MATHEMATICS, REALITY, AND GOD

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Sumário. Este é um capítulo escrito para o livro **Teaching the Tradition** sobre a tradição intelectual católica publicado pela Oxford University Press em 2012, com redatores John Piderit S.J. e Melanie Morey. Apresenta vários aspectos culturais da matemática e sua história que têm alguma ligação—talvez tênue—com a fé cristã.

Abstract. This is a chapter written for the book **Teaching the Tradition** on the Catholic intellectual tradition published by Oxford University Press in 2012, edited by John Piderit S.J. and Melanie Morey. It presents various cultural aspects of mathematics and its history that have a connection—perhaps tenuous—with Christian faith.

1. Beauty and esthetics in mathematics. Mathematical beauty is the principal virtue that attracts people to mathematics. There are various ways of expressing why some mathematical objects, figures, and proofs are beautiful, but simplicity and symmetry are among the reasons. A certain completeness is perceived, so that no part can be removed without its absence being felt as a flaw. There is a necessity; the mathematical object must be as it is, if it is not to be damaged. This beauty has an innocence and purity that suggests its divine origin. Let us see some examples.

As a first example, consider the golden mean φ, the ratio of the base to the height of a rectangle pleasing to the eye. It has the property that if a square of the same height is removed and the remaining rectangle is rotated 90^o, it will have the same proportion (See Figure 1). If the height is 1, then the length of the base is φ. If we remove the square on the right of side 1, the base of the remaining rectangle will be $\varphi-1$. Then the basic property of φ states that φ is to 1 as 1 is to φ-1, i.e., $\varphi/1 = 1/(\varphi-1)$. Thus $\varphi^2 - \varphi - 1 = 0$ so the quadratic formula gives $\varphi = (\sqrt{5} + 1)/2$. The value of the golden mean φ is approximately 1.6180339887. A rectangle with these proportions is esthetically pleasing, in either horizontal or vertical form, so that it is the shape usually used for the frames of paintings. Repeatedly cutting off squares with inscribed circular arcs produces a lovely spiral in Figure 1. The golden mean also occurs in three different ratios of segments in the pentagram, the regular five-pointed star. In Figure 2, the ratios of the segments AD:AC, AC:AB, and AB:BC are each given by the golden mean.

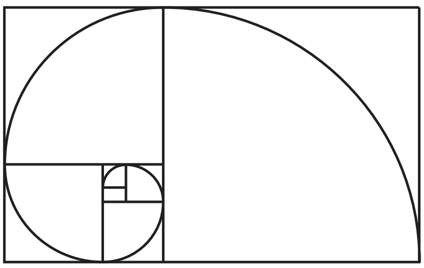

Figure 1. The golden mean and its spiral.

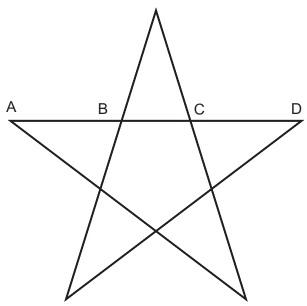

Figure 2. The golden mean in the five-pointed star

Figure 3. The Fibonacci numbers in the sunflower.

The golden mean φ is also related to the sequence of Fibonacci numbers, named after the Italian mathematician Leonardo da Pisa (1171 – 1249, 'figlio de Bonacci'). The sequence begins 1, 1, 2, 3, 5, 8, 13, 21, 34, 55, 89, ..., each term being the sum of the preceding two terms. If we denote the nth term by a_n, so that $a_1 = 1$, $a_2 = 1$, $a_3 = 2$, $a_4 = 3$, etc., then the ratios a_{n+1}/a_n of successive terms converge to φ as n tends to infinity. This sequence occurs in nature in the spirals visible in the core of the sunflower, as in Figure 3, where the numbers of clockwise and counterclockwise spirals in the sunflower are the Fibonacci numbers 34 and 55, respectively.

The feeling for mathematical beauty is a quality shared by outstanding mathematicians. Leonhard Euler (1707-1783) appreciated the beauty of his formula $e^{\pi i} + 1 = 0$, which brings together the five most important numbers – 0, 1, e, π, and i – in impressive harmony. The prolific but eccentric Hungarian mathematician Paul Erdös (1913-1996) liked to talk about "The Book," in which God (whose existence he doubted) would have gathered together all of the most beautiful, surprising, elegant, and simple proofs in all of mathematics. I remember fondly how one of my mathematics professors at Princeton University, Salomon Bochner (1899-1982), after writing some especially beautiful part of mathematics on the blackboard, would turn around to the class, beaming with an intense and contagious delight shared by all.

The esthetic sense of what is fitting and beautiful guides research in physics as well as in mathematics. In 1928, the eminent physicist Paul Dirac (1902-1984) arrived at his celebrated Dirac equation describing the electron and similar particles because it was so beautiful. Its symmetry suggested the possibility of the existence of a particle similar to the electron, but with negative energy. This was the positron, the antiparticle of the electron, discovered in 1932.

The theory of general relativity of Albert Einstein (1879-1955), explaining the gravitational field as the curvature of the four-dimensional continuum of space-time, is convincing because of its simplicity and beauty. In 1919, after the theory was confirmed during an eclipse of the sun observed in Sobral and Principe, Einstein was asked what would have been his reaction if the observations had not confirmed his theory. He replied, "Then I would feel sorry for the dear Lord. The theory is correct anyway." The theory was so elegant that its correctness could not be doubted.

Figure 4. Archimedes' sphere and circumscribed cylinder. The oil painting "Cicero Discovering the Tomb of Archimedes" painted in 1797 by the American painter Benjamin West (1738-1820).

In his work *On the Sphere and Cylinder,* Archimedes (287-212 BC) proved the beautiful theorem that the ratio of the volumes of a sphere and the cylinder circumscribed about it is 2:3, and the ratio of their areas is the same. Plutarch (45-120 AD), in *Parallel Lives, Marcellus*,[i] comments about Archimedes that "although he made many excellent discoveries, he is said to have asked his kinsmen and friends to place over the grave where he should be buried a cylinder enclosing a sphere, with an inscription giving the proportion by which the containing solid exceeds the contained." Cicero (106-43 BC) recorded that when he was quaestor in Sicily in 75 BC he managed to track down Archimedes' grave. It was a small column with the sphere and cylinder above it almost covered by bushes, but after the brush was cleared away, Cicero was able to read what was left of the inscription. In his reverence for the great mathematician and scientist, he donated funds to restore the tomb.

The musical scale is another example of beauty linked to mathematics. An octave in the musical scale has seven tones and five more half tones, making a total of twelve half tones. An octave corresponds to doubling the frequency of a tone. Consequently, in the tempered scale, each half tone corresponds exactly to a ratio of frequencies equal to the twelfth root of 2. By a marvelous coincidence, the powers of this twelfth root of 2 are very close to fractions with small numerator and denominator. For example, the fourth power, which is approximately 1.25992 (corresponding to the interval from do to mi), is very close to 5/4=1.25, and the seventh power, approximately 1.49831 (corresponding to the interval from do to sol), very close to 3/2=1.5. The harmonies of the overtones of a fundamental frequency are therefore close to powers of the twelfth root of 2, contributing to the perceived harmony. The slight differences between these values is what determines the qualitative difference among the various keys of musical compositions. The intimate relationship of music to mathematics was recognized in higher education in the Middle Ages, when music was one of the four disciplines of the quadrivium, along with arithmetic, geometry and astronomy, to be studied after the trivium, which consisted of grammar, logic (then called "dialectic") and rhetoric.

After all these instances of how beauty is everywhere present in mathematics, let us ask what the significance and meaning of this beauty may be. There is a pristine quality of mathematical beauty that avoids the blemishes and faults that seem to appear in the material universe and in human society. The poet Edna St. Vincent Millay (1892-1950) wrote, "Euclid alone has looked on Beauty bare." Mathematics has escaped from original sin. We seem to be viewing the framework of existence before any defects could appear. This unsullied beauty is a reflection of the beauty of the Creator and of his original creation. The transcendental categories of being—one, good, true, and beautiful—can be seen in mathematics, and they show forth the marvelous qualities of the absolute Being, our creator God. Just as when the beauty of the lilies of the field, the songs of birds, or the smile of a child overwhelm us, in the contemplation of mathematical beauty a window opens onto eternity and one can sense the holy presence of our loving God.

2. Mathematics in religious art. The Catholic Catalan painter Salvador Dali (1904-1989) painted two famous paintings[ii] that mix mathematics and Christian faith, "Crucifixion—Corpus Hypercubus" and "The Sacrament of the Last Supper." In "Corpus Hypercubus," the cross has the form of the 3-dimensional surface of a 4-dimensional hypercube, unfolded so as to fit into 3-dimensional space. To understand this, consider how one can make a 3-dimensional cube out of a sheet of construction paper cut into the form of a cross composed of six squares, four of them vertical and one on each side, as in Figure 7. By folding the paper along the edges separating the squares, as many children know, the result will be a 3-dimensional cube with six squares as faces.

Figures 5 and 6. Dali's paintings *Corpus Hypercubus* and *The Last Supper*

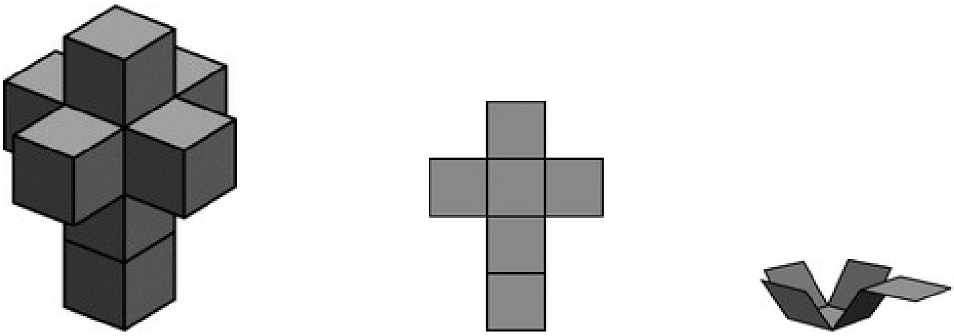

Figure 7. The hypercube and the cube, both unfolded.

Dali's painting is based on the analogous construction in one higher dimension. In 4-dimensional Euclidean space, endowed with coordinates (x_1, x_2, x_3, x_4), the hypercube is formed by the points whose coordinates x_i are each between 0 and 1, for i = 1, 2, 3, 4. This is analogous to the way that a 3-dimensional cube in 3-dimensional space with coordinates (x, y, z) is obtained by restricting the values of each of the coordinates x, y, and z to lie between 0 and 1. Just as the cube has six squares as faces, corresponding to making each coordinate either 0 or 1, the hypercube has eight cubes as faces, corresponding to making each of the four coordinates x_i equal to 0 or 1. We can imagine the boundary of a hypercube to be unfolded into a shape of 3-dimensional construction 'paper' cut into eight cubes, four in a vertical sequence with one on each side, left and right, and two others, front and back, as in Figure 7. Now looking at "Corpus Hypercubicus," we see that it is exactly this unfolded hypercube that Dali has portrayed as the cross. The eighth cube, in front, is merely outlined by the four nails, in order not to hide Christ's body.

What is the message of this presence of the 4-dimensional hypercube in Dali's painting? The reference to the fourth dimension evokes Einstein's special theory of relativity, in which the fourth dimension, time, is joined to the three spatial dimensions to form the 4-dimensional space-time continuum. Dali places the crucifixion of Christ at the center of all of space-time, as the pivotal event in the history of the whole universe. Christ's body floats in space, freed from suffering and death by the resurrection. The woman in the painting contemplates Christ in the full Paschal mystery of our faith—the crucifixion, death, and resurrection of the Lord, which opens for us the way of grace and salvation.

Dali's painting of "The Last Supper" also uses a geometrical image to express the meaning of Christ's farewell dinner with his apostles in which he instituted the Eucharist. The frame of the room is clearly a 12-sided regular polyhedron, the dodecahedron, the fourth of the Platonic solids portrayed in Figure 8. In the section below on Numerology and Symbolism, we shall see that the numbers 7 and 12 connote fullness and plenitude. The twelve apostles united around the Lord are going to preach the Gospel to the whole known world, totality and plenitude. Dali associates them symbolically with the faces of the dodecahedron. Their cowls even have a pentagonal shape. The fullness, beauty, and symmetry of the dodecahedron are evoked by Dali to symbolize the completion of Christ's own work on earth, to be continued by the Twelve and their successors. The arms of Christ are spread above, inscribed in the dodecahedron, as they will soon be spread on the cross, suggesting blessing with the fruits of the cross and the sending of the Holy Spirit, recalling Christ's promise "I shall be with you always, until the end of the world" (Mt. 28: 20).

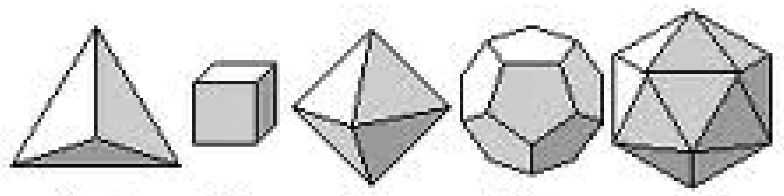

Figure 8. The five Platonic solids—the regular polyhedra.

Dali's use of mathematical symbols to express profound religious realities is an example of what Thomas Rausch refers to as the "sacramental imagination" of Catholicism.[iii] The young and handsome Christ in both paintings suggests his victory over evil, sin and death. The first light of dawn in the Crucifixion and the horizon opening out onto the world in the Last Supper symbolize that victory as it spreads out over the whole earth and even to all of space-time.

3. Mathematics as science—The search for truth. Mathematical discoveries date from the early ages of human civilization. The Pythagorean theorem, which states that the area of the square on the hypotenuse of a right triangle is the sum of the areas of the squares on the two legs—was apparently used by the builders of the pyramids in ancient Egypt. The Rhind papyrus, an Egyptian scroll copied by the scribe Ahmes from an earlier work dated around 1850 B.C., has solutions to various mathematical problems.[iv]

Cuneiform baked clay tablets from ancient Babylonia (1900-1600 B.C.) contain multiplication tables, probably used by children learning their lessons.[v]

Since these ancient times, mathematics has investigated and studied a great variety of numerical and geometrical problems. In the third century before Christ, Archimedes made significant advances in geometry and used mathematics to study statics. In the late Middle Ages there were advances in the solution of polynomial equations. At the dawn of modern science towards the end of the 16th century, Galileo introduced precise mathematical measurements into his studies of physics, and mathematics became the language of science. Nobel Prize physicist Eugene Wigner, in his fascinating essay on "The unreasonable effectiveness of mathematics..." mentioned under "Helpful Resources" at the end of this chapter, analyzes our surprising capacity to describe and understand natural phenomena through mathematics.

As a science, mathematics attains the highest degree of precision, since arithmetical calculations can be exact. Measures of weights, lengths, or volumes are only approximations, to a greater or lesser degree of accuracy, but counting a number of objects can give a precise answer. Even though arithmetic is abstracted from experience of nature and daily life, it reaches a degree of absolute precision that no other domain of human activity can rival.

4. Axiomatization of mathematics. A new dimension of logical structure was introduced into mathematics by Euclid around 300 B.C., when in the book *Elements of Geometry* he gave a system of axioms and postulates and used them to prove many basic theorems about geometric figures in the plane. Somewhat earlier, in the fourth century B.C., Aristotle had introduced principles of logic. Many of the theorems of Euclid were already known facts, but the logical structure of giving cogent proofs for them in a logical order, starting from the axioms and postulates, was an important advance.

It later turned out that Euclid's assumptions—his axioms and postulates—were not complete. A completely formal presentation of geometry requires axioms dealing with the order of points on a line and of the location of points inside or outside of a polygon. For example, the fact that the bisector of the angle at one vertex of a triangle meets the opposite side between the other two vertices is clear from our geometrical intuition, but it does not follow from Euclid's assumptions. There is also an amusing—but false!—proof that every triangle ABC is isosceles (i.e., has the two sides AB and AC equal in length), if one supposes that the bisector, AE, of the angle A meets the perpendicular bisector, DE, of the opposite side, BC, *inside* the triangle at the point E (See Figure 10, for which I thank my colleague Fabio Souza). One argues that the triangles BDE and CDE are congruent, so BE = CE; then the triangles AEB and AEC are congruent (using the angle-side-side argument, which is valid in this case since the angles AEB and AEC are both obtuse), so AB = AC! The fallacy lies in the fact that if the sides AB and AC have different lengths, then the intersection point will actually lie *outside* the triangle. It was only in 1899 that the great German mathematician David Hilbert (1862-1943) published a complete set of axioms for planar Euclidean geometry including axioms involving order of points on a line and the location of points on one side or the other of a line.[vi]

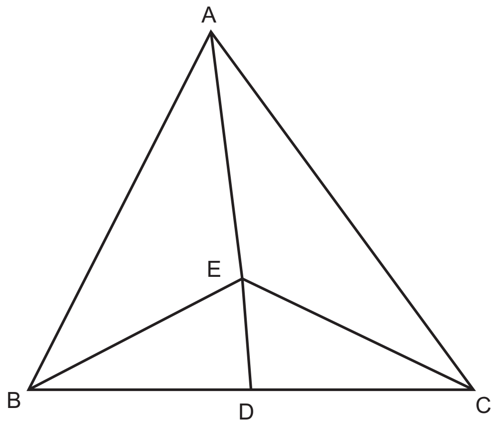

Figure 9. A triangle with false bisectors AE and DE.

There were two important advances in the axiomatization of mathematics in the 19th century. First of all, non-Euclidean geometry was discovered by Carl Friedrich Gauss (1777-1855), Janos Bolyai (1802-1860), and Nikolai Lobachevsky (1792-1856), who showed that there was a logically consistent geometry, now called hyperbolic geometry, that did not satisfy Euclid’s fifth postulate. For centuries mathematicians had tried without success to prove the fifth postulate, also known as the parallel postulate, from Euclid’s other axioms and postulates. It states that if two straight lines meet a third one in so that the sum of the interior angles on one side of the third line is less than two right angles, then the first two lines will meet on that side, provided that they are extended sufficiently far. In 1733 the book *Euclides ab omni naevo vindicatus* (“Euclid freed from every blemish”) authored by the Jesuit priest Girolamo Saccheri (1667-1733) was published. In it he proved many results that follow from supposing the parallel postulate to be false. He concluded that the results were so counter-intuitive that Euclid’s parallel postulate would have to be accepted. This was wrong, but in his work he had discovered many of the most basic properties of non-Euclidean geometry. Almost a century later, when Gauss realized that this non-Euclidean hyperbolic geometry was logically consistent, he hesitated to publish this conclusion for fear of being ridiculed. The method by which the three geometers Gauss, Bolyai, and Lobachevsky showed that hyperbolic (or Lobachevskian) geometry is as logically consistent as Euclidean geometry was by constructing a model of hyperbolic geometry inside Euclidean geometry. It follows that if there is no inherent contradiction in Euclidean geometry, then there is no contradiction in hyperbolic geometry. It is interesting to note that the underlying geometry used by Albert Einstein in his general theory of relativity is a version of non-Euclidean geometry.

The second advance in axiomatization of mathematical theories was in mathematical analysis, the branch of mathematics which grew out of the differential and integral calculus discovered by Isaac Newton (1643-1727) and Gottfried Leibniz (1646-1716) in the second half of the 17th century. Problems and basic questions arose in the late 18th and early 19th centuries. Exactly what properties should be supposed to hold for a numerical function? When should an infinite sequence of numbers be said to converge? When should an infinite series (or sum) of terms have a limiting value? It was necessary to decide exactly what were the assumptions, since different assumptions would lead to different conclusions. This led to the creation of systems of axioms for the real numbers, which correspond to the points on the line and form the basis for mathematical analysis.

Since the end of the 19th century, mathematics has been organized like Euclid’s geometry around systems of axioms that reflect properties of the structures that are being studied. From the axioms new statements, called theorems, are deduced by strict rules of logic. This procedure is so precise and secure that there is essentially no disagreement about what results follow from the axioms, although there are different positions concerning which axioms should be assumed. The process of proving theorems in a formal system is very precisely determined. Determining which sequences of propositions constitute a proof—the syntactic aspect of the theory—can be checked without any need for creativity. It can be completely mechanized and done on a computer. On the other hand, discovering which propositions are true and finding proofs of them requires intelligence, sometimes even genius!

The precision of mathematical argument can function as an incentive and as a goal to strive for in other sciences and in other branches of human thought. It is obviously desirable that rational discourse and arguments have criteria by which their validity can be analyzed and decided. Nevertheless, it is clear that most judgments in many fields of thought, as well as in the decisions of daily life, do not admit criteria as clear and precise as in mathematics. The rules for arguments in theology and humanities have to be very different in nature from those of mathematics. Every social science must have its own criteria and means of verifying its conclusions. Even in the physical sciences, such deductive arguments as in mathematics are often not possible. In mathematics, one argues that if the axioms A, B, C, and D are assumed, one can prove theorems J, K, and L, and rigorous proofs are given. The process of induction in physics and other empirical sciences, on the other hand, goes in the opposite direction. If J, K, and L are known to follow from A, B, C, and D, then when J, K, and L are observed in a sufficient number of experiments, this is interpreted as support for affirming A, B, C, and D. Observing that the conclusions are satisfied in sufficiently varied laboratory experiments gives credence to the hypotheses. Even though the precision and deductive certainty achieved in mathematics are positive qualities, they cannot be required in the same degree in other disciplines. One should not reject arguments in philosophy and theology, for example, because they do not exhibit mathematical rigor; they must be judged by their proper principles and methods.

5. Limitations present in mathematical discourse itself. As we have just noted, the demanding requirements of precision in mathematical discourse and proofs cannot be applied to other fields, but as we shall now see, these requirements have intrinsic limitations *even in their application to mathematics itself.* This amazing discovery from the 1930's has far-reaching implications.

The axiomatization of mathematical theories in the 19th century and the development of set theory as a foundation of mathematical theory led naturally to the hope of finding an axiomatic basis for all of mathematics. This was the goal of Alfred North Whitehead (1861-1947) and Bertrand Russell (1872-1970) in their enormous three volume work *Principia Mathematica*, published in 1910, 1912, and 1913 and in a revised second edition in 1927. Despite their heroic effort, their goal was proven to be unattainable by Kurt Gödel (1906-1978), a young Austrian mathematician, in his paper "On formally undecidable propositions of *Principia Mathematica* and similar systems" (*Über formal unentscheidbare Sätze der Principia Mathematica und verwandter Systeme,* 1931),[vii] one of the most important intellectual advances of the 20th century and even of the whole history of human thought! Gödel showed that if a finite set of axioms is sufficiently powerful to be able to deal with the basic properties of arithmetic, then it must be either inconsistent or incomplete. This means that if there is no proposition such that the proposition and its negation can both be proven from the axioms (consistency), then there must be a meaningful proposition in the system—one that must either be true or false in any interpretation of the axioms—such that neither the proposition nor its negation can be proven from the axioms (incompleteness). Hence mathematical thought must always look beyond the axioms to the basic insights behind them. Furthermore, there will always be new questions in mathematics that have not yet been answered. Mathematics in itself reveals an opening to the infinite, to the transcendent, to a tendency that reaches beyond all its conquests. Intuition can never be replaced by formalism, even in this most formal of fields, and certainly not in any other field, such as the social sciences, philosophy, and theology. This is not a flaw, but a fundamental limitation of human thought, an invitation to contemplation.

Gödel's idea is so brilliant that we should give an idea of how it works. He assigned numbers to propositions and to proofs. Then he constructed a proposition—let us call it *P*— that asserts that the proposition assigned a certain number *n* cannot be proven from the axioms. (The construction of *P*, which depends on the set of axioms chosen, is lengthy and complicated.) When we calculate the number assigned to the proposition *P*, we discover that its number is *n*. Thus in a certain sense, *P* asserts that *P* itself cannot be proven! Now if the system is consistent and *P* could be proven, it would be true, so it could not be proven—a contradiction! Hence if the system of axioms is consistent, *P* cannot be proven. But then what *P* affirms is true. Thus *P* is a meaningful proposition that is true but cannot be proven from the axioms. In particular, neither *P* nor its negation can be proven from the given axioms. We could add the true proposition *P* as a new axiom, but then Gödel's construction would give a new proposition *P'* to which his argument would apply.

In 1937, a few years after the appearance of Gödel's paper, another epochal work in the same direction, "On Computable Numbers, with an Application to the *Entscheidungsproblem,*" was published by a young English mathematician, Alan Turing (1912-1954).[viii] In this paper Turing invented a kind of computer, now called a "Turing machine," capable of performing any calculation that could be explicitly and completely described. The machine has a finite tape for memory, but the tape can be extended as far as the calculation requires in both directions, so that arbitrarily large computations can be carried out. All contemporary digital computers are fundamentally Turing machines, so in a certain sense he was the inventor of the computer. He showed that there exists a universal Turing machine, which can do all the calculations that any other Turing machine can do. A fundamental question about the calculation done by a given Turing machine with a given input is whether the machine will ever stop, or whether it will keep on computing forever. Turing showed in his article that the stopping question is undecidable, thus giving another version of Gödel's theorem extended into the field of computability. During the Second World War, Turing, an extraordinarily brilliant man, invented machines that decrypted intercepted messages of the German armed forces, thus contributing significantly to the defeat of Hitler's Germany. Unfortunately, Turing's important contributions to winning the war were not sufficient to avoid his prosecution for being openly homosexual; he was subjected to an invasive treatment with hormones, and tragically took his own life in his early forties.

Perhaps one can see a "tragic flaw" of pride (the "hamartia" of classical Greek theater) in the definitive failure of Whitehead and Russell's project, as shown by Gödel's theorem. With a surprising lack of modesty they took their title, *Principia Mathematica,* from *Philosophiae Naturalis Principia Mathematica*, the title of Newton's great work in which he created the calculus and presented his theory of gravitation. Implicitly they put their work on the same level of importance as Newton's. But however we may judge the intentions of Whitehead and Russell, the failure of their undertaking teaches us a clear lesson. Formal logical arguments from axioms are not sufficient to decide important questions in mathematics, let alone in other fields. Intuition and reasoning transcend the possibilities attainable by formal axiomatic arguments. We must respect the varied nature of different forms of human reasoning, and not try to limit them to a mathematical form. Every type of rational discourse must be judged according to its own procedures and limitations. Decisions involving common sense cannot be reduced to syllogisms. This conclusion is especially important when we deal with fundamental questions of values and religious truths, where the basic premises are much more difficult to reach.

A further example of the limits of axioms in mathematics is a proposition called the "continuum hypothesis." It states that any infinite set S of real numbers must either be enumerable (i.e., there must exist a one-to-one correspondence between the natural numbers and the elements of S) or have a one-to-one correspondence with the set of all the real numbers (expressed by saying "S has the cardinal of the continuum"). Gödel and Paul Cohen (1934-2007) showed that the continuum hypothesis is independent of the other axioms in the usual (Zermelo-Frankel) axiomatization of set theory, which remains equally consistent if either the continuum hypothesis or its negation is added as a new axiom. In other words, the continuum hypothesis can neither be proven nor disproven within Zermelo-Frankel set theory. Nevertheless, Gödel considered this an open problem that may be decided some day by new mathematical insights. Perhaps someone will discover a new proposition which will be generally recognized as true, and such that adding it to the Zermelo-Frankel axioms makes is possible to prove the continuum hypothesis (or its negation!). In Gödel's view, here again mathematics passes beyond the limits of the axiomatic method.

6. Where do mathematical objects come from? If mathematical truths cannot simply be reduced to a set of axioms, then we must ask what mathematical objects are and where they come from. This question cannot be evaded. With a little reflection, it is obvious that our mathematical concepts had their origin in abstraction from our perception of reality. The counting numbers, now known as the "natural numbers," arose at the dawn of human consciousness, to make it possible to number the oxen in a herd, or the number of coins in a purse, or the number of people in a tribe. Thus numbers were abstracted from concrete reality.

Systems of representing numbers also were developed very early on. We have already mentioned Sumerian clay tablets with multiplication tables. In ancient Sumer, numbers were measured in multiples of 60, a custom which has left its traces in contemporary measures: an hour has 60 minutes, a minute has 60 seconds, an angle of an equilateral triangle has 60 degrees. In ancient Israel and Greece, letters had numerical values on the base ten, a consequence of counting on one's fingers. Some learned rabbis know the Old Testament text by heart, so they can recite lengthy lists of numbers for hours on end by interpreting the text numerically. In Greece, α′ represented 1, β′ was used for 2, γ′ for 3, etc. Then ι′ was used for 10, ια′ for 11, ιβ′ for 12, then ρ′ for 100, σ′ for 200, and so on. Three obsolete letters were also used, since the later Greek alphabet had only 24 letters. For thousands, a prime was put on the lower left rather than on the upper right, so numbers up to 999,999 could be represented. Archimedes (287-212 BC), the great mathematician and scientist from Syracuse in Sicily, created a new system of numbers of astronomical size in his essay *The Sand Reckoner*.[ix] This permitted him to calculate the approximate number of grains of sand that would be needed to fill up the whole visible universe! He did this to show that the expression "as infinite as the number of grains on the beaches of the sea" did not make any sense, since that would be only a very large but still finite number.

The invention of the numeral zero "0" in India, transmitted to Europe by the Arabs, made our modern positional notation possible, and greatly simplified calculations. (Zero and positional notation were invented independently in the pre-Columbian Mayan culture in Central America, but they used the base twenty, apparently since in their tropical climate they used both fingers and toes for counting.) The binary system, with only 0 and 1 as numerals, was fundamental in the development of digital computers, where components have only two positions (on or off) and most computations are carried out in binary. Just as in the decimal system '269' represents 9 units plus 6 tens plus 2 hundreds, counting from the right, so in the binary system '10011' represents 1 plus 2 plus 16, giving 19 (with 4 and 8 omitted because of the 0's in their positions). The binary cases have the values 1, 2, 4, 8, 16, 32, etc., counting from the right. Each case has as its value the double of the case to its right.

Fractions, known as "rational numbers," were necessary to measure quantities that are not given by whole numbers, such as the result of dividing two loaves of bread among five people, so that each one receives two fifths of a loaf, or measuring lengths, volumes or weights to a reasonable approximation.

There are also "irrational numbers" that cannot be expressed as the quotient of two whole numbers. For example, the square root of 2, $\sqrt{2}$, which is the ratio of the length of the diagonal of a square to its sides, cannot be written as p/q. (If it could be so represented, then we could choose p and q so that at least one of them would not be even. Then p^2/q^2 would be 2, so $p^2=2q^2$, and consequently p must be even, say p=2k. But then we would have $4k^2=2q^2$ and so $q^2=2k^2$, and q would also be even, contrary to assumption.) Ancient historians relate that when this was discovered and disclosed by a member of the Pythagorean school in the fifth century B.C., it was such a scandal that he was put to death. The story is probably legendary, but it shows motivation for the term "irrational." In any case, to measure lengths—or areas, volumes, or other continuous sizes—we need what are called "real numbers," which correspond to all the points on a line once we have marked two points as 0 and 1. Every point on the line, like every real number, can be represented by an infinite decimal expansion. For example, 1/3 = 0.33333... where the 3's repeat infinitely often, while 2 = 2.0000... and 16/7 = 2.2857142857... where the block 142857 is repeated *ad infinitum*. A real number will be irrational if there is no such eventually repeating block in its decimal expansion. The golden mean φ described above is such an irrational number.

In the late Middle Ages, "complex numbers" of the form $a + bi$ ("*a* plus *b* times *i*"), where i is an "imaginary number" satisfying $i^2 = -1$ and a and b are real numbers, were invented to provide solutions of certain equations with no real solutions. Today complex numbers are extremely important in almost all branches of mathematics and their applications. Many fields of physics, such as electromagnetic theory and quantum theory, use complex numbers in an essential way. It is fascinating to note that the creation of "imaginary" numbers to provide solutions for unsolvable equations ended up as a fundamental advance in mathematics, without which modern technology would not be possible.

Geometrical figures and spaces are also abstracted from observations of reality, but in this case from the spatial relations between objects. There were many advances in Babylonia, Egypt, and Greece leading up to the organization and axiomatic presentation in Euclid's *Elements*. In the 17th century, Fermat (1601?–1665), Descartes (1596–1650), and other mathematicians discovered that each point in the plane could be represented by an ordered pair of real numbers (x,y), using the horizontal x-axis and the vertical y-axis. These "coordinates" x and y made it possible to translate geometric statements into algebraic ones, relating y to x and leading to great progress in geometry. It is also possible to represent a point in three-dimensional space by an ordered triple (x,y,z) and so spatial geometry can also be translated into algebra and mathematical analysis. It was a natural further step to consider spaces of four, five, or even n dimensions. A point in n-dimensional Euclidean space is defined to be an ordered sequence of n real numbers $(x_1, x_2, ..., x_n)$. The geometrical properties of figures in n-dimensional space, such as lengths and angles, can then be formulated in terms of the coordinates $x_1, x_2, ...,$ and x_n As was the case for complex numbers which were created as extensions of other mathematical numbers, but not abstracted from reality, the creation of n-dimensional space has led to extremely important practical and theoretical applications. Here we see a dimension of nobility in human rationality; the human mind shows that it is in the image of God as it creates these new mathematical objects (see Gen 1: 26-27).

7. What existence do mathematical objects have? As we have just seen, mathematical objects arise in human thought by abstraction from observations of reality. Yet most practicing mathematicians consider them to have some kind of existence in themselves, since they are not amenable to mathematicians' desires and do not yield to our wishes. The question of whether a certain conjecture is true is an objective question which must be investigated to be decided, independently of whether one would like it to be true or false. In what sense, then, can mathematical objects like numbers, sets, and geometrical figures be said to exist?

Plato (428/427 BC – 348/347) thought mathematical objects, like other ideas, had a real existence in some ideal world. St. Augustine (354-430) gave a Christian interpretation of Plato's position by claiming Platonic ideas exist in the mind of God. For the believing Christian, this claim yields two possible implications: either numbers are inherent in any possibility of existence, before God created anything, or else they are in God's mind as creator. Perhaps these two positions are just two facets of the same basic belief, for God is the creator of all that is except himself. All created reality existed in the mind of God prior to its creation, and God is pure existence ("*summum esse*," as St. Thomas Aquinas expresses it). All that exists is derivative from God, numbers and other mathematical objects as well.

An interesting application of the belief that numbers exist in some objective sense is the way they are being used in the Search for Extra-Terrestrial Intelligence, SETI. The prime numbers 2, 3, 5, 7, 11, 13, 17, 19, ..., those positive whole numbers (excluding 1 itself) that cannot be factored into factors greater than 1, are fundamental in studies of arithmetic. It is difficult to imagine any process not due to intelligent beings that could produce the ordered sequence of prime numbers, so this list has been radioed into outer space in hopes that another intelligent civilization may observe it and send back some similar reply. (Unfortunately, if there are intelligent extraterrestrials, they are probably hundreds or thousands of light-years away, so no quick reply can be hoped for.)

8. Some analogies between mathematics and theology. Mathematics advances with an enormous dynamism. A research mathematician is often asked by people who have never encountered advanced mathematics how it is possible to do research. "Haven't all the problems been solved by now?" In reality, as old mathematical problems are resolved, many more new ones keep appearing. The proliferation of new questions and ideas has also been taking place in Christian theology throughout its history. There were times when many people thought that the essential problems of our faith had been solved, and there were only secondary questions to be attacked. Yet as Blessed Cardinal John Henry Newman (1801-1890) explained in his revolutionary work, *An Essay on the Development of Christian Doctrine*, the Holy Spirit keeps giving new insights to help theologians advance in understanding the Christian faith. The way that both mathematics and theology keep advancing and deepening their understanding is a first common trait.

A second similarity in the progress of mathematics and theology is the strong interlinking of their various subordinate disciplines. Perhaps this unity is easier to understand in theology, where there is a common focus on God's salvific action in the world. In mathematical research, on the other hand, new theories and domains of mathematical research are constantly appearing. One would expect that these theories would diverge and become independent of each other, but that is not what happens. The history of mathematics is full of surprising connections between what were thought to be independent fields. The studies of the young French mathematician Evariste Galois (1811-1832) on permutation of roots of polynomial equations led to the concept of a group which has since become a unifying structure in essentially all branches of mathematics and science. The Atiyah-Singer index theorem, proven in various versions in the 1960's, showed that deep invariants in the algebraic topology of manifolds and in the analysis of differential operators were actually the same! The examples can be multiplied without end. Despite the expansion of mathematical research into new and apparently divergent fields, they often come together and form one closely integrated field of study.

Not only are mathematics and theology integrated disciplines in themselves, but a third similarity is that they both perform a function of integrating widely different areas of human thought. Throughout the Middle Ages, theology was considered the "queen of the sciences," though the word "science" meant knowledge and referred to all domains of human thought, and not just empirical science, as we use the word today. All forms of learning were integrated together in the almost universal belief in God who created everything and gave human beings the intelligence to study and understand all of creation in all its aspects.

In contemporary secular society, it is now mathematics which is considered the "queen of the sciences," and "science" is now understood as the empirical study of nature. The physical and social sciences all depend on mathematical measurement and analysis. This common dependence on mathematics makes it the "language of science." It has become a common structure that relates and integrates all forms of precise reasoning.

A fourth fundamental analogy between mathematics and theology can be seen in the observation of the Jesuit theologian Gerald O'Collins, cited in Lawrence Cunningham's chapter, that there are "three fundamental strands of contemporary theology: academic, practical and contemplative." He quotes O'Collins' comment that "we might contrast faith seeking scientific understanding with faith seeking social justice and with faith seeking adoration.[x] These three strands have analogous lines of force in mathematics. Like theology, mathematics seeks scientific understanding in resolving open problems and organizing the discoveries in coherent and simple theories. The practical applications of mathematics as the "language of science" serve other branches of science and technology, thus contributing significantly to social organization and advance. Finally, there is a dimension of mathematical activity that shares the humble and unselfish attitude of contemplation of God. The pursuit of understanding and new advances in mathematics seeks insight and knowledge for its intrinsic value in itself, rather than for applications or other advantages. There is a dimension of unselfish asceticism in the long hours and struggles of research. Just as faith seeking adoration reaches out to God himself, so mathematics—as viewed in St. Augustine's version of Platonic philosophy—seeks to contemplate the mind of God.

9. Numerology and symbolism. From ancient times, numbers have had a certain symbolic significance, although today they have lost their aura of mysticism. Let us consider the traditional symbolic significance of several numbers and how that symbolism is reflected in Judeo-Christian faith. "One" represents unity and existence, and is affirmed of God in the profession of Israelite faith, the Shema Yisrael prayed daily by pious Jews: "Hear, O Israel: the Lord our God, the Lord is one" (Deuteronomy 6:4). But "one" also denotes solitude, and seeks a partner, leading to "two," a couple. In Christian theology, God the Son proceeds from God the Father. Then the love of "two" generates a third, as the Holy Spirit proceeds from the Father and the Son, or as sexual relations procreate a child. But "three" exhibits some incompleteness, lacking the eight-fold symmetries of the four points of the compass, the symmetries of the square and mandala. So "three" leads to "four," which represents fullness or completeness. "Three" and "four" put together by addition or multiplication, give "seven" and "twelve," numbers which suggest totality or plenitude, as in the seven cardinal virtues, the seven gifts of the Holy Spirit, the seven sacraments, the twelve tribes of Israel, and the twelve apostles. For us today, the symbolic values of numbers have lost their importance. Perhaps we might suggest that cultivation of this mystical dimension of numbers is analogous to the contemplative life of prayer and meditation that opens our consciousness to profound and mystical aspects of reality and of our relationship with God.

Number theory is still one of the most popular and attractive branches of mathematics. It has received the attention and contributions of many of the greatest mathematicians through the centuries. Carl Friedrich Gauss (1777-1855), considered the greatest number theorist of all time, referred to number theory as the "Queen of Mathematics". The oldest unsolved mathematical problem is a question from number theory: does there exist an odd perfect number? A natural number is said to be *perfect* if it is equal to the sum of all its divisors, except itself, such as 6=1+2+3 and 28=1+2+4+7+14. The even perfect numbers are known to be precisely the numbers of the form $2^{p-1}(2^p-1)$ where 2^p-1 is a prime number, but the conjecture that no odd perfect numbers exist remains unproven after at least two and half millennia of efforts.

There is an interesting anecdote about the Cambridge and Oxford don G.H. Hardy (1877-1947) and his collaborator Svinrasa Ramanujan (1887-1920), both eminent number theorists. When Hardy visited Ramanujan in a hospital, Hardy disagreed with Ramanujan's statement that every number was interesting, asking what was special about the number of the taxicab that he had taken, 1729. Ramanujan answered immediately that it was the smallest number that could be expressed as the sum of two cubes in two different ways, $1729 = 8^3 + 9^3 = 12^3 + 1^3$. Upon hearing of this incident, another famous number theorist and collaborator of Hardy, J.E. Littlewood (1885-1977), commented that "Every positive integer is one of Ramanujan's personal friends."

10. Mathematics and proofs for God's existence. The pristine beauty of mathematical figures and structures can be seen as a reflection of the beauty of the Creator, as we have commented. Harmony and symmetry in nature have a similar impact. In contemplating many aspects of nature we are led to believe in the presence of God. But in this section we shall consider two classical philosophical proofs of God's existence to see whether mathematics can shed some light on them. Christians usually do not believe in God the Father and in Jesus Christ because of formal logical proofs; nevertheless, proofs are important and deserve our attention. We shall comment on the second and third ways of showing God's existence of St. Thomas Aquinas (1225-1274) and also on the ontological argument as reformulated by Kurt Gödel.

Aquinas' second and third ways of proving God's existence use arguments from efficient causality and from contingency and necessity. He argues that an infinite regression does not give a sufficient explanation (*Summa Theologiae*, I^a, q. 2, a. 3). Sometimes his argument is criticized by saying that it would be possible to have an infinite regression of efficient causes, say $C_1, C_2, C_3, \ldots, C_n, C_{n+1}, \ldots$, such that C_{n+1} is the cause of C_n, C_n the cause of C_{n-1}, ... C_3 the cause of C_2, and C_2 the cause of C_1, as if Aquinas had not considered this possibility. The causes would have an ordering like the negative numbers. This criticism is based on a misunderstanding of his argument. He explicitly recognized the logical possibility of a universe that had always existed, so that there could be an infinite regression of generations of men or other organisms, but he discarded this possibility because of the Christian belief that God created the universe, which he interpreted as necessarily a creation at a determined moment in time. (Today I think that there would be no theological difficulty in holding that God's creation may have been before time, from all eternity. Furthermore, in the standard model of cosmology, no one knows what might have existed 'before' the big bang—if it makes any sense to talk about "before"!) Aquinas' argument is not about partial efficient causes, but about the total cause of the existence of a being. He claims that if C_3 is the total cause of C_2 and C_2 is the total cause of C_1, then C_1 depends for its existence directly on C_3, since the causality exercised by C_2 on C_1 depends totally on C_3. However one judges the validity of Aquinas' metaphysical argument, the mathematical possibility of a sequence of causes or events in an infinite regression did not escape his notice. It is not a refutation of his argument. Bringing in the mathematical concept of infinite ordered sets sheds no light on the question. For Aquinas, the existence of a non-necessary being not only requires an efficient cause at the onset of its existence, but the continuing action of such a cause is necessary as long as the being continues to exist. The same analysis applies to the third way, which argues from necessity and contingency.

If mathematics does not disprove St. Thomas Aquinas' five ways, it does shed some light on the ontological argument for God's existence, first formulated by St. Anselm of Canterbury (c.1033-1109), taken up by Leibniz, but rejected by Aquinas, Kant, and most other philosophers. Anselm considered "that than which nothing greater can be conceived" ("*id quod magis cogitari nequit*"), and argued that such an existing being would be greater than one that did not exist. His argument was rejected by Gaunilo, a monk who was Anselm's contemporary. Descartes based his philosophy on an argument analogous to Anselm's. In recent decades, there has been considerable interest in variations of the ontological argument.

After the death of the great mathematician, physicist and logician Gödel, an unpublished version of the ontological argument that he had composed was found among his papers. It has been published in his Complete Works.[xi] Gödel had discussed his argument with several logician colleagues and considered it to have value. According to his widow, he read the Bible every morning, but apparently he hesitated to publish his argument since he did not want to be known as a religious person. Since he was arguably the greatest logician in human history, his argument carries weight, so we shall examine it briefly.[xii]

The argument uses modal logic, the logic of possibility and necessity. Gödel considers what he calls "positive properties," and the essence of God is to have all positive properties and no others. According to Gödel, there can be no incompatibility among different positive properties. In particular, a property and its negation cannot both be positive. Necessary existence is a positive property, as is the property of being God-like. For any property P, if P is positive, then being necessarily P is positive. The final result is that, necessarily, the property of being God-like is exemplified, i.e., there necessarily exists a God-like being, a being with all positive properties. Thus God exists!

There is not enough space to give the complete argument, much less to analyze it in depth. The logic seems to be correct, if one accepts the axioms of modal logic that Gödel uses. There is room for disagreement in the premises, the axioms that he assumes. Nevertheless, in an age when many people think that God's existence has been disproved, that science leaves no room for God, that there is an irreducible conflict between science and religious faith, we should remember that one of the greatest scientists and mathematicians of the twentieth century constructed a logical proof for the existence of God that he apparently considered to be valid.

Helpful Resources

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Brief Bio

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Notes

[i] English translation and Greek text edited by Bernadotte Perrin in *Plutarch's Lives,* Volume V, The Loeb Classical Library, G. P. Putnam's Sons, New York, 1917 (see pages 480-1), cited on the site of Prof. Chris Rorres, http://www.math.nyu.edu/~crorres/Archimedes/Tomb/Cicero.html.

[ii] Color reproductions of Salvador Dali's paintings Crucifixion—Corpus Hypercubicus and The Sacrament of Last Supper are easily accessible via Google Images on the internet. The originals are in the Metropolitan Museum of Art in New York and the National Gallery of Art in Washington, respectively.

[iii] Thomas P. Rausch, S.J. Catholic Anthropology and the Good Life, in this volume, p. 6.

[iv] The sites http://library.thinkquest.org/25672/areasand.htm, http://www.math.tamu.edu/~don.allen/history/egypt/node3.html, and http://www-history.mcs.st-andrews.ac.uk/HistTopics/Egyptian_papyri.html contain information about the Rhind papyrus.

[v] Jöran Friberg, A Remarkable Collection of Babylonian Mathematical Texts, *Notices of the American Mathematical Society* 55 (October, 2008), 1076-1086.

[vi] David Hilbert, *The Foundations of Geometry*, (1899); 2nd ed. Chicago: Open Court (1980).

[vii] Kürt Gödel, Über formal unentscheidbare Sätze der Principia Mathematica und verwandter Systeme I, *Monatshefte für Mathematik und Physik* v.38 (1931), p.173-198, Leipzig. Translated in K. Gödel, Collected Works I, p. 30-35. Oxford: Oxford University Press, 1986.

[viii] Alan Turing, "On Computable Numbers, with an Application to the Entscheidungsproblem," *Proceedings of the London Mathematical Society*, 42 (1936), 230–265; correction *ibid*., 43 (1937), 544–546.

[ix] Archimedes, *The Sand Reckoner*, in T.L. Heath (translator and editor), The Works of Archimedes, (London: C.J. Clay and Sons, Cambridge University Press, 1897), 221-232.

[x] L.S. Cunningham, Academic Theology and the Catholic University, in this volume, p.3.

[xi] K. Gödel, Ontological Proof, in *Collected Works* vol. III, Unpublished Essays and Lectures, ed. S. Feferman *et alii*, p. 403-404 (with an excellent introduction by R.M. Adams, p. 388-402), (Oxford: Oxford University Press, 2001).

[xii] See the online *Stanford Encyclopedia of Philosophy*: G. Oppy, "Ontological Arguments", for a good discussion of the ontological argument, a detailed analysis of Gödel's argument, and recent references, at http://plato.stanford.edu/entries/ontological-arguments/.