# Twists for duplex regions

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**Abstract**

This note relies heavily on arXiv:1404.6509 and arXiv:1410.7693. Both articles discuss domino tilings of three-dimensional regions, and both are concerned with *flips*, the local move performed by removing two parallel dominoes and placing them back in the only other possible position. In the second article, an integer $\mathrm{Tw}(t)$ is defined for any tiling $t$ of a large class of regions $\mathcal{R}$: it turns out that $\mathrm{Tw}(t)$ is invariant by flips. In the first article, a more complicated polynomial invariant $P_t(q)$ is introduced for tilings of two-story regions. It turns out that $\mathrm{Tw}(t) = P_t'(1)$ whenever $t$ is a tiling of a *duplex region*, a special kind of two-story region for which both invariants are defined. This identity is proved in arXiv:1410.7693 in an indirect and nonconstructive manner. In the present note, we provide an alternative, more direct proof.

## 1 Introduction

We assume that the reader is familiar to the notations of [2, 1]. In particular, a *multiplex region with axis* $\vec{\mathbf{k}}$ (or $\vec{\mathbf{k}}$*-multiplex*) is a region of the form $\mathcal{D} + [0, N]\vec{\mathbf{k}}$, where $\mathcal{D} \subset \mathbb{R}^2 \times \{0\}$ is simply connected and has connected interior; if $N = 2$, the region is a ($\vec{\mathbf{k}}$-)*duplex region*. The twist $\mathrm{Tw}(t)$ is an integer defined below for tilings of a multiplex, and the polynomial $P_t(q) \in \mathbb{Z}[q, q^{-1}]$ is defined for tilings of a duplex region via the formula (1) below.

Recall that given $\vec{u} \in \Phi = \{\pm\vec{\mathbf{i}}, \pm\vec{\mathbf{j}}, \pm\vec{\mathbf{k}}\}$, we define the $\vec{u}$*-pretwist* of a tiling $t$ of a region $\mathcal{R}$ as $T^{\vec{u}}(t) = \sum_{d_0, d_1 \in t} \tau^{\vec{u}}(d_0, d_1)$, where $\tau^{\vec{u}}$ denotes the effect along $\vec{u}$, or

$$\tau^{\vec{u}}(d_0, d_1) = \begin{cases} \frac{1}{4}\det(\vec{v}(d_1), \vec{v}(d_0), \vec{u}), & d_1 \cap \mathcal{S}^{\vec{u}}(d_0) \neq \emptyset \\ 0, & \text{otherwise.} \end{cases}$$

Here $\mathcal{S}^{\vec{u}}(d_0)$ denotes the open $\vec{u}$-shade of $d_0$, as defined in Section 3 of [1], and $\vec{v}(d) \in \Phi$ denotes the center of the black cube contained in $d$ minus the center of the white one.

**Proposition 1.** *If $\mathcal{R}$ is a duplex region, then, for any tiling $t$ of $\mathcal{R}$,*

$$P_t'(1) = T^{\vec{\mathbf{i}}}(t) = T^{\vec{\mathbf{j}}}(t) = T^{\vec{\mathbf{k}}}(t).$$

The equality $T^{\vec{\mathbf{i}}}(t) = T^{\vec{\mathbf{j}}}(t) = T^{\vec{\mathbf{k}}}(t)$ above is a special case of Proposition 3.3 in [1], and this value is, by definition, the twist $\mathrm{Tw}(t)$. In this note, we give an independent proof of this equality in the particular case of duplex regions. In the aforementioned article, we present a different, shorter proof of the equality $P_t'(1) = \mathrm{Tw}(t)$ using the connectivity of the space of domino tilings of a duplex region by flips and trits. The proof here presented is longer, but more direct.

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## 2 Socks and winding numbers

Let $\mathcal{R}$ be a duplex region. Consider the undirected plane graph $G$ whose vertex set is

$$\left\{(\lfloor x \rfloor, \lfloor y \rfloor) : (x^\sharp, y^\sharp, z^\sharp) \text{ is the center of a cube in } R\right\},$$

and where two vertices are joined by an edge if their Euclidean distance is 1. A *system of cycles*, or *sock*, in $G$ is a directed subgraph of $G$ consisting solely of oriented simple cycles. A *jewel* of a sock is a vertex of $G$ that is not contained in the sock. A vertex $v = (x, y) \in \mathbb{Z}^2$ is called *white* (resp. *black*) if $x + y$ is even (resp. *odd*). We set $\mathrm{color}(v) = 1$ if $v$ is black, and $-1$ if it is white.

Each tiling $t$ of $\mathcal{R}$ has a unique corresponding sock in $G$, where trivial cycles in the associated drawing of $t$ are represented as pairs of adjacent jewels: this is illustrated in Figure 1. Each sock may refer to a set of tilings, all in the same flip connected component (therefore, with the same $P_t'(1)$ and the same $\mathrm{Tw}(t)$).

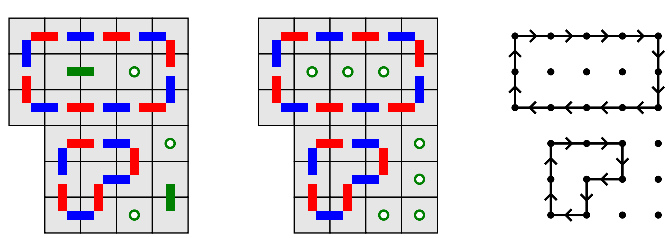

Figure 1: A tiling with two trivial cycles; the same tiling with the two trivial cycles flipped into jewels; and the sock that corresponds to both of them.

Let $t$ be a tiling of $\mathcal{R}$, and let $s$ be its corresponding sock in $G$. For $p \in \mathbb{R}^2$ and a cycle $\gamma$ of $s$, let $\mathrm{wind}(\gamma, p)$ be the winding number of $\gamma$, thought of as a curve in $\mathbb{R}^2$, around $p$. Clearly we can write our invariant $P_t(q)$ as

$$P_t(q) = \sum_{v \in \mathbb{Z}^2} \operatorname{color}(v) q^{\sum_{\gamma, v \notin \gamma} \operatorname{wind}(\gamma, v)}, \tag{1}$$

where the sum in the exponent of $q$ is taken over all the cycles in $s$ that do not contain $v$.

**Lemma 2.** *If $\mathcal{R}$ is a $\vec{\mathbf{k}}$-duplex region with associated graph $G$, $\vec{u} \in \{\pm\vec{\mathbf{i}}, \pm\vec{\mathbf{j}}\}$ and $t$ is a tiling of $\mathcal{R}$ with corresponding sock $s$, then $T^{\vec{u}}(t) = P'_t(1)$.*

*Proof.* Two dominoes that are not parallel to $\vec{\mathbf{k}}$ have no effect along $\vec{u}$ on one another. Therefore, we only consider pairs of dominoes where one is parallel to $\vec{\mathbf{k}}$, that is, refers to a jewel of $s$.

If $\gamma$ is a cycle of $s$ and $v$ is a jewel, one way of computing $\operatorname{wind}(\gamma, v)$ is to count (with signs) the intersections of $\gamma$ with the half-line $v + [0, \infty)\vec{u}$. Thus, if $d_v$ denotes the domino containing $v$ and $d \in \gamma$ means that $d$ refers to an edge of $\gamma$, then $\operatorname{color}(v) \operatorname{wind}(\gamma, v) = 2\sum_{d \in \gamma} \tau^{\vec{u}}(d, d_v) = 2\sum_{d \in \gamma} \tau^{\vec{u}}(d_v, d)$. Thus,

$$P'_t(1) = \sum_{\gamma, v} \operatorname{color}(v) \operatorname{wind}(\gamma, v) = \sum_{\substack{\gamma, v \\ d \in \gamma}} (\tau^{\vec{u}}(d, d_v) + \tau^{\vec{u}}(d_v, d)) = T^{\vec{u}}(t),$$

completing the proof. □

# 3 Charges and weights

We now consider $T^{\vec{\mathbf{k}}}$. Again, let $t$ be a tiling of a duplex region with corresponding sock $s$. Let the *charge enclosed* by a cycle $\gamma$ of $s$ be

$$\operatorname{charge}_{\text{int}}(\gamma) = \sum_{v \notin \gamma} \operatorname{color}(v) \operatorname{wind}(\gamma, v),$$

so that $P'_t(1) = \sum_{\gamma \text{ cycle of } s} \operatorname{charge}_{\text{int}}(\gamma)$. Charges can be looked at from a point of view that is more interesting for our purposes. Given $v \in \mathbb{R}^2$, consider the set of four points $\mathcal{N}_v = \{v + (\frac{k}{2}, \frac{l}{2}) | k, l \in \{-1, 1\}\}$, i.e., the set of points of the form $v + (\pm\frac{1}{2}, \pm\frac{1}{2})$. The *metric weight* of a vertex $v \in \mathbb{Z}^2$ with respect to a cycle $\gamma$ of $s$ is given by

$$\mathrm{w}_{\text{metric}}(\gamma, v) = \frac{1}{4} \sum_{u \in \mathcal{N}_v} \operatorname{wind}(\gamma, u),$$

while the *topological weight* $\mathrm{w}_{\text{top}}(\gamma, v)$ of $v$ is the (arithmetic) average of the set $\operatorname{wind}(\gamma, \mathcal{N}_v) = \{\operatorname{wind}(\gamma, u) | u \in \mathcal{N}_v\}$ (see Figure 2).

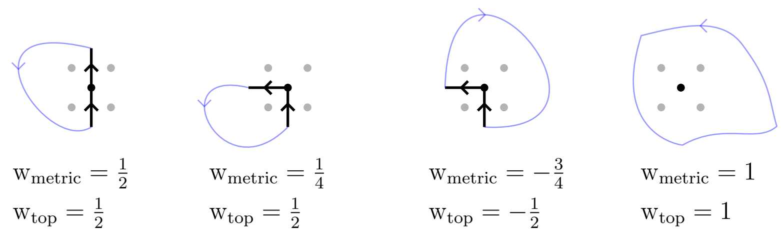


Figure 2: Illustration of topological and metric weights. The points in $\mathcal{N}_v$ are in grey.

**Lemma 3.**

$$\mathrm{charge}_{\mathrm{int}}(\gamma) = \sum_{v \in \mathbb{Z}^2} \mathrm{color}(v)\, \mathrm{w}_{\mathrm{top}}(\gamma, v).$$

*Proof.* Notice that

$$\mathrm{w}_{\mathrm{top}}(\gamma, v) = \begin{cases} \mathrm{wind}(\gamma, v), & \text{if } v \notin \gamma, \\ \frac{1}{2}, & \text{if } v \in \gamma \text{ and } \gamma \text{ is counterclockwise oriented,} \\ -\frac{1}{2}, & \text{if } v \in \gamma \text{ and } \gamma \text{ is clockwise oriented.} \end{cases}$$

In particular, $\sum_{v \in \gamma} \mathrm{w}_{\mathrm{top}}(\gamma, v)\, \mathrm{color}(v) = \pm\frac{1}{2} \sum_{v \in \gamma} \mathrm{color}(v) = 0$. Hence,

$$\mathrm{charge}_{\mathrm{int}}(\gamma) = \sum_{v \in \mathbb{Z}^2} \mathrm{color}(v)\, \mathrm{w}_{\mathrm{top}}(\gamma, v).$$

□

If $s$ is the corresponding sock of a tiling $t$ and $\gamma$ is a cycle of $s$, then the angle $\angle(\gamma, v)$ of a vertex $v \in \gamma$ is the difference between the angle of the edge of $\gamma$ leaving $v$ and the angle of the edge of $\gamma$ entering $v$, counted in counterclockwise laps. In other words, a vertex $v$ where the curve goes straight has angle 0, whereas a vertex where a left (resp. right) turn occurs has angle $1/4$ (resp. $-1/4$), as shown in Figure 3.

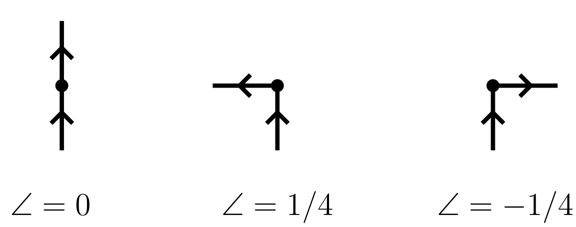


Figure 3: Illustration of the angle of a vertex.

The *boundary charge* of a curve $\gamma$ is $\operatorname{charge}_\partial(\gamma) = \sum_{v\in\gamma} \angle(\gamma, v)\operatorname{color}(v)$. Notice that

$$T^{\vec{\mathbf{k}}}(t) = \sum_{\gamma \text{ cycle of } s} \operatorname{charge}_\partial(\gamma). \tag{2}$$

We now set out to prove that, for each $\gamma$, $\operatorname{charge}_\partial(\gamma) = \operatorname{charge}_{\mathrm{int}}(\gamma)$, which will complete the proof of Proposition 1.

**Lemma 4.** *For each cycle $\gamma$ of $s$,*

$$\sum_{v\in\mathbb{Z}^2} \mathrm{w}_{\mathrm{metric}}(\gamma, v)\operatorname{color}(v) = 0.$$

*Proof.*

$$\sum_{v\in\mathbb{Z}^2} \mathrm{w}_{\mathrm{metric}}(\gamma, v)\operatorname{color}(v) = \sum_{u\in\mathbb{Z}^2+(\frac{1}{2},\frac{1}{2})} \left(\frac{1}{4}\operatorname{wind}(\gamma, u)\sum_{v\in\mathcal{N}_u}\operatorname{color}(v)\right) = 0.$$

□

**Lemma 5.** *If $v \in \mathbb{Z}^2$,*

$$\mathrm{w}_{\mathrm{top}}(\gamma, v) - \mathrm{w}_{\mathrm{metric}}(\gamma, v) = \begin{cases} \angle(\gamma, v), & \textit{if } v \in \gamma, \\ 0, & \textit{otherwise.} \end{cases}$$

*Proof.* Figure 2 illustrates most of the elements needed in this proof. If $v \notin \gamma$, then $\operatorname{wind}(\gamma, u) = \operatorname{wind}(\gamma, v)$ for each $u \in \mathcal{N}_v$, so $\mathrm{w}_{\mathrm{metric}}(\gamma, v) = \mathrm{w}_{\mathrm{top}}(\gamma, v) = \operatorname{wind}(\gamma, v)$.

If $v \in \gamma$ and the curve goes straight at $v$, then, for some $k$, two points in $\mathcal{N}_v$ have winding number $k$ and the other two have winding number $k+1$, so $\mathrm{w}_{\mathrm{metric}}(\gamma, v) = \frac{1}{4}(k + k + (k+1) + (k+1)) = \frac{1}{2}(k + (k+1)) = \mathrm{w}_{\mathrm{top}}(\gamma, v)$.

If $v \in \gamma$ and the curves turns left at $v$, then the winding numbers are $k, k, k, k+1$, so $\mathrm{w}_{\mathrm{top}}(\gamma, v) - \mathrm{w}_{\mathrm{metric}}(\gamma, v) = 1/4 = \angle(\gamma, v)$. Analogously, $\mathrm{w}_{\mathrm{top}}(\gamma, v) - \mathrm{w}_{\mathrm{metric}}(\gamma, v) = -1/4 = \angle(\gamma, v)$ if $\gamma$ turns right at $v$. □

**Lemma 6.** *For each cycle $\gamma$ of $s$,* $\operatorname{charge}_\partial(\gamma) = \operatorname{charge}_{\mathrm{int}}(\gamma)$.

*Proof.* By Lemma 5, the boundary charge of $\gamma$ can also be written as

$$\begin{aligned}
\operatorname{charge}_\partial(\gamma) &= \sum_{v\in\gamma} \angle(\gamma, v)\operatorname{color}(v) \\
&= \sum_{v\in\mathbb{Z}^2} (\mathrm{w}_{\mathrm{top}}(\gamma, v) - \mathrm{w}_{\mathrm{metric}}(\gamma, v))\operatorname{color}(v) \\
&= \sum_{v\in\mathbb{Z}^2} \mathrm{w}_{\mathrm{top}}(\gamma, v)\operatorname{color}(v) - \sum_{v\in\mathbb{Z}^2} \mathrm{w}_{\mathrm{metric}}(\gamma, v)\operatorname{color}(v) \\
&= \sum_{v\in\mathbb{Z}^2} \mathrm{w}_{\mathrm{top}}(\gamma, v)\operatorname{color}(v) = \operatorname{charge}_{\mathrm{int}}(\gamma),
\end{aligned}$$

the fourth equality holding because of Lemma 4; the last equality is Lemma 3. □

*Proof of Proposition 1.* Lemma 6 and Equation (2) imply that $P_t'(1) = T^{\vec{\mathbf{k}}}(t)$. Together with Lemma 2, this proves the result. □

**Remark 7.** Proposition 1 and Proposition 3.6 in [1] establish Remark 10.1 in [2].

# References

[1] Pedro H Milet and Nicolau C Saldanha. Domino tilings of three-dimensional regions: flips, trits and twists. *arXiv preprint arXiv:1410.7693*, 2014.

[2] Pedro H Milet and Nicolau C Saldanha. Flip invariance for domino tilings of three-dimensional regions with two floors. *arXiv preprint arXiv:1404.6509*, 2014.

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