Ergodic measures with only (and many) zero exponents

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PUC-Rio

Luisenthal, October 1st

joint work with J. Bochi (PUC-Rio) and Ch. Bonatti (Dijon).

Old title:
Robust vanishing of all Lyapunov exponents.

Hidden/implicit part of the title:
for Iterated Function Systems or One Step skew products.

Important: the multiple zero exponents hold for open sets of systems (robust)

are the measures robust ? (dependence of measures on maps)

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Construction of ergodic measures with some exponent zero **(non-hyperbolic measures)**.

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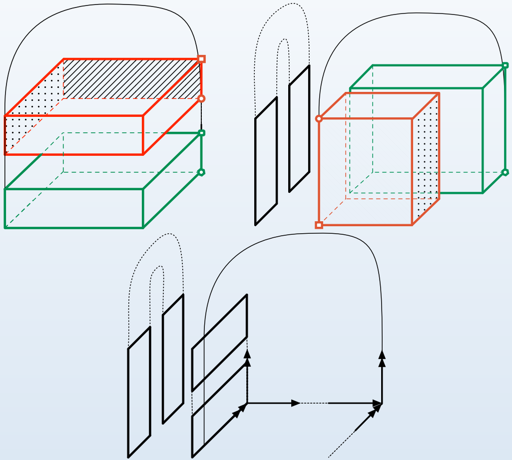

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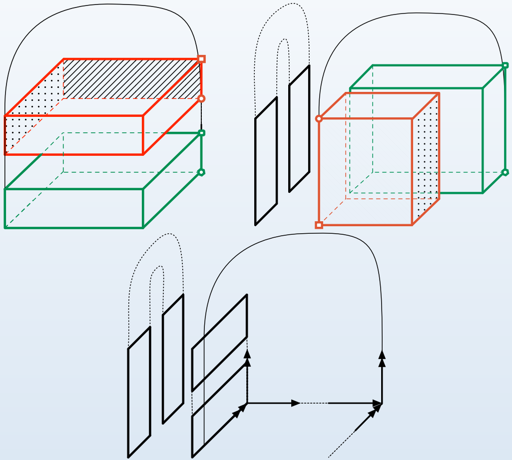

Skew product and partial hyperbolicity

$$E^s \oplus E^c \oplus E^u$$

E^s stable, E^u unstable, E^c central.

1. Q is expanding in the E^c-direction,
2. all other periodic points are contracting along E^c,
3. Ergodic measures (not Dirac at Q) are hyperbolic:
   two negative exponents (E^c, E^s) and a positive one (E^u).

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$f : M \to M$, μ ergodic invariant measure:

Oseledts splitting $T_x(M) = E_1(x) \oplus E_2(x) \oplus \cdots \oplus E_k(x)$ defined μ-a.e., dimension independent of x.

and Lyapunov exponents $\chi_1 < \chi_2 < \cdots < \chi_k$

$$\lim_{n\to\infty} \frac{\log ||Df^n(v)||}{n} = \chi_j, \quad \forall v \in E_j(x) \setminus \{0\}, \quad \mu - \mathrm{a.e.}\, x$$

$\dim E_j$ is the multiplicity of χ_j.

μ is hyperbolic if all exponents are non-zero.

number of zero exponents:

Λ transitive set with a Df-invariant splitting:

$$T_\Lambda M = E^s \oplus E^c \oplus E^u.$$

E^s uniformly contracting, E^u uniformly expanding,
E^c central non-hyperbolic part with finest dominated splitting

$$E^c = E_1 \oplus E_2 \oplus \cdots \oplus E_k$$

μ ergodic, χ_i exponent of μ relative to E_i,

$$\chi_i = 0 \implies \chi_{j \neq i} \neq 0.$$

Conclusion: The number of zero exponents of an ergodic measure is $\leq \dim(E^c)$.

Constrains on the type of dominated splitting of E^c.

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$f_0,\ f_1\colon M \to M, \qquad \sigma\colon \{0,1\}^{\mathbb{Z}} \to \{0,1\}^{\mathbb{Z}}$ the shift map,

one-step skew products:

$$F\colon \{0,1\}^{Z} \times M \to \{0,1\}^{\mathbb{Z}} \times M, \quad (\alpha, x) \mapsto (\sigma(\alpha), f_{\alpha_0}(x)).$$

Family of maps $(f_\alpha)_{\alpha\in\{0,1\}^{\mathbb{Z}}}$, $f_\alpha\colon M \to M$, (nice dependence on α).

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1. [+] full support
2. [+] constructive (limit of periodic measures)
3. [−] zero entropy

C^1 dynamics

1. [−] support?
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$\sigma\colon \{0,1,\dots,\ell-1\}^{\mathbb{Z}} \to \{0,1,\dots,\ell-1\}^{\mathbb{Z}}$ is the shift map.

$g_0,\dots,g_{\ell-1}$ diffeomorphisms $g_i\colon M \to M$,

let $G=(g_0,\dots,g_{\ell-1})$ and the one-step skew product map

$$\varphi_G\colon \{0,1,\dots,\ell-1\}^{\mathbb{Z}} \times M \to \{0,1,\dots,\ell-1\}^{\mathbb{Z}} \times M,$$

$$\varphi_G(\alpha,x)=(\sigma(\alpha),g_{\alpha_0}(x))$$

all zero exponents (I)

Ergodic measures with all zero exponents and full support

Given any closed and compact M, $\dim M \geq 2$, there are ℓ and open set $\mathcal{U} \subset (\mathrm{Diff}^2(M))^\ell$: for every $G = (g_0, \dots, g_{\ell-1}) \in \mathcal{U}$ the map

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has an ergodic measure with full support and whose exponents are all zero. This measure is a limit of periodic measures.

Warning: Exponents refer to fibered exponents along M.

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1. all exponents of any ergodic ν $(\mathrm{supp}(\nu) \subset \Lambda_G)$ are zero,
2. $h_{\mathrm{top}}(G_{|\Lambda_G}) > 0$. So there is an ergodic μ with positive entropy and only zero exponents $(\mathrm{supp}(\mu) \subset \Lambda_G)$.

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Properties of the maps $(g_0, \dots, g_{\ell-1})$

- existence of an attracting point with simple spectrum - all exponents positive and different,
- forward minimality,
- maneuverability: "minimality in the space of directions',
- implicit: there are no invariant directions, no-domination.
- warning: possibly the number ℓ is very big (!), this does not seem to be important in applications (we have in mind).
- consequence of a stronger result on flag dynamics (later on)

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Intermingled horseshoes of different indices

Method for constructing ergodic measures with a zero exponent. [Gorodetski-Ilyashenko-Kleptsyn-Nalsky]

Ergodic measures as limit of periodic measures.

$f_0 : \mathbb{S}^1 \to \mathbb{S}^1$ pole north - pole south map,
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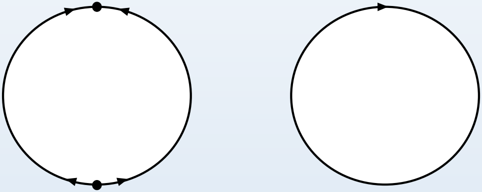

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key feature Minimality: the system f_0, f_1 is forward minimal (the forward orbit of any point is dense in $\mathbb{S}^1$)

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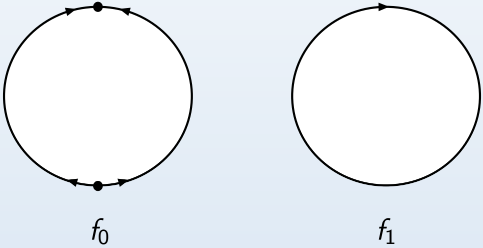

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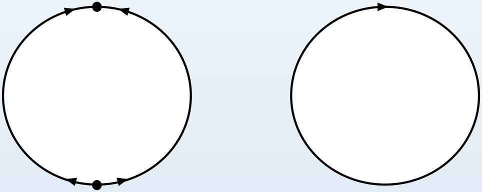

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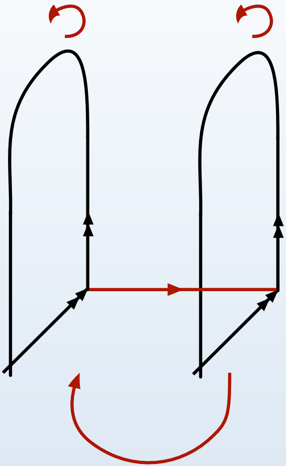
naive representation



Summary of previous results:

1-dimensional central direction.

- some one-step skew products over $\mathbb{S}^1$ [GIKN]
- general skew products over $\mathbb{S}^1$ [KN]
- open sets of diffeomorphisms in dimension ≥ 3 [KN]
- generic non-hyperbolic diffeomorphisms [DG], [BonattiDG]

have an ergodic measure with full support with a zero exponent. This measure is a weak-limit of periodic measures.

- The ergodic measure is the limit of a sequence of periodic measures

$$\mu_n \to^* \mu.$$

- key one dimensional ingredient: the exponent is an integral.

$$\chi_c(\mu_n) \to \chi_c(\mu).$$

- This does not hold in higher dimensions.
  Difficulty for obtaining measures with several zero exponents.
- key ingredient: minimality in the central directions (jump in finite time from an repeller to an attractor and vice-versa).
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Several zero exponents: strategy

1. **Problem:** Consider skew products with higher dimensional central direction and construct systems with ergodic measures with all exponents equal to 0.
2. **Difficulty:** Exponents are not given by integral and thus they are not limits.
3. **Trick:** Rewrite exponents as integrals (recovering continuity).
4. **Ingredient:** Flag dynamics (a dynamics induced in the space of flags of the tangent bundle).
5. **Price:** Increase differentiability. C^1-dynamics in the space of flags $\implies$ C^2-dynamics in the ambient.

Programm: Repeat the one-dimensional procedure: one-step skew products $\to$ skew products $\to$ open sets $\to$ generic dynamics.

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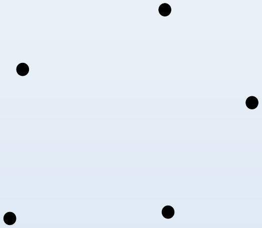
ergodic limit of periodic measures (I)

improved method of Gorodetski-Ilyashenko-Kleptsyn-Nalsky for constructing ergodic measures as limits of periodic measures (version in [BDG]).

initial periodic orbit (and periodic measure)



ergodic limit of periodic measures (I)

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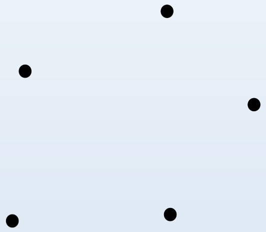

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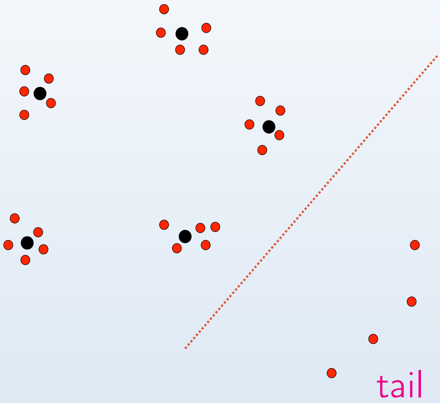

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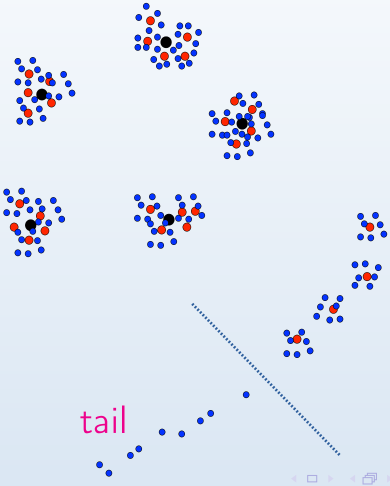

two effects of the tail:

- spread the support of the limit measure: the support of the measure is
$$\bigcap_n \overline{\bigcup_{k\geq n} \mathcal{O}(P_n)}.$$
- approach the exponent to zero:
  1. n-th orbit P_n with exponent χ_n, a pivot auxiliary orbit with exponent $\simeq 0$
  2. $(n+1)$th orbit P_{n+1} mimics P_n 90% of time and Q 10% of time,
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ergodic limit of periodic measures (IV)

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Everything You Always Wanted to Know About Flags But Were Afraid to Ask.

Flags in 5 minutes!

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$\mathbb{F}_1 \subset \mathbb{F}_2 \subset \cdots \subset \mathbb{F}_n = \mathbb{V}$, $\dim \mathbb{F}_i = i$.

Flag $\mathbb{F} = (\mathbb{F}_1, \mathbb{F}_2, \ldots, \mathbb{F}_d)$.

(Forgetting orientation) flag $\simeq$ orthonormal basis:
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induced map in the flag space:

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Using orthonormal basis the flag action of L has a triangular form. This will simplify calculations.

TM, T_xM

flag manifold $\mathcal{F}M$, the fiber of x consists of the flags of T_xM.

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All zero exponents

Reformulation of the main result for flags (stronger version):

Ergodic measures with all zero exponents

Given any closed and compact M, $\dim M \geq 2$, there are ℓ and open set $\mathcal{U} \subset (\mathrm{Diff}^2(M))^\ell$: for every $G = (g_0, \dots, g_{\ell-1}) \in \mathcal{U}$ the map

$$\varphi_{\mathcal{F}G} \colon \{0, \dots, \ell-1\}^{\mathbb{Z}} \times M \to \{0, \dots, \ell-1\}^{\mathbb{Z}} \times M,$$

$$\varphi_{\mathcal{F}G}(\alpha, x) = (\sigma(\alpha), \mathcal{F}g_{\alpha_0}(x))$$

has an ergodic measure with full support and whose exponents are all zero. This measure is a limit of periodic measures.

Notations, ingredients:

X compact metric space, $T\colon X \to X$, homeomorphism.

$\mathbb{V}$ vector bundle over X.

a projection $\pi\colon \mathbb{V} \to X$.

$S\colon \mathbb{V} \to \mathbb{V}$ vector bundle linear isomorphism, $\pi \circ S = T \circ \pi$.

$\mathbb{V}_x = \pi^{-1}(x)$, $\qquad S_x^{(n)}\colon \mathbb{V}_x \to \mathbb{V}_{T^n(x)}$.

flag dynamics (II)

S induces the map $\mathcal{F}S$... acting on the flag bundle $\mathcal{F}\mathbb{V}$.

ν ergodic measure (in the flag bundle $\mathcal{F}\mathbb{V}$) of $\mathcal{F}S$.

Furstenberg vector:

$$\overrightarrow{\Lambda(\nu)} = (\Lambda_1(\nu), \dots, \Lambda_d(\nu))$$

$$\Lambda_j(\nu) = \int_{\mathcal{F}\mathbb{V}} \log |\det S_x|_{\mathbb{F}_j} \, d(\nu, \mathbb{F}), \quad \mathbb{F} = (\mathbb{F}_1, \dots, \mathbb{F}_d).$$

key move! the numbers $\Lambda_j(\nu)$ (thus $\overline{\Lambda(\nu)}$) are defined as integrals and thus depend continuously on the weak* topology.

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Exponents of the flag dynamics

Exponents and determinants

Let $\chi_1, \ldots, \chi_d$ the exponents of S. Then there is a permutation $i_1, \ldots, i_d$ of $(1, \ldots, d)$ such that

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Exponents of the flag maps

The exponents of $\mathcal{F}S$ are of the form $\chi_{i_j} - \chi_{i_k}$.

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End of the proof....

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$0 > \chi_1(z) > \chi_2(z) > \cdots > \chi_d(z)$.

$E_1(z), E_2(z), \ldots, E_d(z)$ eigendirections.

stable flag of z (an attracting flag):

$$S(z) = (E_1(z), E_1(z) \oplus E_2(z), \ldots, E_1(z) \oplus \cdots \oplus E_d(z)).$$

attracting points z_n with attracting flags $S(z_n)$ such that

$$\overrightarrow{\Lambda(\delta_{S(z_n)})} \to \vec{0}, \quad \delta_{S(z_n)} \text{ periodic Dirac.}$$

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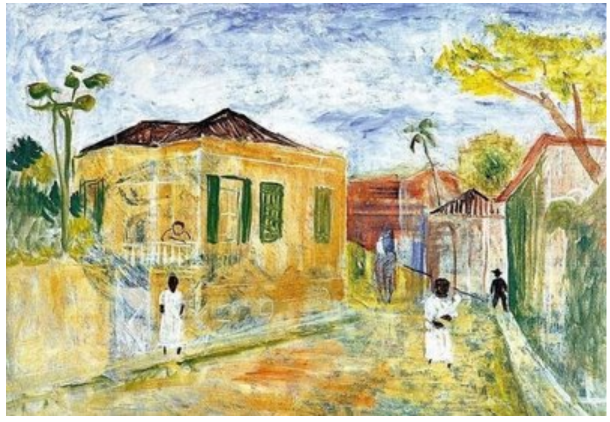

technical step: construction of orbits

cone $C = \{\vec{\lambda} = (\lambda_1, \dots, \lambda_d) \colon 0 > \lambda_1 > \cdots > \lambda_d\}$.

projective map $\Gamma \colon C \to \mathbb{R}^+$, $\quad \Gamma(t\,\vec{\lambda}) = \Gamma(\vec{\lambda})$

generation of orbits

Given $(z, S(z))$ stable flag, $\vec{\chi}(z)$ (Lyapunov vector)
There is $(\bar{z}, S(\bar{z}))$ stable flag, $\vec{\chi}(\bar{z})$ such that
(there are some quantifiers.... ϵ, δ, κ....)

1. the angle $\chi(z), \chi(\bar{z})$ is small: $\chi(\bar{z}) \in C$ and $S(\bar{z})$ is defined.
2. comparison of (decreasing) exponents:

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1. allows to repeat the procedure,
2. exponents go to 0^d,
3. criterium of the ergodic measures,
4. as $\delta \to 0$ the supp periodic measures increase.