Stability of Compact Actions of $\mathbb{R}^n$ of Codimension One

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Abstract: In this paper we study the C^r stability of locally free compact actions of $\mathbb{R}^n$ with compact orbits over manifolds of dimension $n+1$. More precisely, we show that in many cases a C^1 perturbation of an action with all orbits compact must also have all orbits compact and a C^0 perturbation usually has many compact orbits.

0. Introduction

In this paper we study the C^r stability of locally free compact actions of $\mathbb{R}^n$ over a manifold of dimension $n+1$. More precisely, let θ be locally free smooth action of $\mathbb{R}^n$ over a manifold of dimension $n+1$ such that all orbits are compact; θ is a family of n commuting linearly independent vector fields. Let $\tilde{\theta}$ be another such nearby family. We want to know if $\tilde{\theta}$ also has compact orbits only: with the C^0 topology for corresponding vector fields, this is not necessarily the case. With the C^1 topology, however, the situation is much more interesting: in most cases (which we characterize) we *can* guarantee that all orbits of $\tilde{\theta}$ are compact. All actions and vector fields are assumed to be smooth.

Let θ be a smooth action of $\mathbb{R}^n$ over $M = \mathbb{T}^n \times (-\epsilon, \epsilon)$ given by commuting vectors fields $X_1, \ldots, X_n$. Using $\mathbb{T}^n = \mathbb{R}^n/\mathbb{Z}^n$, we give coordinates $(y_1, \ldots, y_n, z)$ to M; let $Y_1, \ldots, Y_n, Z$ be the corresponding vector fields. Assume that θ is such that $X_j(y_1, \ldots, y_n, z) = \sum_i a_{ij}(z) Y_i$ (we call such an action *homogeneous horizontal*; actually, as we shall see, this is really a matter of choosing the appropriate coordinates) and let $A(z)$ be the matrix with entries a_{ij}; the j-th column of A gives the coordinates of X_j in the $Y_1, \ldots, Y_n$ basis. We assume of course $A(z)$ invertible so that the action is locally free. For homogeneous horizontal actions our main result is the following (but the reader is invited to read examples 1.4, 1.5, 1.8 and 2.5 right now):

Theorem:

(a) *If* $\operatorname{rank}(A'(0)) > 1$, *the orbit* $\mathbb{T}^n \times \{0\}$ *is* C^1*-stable in the following sense: for all sufficiently* C^1*-close families of commuting vector fields* $\tilde{\theta}$, *the orbits of* $\tilde{\theta}$ *intersecting* $\mathbb{T}^n \times \{0\}$ *are all compact.*

(b) *If* $\operatorname{rank}(A'(0)) \leq 1$, *the orbit* $\mathbb{T}^n \times \{0\}$ *is* C^1*-unstable in the following sense: for all* $\delta > 0$ *there exists a family of commuting vector fields* $\tilde{\theta}$, *coinciding with* θ *outside* $\mathbb{T}^n \times (-\delta, \delta)$, *at a distance less than* δ *(in the* C^1 *metric) from* θ, *such that all orbits of* $\tilde{\theta}$ *intersecting* $\mathbb{T}^n \times \{0\}$ *are non-compact.*

Although actions of $\mathbb{R}^n$ over manifolds have been studied by several authors ([AS], [CRW]), the questions raised in this paper appear to be new.

In the first section we restate this theorem somewhat more carefully, discuss the exact relationship of this result to the questions of the previous paragraph, give examples of applications of the theorem, and describe the situation concerning the C^0 topology (propositions 1.6 and 1.7). In the second section we give the proof of the main theorem and a few auxiliary results, including proposition 2.2, a result about foliations, and lemma 2.4, which shows that our most important tool, the linear transformation τ, is constant on orbits. In the third section we discuss possible generalizations of these results: this amounts mostly to stating unanswered questions.

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1. Statement of the Theorem and Applications

Let $\theta : \mathbb{R}^n \times M^{n+m} \to M^{n+m}$ be a locally free action. *All actions are assumed to be smooth (i.e., C^∞).* We shall call n the *dimension* and m the *codimension* of the action; in this paper we are almost exclusively concerned with the case $m = 1$. We call $\mathbb{R}^n$ the *domain* of the action and sometimes denote it by D. For a basis of D, the action θ is defined by a family of commuting vector fields X_i, $1 \leq i \leq n$, which are linearly independent at every point. On an open manifold, such a family of vectors in general defines what we call a *local action*.

Given $p \in M$, define $\mathrm{Iso}_p = \{x \in \mathbb{R}^n \mid \theta(p, x) = p\}$, the *isotropy group* of p: Iso_p is a discrete subgroup of $\mathbb{R}^n$. If the rank of Iso_p is k, the orbit of p is isomorphic to $\mathbb{T}^k \times \mathbb{R}^{n-k}$. An action all of whose orbits are compact will be called a *compact* action.

If M is compact and has a smooth Riemannian metric, the space of actions over M has a metric given by the corresponding C^r metric over families of vector fields. More exactly, for this metric, a perturbation $\tilde{\theta}$ of θ is such that the partial derivatives up to order r of $(\tilde{X}_i)$ and (X_i) are close to each other. If M is not compact, we have to allow for infinite distances.

Important examples of manifolds of dimension $n+m$ are $\mathbb{T}^n \times B^m_\epsilon$ and $\mathbb{T}^n \times N^m$ where $\mathbb{T}^n$ is an n-dimensional torus, B^m_ϵ is the open ball of radius ϵ around the origin in $\mathbb{R}^m$ and N^m is a compact manifold of dimension m. We shall often think of $\mathbb{T}^n$ as $\mathbb{R}^n/\mathbb{Z}^n$. In any case, $\mathbb{T}^n$ will have coordinates $y_1, \ldots, y_n$ and B^m_ϵ will have coordinates $z_1, \ldots, z_m$. We shall call the directions corresponding to $\mathbb{T}^n$ *horizontal* and the directions corresponding to B^m_ϵ or N^m *vertical*. We shall consider vector spaces $H = \mathbb{R}^n$ and $V = \mathbb{R}^m$ with coordinates y_i and z_j, respectively: the spaces H and V can be thought of as two components in the tangent space to the manifold; the space H can also be thought of as $H_1(\mathbb{T}^n; \mathbb{R})$.

The coordinates of the domain $D = \mathbb{R}^n$ of an action will be called $x_1, \ldots, x_n$. Consider an action θ of $\mathbb{R}^n$ on $\mathbb{T}^n \times N^m$; θ is defined by n vector fields $X_1, \ldots, X_n$. If θ is such that we can write

$$X_j = \sum_{1 \leq i \leq n} f_{ij}(z) Y_i + \sum_{1 \leq k \leq m} g_{kj}(z) Z_k$$

for all j, $1 \leq j \leq n$, we call θ *homogeneous*; if the g_{kj} above are all identically zero we call θ *horizontal* (notice the somewhat unusual indexes). A homogeneous horizontal action θ is obviously compact and can be naturally identified with a map A from N^m to $GL(n, \mathbb{R})$ given by

$$A(z) = \begin{pmatrix} f_{11}(z) & \dots & f_{1n}(z) \\ \vdots & \ddots & \vdots \\ f_{n1}(z) & \dots & f_{nn}(z) \end{pmatrix}.$$

The j-th column of A gives the coordinates of X_j in the $Y_1, \dots, Y_n$ basis. The matrix A thus represents (in the canonical basis) an invertible linear transformation from D to H. Thus, the columns of A^{-1} form a natural basis for Iso_p (with the canonical basis in D). Our interest in homogeneous actions is justified by the following proposition:

Proposition 1.0: *Let θ be a compact action of $\mathbb{R}^n$ on M^{n+1} and U be a compact orbit of θ. Then there are arbitrarily small open neighbourhoods B of U in M such that B is invariant by θ. Furthermore, if U has trivial holonomy, then B can be chosen so that there exists a smooth bijection ψ between B and $\mathbb{T}^n \times (-\epsilon, \epsilon)$ such that if ψ is used to define an action $\tilde{\theta}$ on $\mathbb{T}^n \times (-\epsilon, \epsilon)$ then $\tilde{\theta}$ is horizontal homogeneous.*

This proposition says that if we want to look only at the neighbourhood of one compact orbit then, if the orbit has trivial holonomy, we may just as well assume that we are dealing with a horizontal homogeneous action on $\mathbb{T}^n \times (-\epsilon, \epsilon)$. This kind of assumption shall be made from now on without explicit mention of this proposition. Notice that if our action has codimension one and M is orientable, then all orbits have trivial holonomy.

Proof: The first statement has nothing to do with actions: it is a known fact about foliations. Notice that it does not hold in general for larger values of m (see [Su]).

The second statement is proved by constructing one such ψ. Choose a smooth line in B, transverse to the action at every point. Take this line to $\{0\} \times (-\epsilon, \epsilon)$ by a smooth bijection. This bijection can now be extended to define our ψ in a unique way. It is straightforward to check that ψ is smooth. Notice also that all homogenizations are constructed as above. ∎

Let us define when we will call an orbit *C^1-stable*. More than one definition is plausible; we may perturb the action globally or locally, and this second possibility is open to more than one interpretation. Also, we may restrict ourselves to homogeneous perturbations. We shall now give two rather extreme definitions: so extreme that any 'plausible' definition of C^1-stability must imply one and be implied by the other. In fact, they are so extreme that the surprise may come when we see that, at least in our context, they are equivalent!

Definition 1.1: *Let θ be an action of $\mathbb{R}^n$ on M^{n+1} and U be a compact orbit contained in an open set of compact orbits.*

(a) *The orbit U is said to be C^1* totally stable, *or C^1-*T-stable, *if for any neighbourhood B of U there exists $\delta > 0$ such that any local action defined on B which is a distance less than δ away from θ restricted to B (in the C^1 metric for families of vector fields) has* *the property that all orbits intersecting U are contained in B and compact. Otherwise, U is said to be* C^1-T-unstable.

(b) *The orbit U is said to be C^1* homogeneously locally unstable, *or* C^1-HL-unstable, *if for any homogenization of the action near U, any neighbourhood B of U and any $\epsilon > 0$ there is an action $\tilde{\theta}$ which coincides with θ outside B, is homogeneous for the given homogenization inside B, is at a distance less than ϵ from θ (again, in the C^1 metric for families of vector fields) and such that the orbits of $\tilde{\theta}$ intersecting U are not compact. Otherwise, U is said to be* C^1-HL-stable.

Clearly, the freedom to choose the homogenization in item (b) above is totally superfluous given the strong relationship between different homogenizations.

Definition 1.2: *Let θ be an action of $\mathbb{R}^n$ on M^{n+1}, U be a compact orbit contained in an open set of compact orbits and $\gamma : (-\epsilon, \epsilon) \to M$ be a smooth regular path transversal to orbits with $\gamma(0) \in U$. Let $A : (-\epsilon, \epsilon) \to GL(n, \mathbb{R})$ be a smooth function such that the columns of $A^{-1}(t)$ form a basis for $\mathrm{Iso}_{\gamma(t)}$ in some fixed basis in D. We define the* rank *of U, or* $\mathrm{rank}(U)$, *to be* $\mathrm{rank}(A'(0))$.

The reader can easily verify that the choice of γ and A does not affect $\mathrm{rank}(U)$; it is also clear that $A(z)$ as in the introduction is a special case of this construction.

We are now ready to state our theorem in full generality.

Theorem: *Consider a compact action θ of $\mathbb{R}^n$ on M^{n+1} and let U be any orbit with trivial holonomy. The following conditions are equivalent:*

(a) $\mathrm{rank}(U) \geq 1$.

(b) *The orbit U is C^1-T-stable.*

(c) *The orbit U is C^1-HL-stable.*

It should be clear from previous considerations that this last theorem is indeed equivalent to the only apparently less general version given in the introduction.

Now that we have stated a very general result let us move in the opposite direction and see what this theorem boils down to in simple cases. For homogeneous horizontal actions on $\mathbb{T}^n$, we assume that the z-coordinate of $\gamma(t)$ is t, so that $z = t$ and we can dispense with γ and talk about $A'(z)$.

Corollary 1.3: *Consider a homogeneous horizontal action θ on $\mathbb{T}^3$. Then the following conditions are equivalent:*

(a) *For any z,* $\det(A'(z)) \neq 0$.

(b) *Any C^1 perturbation of θ is compact.*

(c) *Any homogeneous C^1 perturbation of θ is compact.*

Example 1.4: *All sufficiently small perturbations (in the C^1 metric for corresponding vector fields) of the action of $\mathbb{R}^2$ on $\mathbb{T}^3$ given by*

$$\begin{aligned} X_1 &= \cos(2\pi z)Y_1 + \sin(2\pi z)Y_2 \\ X_2 &= -\sin(2\pi z)Y_1 + \cos(2\pi z)Y_2 \end{aligned}$$

are compact.

Indeed, from the above results it suffices to show that $\det(A'(z)) \neq 0$, a straightforward computation.

Notice that a generic compact action of $\mathbb{R}^2$ on $\mathbb{T}^3$ does not necessarily satisfy item (a) of Corollary 1.3: we can thus easily construct open sets of examples for which C^1 perturbations may create bands of non-compact orbits. Indeed, each point where the determinant changes sign corresponds to a C^1 unstable orbit where we can destroy the compactness.

Example 1.5: *If θ is a generic compact action of $\mathbb{R}^n$ on $\mathbb{T}^{n+1}$, $n > 2$, then all sufficiently small perturbations of θ (in the C^1 metric for corresponding vector fields) are compact.*

The argument is similar to the previous example if we notice that generically the rank of A' is always greater than or equal to $n-1$: this follows from a transversality argument since the set of matrices with rank strictly less than $n-1$ has codimension greater than one.

We conclude this section by taking a look at C^0 perturbations.

Proposition 1.6: *Suppose M compact. Given any compact action θ of codimension 1 there exist actions $\tilde{\theta}$, arbitrarily near θ in the C^0 metric for corresponding vector fields, such that $\tilde{\theta}$ has only a finite number of compact orbits.*

If this says that C^0 perturbations are never guaranteed to preserve compact actions, the next proposition says that usually there are still going to be plenty of compact orbits left.

Proposition 1.7: *Let θ be a compact homogeneous action of $\mathbb{R}^n$ over $\mathbb{T}^n \times (-\epsilon, \epsilon)$ defined by matrices $A(z)$ as above. Suppose z_1 and z_2 are such that $\operatorname{rank}(A(z_1) - A(z_2)) > 1$. Then any sufficiently C^0-close family of commuting vector fields has a compact orbit contained between z_1 and z_2.*

The proof of proposition 1.7 depends on the same techniques as the proof of the theorem; it is therefore postponed to the end of section 2. The much easier proof of proposition 1.6 is given at the end of this section. The following example follows immediately from proposition 1.7.

Example 1.8: *Given $\epsilon > 0$, for any sufficiently small perturbation $\tilde{\theta}$ (in the C^0 metric for corresponding vector fields) of the action θ of $\mathbb{R}^2$ on $\mathbb{T}^3$ in example 1.4 above and for any open ball B in $\mathbb{T}^3$ of radius ϵ there exists a compact orbit of $\tilde{\theta}$ intersecting B.*

Proof of Proposition 1.6: Let N be the quotient of M by the action. If M is orientable and connected, we must have $N = S^1$; in the general case, N will be a 1-dimensional orbifold. The reader may keep in mind the situation $N = S^1$ during the proof if he wishes to, the general case being analogous. We have that M is a torus bundle over N. The action is given by a function A that takes each point of N to some point in the space of invertible linear transformations from $\mathbb{R}^n$ to the tangent space to the corresponding fiber.

Let us now consider the case $M = \mathbb{T}^n \times N$. The function A is now simply a function from N to $GL(n, \mathbb{R})$. This function can be thought of as a function from N to n-tuples of vectors in $\mathbb{R}^n$. By a small perturbation of θ, we can suppose that N is divided into a finite number of disjoint intervals such that inside each interval all but one of the vectors in $\mathbb{R}^n$ given by A is constant. We can now, using bump functions, introduce a small vertical component to the changing vector in the interior of each interval. (Remember that by 'vertical' we mean the N direction.) It is now straightforward to check that the resulting vector fields commute and that the corresponding action only has compact orbits at points separating two intervals.

The general case is similar, except that we have to give a meaning to the phrase "all but one of the vectors ... is *constant*". This, however, is no serious problem since we can find local trivializations for the bundle. We can assume that the trivializations cover M with a finite number of sets and that the trivializations coincide in the intersections up to a fixed linear transformation. ■

2. Proof of the Theorem

This section is devoted to the proof of our main theorem and of proposition 1.7, together with a few auxiliary results. We turn back to the formulation given in the introduction, beginning with the easier proof of item (b).

Proof of item (b) of the Theorem: The hypothesis $\operatorname{rank}(A'(0)) \leq 1$ allows us to change the basis of $\mathbb{R}^n$ (the domain of the action) so that we have $X_i'(0) = 0$ for $i = 1, \ldots, n-1$ (the derivative is of course with respect to z: X_i depends on z only). For all $\delta > 0$ it is now easy to construct (using bump functions) another horizontal homogeneous action $\hat{\theta}$ given by vectors $\hat{X}_i$ such that:

- θ and $\hat{\theta}$ coincide outside of $\mathbb{T}^n \times (-\delta, \delta)$,
- the C^1-distance between θ and $\hat{\theta}$ is smaller than $\delta/2$,
- there exists $\epsilon_1 > 0$ such that $\hat{X}_i(z) = X_i(0)$ for all $i = 1, \ldots, n-1$ and all $z \in (-\epsilon_1, \epsilon_1)$.

We can now define $\tilde{\theta}$ by the vector fields $\tilde{X}_i = \hat{X}_i$, $i = 1, \ldots, n-1$ and $\tilde{X}_n = \hat{X}_n + \psi(z)Z$ where ψ is a small bump function with $\psi(0) > 0$ and support contained in $(-\epsilon_1, \epsilon_1)$. It is easy to check that $\tilde{X}_i$ indeed commute but that no orbit of $\tilde{\theta}$ intersecting $\mathbb{T}^n \times \{0\}$ is compact. If ψ is small enough, the C^1-distance between $\tilde{\theta}$ and $\hat{\theta}$ is less than $\delta/2$. This concludes the proof of item (b). The reader will have noticed the similarity between this proof and that of proposition 1.6. ■

We give a brief informal sketch of the proof of (a). From now on, let $D = \mathbb{R}^n$ stand for the domain of the action, $H = \mathbb{R}^n$ for the tangent plane to $\mathbb{T}^n$ generated by $Y_1, \ldots, Y_n$ and $V = \mathbb{R}$ for the line generated by Z. Any locally free smooth (local) action thus defines at each point a linear injective map, which we also denote by θ, from D to $H \oplus V$.

Consider the perturbed actions constructed in the above proof: for them, there is clearly some interesting structure which we may want to take into account. Namely, the action defines a linear bijection between $D' = \mathbb{R}^{n-1} \subset D$ and the subspace H' of H generated by $X_1(0), \ldots, X_{n-1}(0)$. Such subspaces clearly have a lot of geometric meaning: you can walk in $\mathbb{T}^n \times (-\epsilon, \epsilon)$ horizontally *and* on an orbit if you follow a direction in H'.

Furthermore, the isotropy group (if non-trivial) is contained in D'; it generates D' if the non-compact orbit is homeomorphic to $\mathbb{R} \times \mathbb{T}^{n-1}$.

We want to extend these notions to more general perturbed actions. More precisely, for $\tilde{\theta}$ a good perturbation (to be defined) and any point $(y_1, \ldots, y_n, z)$ of $M = \mathbb{T}^n \times (-\epsilon, \epsilon)$ with $|z| < \epsilon_1/2$ on a non-compact orbit of $\tilde{\theta}$ (ϵ_1 is to be defined in the next paragraph) we shall define subspaces $D' \subset D$ and $H' \subset H$ of codimension 1 and a linear bijection $\tau : H' \to D'$. We prove that D', H' and τ are constant on orbits; these concepts are closely related with that of asymptotic cycles (see [Sc]). Only then we use the hypothesis $\operatorname{rank}(A'(0)) > 1$ in order to derive a contradiction with the existence of perturbations with non-compact orbits. By the way, it clearly suffices to show that the point $(0, \ldots, 0, 0)$ can not be on a non-compact orbit: this observation leads us to a slight simplification in the notation. This concludes our sketch of the proof of (a).

We first of all make precise the notion of an action being close enough to θ. It is interesting to define the related concepts for foliations. Let $\mathcal{F}_0$ be the codimension 1 foliation of $\mathbb{T}^n \times (-\epsilon, \epsilon)$ defined by projection onto the second coordinate, so that $\mathcal{F}_0$ is the foliation corresponding to θ. By the C^r metric for foliations we mean the C^r metric for plane fields.

Definition 2.0: *A foliation $\mathcal{F}_1$ of M is called* acceptable *if its leaves are always transversal to vertical lines; an action is* acceptable *if its corresponding foliation is acceptable.*

It is clear that any $\mathcal{F}_1$ (resp., $\tilde{\theta}$) sufficiently C^0-close to $\mathcal{F}_0$ (resp., θ) is acceptable. An acceptable perturbation $\tilde{\theta}$ of θ naturally defines at each point a linear bijection $\tilde{\vartheta} : H \to D$, the inverse of the projection of $\tilde{\theta}$ onto H; of course, for $\tilde{\theta} = \theta$, $\tilde{\vartheta}$ is given (in the canonical basis) by $(A(z))^{-1}$. Also, given a path $\gamma : [a, b] \to H$ and a point $(\gamma(a), z_a)$ there exists, for sufficiently small δ, a unique path $\tilde{\gamma} : [a, a + \delta) \to M$ with $\tilde{\gamma}(a) = (\gamma(a), z_a)$ whose image is contained in an orbit of $\tilde{\theta}$; we say $\tilde{\gamma}$ is obtained by *lifting* γ; in other words, $\tilde{\gamma}$ must remain inside the same leaf of the foliation. The reason we may not be able to define $\tilde{\gamma}$ on all of $[a, b]$ is that it may "fall out" of M if z becomes too large or too small. Going back to A for a moment, there clearly exists $r_1 > 0$ such that the open ball of radius r_1 around $(A(0))^{-1}$ consists of invertible matrices only. Let $\epsilon_1 > 0$, $\epsilon_1 \leq \epsilon$, be such that $|z| < \epsilon_1$ implies $|(A(z))^{-1} - (A(0))^{-1}| < r_1/2$.

Definition 2.1: *We say an acceptable $\mathcal{F}_1$ is* good *if the following condition holds:*

- *For all γ of length less than $4n^2$ and $|z_a| < \epsilon/2$, γ can be lifted on its entire domain and the resulting curve is entirely contained in the region $|z| < \epsilon$.*

We say an acceptable $\tilde{\theta}$ is good *if the following conditions hold:*

- *If $|z| < \epsilon_1$ then $|\tilde{\vartheta}(y_1, \ldots, y_n, z) - (A(0))^{-1}| < r_1$.*
- *For all γ of length less than $4n^2$ and $|z_a| < \epsilon_1/2$, γ can be lifted on its entire domain and the resulting curve is entirely contained in the region $|z| < \epsilon_1$.*

The term $4n^2$ in this definition is actually a bit arbitrary. Again, it is clear that any $\mathcal{F}_1$ (resp., $\tilde{\theta}$) which is sufficiently C^0-close to $\mathcal{F}_0$ (resp., θ) is good.

We now state and prove an auxiliary result about perturbations of foliations.

Proposition 2.2: *Let $\mathcal{F}_1$ be a good perturbation of $\mathcal{F}_0$ and $p_0 = (y_1, \ldots, y_n, z)$ with $|z| < \epsilon/2$ be such that the leaf passing through p_0 is non-compact. Then there exists a unique hyperplane $H' \subseteq \mathbb{R}^n$ with the following properties:*

- *The natural projection $H' \to \mathbb{T}^n$ taking 0 to $(y_1, \ldots, y_n)$ can be globally lifted to a function $H' \to \mathbb{T}^n \times (-\epsilon, \epsilon)$ in such a way that 0 is taken to p_0.*
- *If H' is irrational (i.e., does not admit an equation with rational coefficients) then the closure of the image is a topological n-torus T_0 and its projection onto $\mathbb{T}^n$ is a homeomorphism; T_0 is contained in a union of non-compact leaves.*
- *If H' is rational then the supremum and infimum of the intersection of the image with each vertical line form two (possibly identical) smooth $(n-1)$-tori T_1^+ and T_1^- contained in non-compact leaves and such that their projections onto $H'/\mathbb{Z}^n \subseteq \mathbb{T}^n$ are diffeomorphisms.*

Furthermore, H' does not change if we substitute p_0 for another point on the same leaf.

Proof: We adopt all the terminology defined for actions; assume without loss that $p_0 = (0, \ldots, 0, 0)$. Given a good $\mathcal{F}_1$, we now define smooth strictly increasing functions

$$f_1^+, f_1^-, f_2^+, \ldots, f_n^- : (-\epsilon/2, \epsilon/2) \to (\epsilon, \epsilon).$$

Lift the path $\gamma_i^s(t) = (0, \ldots, st, \ldots, 0)$, $s = \pm 1$, $0 \le t \le 1$, where the only non-zero coordinate is in the i-th position, starting from the point $(0, \ldots, 0, z)$: call the end-point $(0, \ldots, 0, f_i^s(z))$. The above claims about our functions are trivial, as is the fact that f_i^+ and f_i^- are inverses of each other and that all these f's *commute*. The f's describe the *holonomy* of the foliation $\mathcal{F}_1$. If $f_i^s(0) = 0$ for all s and i, the orbit of $\tilde{\theta}$ through $(0, \ldots, 0, 0)$ is compact, contradicting the hypothesis. Assume, therefore, without loss of generality, $f_n^-(0) < 0 < f_n^+(0)$ and $f_n^-(0) \le f_i^-(0) \le 0 \le f_i^+(0) \le f_n^+(0)$ for all i, $0 \le i \le n$. It will be useful to consider an open subset $N \subseteq M = \mathbb{T}^n \times (-\epsilon, \epsilon)$ inside which our constructions happen: lift the cube $(-3n, 3n)^n$ around $(0, \ldots, 0, 0)$ (which we can do because $\mathcal{F}_1$ is good) and let N be the set of points p of M such that, if we draw a vertical line through p, we see points of the lifted square both above and below p.

We can now build a copy of S^1 by taking a quotient of the interval $(f_n^-(0), f_n^+(0))$: identify z with $f_n^+(z)$. The functions $f_1^+, f_2^+, \ldots, f_{n-1}^+$ naturally induce smooth functions $g_1, g_2, \ldots, g_{n-1}$ from this S^1 to itself: the formula $g_i([z]) = [f_i^+(z)]$ is unambiguous (f_n^+ and f_i^+ commute). Let h_i be the rotation number of g_i, with $0 \le h_i \le 1$, $h_i = 0$ iff $f_i^+(z) = z$ for some z and $h_i = 1$ iff $f_i^+(z) = f_n^+(z)$ for some z; for consistency in notation, let $h_n = 1$. We define H' as the space generated by $Y_i - h_i Y_n$, $1 \le i < n$; equivalently, H' is orthogonal to $(h_1, h_2, \ldots, h_n)$. Since conjugate functions have the same rotation number, this hyperplane is constant on orbits; we have to prove that it satisfies all the properties in the statement of the proposition.

We now prove that we can lift all of H' and that the resulting set is contained in N. In fact, consider the subset of H defined by the following inequality: $-n-1 \le \sum_i h_i \lfloor y_i \rfloor \le 1$. This set is a union of unit cubes (defined by the lattice $\mathbb{Z}^n$); in order to show that this can be lifted inside N we can proceed by induction on the distance of the cube from the origin. By connectivity of the set, it suffices at each step to prove that the next desired cube has gone neither above nor below N. This, however, follows easily from properties of rotation numbers. More precisely, let $G_1, \ldots, G_{n-1}, G_n$ be liftings of $g_1, \ldots, g_{n-1}, g_n$ to $\mathbb{R}$, with $G_n(t) = t + 1$. These functions correspond to the f_i up to a fixed change of coordinates but, unlike the f_i, are defined on all of $\mathbb{R}$. Suppose we want to reach a cube of coordinates $y_1, \ldots, y_n$ (these are now integers) contained in the above set. We clearly have $-n-2 \leq (G_1^{y_1} \circ \cdots \circ G_n^{y_n})(0) \leq n+2$. Furthermore, the order of the composition can be changed so that intermediate expressions also satisfy the same inequalities (remember that these functions commute). This not only shows that the desired cube lifts in N but also exhibits a path of cubes, also with lifts in N, which takes us there. Finally, since H' is contained in the above set, H' lifts to a subset of N.

Let us consider this lifting of H' to N and its projection to $\mathbb{T}^n$. If at least one of the h_i is irrational (case (i)), this projection is dense in $\mathbb{T}^n$; on the other hand, if all h_i are rational (case (ii)), this projection is a $(n-1)$-torus contained in $\mathbb{T}^n$. This classification will be useful in order to study the lifting of H' to N (and later in order to define D' and τ).

In case (i), assume without loss of generality that h_1 is irrational. Using Denjoy's theorem (see [D]), there exists a continuous conjugation between g_1 and the corresponding irrational rotation (the conjugation need *not* be smooth, or even C^1). By lifting the conjugation we have an increasing homeomorphism $\phi : (f_n^-(0), f_n^+(0)) \to (-1, 1)$ with $\phi(f_n^+(z)) = \phi(z) + 1$ and $\phi(f_1^+(z)) = \phi(z) + h_1$. Since all f's commute, we have $\phi(f_i^+(z)) = \phi(z) + h_i$. Clearly, ϕ can be naturally extended to, say, $((f_n^-)^{3n}(0), (f_n^+)^{3n}(0))$. With the help of ϕ, we shall now define a (C^0) n-dimensional torus $T_0 \subseteq N$ which plays a role similar to that of the torus $z = 0$ for the simple examples of item (b). We first define $\Phi : N \to \mathbb{R}$, a continuous function which is strictly increasing on vertical straight lines. Consider $p = (y_1, \ldots, y_n, z) \in N$, $0 \leq y_i < 1$, and let $\gamma : [0, 1] \to H$ be the straight line joining $(y_1, \ldots, y_n)$ to 0. Lift γ starting from p, which takes us to a point of the form $(0, \ldots, 0, z')$: define $\Phi(p) = \phi(z') - h_1 y_1 - \ldots - h_n y_n$. The continuity of Φ follows from the identities $\phi(f_i^+(z)) = \phi(z) + h_i$. Let $T_0 = \Phi^{-1}(\{0\})$; T_0 is the closure of the lifting of H' to N.

In case (ii), we define two $(n-1)$-dimensional tori T_1^+ and T_1^-. For each $(y_1, \ldots, y_n)$ in the projection to $\mathbb{T}^n$ of the lifting of H' to N, consider the set of corresponding z in the lifting and set z^+ and z^- to be the supremum and infimum of this set, respectively. Let T_1^+ be the set of points of the form $(y_1, \ldots, y_n, z^+)$ and similarly for T_1^-. Clearly, these tori are contained in non-compact orbits.

In any case, the uniqueness of H' is easy to verify. Indeed, any other hyperplane contains elements not in H': following such a direction, again by rotation numbers, we are "going up" and must eventually either leave M or accumulate on a compact orbit, in any case contradicting the conditions. ∎

We are ready to give the definition of one of our main tools. On each point p of T_0, T_1^+ or T_1^-, the action $\tilde{\theta}$ induces a linear map $\tilde{\tau}_p : H' \to D$. The torus T_0 naturally receives a unit measure by lifting the unit Lebesgue measure on $\mathbb{T}^n$ via the projection. In case (ii), the Lebesgue $(n-1)$-dimensional measure on the vertical projection of T_1^+ can be multiplied by a constant and lifted by the projection in order to give unit measures on T_1^+ and T_1^-. Whenever we integrate over these tori, we do it with respect to these measures.

Definition 2.3: *Let* $\tau = \int_{T_0} \tilde{\tau}_p dp$ *in case* (i) *and* $\tau = \int_{T_1^+} \tilde{\tau}_p dp$ *in case* (ii); D' *is the image of* τ.

In order to prove that τ behaves well we need a more geometric interpretation for it. We introduce an auxiliary function ξ. In case (i), the lifting of H' (which is contained in T_0) is contained in an orbit and can therefore be brought back to D. In other words, this defines a continuous map $\xi : H' \to D$ whose image is the pre-image of T_0 by the action $\tilde{\theta}$. In case (ii), consider the lifting of H' starting from the point $(0, \ldots, 0, z^+) \in T_1^+$ and do the same as above in order to define ξ. The next lemma relates τ and ξ.

Lemma 2.4: *Assume* $\tilde{\theta}$ *good. The linear transformation* τ *is injective, constant on orbits and satisfies* $\tau(v) = \lim_{t\to\infty} \frac{1}{t}\xi(tv)$.

Proof: Since $\tilde{\theta}$ is good, ξ can be extended to the neighbourhood of H' of radius 1. The compactness of the closure of N (as in proposition 2.2) implies that ξ is uniformly continuous even on this larger domain.

We first prove the formula for τ. Notice that $\xi(tv) = \int_0^t \tilde{\tau}_{p(s)}(v)ds$ where $p(s)$ is obtained by lifting sv to T_0 or T_1^+. Let us start by proving the formula for τ if v is such that the line tv is dense in the vertical projection to T_0 or T_1^+. Indeed, consider the vector field given by v on T_0 or T_1^+: it is uniquely ergodic, with our measure on T_0 or T_1^+ being the only invariant measure (see [M]); from Birkhoff's theorem, the above limit exists and its value is $\tau(v)$. For general v, a similar argument shows that the limit also exists and is equal to the average of $\tilde{\tau}_p(v)$ where p ranges over the closure of the line tv, a sub-torus. Furthermore, the average over the original torus is the average of averages over parallel sub-tori; if the averages over such tori are equal, we have the formula for arbitrary v. It suffices therefore to show that $\lim_{t\to\infty} \frac{1}{t}\xi(tv) = \lim_{t\to\infty} \frac{1}{t}\xi(tv + w)$ for any w: these other limits correspond to the averages of $\tilde{\tau}_p(v)$ over the parallel sub-tori and are thus known to exist. The equality of the two limits follows from the uniform continuity of ξ: the distance between tv and $tv + w$ is fixed, that between $\xi(tv)$ and $\xi(tv + w)$ is bounded and that between $\frac{1}{t}\xi(tv)$ and $\frac{1}{t}\xi(tv + w)$ tends to zero.

Consider now two points on the same orbit; we want to prove that τ is the same for these two points. Without loss of generality, they are at a distance smaller than 1 from each other. Take one of these to be the base point 0 in the above discussion and let $w \in H$ be a vector taking 0 to the other point in consideration; call the τ map for this second point τ_2. We already saw that H' is the same for both points. Let $v \in H'$ be arbitrary. As we saw above, $\tau(v) = \lim_{t\to\infty} \frac{1}{t}\xi(tv)$. Similarly, $\tau_2(v) = \lim_{t\to\infty} \frac{1}{t}\xi(tv + w)$. As before, the uniform continuity of ξ tells us that these limits are equal. The injectivity of τ follows directly from its definition and the hypothesis that $\tilde{\theta}$ is good: τ is an average of $\tilde{\tau}_p$, $\tilde{\tau}_p$ is the restriction of $\tilde{\vartheta}$ to H' and, again since $\tilde{\theta}$ is good, $\tilde{\vartheta}$ must always lie inside a convex ball of invertible linear transformations. ■

We now have all the necessary tools to handle item (a) of the theorem and proposition 1.7. The two proofs are somewhat similar: they are in a sense the finite and differential versions of the same fact. We begin with the easier proof.

Proof of Proposition 1.7: Notice first that if A_1 and A_2 are invertible, $\mathrm{rank}(A_1 - A_2) = \mathrm{rank}(A_1^{-1} - A_2^{-1})$. Thus $\mathrm{rank}(A(z_1) - A(z_2)) > 1$ implies $\mathrm{rank}(A^{-1}(z_1) - A^{-1}(z_2) > 1$. Assume without loss $z_1 < z_2$; since having high rank is an open property of matrices let $z_3 < z_4$ be such that $z_1 < z_3 < z_4 < z_2$ and $\mathrm{rank}(A^{-1}(z_3) - A^{-1}(z_4)) > 1$. It is clearly enough to prove that any $\tilde{\theta}$ sufficiently C^0 close to θ has a compact orbit touching some point with z coordinate between z_3 and z_4, since being nearly horizontal this compact orbit will then be contained between z_1 and z_2.

Assume by contradiction that this is not the case and take any point with z coordinate z_3. The orbit passing through it must be non-compact and must cross the z_4 level: otherwise, the supremum of that orbit would be a compact orbit between z_3 and z_4. There are therefore two points on the same non-compact orbit, one with z coordinate z_3 and the other with z coordinate z_4. On these two points τ must therefore be identical. But τ, being an average of $\tilde{\tau}_p$'s, is close to the restriction of A^{-1} to H'. Perturbations of $A^{-1}(z_3)$ and $A^{-1}(z_4)$ can be assumed, however, to have difference of rank greater than 1 and when restricted to a hyperplane would still be different. ■

Proof of item (a) of the Theorem: Up to this point we never bothered about the rank of $A'(0)$. Notice that the rank of $A'(0)$ equals that of $(A^{-1})'(0)$. From now on we suppose $\mathrm{rank}(A'(0)) > 1$ and that there exist arbitrarily good C^1-perturbations of θ such that the orbit through $(0, \ldots, 0, 0)$ is non-compact — and get a contradiction.

Let $U \subseteq H$ be a subspace of dimension 2 such that $(A^{-1})'(0)$ is injective when restricted to U. For any $\tilde{\theta}$ such that $(0, \ldots, 0, 0)$ is on a non-compact orbit we can define H' as above and $L \subseteq U \cap H'$ of dimension 1. By compactness of S^1, consider a sequence $\tilde{\theta}_k$ of perturbations of θ such that their C^1 distances to θ tend to zero and such that their versions L_k of L tend to a fixed line L_0. Let $v_0 \in L_0$ be a fixed non-zero vector, $w_0 = (A^{-1})'(0)(v_0)$ and $v_k \in L_k$ be non-zero vectors tending to v_0. Let τ_k be τ (as defined above) for $\tilde{\theta}_k$. We can compute $\tau_k(v_k)$ starting from $(0, \ldots, 0, 0)$ or from $(0, \ldots, 0, f_i^+(0))$, which, according to lemma 2.4, should give us the exact same answer: call these two (equal?) vectors q_k^- and q_k^+ (i is any index, $1 \leq i \leq n$, with $f_i^+(0) > 0$).

We now show that, for sufficiently large k, $(q_k^+ - q_k^-) \cdot w_0 > 0$, which is of course the desired contradiction. Let T^- and T^+ be the tori as described above for $(0, \ldots, 0, 0)$ and $(0, \ldots, 0, f_i^+(0))$, respectively (we sometimes omit the index k for brevity). We have $q_k^\pm = \int_{T^\pm} \tilde{\tau}_p(v) dp$, whence $q_k^\pm \cdot w_0 = \int_{T^\pm} \tilde{\tau}_p(v) \cdot w_0 dp$. Subtracting, we have

$$\begin{aligned}
(q_k^+ - q_k^-) \cdot w_0 &= \int_T (\tilde{\tau}_{p^+} - \tilde{\tau}_{p^-})(v) \cdot w_0 \, dp \\
&= \int_T \left(\int_{z^-}^{z^+} \frac{\partial}{\partial z} \vartheta(p, z) dz \right)(v) \cdot w_0 \, dp \\
&= \int\!\!\int \frac{\partial}{\partial z} \vartheta(p, z)(v) \cdot w_0 \, dz \, dp.
\end{aligned}$$

For our good C^1 perturbations, however, $\frac{\partial}{\partial z}\vartheta$ is close to $(A^{-1})'(0)$ and v is close to v_0. In other words, the expression inside the last integral is close to $(A^{-1})'(0)(v_0) \cdot w_0 = w_0 \cdot w_0$ and therefore strictly positive. This concludes the proof of the inequality, which, as we have seen, contradicts previous results. This finishes the proof of item (a) and of the theorem. ∎

We finish this section by returning to our favorite action, as introduced in example 1.4.

Example 2.5: *Let θ be as in example 1.4; there exists $\epsilon > 0$ such that any real analytic $\tilde{\theta}$ which is at a distance less than ϵ from θ in the C^0 topology for vector fields is a compact action.*

If there are non-isolated compact orbits, our claim follows from analyticity of $\tilde{\theta}$. Furthermore, $\tilde{\theta}$ can be assumed to be globally good. Thus, H' is defined almost everywhere and is locally constant. Its definition makes it clear that it varies analytically when restricted to a vertical circle, i.e., is the same on both sides of a compact orbit. Thus, H' can be extended so as to be constant on $\mathbb{T}^3$. The same kind of considerations show that τ is the same on both sides of a compact orbit, thus constant. But this contradicts the fact that τ is near A^{-1}.

3. Generalizations

There are two natural directions along which our theorem could be generalized. One of them is by considering other kinds of perturbations, more precisely, by considering C^r perturbations for other values of r. Another one is by considering higher codimensions. We shall briefly consider such questions in this final section.

The first obvious thing to notice about C^r perturbations, $r > 1$, is that they are special cases of C^1 perturbations. It is therefore clear that things are more stable for higher values of r. If there exists a codimension 1 subspace of D on which the action is constant up to order r the construction for item (b) of the theorem can be applied and there is no stability of compact orbits. If this is not the case, the question on whether some sort of stability phenomenon occurs is unanswered: it is easy to construct τ and other objects as in Section 2 but it is not clear what to do about the proof of item (a), which is in fact a motonotonicity argument. In this case, however, the compact orbit is *homogeneously* stable, i.e., there are no nearby homogeneous actions with non-compact orbits. Indeed, for homogeneous actions the tori T_0 or $T_1^{\pm}$ are horizontal and τ is computed as an average of a constant function: the perturbation of the action gives us therefore a perturbation of A which is constant on a small interval. In particular, a generic horizontal homogeneous action of $\mathbb{R}^2$ on $\mathbb{T}^3$ is C^2 homogeneously stable; is it C^2 stable?

The higher codimension case seems to be much more complicated. Indeed, if we believe that homogeneous perturbations are representative, there exist stability phenomena even in high codimension, as the following considerations show.

Consider a homogeneous horizontal action of dimension n and codimension m. Let $X_i = \sum_j f_{ij} Y_j$ and let $H^\ell, \ell = 1, \ldots, n$ be matrices with $h^\ell_{kj} = \frac{\partial f_{j\ell}}{\partial y_k}$. If it is possible to *homogeneously* perturb this action in order to destroy compact orbits we can consider a perturbation of the form

$$\tilde{X}_i = \sum_j \tilde{f}_{ij} Y_j + \sum_k g_{ik} Z_k.$$

A straightforward computation shows that

$$[\tilde{X}_i, \tilde{X}_{i'}] = \sum_{\ell,k} (g_{ik} \tilde{h}^{\ell}_{ki'} - g_{i'k} \tilde{h}^{\ell}_{ki}) Y_\ell.$$

For the perturbed matrices $\tilde{H}^\ell$, therefore, there exists a non-zero $n \times m$ matrix G such that all of the matrices $G\tilde{H}^\ell$ are symmetric. Since this is a closed property, there must exist a non-zero G such that the matrices GH^ℓ are symmetric for all ℓ.

As an example, an action of $\mathbb{R}^4$ on $T^4 \times B^4_\epsilon$ with matrices given by

$$H_0 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \qquad H_1 = \begin{pmatrix} 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \end{pmatrix}$$

$$H_2 = \begin{pmatrix} 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{pmatrix} \qquad H_3 = \begin{pmatrix} 0 & 0 & 0 & -1 \\ 0 & 0 & -1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}$$

is C^1 homogeneously stable, since a straightforward computation will show that the only G such that GH^ℓ is symmetric for all four matrices is the zero matrix.

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