta(3) + 18\zeta(5)$, $-36\zeta(3)^2 - 146\zeta(6)$, $36\zeta(3)^2 + 2\zeta(6)$, $24\zeta(2)\zeta(5) + 24\zeta(3)\zeta(4) + 76\zeta(7)$, $\frac{360\zeta(3)^2\zeta(2)-1080\zeta(3)\zeta(5)+1176\zeta(8)}{11}$, $803\zeta(3)^3 - 528\zeta(2)\zeta(7) + 318\zeta(3)\zeta(6) - 244\zeta(4)\zeta(5) - 35\zeta(9)$, $75\zeta(3)^3 - 336\zeta(2)\zeta(7) - 395\zeta(3)\zeta(6) - 22\zeta(4)\zeta(5) - 70\zeta(9)$,...
$B(6,4)$	18	$-1\zeta(2)$, $-10\zeta(3)$, $-\frac{17}{4}\zeta(4)$, $-14\zeta(4)$, $5\zeta(2)\zeta(3)+19\zeta(5)$, $50\zeta(3)^2 + 317\zeta(6)$, $-50\zeta(3)^2 - \frac{4135}{8}\zeta(6)$,
$B(6,5)$	14	$-8\zeta(2)$, $-8\zeta(3)$, $-84\zeta(4)$, $64\zeta(2)\zeta(3) + 16\zeta(5)$, $-64\zeta(2)\zeta(3)$, $\frac{80}{3}\zeta(3)^2 + 24\zeta(6)$, $110\zeta(2)\zeta(5) + \frac{49}{2}\zeta(3)\zeta(4) + \frac{101}{2}\zeta(7)$,
$B(6,6)$	5	$6\zeta(3)$, $18\zeta(5)$, $-18\zeta(3)^2-60\zeta(6)$, $36\zeta(3)^3+360\zeta(3)\zeta(6)+332\zeta(9)$
$B(7,2)$	6	$7\zeta(2)$, $\frac{211}{4}\zeta(4)$, $\frac{1733}{8}\zeta(6)$, $-\frac{76699}{96}\zeta(8)$, $-\frac{5368203}{640}\zeta(10)$.
$B(7,7)$	8	$8\zeta(3)$, $16\zeta(5)$, $-30\zeta(3)^2 - 60\zeta(6)$, $-112\zeta(7)$, $\frac{256}{3}\zeta(3)^3 + 480\zeta(3)\zeta(6) + \frac{992}{3}\zeta(9)$, ...

X	μ	Apery numbers
$C(3,2)$	2	$2\zeta(2)$.
$C(3,3)$	2	$\frac{7}{2}\zeta(3)$.
$C(4,2)$	3	$4\zeta(2)$, $16\zeta(4)$.
$C(4,3)$	4	$\zeta(2)$, $-9\zeta(3)$, $-\frac{9}{2}(\zeta(2)\zeta(3)+\zeta(5))$.
$C(4,4)$	2	$4\zeta(3)$.
$C(5,2)$	4	$6\zeta(2)$, $42\zeta(4)$, $108\zeta(6)$.
$C(5,3)$	8	$3\zeta(2)$, $-11\zeta(3)$, $\frac{27}{4}\zeta(4)$, $-\frac{33}{2}\zeta(2)\zeta(3)-\frac{11}{2}\zeta(5)$, $\frac{242}{3}\zeta(3)^2+\frac{2383}{48}\zeta(6)$, $-11\zeta(2)\zeta(5)-\frac{99}{4}\zeta(3)\zeta(4)-\frac{11}{3}\zeta(7)$, $108\zeta(3)^3-38\zeta(2)\zeta(7)+\frac{309}{4}\zeta(3)\zeta(6)-\frac{41}{4}\zeta(4)\zeta(5)+36\zeta(9)$
$C(5,4)$	8	0_2, $-10\zeta(3)$, $30\zeta(4)$, $-5\zeta(5)$, $\frac{250}{3}\zeta(3)^2+\frac{175}{3}\zeta(6)$, $-\frac{100}{3}\zeta(3)\zeta(4)-\frac{10}{9}\zeta(7)$, $\frac{2500}{21}\zeta(3)^3+250\zeta(3)\zeta(6)-\frac{150}{7}\zeta(4)\zeta(5)-\frac{10}{21}\zeta(9)$.
$C(5,5)$	3	$\frac{9}{2}\zeta(3)$, $-\frac{21}{2}\zeta(5)$.
$C(6,2)$	5	$8\zeta(2)$, $78\zeta(4)$, $328\zeta(6)$, $768\zeta(8)$.
$C(6,3)$	12	$5\zeta(2)$, $-13\zeta(3)$, $\frac{111}{4}\zeta(4)$, $-\frac{65}{2}\zeta(2)\zeta(3)-\frac{13}{2}\zeta(5)$, $-\frac{169}{2}\zeta(3)^2+\frac{155}{16}\zeta(6)$, $\frac{169}{2}\zeta(3)^2+\frac{65}{2}\zeta(6)$, $\frac{169}{2}\zeta(3)^2+\frac{65}{2}\zeta(6)$,
$C(6,6)$	4	$\zeta(3)$, $-11\zeta(5)$, $-25\zeta(3)^2-\frac{15}{2}\zeta(6)$, $\frac{500}{3}\zeta(3)^3+150\zeta(3)\zeta(6)-\frac{131}{3}\zeta(9)$.
$C(7,2)$	6	$10\zeta(2)$, $124\zeta(4)$, $695\zeta(6)$, $\frac{7664}{3}\zeta(8)$, $5760\zeta(10)$.
$C(7,7)$	8	$\frac{11}{2}\zeta(3)$, $-\frac{23}{2}\zeta(5)$, $-\frac{121}{4}\zeta(3)^2-\frac{15}{2}\zeta(6)$, $\frac{71}{2}\zeta(7)$, $\frac{1331}{6}\zeta(3)^3+165\zeta(3)\zeta(6)-\frac{263}{6}\zeta(9)$, $\frac{781}{12}\zeta(3)\zeta(7)-\frac{529}{12}\zeta(5)^2-\frac{63}{2}\zeta(10)$,...

$C(n,2) \simeq Gr(2,2n)$ by quantum Lefschetz.

X	μ	Apery numbers
$D(4,2)$	4	$0, 0, -24\zeta(4)$.
$D(5,2)$	5	$2\zeta(2)$, 0, $-12\zeta(4)$, $-144\zeta(6)$.
$D(5,3)$	9	$-\zeta(2)$, $-\zeta(2)$, $-6\zeta(3)$, 0_4, $-\frac{45}{2}\zeta(4)$, $3\zeta(2)\zeta(3)+21\zeta(5)$, 0_5, $12\zeta(3)^2+\frac{275}{24}\zeta(6)$.
$D(5,4)$	2	$2\zeta(3)$.
$D(6,2)$	6	$4\zeta(2)$, $10\zeta(4)$, $10\zeta(4)$, $-124\zeta(6)$, $-960\zeta(8)$.
$D(6,3)$	14	$\zeta(2)$, $-5\zeta(3)$, $-5\zeta(3)$, $-\frac{41}{2}\zeta(4)$, 0, $-5\zeta(2)\zeta(3)+19\zeta(5)$, $\frac{25}{2}\zeta(3)^2+\frac{953}{16}\zeta(6)$, $\frac{25}{2}\zeta(3)^2-\frac{937}{16}\zeta(6)$, 0,
$D(6,5)$	3	$4\zeta(3)$, $20\zeta(5)$.
$D(7,2)$	7	$6\zeta(2)$, $36\zeta(4)$, 0, $50\zeta(6)$, $-1072\zeta(8)$, $-6912\zeta(10)$.
$D(7,6)$	5	$6\zeta(3)$, $18\zeta(5)$, $-18\zeta(3)^2-60\zeta(6)$, $36\zeta(3)^3+360\zeta(3)\zeta(6)+332\zeta(9)$.

X	μ	Apery numbers
$E(6,6)$	3	$6\zeta(4)$, 0_8.
$E(6,2)$	6	0_3, $18\zeta(4)$, $90\zeta(6)$, 0_7, $-3456\zeta(10)$.
$E(7,7)$	3	$-24\zeta(5)$, $168\zeta(9)$.
$E(8,8)$	11	$120\zeta(6)$, $-1512\zeta(10)$, $\ldots$ (of degrees 12, 16, 18, 22, 28).
$F(4,1)$	2	$21\zeta(4)$.
$F(4,3)$	8	$-4\zeta(2)$, 0_3, $-2\zeta(4)$, $-24\zeta(5)$, $-246\zeta(6)$, $32\zeta(2)\zeta(5)+60\zeta(7)$, $2160\zeta(2)\zeta(7)-144\zeta(4)\zeta(5)$.
$F(4,4)$	2	$6\zeta(4)$.

One may consider the same question for varieties X with higher Picard group. Canonicaly we should put $H = -K_X$, but if we like, we could choose any $H \in \mathrm{Pic}(X)$.

Even for such simple spaces as products of projective spaces one immidiately calculates some non-trivial Apery constants.

X	μ	Apery numbers
$\mathbb{P}^2 \times \mathbb{P}^2$	3	$0_1,\ 6\zeta(2)$.
$\mathbb{P}^2 \times \mathbb{P}^3$	3	$0_1,\ \frac{14}{3}\zeta(2)$.

In all these cases Apery numbers corresponding to all primitive divisors vanish. Van Straten's calculation for Fano fourfolds: monodromies of QDE are related to to Chern numbers of it's anticanonical Calabi-Yau hyperplane section Y.

C-factors correspond to c_1-factors in the Chern number,
and since for Calaby-Yau $c_1(Y) = 0$ we observe Euler constant is not involved.
So one should consider something non-anticanonical.

Test the case $H = \mathcal{O}(1,1)$ on $\mathbb{P}^2 \times \mathbb{P}^3$. Being exact, we restrict D-module to subtorus corresponding to H, and consider operator of quantum multiplication by H on it (subtorus associated with H is invariant with respect to vector field associated with H).

(X, H)	μ	Apery numbers
$(\mathbb{P}^2 \times \mathbb{P}^3, \mathcal{O}(1,1))$	3	$-C,\ \frac{C^2+7\zeta(2)}{2}$.

Apery recursions are essentially cases of $Gr(2,5)$ and $OGr(5.10)$.

Proposition. *Let X be a subcanonically embedded smooth Fano variety of index $r > 1$ i.e. X is embedded to the projective space by a linear system $|H|$, and $-K_X = rH$. Consider a general hyperplane section Y — a subcanonically embedded smooth Fano variety of index $r-1$. There is a restriction map $\gamma \to \gamma \cap H$ from cohomologies of X to cohomologies of Y and by Hard Lefschetz theorem except possible of intermideate codimension all primitive classes of Y are restricted primitive classes of X. Consider a homogeneous primitive class of nonintermediate codimension $\gamma \in H^{\bullet}(X)$. Then Apery numbers for γ calculated from QDE of X and Y coincide.*

Apery class is functorial with respect to hyperplane sections.

If X is quantum minimal then Apery class is equal to 1.