Spectra of Regular Polytopes

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Introduction

As is well known, the combinatorial problem of counting paths of length n between two fixed vertices in a graph reduces to raising the adjacency matrix A of the graph to the n-th power ([B], p. 11). For an undirected graph, A is symmetric and the problem above simplifies considerably if its spectrum $\sigma(A)$ is known and contains few distinct elements. Spectra of graphs, meaning spectra of the corresponding adjacency matrices, have been an active subject for decades ([CDS]). In this paper, we compute the spectra with multiplicities of the adjacency graphs of all regular polytopes in $\mathbb{R}^n$, or, for brevity, spectra of polytopes. We begin with a short theoretical set-up and then proceed to the computations.

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Theoretical Remarks

Let P be a regular polyhedron in $\mathbb{R}^n$ and let V be its set of vertices. The symmetry group G of P, which includes orientation preserving and orientation reversing isometries, acts transitively on V. Given a vertex v of P, let H_v be the subgroup of G fixing v. The orbits of the action of H_v on V will be called *levels* of V with respect to v. Let $A = A_P$ be the adjacency matrix of P, with an eigenvalue λ associated to an eigenvector $u \in U$, the vector space of functions from V to $\mathbb{C}$. By the transitivity of G, we can assume $u(v) \neq 0$. By averaging with respect to H_v, we can take u to be non-zero and constant on levels. Moreover, the subspace $S \subseteq U$ of vectors taking constant values on levels is invariant under A. In particular, A induces a linear transformation $\tilde{B}$ from S to itself, and the spectra of A and $\tilde{B}$ coincide.

For convenience, we represent $\tilde{B}$ by a matrix B by choosing a normalized ℓ^∞ basis of S consisting of vectors taking the value 1 in one level and 0 everywhere else. In this basis, $b_{i,j}$ is given by the number of neighbours in level j of an arbitrary element in level i. This implies that the sum of entries of any line of B is constant equal to the number of neighbours of an arbitrary vertex. There are two other natural basis, differing from the previous one by normalization: the ℓ^1 basis, whose elements have entries adding to one and the ℓ^2 basis, whose elements have Euclidean norm one. In the ℓ^2 basis, the transformation $\tilde{B}$ is symmetric, as can be seen by counting the edges between two given levels from the point of view of each level. In particular, B is diagonalizable as well as its restrictions B_e and B_o.

In a language familiar to graph theorists, the multi-digraph associated with B is a *front divisor* of the graph associated with A, i. e., the graph consisting of the vertices and edges of P ([CDS]).

Whenever P has central symmetry, B shall also have central symmetry provided levels are properly ordered. In this case, the linear involution R taking points to their antipodes on S commutes with $\tilde{B}$ and permutes elements of the chosen basis of S. The eigenspaces S_e and S_o associated to $+1$ and -1 of R are invariant under $\tilde{B}$, and for $\tilde{B}_e = \tilde{B}|_{S_e}$ and $\tilde{B}_o = \tilde{B}|_{S_o}$ we must have $\sigma(\tilde{B}) = \sigma(\tilde{B}_e) \cup \sigma(\tilde{B}_o)$ since $S = S_e \oplus S_o$.

We remind the reader that the computation of a high power n of a diagonalizable matrix M is easily performed once its spectrum (but not the multiplicities) is known. Indeed, the matrix M^n equals $p(M)$, where p is any polynomial taking each eigenvalue to its n-th power. In particular, any power of A, B, B_e and B_o can be computed from a polynomial with integral coefficients of degree smaller than the number of its distinct eigenvalues.

The symmetry of the polytope allows us to count paths of length n from one vertex to another simply by looking at the first column of B^n. Indeed, without loss, assume one of the vertices to be v, the vertex at the top level. Then the number of paths of length n from v to a vertex in level i is given by the i-th element of the first column of B^n. The proof is similar to the usual argument which identifies entries of powers of adjacency matrix to numbers of paths of prescribed length from one vertex to another. Also, B^n can be obtained from B_e^n and B_o^n, which simplifies computations even further.

We now indicate how to compute the multiplicities m_i of the distinct eigenvalues λ_i of A. Let ℓ_k be the number of closed loops of length k, with base point v. Then

$$|V| \cdot \ell_k = \operatorname{tr}(A^k) = \sum_{\lambda_i \neq \lambda_j} m_i \lambda_i^k.$$

Once the λ_i are known, the m_i are solutions of a linear system obtained by inserting small values of k.

We can also study the eigenspaces in U of the adjacency matrix A; remember that all vector spaces are complex. Representation theory now comes in naturally, since these eigenspaces are invariant under the action of the full symmetry group. Do the eigenspaces break into smaller group invariant subspaces? This is of course equivalent to asking whether the action restricted to the eigenspaces is irreducible. For eigenvalues of A which have multiplicity one as eigenvalues of B, this is true. Indeed, the averaging process described above can be performed within any invariant subspace, yielding a non-trivial subspace in the quotient S. If all eigenspaces are irreducible, the multiplicities of the eigenvalues correspond to the dimensions of the irreducible parts of the representation of the symmetry group in U. These dimensions could of course have been computed using the character table of the group, but linear algebra obtains the answer in a computationally simpler way.

Eigenvalues

The list of all regular polytopes in all dimensions is well known ([C]) and we shall almost always use Schläfli symbols to denote them. In dimension 2 we have the regular n-gons $\{n\}$, for $n \geq 3$. In dimension 3 we have the tetrahedron $\{3,3\}$, the octahedron $\{3,4\}$, the cube $\{4,3\}$, the icosahedron $\{3,5\}$ and the dodecahedron $\{5,3\}$. In dimension 4 we have six polytopes denoted by $\{3,3,3\}$, $\{3,3,4\}$, $\{4,3,3\}$, $\{3,4,3\}$, $\{3,3,5\}$ and $\{5,3,3\}$. In dimensions 5 and larger we have only three polytopes: $\{3,\dots,3\}$, $\{3,\dots,3,4\}$ and $\{4,3,\dots,3\}$ where the dots stand for a sequence of 3's. The first of these generalizes the tetrahedron and shall be called T_n where n is the dimension of the ambient space. Likewise, the second and third generalize the octahedron and cube and shall be denoted O_n and C_n respectively.

The spectrum of the regular n-gon is well known ([CDS]):2, $2\cos(2\pi/n)$, $2\cos(4\pi/n)$, $\dots$,$2\cos(2[\frac{n}{2}]\pi/n)$,with multiplicities 1,2,2,$\dots$,where the last number is 1 or 2 depending on the parity of n.

For $P = T_n$, we have

$$B = \begin{pmatrix} 0 & n \\ 1 & n-1 \end{pmatrix}.$$

The spectrum of P is therefore n, -1 and the multiplicities of these eigenvalues are 1 and n, respectively.

For $P = O_n$, we have

$$B = \begin{pmatrix} 0 & 2n-2 & 0 \\ 1 & 2n-4 & 1 \\ 0 & 2n-2 & 0 \end{pmatrix}.$$

The spectrum of P is therefore $2n-2$, -2, 0 and the multiplicities of these eigenvalues are 1, $n-1$ and n, respectively.

For $P = C_n$, we have

$$B = \begin{pmatrix} 0 & n & & & & & \\ 1 & 0 & n-1 & & & & \\ & 2 & 0 & \ddots & & & \\ & & \ddots & & \ddots & & \\ & & & \ddots & 0 & 2 & \\ & & & & n-1 & 0 & 1 \\ & & & & & n & 0 \end{pmatrix}.$$

We could compute the spectrum of this matrix but it is much easier to notice that, if A_n is the matrix A for $P = C_n$,

$$A_{n+1} = \begin{pmatrix} A_n & I_n \\ I_n & A_n \end{pmatrix}$$

for an adequate labeling of the vertices. By induction, the spectrum of A_n is n,$n-2$,...,$-n+2$,$-n$ with multiplicities $\binom{n}{0}$, $\binom{n}{1}$,...,$\binom{n}{n-1}$,$\binom{n}{n}$.

For $P = \{5,3\}$, the dodecahedron, the set of 20 vertices breaks in six levels $\mathbf{a}$, $\mathbf{b}$, $\mathbf{c}$, $-\mathbf{c}$, $-\mathbf{b}$ and $-\mathbf{a}$ with a clear geometric interpretation: hang the solid by a vertex v, and the vertices group at different heights in sets of 1, 3, 6, 6, 3 and 1 elements. The two central subsets are indeed levels, since $H_v = D_3$, the symmetry group of the triangle, contains orientation reversing isometries.

In order to build the matrix B we have to find out in what level lie the neighbours of each vertex. It is clear that the top vertex has all three neighbours in $\mathbf{b}$. Also, a vertex in $\mathbf{b}$ has one neighbour in $\mathbf{a}$ and two in $\mathbf{c}$, and a vertex in $\mathbf{c}$ has one neighbour in $\mathbf{b}$, one in $\mathbf{c}$and one in $-\mathbf{c}$. By symmetry, it is obvious what happens in levels $-\mathbf{c}$, $-\mathbf{b}$ and $-\mathbf{a}$. We then have

$$B = \begin{pmatrix} 0 & 3 & 0 & 0 & 0 & 0 \\ 1 & 0 & 2 & 0 & 0 & 0 \\ 0 & 1 & 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 & 1 & 0 \\ 0 & 0 & 0 & 2 & 0 & 1 \\ 0 & 0 & 0 & 0 & 3 & 0 \end{pmatrix}$$

and for the obvious choice of basis,

$$B_e = \begin{pmatrix} 0 & 3 & 0 \\ 1 & 0 & 2 \\ 0 & 1 & 2 \end{pmatrix} \qquad \text{and} \qquad B_o = \begin{pmatrix} 0 & 3 & 0 \\ 1 & 0 & 2 \\ 0 & 1 & 0 \end{pmatrix}$$

with spectra $3, 1, -2$ and $0, \sqrt{5}, -\sqrt{5}$ respectively. The multiplicities of these eigenvalues in A are 1, 5, 4, 4, 3, 3, in this order.

For $P = \{3,5\}$, the icosahedron, we have

$$B_e = \begin{pmatrix} 0 & 5 \\ 1 & 4 \end{pmatrix} \qquad \text{and} \qquad B_o = \begin{pmatrix} 0 & 5 \\ 1 & 0 \end{pmatrix}$$

with spectra $5, -1$ and $\sqrt{5}, -\sqrt{5}$ respectively. The multiplicities of these eigenvalues in A are 1, 5, 3, 3, in this order.

There are two easy ways to compute the matrices B, B_e and B_o for $P = \{3,4,3\}$. One way would be to study in some detail the combinatorial structure of the polytope, as will be done in the next example. Another way would be to use the following simple coordinates for the vertices ([C], p. 156): they are the 24 permutations of the coordinates of the vectors $(\pm 1, 0, 0, 0)$ and $(\pm\frac{1}{2}, \pm\frac{1}{2}, \pm\frac{1}{2}, \pm\frac{1}{2})$. From the coordinates, the adjacency relations among levels are obvious and, for $P = \{3, 4, 3\}$, we have

$$B_e = \begin{pmatrix} 0 & 8 & 0 \\ 1 & 4 & 3 \\ 0 & 8 & 0 \end{pmatrix} \qquad \text{and} \qquad B_o = \begin{pmatrix} 0 & 8 \\ 1 & 2 \end{pmatrix}$$

with spectra $8, 0, -4$ and $4, 2$ respectively. The multiplicities of these eigenvalues in A are 1, 9, 2, 4, 8, in this order.

In order to obtain B for $P = \{3, 3, 5\}$, we build the polytope from one vertex outwards, by adding pieces. We recall ([C], p. 153) that P has 120 vertices and 600 tetrahedral faces, each edge being surrounded by five tetrahedra. It follows that there are twelve edges incident to each vertex and that there are twenty tetrahedra around each vertex forming an icosahedron. Notice that here, as for any regular polytope, we can think of G as acting not only on vertices, but on k-dimensional faces. In this example, it may sometimes be helpful to think of G as acting on tetrahedral faces. More precisely, there is a unique element in G which takes one source tetrahedron to a target tetrahedron, with arbitrary assignment of vertices.

We begin the construction by calling **a** the level consisting of the single vertex v. We then surround v by twenty tetrahedra labelled α, which give us 12 new vertices in the new level **b**. Notice that **b** is a level: indeed, it is easy to check that **b** is closed and transitive under H_v.

In Figure (1.I), we represent six of the twelve vertices in **b**. From the adjacency relations among vertices in **a** and **b**, we learn that

$$B = \begin{pmatrix} 0 & 12 & \dots \\ 1 & 5 & \dots \\ \vdots & \vdots & \ddots \end{pmatrix}.$$

The five triangular faces in the picture belong to five distinct tetrahedra labelled α. The number of marks along an edge indicates the number of tetrahedra incident to that edge. Next, we glue twenty new tetrahedra labelled β over the exposed faces of the tetrahedra α, obtaining twenty vertices in a new level **c** (Figure (1.II)). Again, **c** is a level, since there is an obvious correspondence between the vertices in **c** and the tetrahedra α. Now, add 30 tetrahedra labelled γ as in Figure (1.III), and edges between elements in **b** get completely surrounded (and omitted from the subsequent pictures). From the construction so far we see that

$$B = \begin{pmatrix} 0 & 12 & 0 & \dots \\ 1 & 5 & 5 & \dots \\ 0 & 3 & 3 & \dots \\ \vdots & \vdots & \vdots & \ddots \end{pmatrix}.$$

At this point, Figures (1.IV) to (1.VII) should be self-explanatory. In the process, we obtained levels **d** and **e**, with 12 and 30 elements, and 60 tetrahedra labelled δ, 60 ϵ's, 60 ζ's and 20 η's.

We are almost halfway through. We can try to complete the polytope by glueing two copies of Figure (1.VII) with levels **a**, **b**, **c**, **d**, **e** and $-$**a**, $-$**b**, **c**, $-$**d**, $-$**e** $=$ **e** to each other along the exposed η-faces. We need 60 additional tetrahedra θ to accomplish this. We insert them in Figure (1.VIII). Notice the presence of 12 vertices belonging to the level $-$**d**. We have to check that we indeed have five tetrahedra around each edge. This follows by adding the numbers attached to corresponding edges in Figures (1.VII) and (1.VIII). We then have the following matrix B, with lines and columns corresponding to **a**, **b**, **c**, **d**, **e** $=$ $-$**e**, $-$**d**, $-$**c**, $-$**b**, $-$**a**:

$$B = \begin{pmatrix} 0 & 12 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 5 & 5 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 3 & 3 & 3 & 3 & 0 & 0 & 0 & 0 \\ 0 & 1 & 5 & 0 & 5 & 1 & 0 & 0 & 0 \\ 0 & 0 & 2 & 2 & 4 & 2 & 2 & 0 & 0 \\ 0 & 0 & 0 & 1 & 5 & 0 & 5 & 1 & 0 \\ 0 & 0 & 0 & 0 & 3 & 3 & 3 & 3 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 5 & 5 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 12 & 0 \end{pmatrix}.$$

From the usual procedure,

$$B_e = \begin{pmatrix} 0 & 12 & 0 & 0 & 0 \\ 1 & 5 & 5 & 1 & 0 \\ 0 & 3 & 3 & 3 & 3 \\ 0 & 1 & 5 & 1 & 5 \\ 0 & 0 & 4 & 4 & 4 \end{pmatrix} \qquad \text{and} \qquad B_o = \begin{pmatrix} 0 & 12 & 0 & 0 \\ 1 & 5 & 5 & 1 \\ 0 & 3 & 3 & 3 \\ 0 & 1 & 5 & -1 \end{pmatrix}$$

with spectra $12, 0, -3, 2+2\sqrt{5}, 2-2\sqrt{5}$ and $3, -2, 3+3\sqrt{5}, 3-3\sqrt{5}$ respectively. The multiplicities of these eigenvalues in A are 1, 25, 16, 9, 9, 16, 36, 4, 4, in this order.

The computation of B for $P = \{5,3,3\}$ is somewhat cumbersome. The combinatorial argument used in the case of $\{3,3,5\}$ can be mimicked and the list of diagrams corresponding to Figure 1 is about twice as long. Instead, for brevity, we compute B by making use of appropriate coordinates for the vertices ([C], p. 240). The coordinates for the 600 vertices of P are listed schematically in Table 2, as well as the adjacency relations among vertices.

The first and second columns in Table 2 contain the name and number of vertices of a level. In order to obtain all vertices in one level from the vertex in the third column of the table keep the entry before the semicolon fixed and permute the remaining entries in all possible ways changing an even number of signs. There are also levels $-$**a** through

Level	#	*Coordinates*	*Neighbours*
a	1	$(a;x,x,x)$	$(b;u,u,u),\ (b;u,-u,-u),\ (b;-u,u,-u),\ (b;-u,-u,u)$
b	4	$(b;u,u,u)$	$(a;x,x,x),\ (c;m,v,v),\ (c;v,m,v),\ (c;v,v,m)$
c	12	$(c;m,v,v)$	$(b;u,u,u),\ (c;m,-v,-v),\ (d;l,q,w),\ (d;l,w,q)$
d	24	$(d;l,q,w)$	$(c;m,v,v),\ (d;q,l,w),\ (e;i,r,-r),\ (g;g,p,t)$
e	12	$(e;i,r,-r)$	$(d;l,q,w),\ (d;l,-w,-q),\ (f;n,n,-n),\ (i;e,r,-r)$
f	4	$(f;n,n,-n)$	$(e;i,r,-r),\ (e;r,i,-r),\ (e;r,r,-i),\ (k;k,k,-k)$
g	24	$(g;g,p,t)$	$(d;l,q,w),\ (g;g,t,p),\ (h;h,h,s),\ (l;d,q,w)$
h	12	$(h;h,h,s)$	$(g;g,p,t),\ (g;p,g,t),\ (j;j,j,j),\ (p;g,g,t)$
i	12	$(i;e,r,-r)$	$(e;i,r,-r),\ (l;d,q,w),\ (l;d,-w,-q),\ (n;f,n,-n)$
j	4	$(j;j,j,j)$	$(h;h,h,s),\ (h;h,s,h),\ (h;s,h,h),\ (s;h,h,h)$
k	4	$(k;k,k,-k)$	$(f;n,n,-n),\ (n;f,n,-n),\ (n;n,f,-n),\ (n;n,n,-f)$
l	24	$(l;d,q,w)$	$(g;g,p,t),\ (i;e,r,-r),\ (m;c,v,v),\ (q;d,l,w)$
m	12	$(m;c,v,v)$	$(l;d,q,w),\ (l;d,w,q),\ (m;c,-v,-v),\ (u;b,u,u)$
n	12	$(n;f,n,-n)$	$(i;e,r,-r),\ (k;k,k,-k),\ (r;e,i,-r),\ (r;e,r,-i)$
p	12	$(p;g,g,t)$	$(h;h,h,s),\ (q;d,l,w),\ (q;l,d,w),\ (t;g,g,p)$
q	24	$(q;d,l,w)$	$(l;d,q,w),\ (p;g,g,t),\ (r;e,i,-r),\ (v;c,m,v)$
r	24	$(r;e,i,-r)$	$(n;f,n,-n),\ (q;d,l,w),\ (r;i,e,-r),\ (-w;d,l,-q)$
s	4	$(s;h,h,h)$	$(j;j,j,j),\ (t;g,g,p),\ (t;g,p,g),\ (t;p,g,g)$
t	12	$(t;g,g,p)$	$(p;g,g,t),\ (s;h,h,h),\ (w;d,l,q),\ (w;l,d,q)$
u	12	$(u;b,u,u)$	$(m;c,v,v),\ (v;c,m,v),\ (v;c,v,m),\ (x;a,x,x)$
v	24	$(v;c,m,v)$	$(q;d,l,w),\ (u;b,u,u),\ (w;d,l,q),\ (-v;c,m,-v)$
w	24	$(w;d,l,q)$	$(t;g,g,p),\ (v;c,m,v),\ (w;d,q,l),\ (-r;e,i,r)$
x	6	$(x;a,x,x)$	$(u;b,u,u),\ (u;b,-u,-u),\ (-u;b,u,-u),\ (-u;b,-u,u)$

where

$$a=4,\ b=\tfrac{1+3\sqrt5}{2}\approx 3.854,\ c=\tfrac{5+\sqrt5}{2}\approx 3.618,\ d=1+\sqrt5\approx 3.236,$$
$$e=3,\ f=\tfrac{-1+3\sqrt5}{2}\approx 2.854,\ g=\tfrac{3+\sqrt5}{2}\approx 2.618,\ h=i=\sqrt5\approx 2.236,$$
$$j=k=l=2,\ m=n=\tfrac{1+\sqrt5}{2}\approx 1.618,\ p=\tfrac{5-\sqrt5}{2}\approx 1.382,\ q=-1+\sqrt5\approx 1.236,$$
$$r=s=1,\ t=u=\tfrac{-1+\sqrt5}{2}\approx 0.618,\ v=\tfrac{3-\sqrt5}{2}\approx 0.382,\ w=x=0.$$

Table 2

$-\mathbf{w}$ (but $-\mathbf{x}=\mathbf{x}$); with vertices obtained by changing the signs of all coordinates of the vertices of the corresponding positive level. The last column lists the four neighbours of the chosen representative of the level. Different levels may well be at the same height. From this information we construct the matrices B, B_e and B_o (Figs. 3 and 4).

The matrices B_e and B_o have spectra 4, 1, 0, -1, $\sqrt5$, $-\sqrt5$, $\frac{5+\sqrt5}{2}$, $\frac{5-\sqrt5}{2}$, $\frac{-3+\sqrt5}{2}$, $\frac{-3-\sqrt5}{2}$, $\frac{-1+\sqrt{21}}{2}$, $\frac{-1-\sqrt{21}}{2}$, a, b, c and -2, $\frac{1+3\sqrt5}{2}$, $\frac{1-3\sqrt5}{2}$, $\frac{1+\sqrt5}{2}$, $\frac{1-\sqrt5}{2}$, $\frac{3+\sqrt{13}}{2}$, $\frac{3-\sqrt{13}}{2}$,

$$
B_e = \begin{pmatrix}
0 & 4 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
1 & 0 & 3 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 1 & 1 & 2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & 1 & 1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 2 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 3 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 & 0 & 1 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 2 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 2 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 3 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 3 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 2 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 2 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 2 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 1 & 1 & 0 & 0 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 3 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 0 & 2 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 2 & 0 & 1 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & 1 & 1 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 1 & 0 & 1 & 1 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 4 & 0 & 0 & 0
\end{pmatrix}
$$

Table 3

$-1+\sqrt{2}$, $-1-\sqrt{2}$, d, e, f respectively, where a, b, c are the roots of $x^3 - x^2 - 7x + 4 = 0$ and d, e, f are the roots of $x^3 - x^2 - 7x + 8 = 0$. These values were first obtained numerically and the exact answers were guessed by matching algebraic conjugates. The fact that many of these eigenvalues are not simple helped a lot; the multiplicities of these eigenvalues in B_e and B_o are 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 2, 2, 3, 3, 3 and 1, 1, 1, 1, 1, 2, 2, 2, 2, 3, 3, 3, in this order. It was then a simple matter to check by exact arithmetic (since all our candidate eigenvalues are algebraic integers) that our guesses were indeed correct. It is rather surprising that all eigenvalues have such small algebraic degree. The multiplicities of these eigenvalues in A are 1, 40, 18, 8, 24, 24, 9, 9, 30, 30, 16, 16, 25, 25, 25, 8, 4, 4, 24, 24, 16, 16, 48, 48, 36, 36, 36, in this order.

Eigenvectors

We now prove that, for any regular polytope in any dimension, the full symmetry group acts irreducibly on all maximal eigenspaces in U.

$$B_o = \begin{pmatrix}
0&4&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
1&0&3&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
0&1&1&2&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&1&1&1&0&1&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&2&0&1&0&0&1&0&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&3&0&0&0&0&0&1&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&1&0&0&1&1&0&0&0&1&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&2&0&0&1&0&0&0&0&1&0&0&0&0&0&0&0\\
0&0&0&0&1&0&0&0&0&0&0&2&0&1&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&3&0&0&0&0&0&0&0&0&0&1&0&0&0&0\\
0&0&0&0&0&1&0&0&0&0&0&0&0&3&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&1&0&1&0&0&0&1&0&0&1&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0&0&2&1&0&0&0&0&0&0&1&0&0\\
0&0&0&0&0&0&0&0&1&0&1&0&0&0&0&0&2&0&0&0&0&0\\
0&0&0&0&0&0&0&1&0&0&0&0&0&0&0&2&0&0&1&0&0&0\\
0&0&0&0&0&0&0&0&0&0&0&1&0&0&1&0&1&0&0&0&1&0\\
0&0&0&0&0&0&0&0&0&0&0&0&0&1&0&1&1&0&0&0&0&-1\\
0&0&0&0&0&0&0&0&0&1&0&0&0&0&0&0&0&0&3&0&0&0\\
0&0&0&0&0&0&0&0&0&0&0&0&0&0&1&0&0&1&0&0&0&2\\
0&0&0&0&0&0&0&0&0&0&0&0&1&0&0&0&0&0&0&0&2&0\\
0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&1&0&0&0&1&-1&1\\
0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&-1&0&1&0&1&1
\end{pmatrix}$$

Table 4

In all cases except $\{5, 3, 3\}$, this follows directly from the fact that the spectrum of B is simple, as can be verified by inspection case by case in the computations above. Indeed, as seen in the theoretical remarks above, a decomposition of the eigenspace corresponding to λ in U into non-trivial invariant subspaces would give a decomposition of the eigenspace corresponding to λ in S into non-trivial subspaces. For the missing polytope, we have twelve non-simple eigenvalues of B, with multiplicities 2 or 3. If one of the associated eigenspaces were reducible, there would be a corresponding one dimensional subspace in S. We explicitly rule out this possibility.

The symmetry group acts on U by permutations, and each element g of the group induces a map P_g from S to itself as follows. Interpret a vector $s \in S$ as a vector in U, and define $P_g(s)$ as what we obtain from s^g by the usual averaging process. Notice that P_g depends only on the level to which the top vertex v is sent by g. We denote by $P_{\mathbf{z}}$ the transformation induced by an element taking the top vertex to the level $\mathbf{z}$. We can compute the entry i, j of $P_{\mathbf{z}}$ in the ℓ^1 basis by considering any isometry $Q_{\mathbf{z}}$ that takes the top vertex to the level $\mathbf{z}$ and then counting how many of the 24 elements of H_v take a fixed vertex in level j to a vertex which is taken by $Q_{\mathbf{z}}$ to level i. Invariant subspaces of U under the group action correspond to subspaces of S invariant under all $P_{\mathbf{z}}$. Therefore, a one dimensional subspace of S of this kind must be an eigenspace for all $P_{\mathbf{z}}$. If such a common eigenspace existed, it would have to be in the kernel of all commutators $[P_{\mathbf{y}}, P_{\mathbf{z}}]$ for arbitrary levels $\mathbf{y}$ and $\mathbf{z}$. We computed $P_{\mathbf{b}}$ and $P_{\mathbf{j}}$ and their commutator turned out to be invertible when restricted to the maximal eigenspace of any non-simple eigenvalue.

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