Nonhyperbolic ergodic measures

Lorenzo J. Díaz

PUC-Rio, Brazil

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Starting point, questions, and summary

"To what extent is the behaviour of a dynamical system hyperbolic?"

[GIKN: Gorodetski, Ilyashenko, Klepsyn, Nalsky 05]

- nonuniform hyperbolicity [Pesin Theory]
- Hénon and Lorenz attractors [nonuniform hyperbolicity]

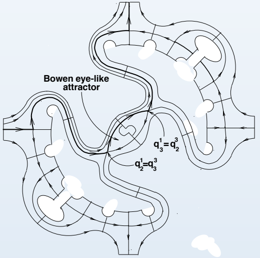
[reformulation:]

To what extent nonhyperbolic dynamics can be detected ergodically?

When **non**hyperbolic systems have nonhyperbolic (**ergodic**) measures?

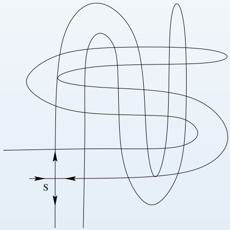
[caveat:]

there are **fragile** nonhyperbolic systems whose ergodic measures **all** are hyperbolic



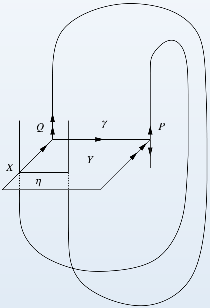
Bowen-eye surgery

[Baladi-Bonatti-Schmitt 99]



Hénon-maps with tangencies

[Cao-Luzzatto-Rios 06]



Horseshoes with internal cycles

[D-Horita-Rios-Samba. 09]

Nonhyperbolic ergodic measures [more questions]

Existence. How are these measures?

- support
- entropy
- number of zero exponents (very few is known)

Methods of construction

[also/related]

- Description of the space of ergodic measures
- Approximation [weak$\star$ and entropy] of nonhyperbolic measures by hyperbolic ones in the spirit of the hyperbolic case: approximation by measures supported on horseshoes [Katok 80] [hyperbolic] [Crovisier 16, Gelfert 17] [dominated]

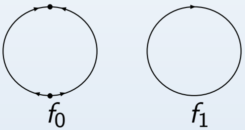
Simple setting: intermingled horseshoes in skew products

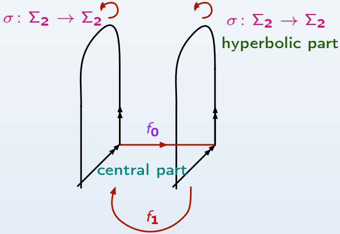
[GIKN examples]

$$F\colon \{0,1\}^Z \times \mathbb{S}^1 \to \{0,1\}^{\mathbb{Z}} \times \mathbb{S}^1, \quad (\alpha, x) \mapsto (\sigma(\alpha), f_{\alpha_0}(x)).$$

$f_0 : \mathbb{S}^1 \to \mathbb{S}^1$ [north pole - south pole]

$f_1 : \mathbb{S}^1 \to \mathbb{S}^1$ irrational rotation (or nearby)





naive idea intermingled horseshoes of different type of hyperbolicity

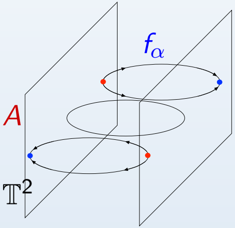
key **minimality** of the IFS generated by f_0, f_1: points have dense orbits in $\mathbb{S}^1$

Differentiable version

$$\underbrace{[\text{Anosov} - \text{horseshoes}]}_{A} \text{ (hyperbolic) } \times \underbrace{[\text{rotations}, \text{Morse} - \text{Smale}]}_{f_\alpha} \text{ (central)}$$

$$F\colon \mathbb{T}^2 \times \mathbb{S}^1 \to \mathbb{T}^2 \times \mathbb{S}^1, \quad (\alpha, x) \mapsto (A(\alpha), f_\alpha(x))$$

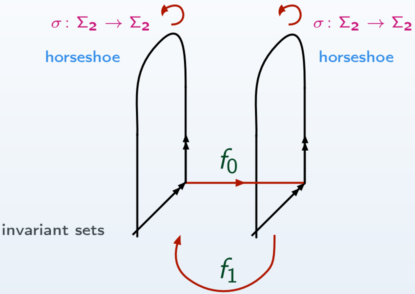
- [a priori] nonrobust method



- transitivity is preserved
- saddles of **indices** 1 y 2 in the same transitive set

  pioneers: [Abraham-Smale 70, Simon 72]

Intermingled horseshoes, one-dim center



$$\mathcal{M}_{\rm erg} = \mathcal{M}^{-}_{\rm erg} \cup \mathcal{M}^{0}_{\rm erg} \cup \mathcal{M}^{+}_{\rm erg}$$

$$\mathrm{Per} = \mathrm{Per}^{-} \cup \mathrm{Per}^{0} \cup \mathrm{Per}^{+}$$

- $-$ negative central exponent
- 0 zero central exponent
- $+$ positive central exponent

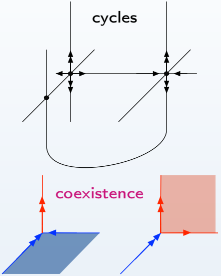
$$\Sigma_2 \times \mathbb{S}^1 = \overline{\mathrm{Per}^{+}} = \overline{\mathrm{Per}^{-}}$$

transitivity

$\underbrace{\text{approximation}}_{\text{weak}*,\ \text{entropy}}$ of measures in $\mathcal{M}^{0}_{\rm erg}$ by measures in $\mathcal{M}^{\pm}_{\rm erg}$

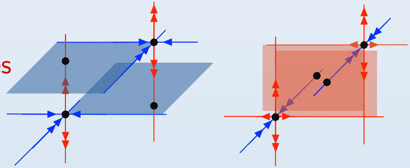
Occurrence of nonhyperbolic dynamics

- critical dynamics
- parabolic dynamics
- cycles
- coexistence of saddles of different indices in the same transitive set



- abundance of homoclinic relations invariant manifolds of periodic points meet cyclically and transversely

homoclinic classes



main objects of study

1. Homoclinic class

closure of the transverse homoclinic points of a saddle

- transitivity (dense orbits)
- density of periodic points
- may fail to be hyperbolic – coexistence of saddles of different indices

2. Robustly transitive diffeos

every nearby diffeo is transitive

Examples

- Anosov [hyperbolic]
- Nonhyperbolic
  - Derived from Anosov [Mañé 78]
  - Perturbations of Anosov $\times$ identity [Shub 71, Bonatti-D 96]
  - Perturbations of time-one Anosov transitive vector field [Bonatti-D 96]
  - New methods [Bonatti-Gogolev-Potrie 16], [G-Hammerlindel-P 18]

2. Robustly transitive diffeos

every nearby diffeo is transitive

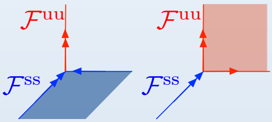
Properties

- Hyperbolic [dim 2] [Mañé 82],
- Partially Hyperbolic [dim 3] [D-Pujals-Ures 99],
- Domination [Bonatti-D-Pujals 03]

- Minimality of strong foliations [Bonatti-D-Ures 02, Ures-RodHertz² 07]

$E^{ss} \oplus E^{c} \oplus E^{uu}$

$\dim(E^{c}) = 1$



existence of a compact central leaf

Focus on

C^1-topology and also general mechanisms

Nonhyperbolic

- homoclinic classes [typically with saddles of different indices]
- robustly transitive diffeos [emphasis on partial hyperbolicity]

Two independent aspects of the constructions of nonhyp. measures

Construction **how?** Sufficient conditions **when ?**

Paradox: To construct the nonhyperbolic measures (with some persistence) some hyperbolicity is needed

General principle: "A little hyperbolicity goes a long way..." [Pugh-Shub]

Some definitions and results

$f : M \to M$ diffeo $d = \dim(M)$.

$\varphi : M \to \mathbb{R}$ continuous (potential)

f-invariant probability measure: $\mu(A) = \mu(f^{-1}(A))$.

- **Ergodicity** $A = f^{-1}(A) \Longrightarrow \mu(A) \in \{0, 1\}$.
- **Birkhoff Theorem**

time average:

$\varphi_n(x) = \frac{1}{n} \sum_{i=0}^{n-1} \varphi(f^i(x))$,

spatial average:

$\int \varphi \, d\mu$.

$$\int \varphi \, d\mu = \varphi_\infty(x), \quad \mu\text{-a.e.} \qquad \varphi_\infty(x) = \lim \varphi_n(x) \quad (\text{if } \exists)$$

- **Oseledets Theorem** μ ergodic, there are

  Lyapunov exponents:

  $$\chi_1(\mu) \geq \chi_2(\mu) \geq \cdots \geq \chi_k(\mu)$$

  ***Df*-invariant splitting:**

  $$TM = E_1 \oplus E_2 \oplus \cdots \oplus E_k$$

  μ-a.e. for all $i \in \{1, \ldots, k\}$ and $v \in E_i \setminus \{\bar{0}\}$:

  $$\lim_{n \to \pm\infty} \frac{\log \|Df_x^n(v_i)\|}{n} = \chi_i(\mu).$$

- **μ hyperbolic** $\chi_i(\mu) \neq 0$ for every i.

  (uniform) hyperbolicity $\Longrightarrow$ all ergodic measures are hyperbolic

GIKN-method of periodic approximations

Sufficient conditions for

a weak$*$ limit of **periodic measures** [suported on a periodic orbit] to be ergodic

$$(\mu_n) \to_* \mu, \quad \text{where } \mu_n \text{ supported on periodic } p_n$$

Setting

partial hyperbolicity with one-dimensional center E^{c}

key property

the "central" exponent is the integral of the continuous map $Df|_{E^{\mathrm{c}}}$

$$\chi_c(\mu_n) \to \chi_c(\mu) \qquad \text{[this fails when } \dim(E^{\mathrm{c}}) \geq 2\text{]}$$

construction of the periodic points p_n

jumps [in finite time] from a center-contracting to a center-expanding region.

minimality

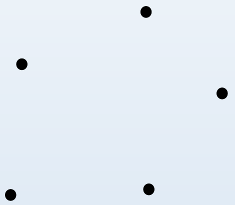
Periodic approximations

Two ingredients:

1. shadowing
2. tails

first generation

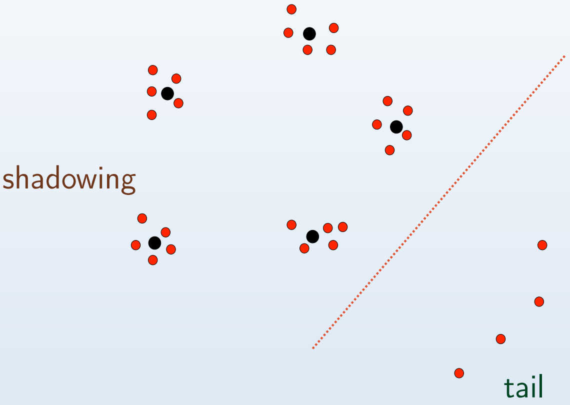
starting orbit •:



second generation

second orbit:

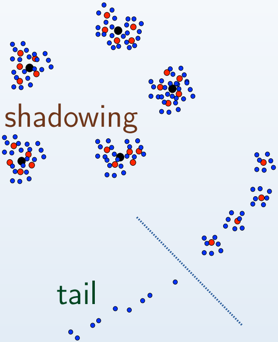
- a new **orbit** • **shadows** the first one [most of time, fixed proportion]
- and adds a **tail**



third generation

third orbit:

- a new orbit • **shadows** the second one [most of time, fixed proportion]
- and adds a new **tail**



and so on.....

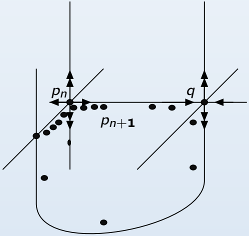
consequences of the tail

- extension of the support of the limit measure μ

$$\sup(\mu) = \bigcap_{n} \overline{\bigcup_{k \geq n} \mathcal{O}(p_n)}$$

- ergodicity of the limit measure
- decreasing the exponent in "our" case

1. p_n with exponent χ_n
2. $\mathcal{O}(p_{n+1})$ shadows $\mathcal{O}p_n$ [90% of the time] and visits a contracting pivot point q [10% of the time]
3. the exponent χ_{n+1} is less than $(9/10)\,\chi_n \to 0$.



Applications of the GIKN-method

Nonhyperbolic ergodic measures with uncountable support for:

- [step] Skew-products [over $\mathbb{S}^1$] [Gorodetski-Ilyashenko-Kleptsyn-Nalsky 05]
- Some open sets of diffeomorphisms in $\mathbb{T}^3$ [Kleptsyn-Nalsky 07]
- Nonhyperbolic homoclinic classes of C^1-generic diffeos:
  - with saddles of different indices [D-Gorodetski 09]
  - Fully supported on the class (partial hyp) [Bonatti-D-Gorodetski 10]
  - General result: no index assumption [Chen-Crovisier-Gan-Wang-Yang 16]

these cases have one zero exponent

several zero exponents?

Applications of GIKN, several zero exponents

$\dim(E^c) \geq 2$ and undecomposable some natural conditions are needed

- Several zero exponents
  - skew-products and IFS's [Bochi-Bonatti-D 14],
  - for homoclinic classes [Wang-Zhang 17]

Caveat on the GIKN-method

GIKN-measure: obtained by the GIKN-method

highly repetitive pattern

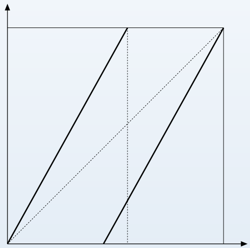
GIKN-measures have zero entropy [Kwietniak-Łącka 18]

Nonhyperbolic ergodic measures with positive entropy?

new methods: blenders and flip-flops

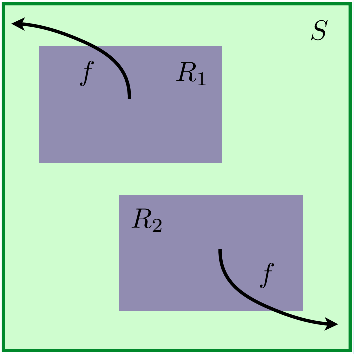
Blenders

one-dimensional version. endomorphisms



Blenders

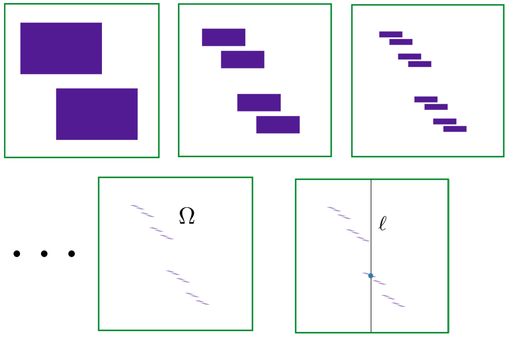
two-dimensional version. endomorphisms



figures from *What is a blender?* [Bonatti-Crovisier-D-Wilkinson 17]

Key property of blenders

the maximal invariant set and the superposition property

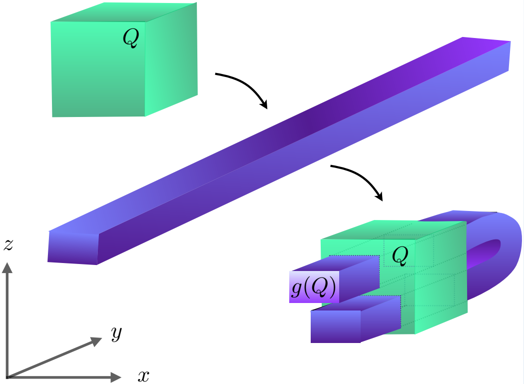


nontransverse intersections are persistent

Blenders

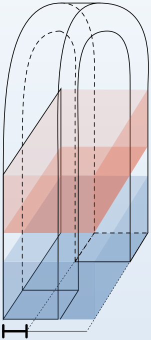
"difeomorphication" of the endomorphism: adding one dimension

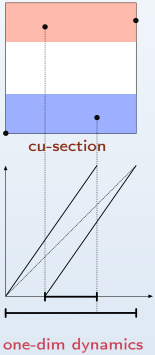
skew horseshoes in dimension three



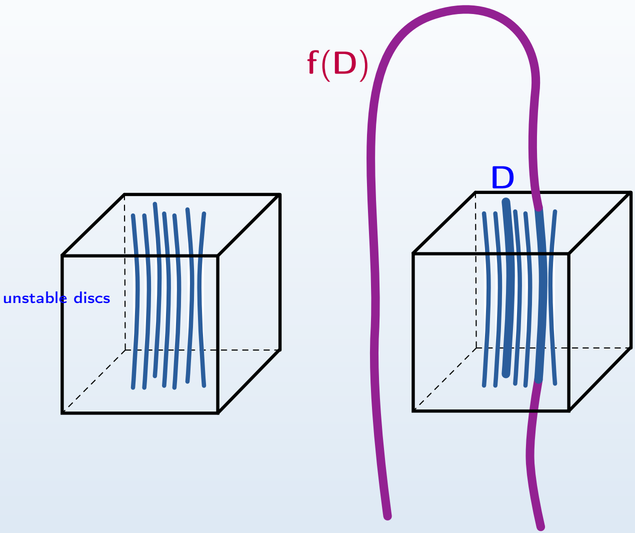
Blenders from another perspective. Summary

perturbations of nonnormally hyperbolic horseshoes



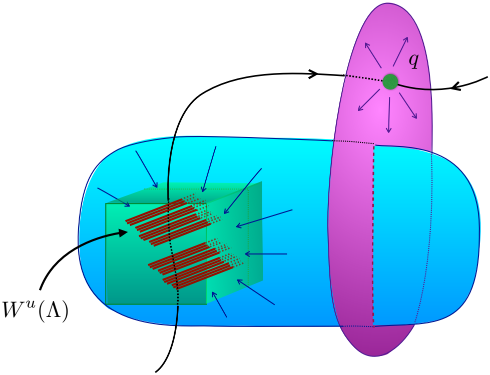


An invariance superposition property



Blenders

mechanism for generating persistent nontransverse intersections (?!)



- Blenders as a mechanism for robust transitivity [Bonatti-D 95]
- Blenders as a mechanism for robust cycles (heterodimensional cycles and tangencies) [Bonatti-D 06, 08]
  Blender-horseshoes (with geometrical data), compare thick horseshoes [Newhouse]
- Dynamical blenders (construction of nonhyperbolic measures) [Bochi-Bonatti-D 16]
- Other blenders, other settings [Nassiri-Pujals 10], [Barrientos-Raibekas-Ki 14], [Avila-Crovisier-Wilkinson 17]
- Superblenders.... [Berger 16]

key for constructing nonhyperbolic ergodic measures with positive entropy

Ergodic nonhyperbolic measures with positive entropy

(I) Flip-flop families

(II) Control of orbits and averages

(III) Flip-flop configurations

(I) Flip-flop families

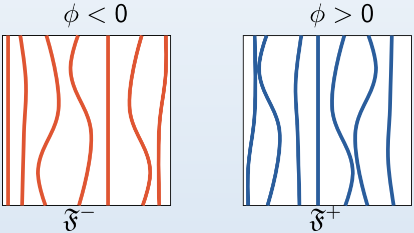
$f\colon X \to X$ homeo., $\phi\colon X \to \mathbb{R}$ cont., (X, d) metric space

Flip-flop family is a family of compact sets **(plaques)**

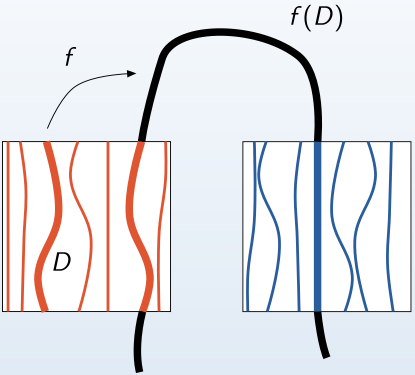
$$\mathfrak{F} = \mathfrak{F}^+ \cup \mathfrak{F}^-$$

- there is $\alpha > 0$ such that for every $D^\pm \in \mathfrak{F}^\pm$ and $x^\pm \in D^\pm$

$$\phi(x^-) < -\alpha < 0 < \alpha < \phi(x^+)$$



- there is $\lambda \in (0,1)$ such that every $D \in \mathfrak{F}$ contains D^+ and D^- with $f(D^\pm) \in \mathfrak{F}^\pm$ and $|D^\pm| < \lambda |D|$



Markovian flavour

Birkhoff averages

$$\varphi_n(x) = \frac{1}{n}\sum_{i=0}^{n-1}\varphi(f^i(x)), \qquad \varphi_\infty(x) = \lim\ \varphi_n(x) \quad \text{if } \exists$$

Flip-flop theorem [Bochi-Bonatti-D 16]

$f\colon X \to X$ homeo with a flip-flop family $\mathfrak{F}$ associated to $\varphi\colon X \to \mathbb{R}$.
Then there is $\Omega = f(\Omega)$ compact such that

- $\varphi_n \to 0$ uniformly on Ω
- $f_{|\Omega}$ has positive entropy

Variational principle

there is ergodic μ with positive entropy and $\int \varphi\, d\mu = 0$

(II) Control of orbits and averages

Given $\beta > 0$, $t \in \mathbb{N}$, $x \in X$

(β, t)-controlled: there is $(\ell_n) \nearrow \infty$ of control times such that

- $\ell_0 = 0$,
- $k_n = \ell_{n+1} - \ell_n \le t$,
- $\frac{1}{k_n} \left|\varphi_{k_n}(f^{\ell_n}(x))\right| \le \beta$

---

control at any scale: there are $(t_i) \nearrow \infty$ and $(\beta_i) \searrow 0$ such that x is (β_i, t_i)-controlled for every i

Control lemma

$$x \text{ controlled at any scale} \implies \frac{1}{n}\varphi_n(y) \to 0 \quad \text{for all } y \in \omega(x)$$

Flip-flop family $\mathfrak{F}$ implies control at any scale

Every $D \in \mathfrak{F}$ contains x controlled at any scale

Generalizations, variations

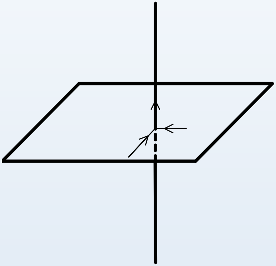
control with tails the control is relaxed in some intervals

this allows to extend the support of the measure [Bochi-Bonatti-D 18]

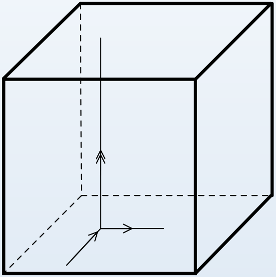
(III) Flip-flop configurations

(a)

center contracting saddle



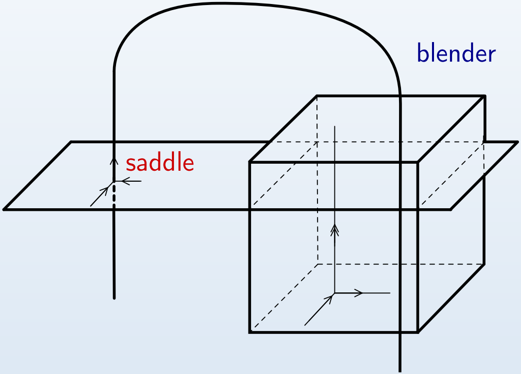
blender



Flip-flop configurations

(b)

the blender and the saddle are heteroclinically and cyclically related



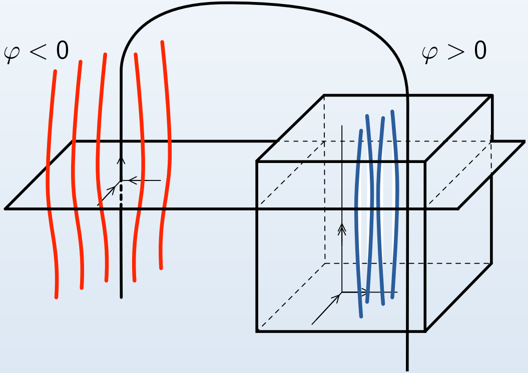
Flip-flop configurations yield flip-flop families (i)

family of plaques

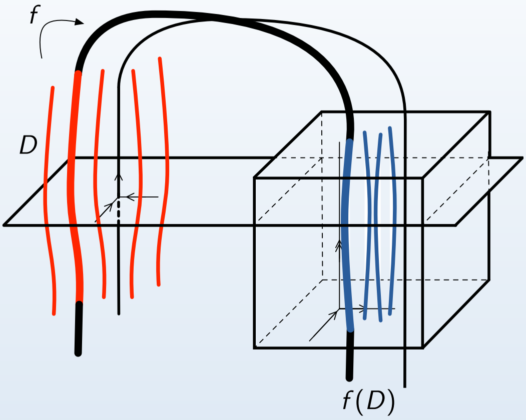
contracting and expanding regions

partially hyperbolic region, one-dim E^{c}

$$\varphi = \log Df|_{E^c}$$



Flip-flop configurations yield flip-flop families (ii)



Applications of the flip-flop method

C^1 robustly nonhyperbolic and transitive diffeos have open and densely

- flip-flop configurations
- nonhyperbolic ergodic measures with positive entropy [Bochi-Bonatti-D 16]

[additional open hypothesis] $\dim(E^c) = 1$ and a central compact leaf

C^1-open and densely

- the measure has full support [Bochi-Bonatti-D 16]
- the measure has full support and positive entropy [Bonatti-D-Kwietniak 18]

Insertion of nonhyperbolic measures

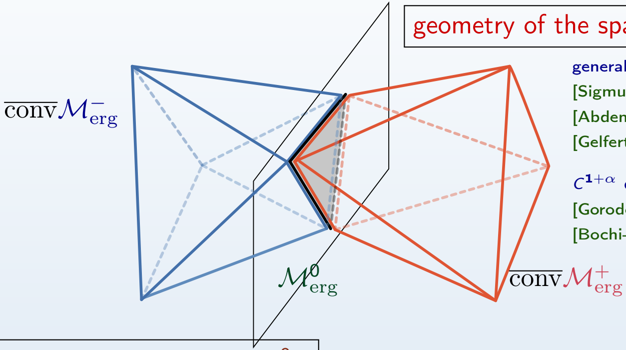
[additionally] $\dim(E^{c}) = 1$, central compact leaf, minimal strong foliations

$$\mathcal{M}_{\mathrm{erg}} = \mathcal{M}_{\mathrm{erg}}^{-} \cup \mathcal{M}_{\mathrm{erg}}^{0} \cup \mathcal{M}_{\mathrm{erg}}^{+}$$

$$\mathcal{M}_{\mathrm{erg}}^{\mp} \stackrel{\mathrm{def}}{=} \{\mu \colon \chi^{c}(\mu) \lessgtr 0\} \text{ hyper.}$$

$$\mathcal{M}_{\mathrm{erg}}^{0} \stackrel{\mathrm{def}}{=} \{\mu \colon \chi^{c}(\mu) = 0\} \text{ nonhyp}$$

3D Picture theorem



geometry of the space of measures

general results
[Sigmund 74,77]
[Abdenur-Bonatti-Crovisier 11]
[Gelfert-Kwietniak 17]

$C^{1+\alpha}$ diffeos, clases homocl.
[Gorodetski-Pesin 17]
[Bochi-Bonatti-Gelfert 17]

weak$*$ and entropy approx. of $\mathcal{M}^{0}_{\text{erg}}$

step skew-products [D-Gelfert-Rams 17]
[one-dim blender $\oplus$ minimality of the ifs]
Also thermodynamical formalism, entropy of level sets...

C^1 diffeos [D-Gelfert-Santiago 18], [Yang-Zhang18]
[blender-horseshoe $\oplus$ minimal strong foliations]
[distortion $\oplus$ fake foliations] [Burns-Wilkinson 10]
Also thermodynamical formalism [D-Gelfert-Santiago]

many thanks!!!