Rich model cycles in skew-product dynamics:

from totally nonhyperbolic to fully prevalent hyperbolic dynamics via heterodimensional cycles

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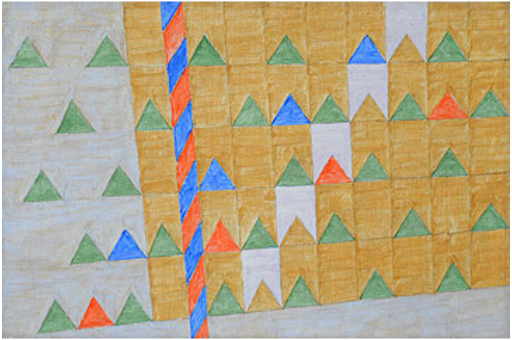
Nizhni Novgorod, July 3th 2013

joint work with

M.S. Esteves (Bragança, Portugal), J. Rocha (Oporto, Portugal).

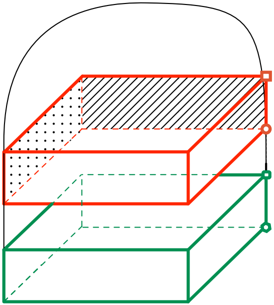
based also in previous works with

J. Rocha (Oporto, Portugal), B. Santoro, (CUNY, USA)



Motivation I

An example: Porcupine-like horseshoes



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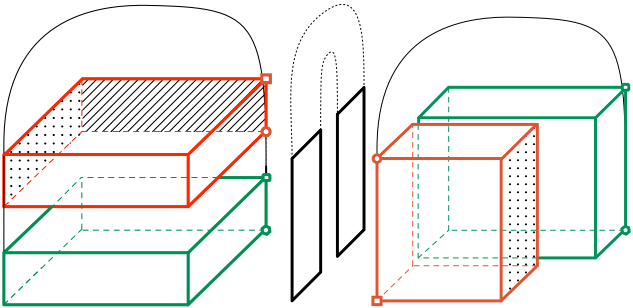

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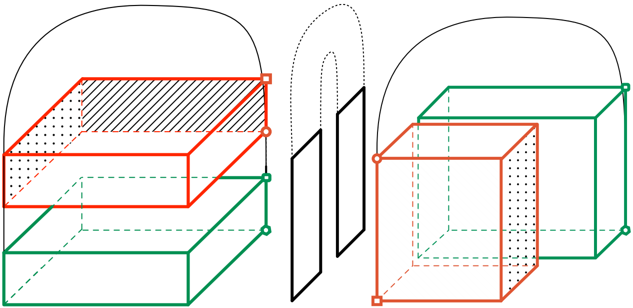

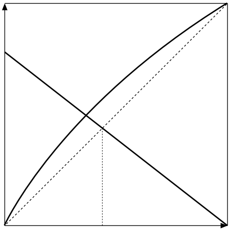

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$$\Lambda = \bigcap_{k\in\mathbb{Z}} F^k(U), \quad U \supset [0,1]^3$$

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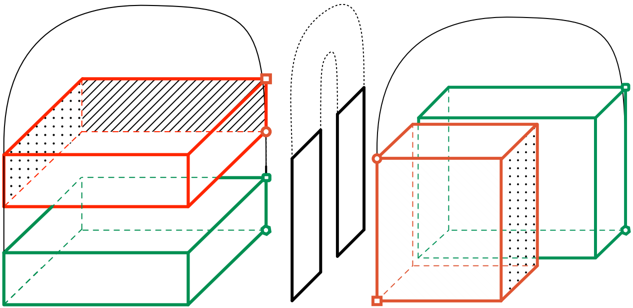

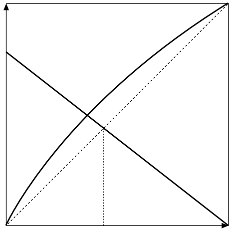

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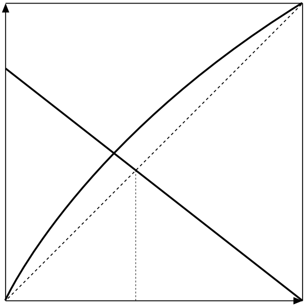
Important fact: the dynamics is determined by an I.F.S..

Porcupine-like horseshoes

Iterated function system



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$q = 0$

$\log \alpha$
$p = 1$

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f_0' decreasing, f_0 "sufficient expansion"
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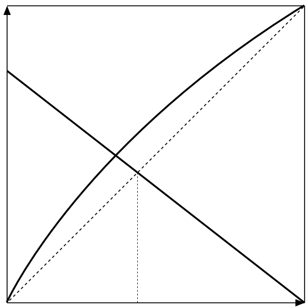

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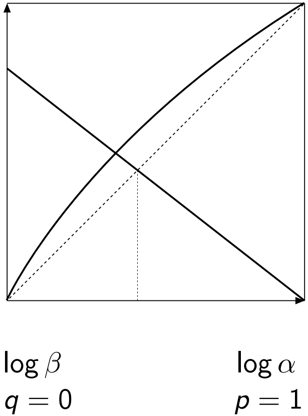

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Given $\xi_i \in \{0,1\}$ let

$$f_{[\xi_0 \ldots \xi_n]} := f_{\xi_n} \circ \cdots \circ f_{\xi_0} : [0,1] \to [0,1]$$
$$f_{[\xi_{-m} \ldots \xi_{-1}.]} := (f_{\xi_{-1}} \circ \cdots \circ f_{\xi_{-m}})^{-1}$$

[Il'yashenko '10] bony sets, [Kudryashov '10]
[D., Horita, Rios, Sambarino '09], [Leplaideur, Oliveira, Rios '11]

[D., Gelfert '12], [D., Gelfert, Rams '11, 13]

Motivation II

Hénon maps and quadratic family

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Quadratic family: important for describing the dynamics close to homoclinic bifurcations via renormalization.

Similar models in non-critical setting.

Homoclinic classes

a key object

R hyperbolic periodic point, define its homoclinic class

$$H(R) := \overline{W^s(O(R)) \pitchfork W^u(O(R))} = \mathrm{cl}\{\text{period pts homoclin. related to } R\}$$

Properties:

- transitivity,
- density of periodic points,
- in general non-hyperbolic, non-locally-maximal....

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In many cases: homoclinic classes are used for structuring the dynamics (elementary pieces of dynamics, spectral decomposition theorems).

Goal: Similar scenario in non-critical dynamics,

Typical features of non-critical dynamics

Bifurcations via heterodimensional cycles

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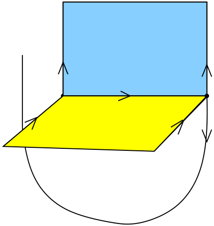

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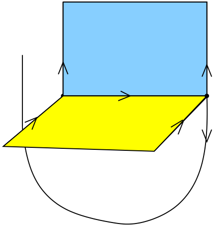

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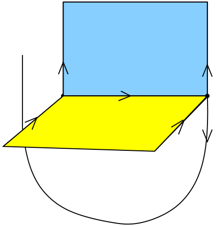

- the dynamics is determined by: central map (f_0) and cycle map ($f_{1,t}$),
- general principle: the dynamics is determined by the central map,
- only some compositions are allowed, iterations by $f_{1,t}$ are very sparse.

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How these features can be detected in a simple model (?)
based on an IFS (two generators) with explicit formulae (explicit iterations).

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one parameter family of skew-product maps, $F_{a,t}$:

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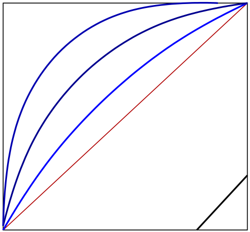
unfolding cycle map $f_{1,t}$ (independent of a) defined

$$f_{1,t}(x) = (x - 1/2) + t.$$

the underlying IFS



Two fixed points in

$$\left(\frac{-1}{2\,(e^a-1)},1\right],$$

0, repelling, $f_a'(0) = e^a > 1$,

1/2, attracting, $f_a'(1/2) = e^{-a} < 1$.

skew product dynamics

step skew-product dynamics

$$F_{a,t} : \Sigma_2 \times \left(\frac{-1}{2(e^a - 1)}, 1 \right] \to \Sigma_2 \times \mathbb{R},$$

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- necessary (sufficient?) condition for hyperbolicity: separate the homoclinic classes of P and Q,
- monotonicity: the (local) dynamics has at most two homoclinic classes,
- intermingled homoclinic classes of P, Q implies non-hyperbolicity.

State previous conditions in terms of the subjacent I.F.S.

Dictionary IFS – skew product dynamics:

- fixed points $Q = ((0), 0)$ (expanding), $P = ((0), 1/2))$ (contracting).
- homoclinic points of P: $f_{[\xi_0 \dots \xi_n],t}(t) \in (0, 1)$
- homoclinic points of Q: pre-orbit of t.
- heterodimensional cycles: $f_{[\xi_0 \dots \xi_n],t}(t) = 0$.

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$$H(P, F_{a,t}) \cap H(Q, F_{a,s}) = \mathcal{O}(S),$$

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- $a \in (0, \log 2)$: full non-hyperbolic dynamics $\mathbf{N}_a \supset [0, t]$,
- $a \in (\log 2, \log 4)$: appearance of hyperbolic parameters,
- $a > \log 4$: prevalence of hyperbolicity

$$\mathbf{F}_a = \liminf_{t \to 0^+} \frac{|\mathbf{H}_a(t)|}{t}, \quad \text{frequency of hyperbolicity}$$

$$\lim_{a \to \infty} \mathbf{F}_a = 1, \quad \text{full hyperbolicity at } \infty$$

intervals of hyperbolicity:

$t_n \to 0^+$ secondary cycles,

$t_n^\star \to 0^+$ collision paramenters,

$t_{n+1} < t_n^\star < t_n$, $(t_n^\star, t_n) \subset \mathbf{H}_a$.

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Study all possible returns to $D_a(t)$: $m \geq 0$, $k \geq 1$,

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interval (maybe empty), m increases then $k = k(m)$ decreases.

Returns (cont)

$$R_{a,t}^{k,m} \colon D_a^{k,m}(t) \to D_a(t), \quad f_a^k \circ f_{1,t} \circ f_a^{n+m}(x) \in D_a(t).$$

- m: contracting iterates,
- k: expanding iterates,

Returns (cont)

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comparison of returns:

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3. adding expanding iterates (k iterates close to 0),

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1. $D_a^{1,0}(t) = \emptyset$ (also $D_a^{1,\ell}(t) = \emptyset$) for $a \in (0, \log 2)$,
2. for returning we need to consider $D_a^{k,0}(t)$, $k > 1$,
3. adding expanding iterates (k iterates close to 0),
4. $D_a^{1,0}(t) \neq \emptyset$ for $a > \log 2$),

Returns (cont)

$$R_{a,t}^{k,m}\colon D_a^{k,m}(t)\to D_a(t),\quad f_a^k\circ f_{1,t}\circ f_a^{n+m}(x)\in D_a(t).$$

- m: contracting iterates,
- k: expanding iterates,
- for large m then $k=0$ and $R_{a,t}^{k,m}$ is contracting.

comparison of returns:

1. $D_a^{1,0}(t)=\emptyset$ (also $D_a^{1,\ell}(t)=\emptyset$) for $a\in(0,\log 2)$,
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3. adding expanding iterates (k iterates close to 0),
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5. existence of returns without adding extra expansion (iterates corresponding to k).

Control of expansion

Large n.

$$t = t_n(a)(1+\mu) \in [t_{n+1}(a), t_n(a)], \quad \mu \leq 0.$$

$$\left(R_{a,t}^{(j,0)}\right)'(x) \geq e^{(j-2)\,a}(1+\mu)^2, \quad \text{for all} \quad x \in D_{a,t}^{j,0}.$$

Induced return map in $D_{a,t}$:

$$\Phi_{a,t}\colon D_{a,t} \to D_{a,t}, \quad \Phi_{a,t}(x) = \begin{cases} R_{a,t}^{i,0}(x) & \text{if } x \in \bigcup_{i\geq 3} D_{a,t}^{i,0}, \\ R_{a,t}^{m,0} \circ R_{a,t}^{2,0}(x) & \text{if } x \in D_{a,t}^{2,0}. \end{cases}$$

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Expansion constant $\kappa(a) \to 1$ as $a \to \log 2$.

Expanding returns

- sweeping property
  Given any interval $H \subset (0,1)$ there exists $(\xi_0 \dots \xi_n)$ such that $f_{[\xi_0 \dots \xi_n]}(H)$ contains a fundamental domain of f_0.
- minimality
  For every $x \in [0,1]$ its forward orbits

$$\mathcal{O}^+(x) := \{f_{[\xi_0 \dots \xi_n]}(x) \colon \xi^+ \in \Sigma_2^+, n \geq 0\}$$

is dense in $[0,1]$.

Thanks!