NONTRANSVERSE HETERODIMENSIONAL CYCLES: STABILISATION AND ROBUST TANGENCIES

LORENZO J. DÍAZ AND SEBASTIÁN A. PÉREZ

ABSTRACT. We consider three-dimensional diffeomorphisms having simultaneously heterodimensional cycles and heterodimensional tangencies associated to saddle-foci. These cycles lead to a completely nondominated bifurcation setting. For every $r \geqslant 2$, we exhibit a class of such diffeomorphisms whose heterodimensional cycles can be C^r stabilised and (simultaneously) approximated by diffeomorphisms with C^r robust homoclinic tangencies. The complexity of our nondominated setting with plenty of homoclinic and heteroclinic intersections is used to overcome the difficulty of performing C^r perturbations, $r \geqslant 2$, which are remarkably more difficult than C^1 ones. Our proof is reminiscent of the Palis-Takens' approach to get surface diffeomorphisms with infinitely many sinks (Newhouse phenomenon) in the unfolding of homoclinic tangencies of surface diffeomorphisms. This proof involves a scheme of renormalisation along nontransverse heteroclinic orbits converging to a center-unstable Hénon-like family displaying blender-horseshoes. A crucial step is the analysis of the embeddings of these blender-horseshoes in a nondominated context.

To Jacob Palis, in the occasion of his 80th birthday

1. INTRODUCTION

Palis' density conjecture [22] claims that bifurcations through *cycles* (either homoclinic tangencies or heterodimensional cycles) associated to *saddles* (hyperbolic periodic points) are the main mechanisms for destroying hyperbolic dynamics: any nonhyperbolic system can be approximated by diffeomorphisms displaying one of those bifurcations. A homoclinic tangency associated to a saddle occurs when the invariant (stable and unstable) sets of a saddle have a nontransverse intersection. A heterodimensional cycle associated with a pair of saddles of different *indices* (dimension of the unstable bundle) occurs when the invariant sets of these saddles intersects cyclically. Note that heterodimensional cycles can only occur in dimension at least three and that there are settings (as the one in this paper) where both types of bifurcations occur simultaneously with overlapping effects.

2020 *Mathematics Subject Classification.* Primary: 37C20. Secondary: 37C29, 37D20, 37D30.

Key words and phrases. Blender-horseshoe, Center-unstable Hénon-like family, Heterodimensional cycle and tangency, Homoclinic tangency, Renormalisation scheme, Stabilisation of a cycle.

This paper is part of the PhD thesis of SP (PUC-Rio) supported by CNPq and CAPES - Finance Code 001 (Brazil). LJD was partially supported by INCTMat-Faperj and CNPq (Brazil). SP was supported by Programa Postdoctorado FONDECYT 3190174 (Chile) and partially supported by CMUP grant PTDC/CTM/BIO-4043-2014 – project UID/MAT/00144/2019 (Portugal). The authors thank the hospitality of CMUP, PUC (Chile), and USACH (Chile).

As a consequence of the Kupka-Smale genericity theorem[1], a cycle associated to saddles is a fragile configuration that can be destroyed by small perturbations. However, these configurations can become robust (indestructible by small perturbations) when these saddles are embedded in some special type of horseshoes. Hence, it is natural to consider also heterodimensional cycles and tangencies associated to (basic) hyperbolic sets (for the precise definition see Section 1.2). One aims to understand when a bifurcation through a fragile cycle associated to saddles can lead to such robust cycles.

Bonatti[2] stated a stronger version of Palis' conjecture using robust cycles: the union of the C^r open sets of hyperbolic diffeomorphisms (satisfying the Axiom A and the no-cycles properties) and of diffeomorphisms with C^r robust cycles is dense in the space of C^r diffeomorphisms, see [8, Conjecture 1.10]. For results and recent progress in the previous conjectures, see [26, 11, 12] for the Palis' one and [21, 19, 1, 8] for Bonatti's one. Some of these results will be discussed below.

The latter conjecture has several motivations, one of them comes from the study of global dynamics of diffeomorphisms when considering the decomposition of the chain recurrence set into its chain of recurrence classes. Note first that two saddles involved in a cycle are always in the same class of recurrence. One aims to put these saddles robustly into the same class. If such saddles are contained in a pair of transitive hyperbolic sets involved in a robust cycle then the continuations of the hyperbolic sets (and hence the ones of the initial saddles) are also in the same class of recurrence. This gives a way to put saddles with different indices into prescribed recurrence classes. This process is known as *stabilisation of a cycle*. More precisely, a heterodimensional cycle of a C^r diffeomorphism f associated to saddles P and Q can be C^r *stabilised* if there are diffeomorphisms arbitrarily C^r close to f with a C^r robust cycle associated to transitive hyperbolic sets containing the continuations of P and Q. The stabilisation of a homoclinic tangency associated to a saddle is defined analogously.

The stabilisation of cycles depends on the type of cycle, differentiability, and dimension. To avoid technicalities, we will restrict our discussion to dimensions two and three[3]. We first consider homoclinic tangencies. For surface diffeomorphisms this question is completely solved: there are no C^1 robust tangencies and hence no homoclinic tangency can be C^1 stabilised, [19]. On the other hand, if $r \geqslant 2$ then every such a tangency can be C^r stabilised, [20]. In dimension three, a combination of [20, 27, 25] and the theory of normal hyperbolicity implies that, every C^r homoclinic tangency can be C^r stabilised for $r \geqslant 2$. In the C^1 case, the stabilisation of homoclinic tangencies involves geometrical constraints and, in

---

[1] Periodic points of generic diffeomorphisms are hyperbolic and their invariant manifolds are in general position (i.e., either they intersect transversely or they are disjoint).

[2] Formulated in Bonatti's talk *The global dynamics of C^1 generic diffeomorphisms or flows*, in the Second Latin American Congress of Mathematicians, Cancún, México (2004). See also [6].

[3] This allows us to skip the technical discussion of the so-called *coindex* of a heterodimensional cycle, since in dimension three the coindex is always one. For phenomena that may occur in higher dimensions, as for instance robust tangencies of large codimension, we refer to [2, 4].

general, it is not known which tangencies can be stabilised (see also [10]). For instance, combining normally hyperbolic surfaces and [19], one can get homoclinic tangencies that cannot be C^1 stabilised, see also [6, Sections 4.3–6].

Consider now heterodimensional cycles. First, every three-dimensional heterodimensional cycle leads to C^1 robust cycles [7], although these cycles may be not related to the saddles in the initial cycle. In [8] there is given a class of heterodimensional cycles that cannot be C^1 stabilised (*twisted* cycles). Finally, in [9] it is proved that every nontwisted cycle can be C^1 stabilised. The techniques used in these works are genuinely C^1. Due to the absence of suitable tools, the stabilisation problem in higher differentiability is widely open.

To explain our results, we recall that, in dimension three, two saddles with different indices have a *heterodimensional tangency* if their two dimensional invariant manifolds have some nontransverse intersection. These tangencies were introduced in [15] as a source of robustly nondominated/wild dynamics, see also [18, 3]. In this paper, we consider a class of three-dimensional C^r diffeomorphisms whose heterodimensional cycles involve heterodimensional tangencies (see Figure 1). For every $r \geqslant 2$, we state the C^r stabilisation of such cycles and show that they also provide C^r robust homoclinic tangencies, see the Stabilisation and Robust tangencies theorems below. Let us now provide further details of our statements.

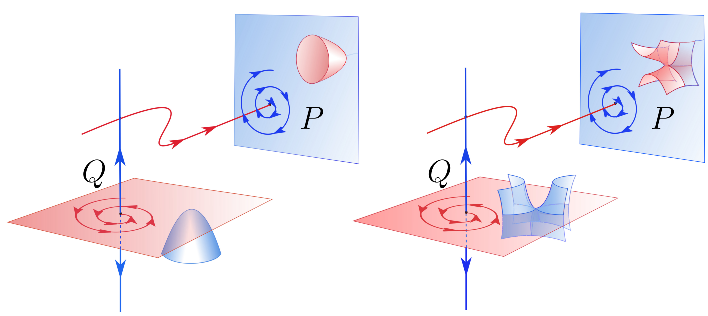

FIGURE 1. Heterodimensional cycles with heterodimensional tangencies.

Let M be a three-dimensional compact manifold. We consider a set $\mathcal{H}^r_{\mathrm{BH}}(M)$ of C^r diffeomorphisms of M having a heterodimensional cycle with a heterodimensional tangency associated to saddle-foci P and Q of indices two and one satisfying the following conditions:

- Linearising assumptions at P and Q and spectral conditions implying some sort of locally dissipative behaviour (Section 2.1.1).
- The one-dimensional invariant manifolds $W^{\mathrm{s}}(P, f)$ and $W^{\mathrm{u}}(Q, f)$ have a quasi-transverse intersection along the orbit of some point X and the two dimensional invariant manifolds $W^{\mathrm{u}}(P, f)$ and $W^{\mathrm{s}}(Q, f)$ have a heterodimensional tangency along the orbit of some point Y. This tangency may be of hyperbolic or elliptic type (Section 2.2). The type of tangency plays an important role in the resulting dynamics.

- Conditions on the "transitions" from P to Q and from Q to P along the orbits of the heteroclinic points X and Y (Section 2.1.2).

The precise description of the set $\mathcal{H}^r_{\mathrm{BH}}(M)$ is given in Section 2. Our main results are the following, see Theorem 1.1 for further details.

Stabilisation of cycles. *Let $r \geqslant 2$. Any cycle in $\mathcal{H}^r_{\mathrm{BH}}(M)$ can be C^r stabilised.*

The next result deals with diffeomorphisms in $\mathcal{H}^r_{\mathrm{BH}}(M)$ whose heterodimensional tangency is of elliptic type (see the lefthand side of Figure 1). This leads to the definition of the subset $\mathcal{H}^r_{\mathrm{BH,e^+}}(M)$ of $\mathcal{H}^r_{\mathrm{BH}}(M)$, see Section 2.2 for the precise definition and a discussion.

Robust tangencies. *Let $r \geqslant 2$. Every diffeomorphism in $\mathcal{H}^r_{\mathrm{BH,e^+}}(M)$ can be C^r approximated by diffeomorphisms with a C^r robust homoclinic tangency associated to a basic set containing the continuation of the saddle-focus of index two.*

1.1. **Our approach: a renormalisation scheme leading to blender-horseshoes.** To explain the strategy of the proof of our results let us first recall the approach in [23, Chapter 6][4] to stabilise homoclinic tangencies of C^2 diffeomorphisms. The construction in [23] has the following main ingredients: **(a)** a renormalisation scheme at a homoclinic tangency, **(b)** convergence of the scheme to a quadratic one-parameter family, **(c)** existence of parameters of the family corresponding to *thick horseshoes* (horseshoes with large "fractal-like dimension"), and **(d)** control of the localisation of the thick horseshoe guaranteeing that it is *homoclinically related*[5] to the continuation of the initial saddle. A key property in this approach is that thick horseshoes are C^2 robust, thus their existence for the limit map extends to nearby systems.

Our strategy to get the C^r stabilisation of cycles in $\mathcal{H}^r_{\mathrm{BH}}(M)$ translates the ideas of [23] to a heterodimensional setting following the approach started in [14]. In our construction, the ingredients **(a)**–**(d)** above are replaced by: **(a')** a renormalisation scheme at a heterodimensional tangency, **(b')** convergence of the scheme to a center-unstable Hénon-like family, **(c')** existence of parameters corresponding to *blender-horseshoes,* **(d')** prove that the blender-horseshoes are homoclinically related to the initial saddle of index two and have a robust cycle with the initial saddle of index one. Let us observe that, in very rough terms, blender-horseshoes are local hyperbolic plugs used to get robust heterodimensional cycles, where they play a role similar to the one of the thick horseshoes for homoclinic tangencies, see Section 5 for details. As above, a key step is to analise how the blender-horseshoes are embedded in the global dynamics.

[4]In [23] it is proved the generic coexistence of infinitely many sinks, in this proof the occurrence of robust tangencies is a key step. In this homoclinic case, these robust tangencies imply the stabilisation of the tangency, defined similarly as in the case of a cycle.

[5]Two hyperbolic sets with the same index are *homoclinically related* if their invariant manifolds intersect cyclically and transversally.

Here we rely on preliminary results in [16, 17] towards the development of the strategy **(a')–(d')**. In our context, the "limit" family is the center-unstable Hénon-like family given by

$$G_{\varpi}(x, y, z) \stackrel{\text{def}}{=} (y, \mu + y^2 + \eta_1\, y\, z + \eta_2\, z^2, \xi\, z + y), \quad \varpi = (\xi, \mu, \eta_1, \eta_2).$$

For diffeomorphisms in $\mathcal{H}^r_{\mathrm{BH}}(M)$, the renormalisation scheme and its convergence to the family G_{ϖ} (steps **(a')** and **(b')**). were obtained in [16]. A crucial property (step **(c')**) is that there is an open set $\mathcal{O}_{\mathrm{BH}}$ of parameters ϖ for which the family G_{ϖ} exhibits blender-horseshoes, see [17]. In the final step **(d')**, we analyse how these blender-horseshoes are embedded in the global dynamics (the blender-horseshoe is homoclinically related to P and has a robust cycle with Q). This is a major difficulty in our nondominated setting. It turns out that the lack of domination is simultaneously a difficulty and, in some sense, an advantage. First, the existence of nonreal multipliers makes the renormalisation scheme and the "existence and localisation" of blenders a difficult task. On the other hand, the dynamics at the bifurcation is very rich and, in particular, enables us to find new homoclinic and heteroclinic orbits close to the initial cycle. As a heuristic principle, this richness allows us to overcome the difficulty of performing C^r perturbations, $r \geqslant 2$, which are notably more problematic than C^1 ones.

The lack of domination also means that there are plenty of possibilities for unfolding the cycles involving many parameters. For instance, comparing with the setting of homoclinic tangencies where any transverse direction of unfolding behaves in the same way, the lack of domination implies that any direction of unfolding is different. Thus we have eight natural parameters: six parameters corresponding to the unfolding of the nontransverse intersections (three for the heterodimensional tangency and three for the quasi-transverse heteroclinic intersection), and two parameters associated to the arguments of the saddle-foci, see Section 6.1. We see that "unfoldings following appropriate directions" lead to robust cycles. However, the complexity of these cycles is huge and a complete description of the bifurcations is beyond reach.

We now recall some definitions and state precisely our results.

1.2. **Stabilisation of cycles and robust tangencies: precise statements.** Let M be a compact boundaryless manifold. Let $\mathrm{Diff}^r(M)$ be the space of C^r diffeomorphisms of M endowed with the C^r uniform topology. Consider $f \in \mathrm{Diff}^r(M)$ and Λ_f a hyperbolic *transitive set* (i.e. with a dense orbit) of f. Recall that there is a C^r neighbourhood $\mathcal{U}_f$ of f such that every $g \in \mathcal{U}_f$ has a hyperbolic set Λ_g that is topologically conjugate to Λ_f called the *continuation* of Λ_f. The *index* of Λ_f is the dimension of its unstable bundle (by transitivity, this number is well defined).

Consider $f \in \mathrm{Diff}^r(M)$ having a pair of transitive hyperbolic sets Λ_f and Υ_f with different indices. These sets form a *heterodimensional cycle* if their invariant stable and unstable sets intersect cyclically, i.e., $W^{\mathrm{s}}(\Lambda_f) \cap W^{\mathrm{u}}(\Upsilon_f) \neq \emptyset$ and $W^{\mathrm{u}}(\Lambda_f) \cap W^{\mathrm{s}}(\Upsilon_f) \neq \emptyset$. This cycle is C^r *robust* if there is a C^r neighbourhood $\mathcal{U}_f$ of f consisting of diffeomorphisms g such that the sets Λ_g and Υ_g have a heterodimensional cycle. The notion of a C^r *robust homoclinic tangency* associated to Λ_f is stated similarly: there is a C^r neighbourhood $\mathcal{U}_f$ of f such that for every $g \in \mathcal{U}_f$ the invariant stable and unstable sets of Λ_g have some nontransverse intersection. Recall that robust cycles cannot be associated to *trivial* hyperbolic sets (i.e., periodic orbits).

A heterodimensional cycle of a C^r diffeomorphism f associated to saddles P_f and Q_f can be C^r *stabilised* if there are diffeomorphisms $g \in \mathrm{Diff}^r(M)$ arbitrarily C^r close f with a C^r robust cycle associated to transitive hyperbolic sets Λ_g and Υ_g containing the continuations P_g and Q_g, respectively.

Our main result is the following theorem.

Theorem 1.1. *Let $r \geqslant 2$ and M be a compact boundaryless three-dimensional manifold. Given $f \in \mathcal{H}^r_{\mathrm{BH}}(M)$, with a cycle associated to saddle-foci P_f and Q_f of indices two and one, there are diffeomorphisms g arbitrarily C^r close to f with a blender-horseshoe Λ_g of index two such that:*

(i) *Λ_g and Q_g has a C^r robust heterodimensional cycle and*
(ii) *Λ_g and P_g are homoclinically related.*

Moreover, if $f \in \mathcal{H}^r_{\mathrm{BH,e^+}}(M)$ then the blender-horseshoe Λ_g has a C^r robust homoclinic tangency.

1.3. **Steps of the proofs.** We now explain the steps of the proof of Theorem 1.1. Consider $f \in \mathcal{H}^r_{\mathrm{BH}}(M)$ with a cycle associated to saddle-foci P_f and Q_f as in the theorem. A preliminary step is to perturb the original cycle to obtain a new diffeomorphism in $\mathcal{H}^r_{\mathrm{BH}}(M)$ (that continue denoting by f) having transverse homoclinic points and new additional quasi-transverse heteroclinic points associated to P_f and Q_f (see Proposition 4.1). We can now apply the renormalisation scheme to this new cycle, getting diffeomorphisms g arbitrarily C^r close to f whose dynamics in a neighbourhood of the cycle is close to a Hénon-like map G_ϖ with $\varpi \in \mathcal{O}_{\mathrm{BH}}$. By Proposition 7.2, each diffeomorphism g has a blender-horseshoe Λ_g of index two. We will see that the following holds:

(ia) *The two-dimensional manifolds $W^{\mathrm{u}}(\Lambda_g, g)$ and $W^{\mathrm{s}}(Q_g, g)$ intersect transversely,* see Proposition 9.1. The difficulty of this step is to control the size of the unstable manifold of Λ_g, assuring that it is sufficiently "large" so that it is connected to the stable manifold of Q_g. We overcome this difficulty with an analysis motivated by the constructions in [23, Section 6.4] for homoclinic tangencies of surface diffeomorphisms.

(ib) *The one-dimensional manifolds $W^{\mathrm{s}}(\Lambda_g, g)$ and $W^{\mathrm{u}}(Q_g, g)$ have nonempty intersection,* see Proposition 10.1. This step is inspired by [14, Theorem 1.4] (see Remark 1.1 for a discussion) and involves quantitative aspects of the renormalisation scheme in [16]. In this step the new quasi-transverse heteroclinic points above play an important role.

(ii) *The saddle P_g and the blender-horseshoe Λ_g are homoclinically related,* see Proposition 11.1. This step is a relatively simple consequence of (ia) and (ib) where the existence of transverse homoclinic points of Q_f is used.

Conditions (ia) and (ii) are C^r open, $r \geqslant 1$, while (ib) is not (due to deficiency of the sum of the dimensions). The blender-horseshoe allows us to make this non-transverse intersection C^r robust, $r \geqslant 1$. Thus conditions (ia) and (ib) give a C^r robust cycle between Q_g and Λ_g. As Λ_g and P_g are homoclinically related, they are contained in a larger hyperbolic set, implying the stabilisation of the initial cycle.

In the second part of the theorem, about robust tangencies, we consider diffeomorphisms with elliptic tangencies in $\mathcal{H}^r_{\mathrm{BH,e^+}}(M)$ (lefthand side of Figure 1) and study the intersections between the two-dimensional manifolds of the saddle-foci in the cycle. We see that these intersections generate "tubes crossing the reference domain of the blender-horseshoe", see Section 5.2. These tubes will provide robust tangencies. This step involves the constructions in [8] using *folding manifolds*.

Remark 1.1. In [14] it is obtained a renormalisation scheme for C^r diffeomorphisms f, $r \geqslant 2$, with a configuration somewhat similar to the one here, where the saddle-foci are replaced by a pair of saddles with real multipliers. In [14] the intersection between the one-dimensional manifolds in (ib) is obtained for $C^{1+\alpha}$ perturbations of f. Let us observe a perhaps counterintuitive fact: the intersections between the "big" two dimensional manifolds in (ia) are more difficult to obtain than the intersections between the "small" one dimensional manifolds in (ib). Indeed, in [14] the intersections (ia) and (ii) were not achieved.

Organisation of the paper. The bifurcation setting is described in Section 2. In Section 3, we introduce the perturbations used in our constructions. In Section 4, we prove that the set of diffeomorphisms having additional "special" homoclinic and quasi-transverse heteroclinic intersections is dense in $\mathcal{H}^r_{\mathrm{BH}}(M)$. These "special" homoclinic and heteroclinic points will play an important role in our proof. Blender-horseshoes and their occurrence in center-unstable Hénon-like families are discussed in Section 5. In Section 6, we review some relevant ingredients renormalisation scheme in [16] used in our constructions. In Section 7, we study the interplay between the blender-horseshoes given by the renormalisation scheme and the additional heteroclinic points. Section 8 deals with orbits and itineraries associated to the renormalisation scheme. The proof of Theorem 1.1 is completed in Sections 9–12. Section 9 deals with the intersections between the two-dimensional invariant manifolds of Q and of the blender-horseshoe. In Section 10, we state the occurrence of robust intersections between the one-dimensional invariant manifolds of Q and of the blender-horseshoe. In Section 11, we see that the saddle P and the blender-horseshoe are homoclinically related. Finally, in Section 12 we prove the part of the theorem corresponding to robust tangencies. Section 13 is an appendix collecting some explicit calculations of the renormalisation scheme borrowed from [16].

2. The bifurcation setting

In this section, we describe precisely the set $\mathcal{H}^r_{\mathrm{BH}}(M)$, see Definition 2.3. We close this section with some comments on the geometry of the cycle. Throughout this section we consider diffeomorphisms f having a pair of saddle-foci of different indices $P = P_f$ and $Q = Q_f$.

2.1. **The set $\mathcal{H}^r_{\mathrm{BH}}(M)$.** We now explain the conditions in the definition of $\mathcal{H}^r_{\mathrm{BH}}(M)$: linearising dynamics and nontransverse intersections and transition maps.

2.1.1. *Linearisable local dynamics.*

(A) *Saddle-foci periodic points*: Let $\pi(P)$ and $\pi(Q)$ be the periods of P and Q. We assume that $f^{\pi(P)}$ and $f^{\pi(Q)}$ are C^r linearisable in small neighbourhoods U_P of P and U_Q of Q. Denote the eigenvalues of $Df^{\pi(P)}(P)$ and $Df^{\pi(Q)}(Q)$ by

$$\begin{aligned}&\big(\lambda_P, \sigma_P\, e^{-2\pi i\varphi_P}, \sigma_P\, e^{2\pi i\varphi_P}\big) \text{ where } \ 0<|\lambda_P|<1<\sigma_P,\ \varphi_P\in[0,1],\\ &\big(\lambda_Q\, e^{-2\pi i\varphi_Q}, \lambda_Q\, e^{2\pi i\varphi_Q}, \sigma_Q\big) \text{ where } \ 0<\lambda_Q<1<|\sigma_Q|,\ \varphi_Q\in[0,1].\end{aligned} \tag{2.1}$$

We assume that

$$0 < \Big|\big||\lambda_P|^{\frac{1}{2}}\,\sigma_P\big|^{\eta}\sigma_Q\Big| < 1, \quad \text{where} \quad \eta = \frac{\log|\lambda_Q^{-1}|}{\log|\sigma_P|}. \tag{2.2}$$

In what follows, we assume that in the linearising local coordinates the sets U_P and U_Q are the "cubes" $[-a_P, a_P]^3$ and $[-a_Q, a_Q]^3$, for some $a_P, a_Q > 0$. For simplicity, we also assume that the periods $\pi(P)$ and $\pi(Q)$ are equal to one.

2.1.2. *Nontransverse intersections and transition maps.*

(B) *Quasi-transverse intersection and its transition map*: The one-dimensional invariant manifolds of P and Q intersect *quasi-transversely* along the orbit of a point $X = X_f$, that is $X \in W^{\mathrm{s}}(P,f)\cap W^{\mathrm{u}}(Q,f)$ and

$$T_X W^{\mathrm{s}}(P,f) + T_X W^{\mathrm{u}}(Q,f) = T_X W^{\mathrm{s}}(P,f) \oplus T_X W^{\mathrm{u}}(Q,f).$$

After replacing X by some iterate, we can assume that $X \in U_Q$. Associated to X there is a *transition map* corresponding to some iterate of f going from U_Q to U_P defined as follows. There are $N_1 \in \mathbb{N}$ such that

$$f^{N_1}(X) \stackrel{\mathrm{def}}{=} \widetilde{X} \in U_P \quad \text{and} \quad f^i(X) \notin U_P \quad \text{for every} \quad 0 \leqslant i < N_1$$

and a small neighbourhood U_X of X contained in U_Q such that

$$f^{N_1}(U_X) \stackrel{\mathrm{def}}{=} U_{\widetilde{X}} \subset U_P.$$

In the local coordinates at P and Q, the restriction $\mathfrak{T}_1$ of f^{N_1} to U_X is of the form:

$$\mathfrak{T}_1(X+W) = f^{N_1}(X+W) = \widetilde{X} + A(W) + \widetilde{H}(W), \tag{2.3}$$

where

$$A = \begin{pmatrix} \alpha_1 & \alpha_2 & \alpha_3 \\ 0 & \beta_2 & 0 \\ 0 & 0 & \gamma_3 \end{pmatrix}, \qquad \alpha_1\beta_2\gamma_3 \neq 0, \tag{2.4}$$

and $\widetilde{H}: \mathbb{R}^3 \to \mathbb{R}^3$ is such that $\widetilde{H}(\mathbf{0}) = \mathbf{0}$ and $D\widetilde{H}(\mathbf{0})$ is the null matrix. Note that $\alpha_1\beta_2\gamma_3 \neq 0$ is not an additional assumption since f^{N_1} is a diffeomorphism.

(C) *Heterodimensional tangency and its transition map*: The two-dimensional invariant manifolds of P and Q intersect along the orbit of a point $Y = Y_f$ that is a *heterodimensional tangency*, that is, the orbit of Y is contained in the set

$$\big(W^{\mathrm{u}}(P,f)\cap W^{\mathrm{s}}(Q,f)\big)\setminus\big(W^{\mathrm{u}}(P,f)\pitchfork W^{\mathrm{s}}(Q,f)\big).$$

As above, after replacing Y by some iterate, we can assume that $Y \in U_P$. Associated to Y there is a *transition map* corresponding to some iterate of f going from U_P to U_Q defined as follows. There are $N_2 \in \mathbb{N}$ such that

$$f^{N_2}(Y) \stackrel{\text{def}}{=} \widetilde{Y} \in U_Q \quad \text{and} \quad f^i(Y) \notin U_Q \quad \text{for every} \quad 0 \leqslant i < N_2$$

and a small neighbourhood U_Y of Y contained in U_P such that

$$f^{N_2}(U_Y) \stackrel{\text{def}}{=} U_{\widetilde{Y}} \subset U_Q.$$

In the local coordinates at Q and P, the restriction $\mathfrak{T}_2$ of f^{N_2} to U_Y is of the form:

$$\mathfrak{T}_2(Y+W) = f^{N_2}(Y+W) = \widetilde{Y} + B(W) + H(W), \tag{2.5}$$

where B is a quadratic map of the form

$$B\begin{pmatrix} x\\ y\\ z\end{pmatrix} = \begin{pmatrix} a_1x + a_2y + a_3z\\ b_1x + b_2y^2 + b_3z^2 + b_4yz\\ c_1x + c_2y + c_3z\end{pmatrix}, \qquad b_1(a_2c_3 - a_3c_2) \neq 0. \tag{2.6}$$

where $H\colon \mathbb{R}^3 \to \mathbb{R}^3$ is a map such that $H(\mathbf{0}) = \mathbf{0}$, $DH(\mathbf{0})$ is the null matrix, and

$$\frac{\partial^2}{\partial y^2}H_2(\mathbf{0}) = \frac{\partial^2}{\partial z^2}H_2(\mathbf{0}) = \frac{\partial^2}{\partial y\partial z}H_2(\mathbf{0}) = \mathbf{0},$$

here H_i is the i-th component of $\widetilde{H}$. Note that $b_1(a_2c_3 - a_3c_2) \neq 0$ is not an additional assumption since f^{N_2} is a diffeomorphism.

The constants $a_1, ..., c_3$ in the definition of B satisfy the following conditions

$$c_2 = c_3, \qquad \gamma_3(a_3 - a_2) > 0 \tag{2.7}$$

that will guarantee the convergence of the renormalisation scheme.

Notation 2.1. Given $f \in \mathcal{H}^r_{\mathrm{BH}}(M)$ we say that P, Q, X, and Y are the *elements of the cycle* of f and that N_1 and N_2 are the *transition times of the cycle.*

Notation 2.2 (Coordinates of the heteroclinic points)**.** In what follows, we will assume that, in our local coordinates, the heteroclinic points above are of the form:

$$\widetilde{X} = (1,0,0), \quad Y = (0,1,1) \quad \text{(in the neighbourhood } U_P),$$
$$X = (0,1,0), \quad \widetilde{Y} = (1,0,1) \quad \text{(in the neighbourhood } U_Q).$$

2.1.3. *Parameters of the transition maps.* To each diffeomorphism f satisfying (**A**)-(**C**) and $\xi > 1$ we associate the following parameters

$$\bar{\varsigma} = \bar{\varsigma}(\xi, f) \stackrel{\text{def}}{=} (\varsigma_1, \varsigma_2, \varsigma_3, \varsigma_4, \varsigma_5) \in \mathbb{R}^5, \tag{2.8}$$

where

$$\begin{aligned} &\varsigma_1 \stackrel{\text{def}}{=} \frac{\beta_2(a_2+a_3)}{\sqrt{2}}, \quad \varsigma_2 \stackrel{\text{def}}{=} \frac{\beta_2^2(b_2+b_3+b_4)}{2}, \quad \varsigma_3 \stackrel{\text{def}}{=} \xi^2 \left(\frac{b_2+b_3-b_4}{(a_3-a_2)^2} \right), \\ &\varsigma_4 \stackrel{\text{def}}{=} \xi\sqrt{2} \left(\frac{\beta_2(b_3-b_2)}{a_3-a_2} \right), \quad \varsigma_5 \stackrel{\text{def}}{=} \frac{\beta_2(c_2+c_3)}{\sqrt{2}}, \end{aligned} \tag{2.9}$$

here β_2 is as in (2.4) and $a_1, \dots, c_3$ are as in (2.6).

Definition 2.3 (The set $\mathcal{H}^r_{\mathrm{BH}}(M)$)**.** The set $\mathcal{H}^r_{\mathrm{BH}}(M)$ consists of the C^r diffeomorphisms f satisfying (**A**)-(**C**) such that

$$(a_2+a_3)(b_2+b_3+b_4) \neq 0 \tag{2.10}$$

and whose vector $\bar{\varsigma}(\xi, f)$ satisfies

$$(\xi, \varsigma_1^2\varsigma_3\varsigma_2^{-1}, \varsigma_1\varsigma_4\varsigma_2^{-1}) \in (1.18, 1.19) \times (-\varepsilon_{\mathrm{BH}}, \varepsilon_{\mathrm{BH}})^2,$$

where $\varepsilon_{\mathrm{BH}}$ is a number fixed in Theorem 5.6.

Remark 2.4. Equations (2.4) and (2.10) implies that $\varsigma_1\, \varsigma_2\, \varsigma_5 \neq 0$. These conditions are used to get blender-horseshoes in the renormalisation scheme.

2.2. Geometry of the cycle: the sets $\mathcal{H}^r_{\mathrm{BH,h}}(M)$, $\mathcal{H}^r_{\mathrm{BH,e}}(M)$, and $\mathcal{H}^r_{\mathrm{BH,e^+}}(M)$.

For $R = P, Q$ consider

$$W^*_{\mathrm{loc}}(R, f) \stackrel{\text{def}}{=} C\big(R, W^*_{\mathrm{loc}}(R, f) \cap U_R\big), \quad * = \mathrm{s}, \mathrm{u},$$

here $C(x, A)$ is the connected component of the set A containing the point x.

The next definition classifies the two types of heterodimensional tangencies that we will consider. Note that given any $f \in \mathcal{H}^r_{\mathrm{BH}}(M)$ the set $U_P \setminus W^{\mathrm{u}}_{\mathrm{loc}}(P, f)$ has two connected components.

Definition 2.5 (Elliptic and hyperbolic tangencies)**.** The heterodimensional tangency at Y is *elliptic* if there is a neighbourhood $\mathcal{P}^{\mathrm{s}}_{\mathrm{Y}}$ of Y in $W^{\mathrm{s}}(Q, f) \cap U_P$ such that the set $\mathcal{P}^{\mathrm{s}}_{\mathrm{Y}} \setminus \{Y\}$ is contained in a connected component of $U_P \setminus W^{\mathrm{u}}_{\mathrm{loc}}(P, f)$. The tangency is *hyperbolic* if every neighbourhood of Y in $W^{\mathrm{s}}(Q, f)$ contains points in both components of $U_P \setminus W^{\mathrm{u}}_{\mathrm{loc}}(P, f)$.

In Figure 1, the heterodimensional tangency in the left-hand side is elliptic while the one in the right-hand side is hyperbolic.

We observe that if $f \in \mathcal{H}^r_{\mathrm{BH}}(M)$ then the heterodimensional tangency at the point Y is either hyperbolic or elliptic. We split the set $\mathcal{H}^r_{\mathrm{BH}}(M)$ in two parts, $\mathcal{H}^r_{\mathrm{BH,h}}(M)$ and $\mathcal{H}^r_{\mathrm{BH,e}}(M)$ consisting of hyperbolic and elliptic heterodimensional tangencies, respectively.

For diffeomorphisms in $f \in \mathcal{H}^r_{\mathrm{BH,e}}(M)$ we need to take in consideration the relative position of the tangency and the quasi-transverse heteroclinic points. We consider the subset $\mathcal{H}^r_{\mathrm{BH,e^+}}(M)$ of $\mathcal{H}^r_{\mathrm{BH,e}}(M)$ such that (with the notation above) the set $\mathcal{P}^{\mathrm{s}}_{\mathrm{Y}} \setminus \{Y\}$ and $\widetilde{X}$ are in the same connected component of $U_P \setminus W^{\mathrm{u}}_{\mathrm{loc}}(P, f)$, see Figure 1. These geometrical considerations have the same flavour of those in [24, Section 2].

3. Translation and rotation-like perturbations

In this section, we describe the two types of C^r perturbations used in our constructions. We start by introducing a class of auxiliary bump functions.

Caveat. For simplicity, throughout this paper, we will use the term *perturbation* to refer to arbitrarily small ones.

3.1. **Auxiliary bump functions.** Consider a family of C^r bump functions b^θ, $\theta > 1$, such that

$$b^\theta(x) = \begin{cases} 0\,, & |x| \geqslant \theta, \\ 0 \leqslant b^\theta(x) \leqslant 1\,, & 1 \leqslant |x| \leqslant \theta, \\ 1\,, & |x| \leqslant 1. \end{cases} \tag{3.1}$$

Associated to b^θ we consider the family of bump functions

$$b^\theta_\rho(x) \stackrel{\mathrm{def}}{=} b^\theta\left(\frac{x}{\rho}\right), \quad \rho > 0$$

and the three-dimensional bump functions

$$\Pi^\theta_\rho \colon \mathbb{R}^3 \to [0,1], \quad \Pi^\theta_\rho(x,y,z) = b^\theta_\rho(x)\, b^\theta_\rho(y)\, b^\theta_\rho(z). \tag{3.2}$$

Denote by $B(x,\tau)$ the open ball in $\mathbb{R}^3$ with center x and radius τ and by $\|\cdot\|_r$ the C^r norm. Note that the support of Π^θ_ρ is the closure of $B(\mathbf{0}, \theta\rho)$ and that

$$\|\Pi^\theta_\rho\|_r \leqslant (\|b^\theta\|_r)^3\, \rho^{-r}. \tag{3.3}$$

In what follows, for simplicity, when $\theta = 2$ we write $b^2_\rho = b_\rho$ and $\Pi^2_\rho = \Pi_\rho$.

3.2. **Translation-like perturbations.** Given a point $Z_0 \in \mathbb{R}^3$, a vector $\widetilde{w} \in \mathbb{R}^3$, and small $\rho > 0$, we consider the C^r map $T_{Z_0,\widetilde{w},\rho} : \mathbb{R}^3 \to \mathbb{R}^3$ defined by

$$T_{Z_0,\widetilde{w},\rho}(Z) = \begin{cases} Z + \Pi_\rho(Z - Z_0)\widetilde{w}, & \text{if } Z \in B(Z_0, 2\rho), \\ Z, \quad \text{if } Z \notin B(Z_0, 2\rho). \end{cases} \tag{3.4}$$

By construction and by (3.3), it holds

$$\big\|T_{Z_0,\widetilde{w},\rho} - \mathrm{id}\big\|_r \leqslant \big\|\Pi_\rho\big\|_r\, ||\widetilde{w}|| \leqslant (\|b\|_r)^3\, \rho^{-r}\, ||\widetilde{w}||.$$

Therefore, for small $||\widetilde{w}||$, the map $T_{Z_0,\widetilde{w},\rho}$ is a C^r perturbation of the identity supported in $B(Z_0, 2\rho)$. Finally observe that

$$T_{Z_0,\widetilde{w},\rho}\big(B(Z_0, 2\rho)\big) = B(Z_0, 2\rho).$$

3.3. **Rotation-like perturbations.** We now consider maps $I^x_\omega, I^y_\omega\colon \mathbb{R}^3 \to \mathbb{R}^3$, $\omega \in [-\pi,\pi]$, defined by

$$I^x_\omega \stackrel{\text{def}}{=} \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos 2\pi\omega & -\sin 2\pi\omega \\ 0 & \sin 2\pi\omega & \cos 2\pi\omega \end{pmatrix}, \qquad I^y_\omega \stackrel{\text{def}}{=} \begin{pmatrix} \cos 2\pi\omega & 0 & -\sin 2\pi\omega \\ 0 & 1 & 0 \\ \sin 2\pi\omega & 0 & \cos 2\pi\omega \end{pmatrix},$$

and for $\theta > 1$ and $\kappa > 0$ their associated C^∞ diffeomorphisms

$$R^*_{\omega,\theta,\kappa}\colon \mathbb{R}^3 \to \mathbb{R}^3, \quad R^*_{\omega,\theta,\kappa}(W) = I^*_{b^\theta(\kappa||W||)\,\omega}(W^T), \quad * = x, y, \tag{3.5}$$

where W^T denotes the transpose of the vector $W \in \mathbb{R}^3$.

Note that the restriction of $R^*_{\omega,\theta,\kappa}$ to the set $[-\kappa^{-1},\kappa^{-1}]^3$ coincides with I^*_ω and $R^*_{\omega,\theta,\kappa}$ is the identity map in the complement of $[-\theta\kappa^{-1},\theta\kappa^{-1}]^3$. Note also

$$R^*_{\omega,\theta,\kappa}\big([-\theta\kappa^{-1},\theta\kappa^{-1}]^3\big) = [-\theta\kappa^{-1},\theta\kappa^{-1}]^3, \quad * = x, y,$$

and that there is a constant $C(\theta,\kappa) > 0$ such that

$$\big\|R^*_{\omega,\theta,\kappa} - \mathrm{id}\big\|_{C^r} < C(\theta,\kappa)|\omega|, \quad * = x, y.$$

Thus, for every ω small enough, the map $R^*_{\omega,\theta,\kappa}$ is a C^r perturbations of identity supported in $[-\theta\kappa^{-1},\theta\kappa^{-1}]^3$.

4. NEW HETEROCLINIC AND HOMOCLINIC INTERSECTIONS

Recall the definitions of the sets $\mathcal{H}^r_{\mathrm{BH}}(M)$, $\mathcal{H}^r_{\mathrm{BH,h}}(M)$, and $\mathcal{H}^r_{\mathrm{BH,e^+}}(M)$ in Section 2.2. The main result of this section is Proposition 4.1 claiming that for every f in $\mathcal{H}^r_{\mathrm{BH,h}}(M)$ (resp. $\mathcal{H}^r_{\mathrm{BH,e^+}}(M)$) there are local C^r perturbations f_ε in $\mathcal{H}^r_{\mathrm{BH,h}}(M)$ (resp. $\mathcal{H}^r_{\mathrm{BH,e^+}}(M)$) of f with pairs of additional quasi-transverse heteroclinic points in $W^{\mathrm{s}}(P,f_\varepsilon) \cap W^{\mathrm{u}}(Q,f_\varepsilon)$ and additional transverse homoclinic points in $W^{\mathrm{s}}(Q,f_\varepsilon) \pitchfork W^{\mathrm{u}}(Q,f_\varepsilon)$. The proof of this proposition is done in Section 4.2. To prove it, in Section 4.1, we state some preliminary results about the invariant manifolds of the saddle-foci in the cycle. In Section 4.3, we study the transitions associated to the new heteroclinic points. Finally, in Section 4.4, we consider parameterisations of special unstable discs throughout the new heteroclinic points contained the unstable manifold of Q. The unfolding the cycle associated to these heteroclinic points will provide unstable discs intersecting robustly the stable manifold of the blender-horseshoes. We now go to the details.

Given $f \in \mathcal{H}^r_{\mathrm{BH}}(M)$ with elements P, Q, X, Y (recall Notation 2.1) define the closed invariant set

$$\Gamma_{P,Q,X,Y}(f) \stackrel{\text{def}}{=} \mathrm{Orb}(X,f) \cup \mathrm{Orb}(Y,f) \cup \{P,Q\}, \tag{4.1}$$

where $\mathrm{Orb}(W,f)$ denotes the f-orbit of the point W.

Recall also the neighbourhoods U_X and U_Y of X and Y in Section 2.1.2.

In what follows, we use the notation $d_r(f,g)$ for the C^r distance between two maps $f, g \in \mathrm{Diff}^r(M)$.

Proposition 4.1. *Let $f \in \mathcal{H}^r_{\mathrm{BH},*}(M)$, $* = \mathrm{h}, \mathrm{e}^+$, with elements P, Q, X, Y and transition times N_1 and N_2. For every $\varepsilon, \delta > 0$ there is $f_\varepsilon \in \mathcal{H}^r_{\mathrm{BH},*}(M)$ with $d_r(f, f_\varepsilon) < \varepsilon$ such that:*

(1) *f_ε coincides with f on the sets $\Gamma_{P,Q,X,Y}(f)$ and*
$$\bigcup_{i=0}^{N_1-1} f^i(U_X) \cup \bigcup_{i=0}^{N_2-1} f^i(U_{Y,\varepsilon})$$
where $U_{Y,\varepsilon}$ is a neighbourhood of Y contained in U_Y depending on ε.

(2) *f_ε has two quasi-transverse heteroclinic points*
$$X_{1,\varepsilon},\, X_{2,\varepsilon} \in f_\varepsilon^{-N_1}\big(W^{\mathrm{s}}_{\mathrm{loc}}(P, f_\varepsilon)\big) \cap W^{\mathrm{u}}(Q, f_\varepsilon) \cap B(X, \delta)$$
such that $\mathrm{Orb}(X_{1,\varepsilon}, f_\varepsilon)$, $\mathrm{Orb}(X_{2,\varepsilon}, f_\varepsilon)$, and $\mathrm{Orb}(X, f_\varepsilon)$ are pairwise disjoint and $X_{1,\varepsilon}, X_{2,\varepsilon} \to X$ as $\varepsilon \to 0$.

(3) *f_ε has two transverse intersection points*
$$Z_\varepsilon^\pm \in W^{\mathrm{s}}(Q, f_\varepsilon) \pitchfork W^{\mathrm{u}}_{\mathrm{loc}}(Q, f_\varepsilon)$$
such that in the local coordinates
$$Z_\varepsilon^\pm = (0, 1 \pm \zeta_\varepsilon^\pm, 0), \quad 0 < \zeta_\varepsilon^\pm < \delta.$$

A preliminary step of the proof of this proposition is Lemma 4.1 in Section 4.1 claiming that the closure of the one dimensional invariant manifold of P (resp. Q) contains the two dimensional invariant manifold of Q (resp. P).

4.1. **Density properties of $W^{\mathrm{s}}(P, f)$ and $W^{\mathrm{u}}(Q, f)$.** Consider $f \in \mathrm{Diff}^r(M)$ with a saddle focus R with $f(R) = R$ such that the eigenvalues of $Df(R)$ are $\lambda \in \mathbb{R}$ and $\sigma\, e^{\pm 2\pi i \varphi} \in \mathbb{C}$, where $0 < |\lambda| < 1 < \sigma$ and $\varphi \in [0, 1)$, and that is C^r linearisable in a neighbourhood U_R of R. We identify U_R with the Cartesian product of the local invariant manifolds of R (where the x- and yz-spaces are the stable and unstable eigenspaces of $Df(R)$, respectively.) We assume that there are (see Figure 2):

- A one-dimensional C^r disc $L \subset U_R$ such that L is quasi-transverse to $W^{\mathrm{s}}_{\mathrm{loc}}(R, f)$ at some point W in interior of L. We let L_+ and L_- the two connected components of $L \setminus \{W\}$.
- A two-dimensional C^r disc $S \subset U_R$ intersecting transverselly $W^{\mathrm{u}}_{\mathrm{loc}}(R, f)$ in a curve γ which is not contained in any radial direction of $W^{\mathrm{u}}_{\mathrm{loc}}(R, f)$ (i.e., a straight-line containing the origin). In this case, we say that the curve γ has a *nontrivial radial projection.*

We need the following simple auxiliary lemma.

Lemma 4.1 (Accelerating angles)**.** *Consider a diffeomorphism f, a saddle R, a disc L, a local surface S, and a curve $\gamma \subset S \cap W^{\mathrm{u}}_{\mathrm{loc}}(R, f)$ as above. Then there is g arbitrarily C^r close to f such that $W^*_{\mathrm{loc}}(R, g) = W^*_{\mathrm{loc}}(R, f)$, $* = \mathrm{s}, \mathrm{u}$, and*

(a) *$W^{\mathrm{u}}_{\mathrm{loc}}(R, g)$ is simultaneously contained in the closure of the sequences of discs $\big(g^j(L_+)\big)$ and $\big(g^j(L_-)\big)$, $j \geqslant 1$, and*

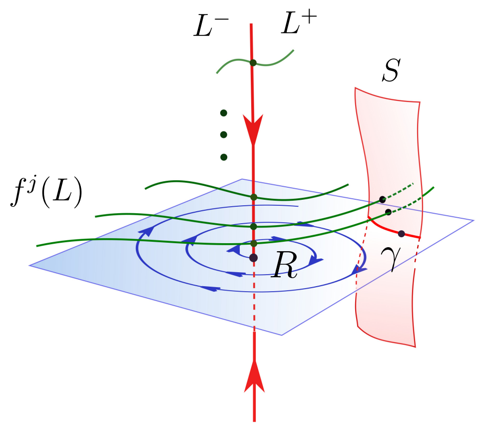

FIGURE 2. The discs $L, L^{\pm}$ and S and the curve γ

(b) *there are infinitely many $j_{\pm} \geqslant 1$ such that $g^{j_{\pm}}(L_{\pm})$ meets transversely S at points arbitrarily close to γ.*

Proof. The result is obvious if the argument φ of $Df(R)$ is irrational, in that case we can take $g = f$. Otherwise, it is enough to consider a sequence $(\alpha_j) \to 0$ such that $\varphi + \alpha_j$ is irrational, rotations $I^x_{\alpha_j}$ (defined on U_R) with argument α_j (recall the definition in Section 3.3), and local perturbations $g = f_j$ of f of the form $f_j = I^x_{\alpha_j} \circ f$ in the set U_R. These perturbations can be chosen supported on an small neighbourhood of the closure of U_R. □

Remark 4.2. There are the corresponding version of Lemma 4.1 for saddle foci with index one.

Remark 4.3. Changing the surface "S" by a one-dimensional disc and "transversality" by "quasi-transversality", the part (b) of Lemma 4.1 can be stated as follows: there is arbitrarily large $j_{\pm}$ such that $g^{j_{\pm}}(L_{\pm})$ meets quasi-transversely S at some point arbitrarily close to γ.

Remark 4.4. Assume that $f \in \mathcal{H}^r_{\mathrm{BH}}(M)$ and that g is obtained perturbing f using Lemma 4.1. Then g can be taken such that $g \in \mathcal{H}^r_{\mathrm{BH}}(M)$ and $\Gamma_{P,Q,X,Y}(g) = \Gamma_{P,Q,X,Y}(f)$, recall (4.1).

4.2. **Proof of Proposition 4.1.** We first consider hyperbolic tangencies, that is, we assume that $f \in \mathcal{H}^r_{\mathrm{BH,h}}(M)$. Consider the points Y and $\widetilde{Y} = f^{N_2}(Y)$ corresponding to the heterodimensional tangency in condition **(C)** in Section 2.1.2 and their neighbourhoods U_Y and $U_{\widetilde{Y}}$ in the definition of the transition map $\mathfrak{T}_2$ in (2.5).

Using the notation in Definition 2.5, we can select small two-discs

$$\mathcal{P}^{\mathrm{s}}_Y \subset W^{\mathrm{s}}(Q,f) \cap U_Y \quad \text{and} \quad \mathcal{P}^{\mathrm{u}}_{\widetilde{Y}} \subset W^{\mathrm{u}}(P,f) \cap U_{\widetilde{Y}}$$

containing Y and $\widetilde{Y}$ (respectively) in their interiors and assume that $\mathcal{P}^{\mathrm{s}}_{Y}$ contains a pair of disjoint surfaces S_+ and S_- intersecting transversely $W^{\mathrm{u}}_{\mathrm{loc}}(P,f)$ throughout curves γ_+ and γ_- with nontrivial radial projection, see Figure 3.

Consider now the quasi-transverse heteroclinic points X and $\widetilde{X} = f^{N_1}(X)$ in condition **(B)** in Section 2.1.2. Fix small $\delta > 0$ such that $B(X,\delta)$ is contained in U_X. Consider small curves

$$L^{\mathrm{u}} \subset U_P \cap f^{N_1}\big(W^{\mathrm{u}}_{\mathrm{loc}}(Q,f) \cap B(X,\delta)\big), \quad L^{\mathrm{s}} \subset W^{\mathrm{s}}(P,f) \cap U_Q$$

containing $\widetilde{X}$ and X in their interiors, respectively. See Figure 3. As above, we let $L^{\mathrm{u}}_{\pm}$ the connected components of $L^{\mathrm{u}} \setminus \{\widetilde{X}\}$.

Fix small $\varepsilon > 0$. Applying item (b) of Lemma 4.1 to P, the surfaces $S_{\pm}$, and the disc L^{u}, we get a diffeomorphism $\widetilde{f}_\varepsilon$ with $d_r(f,\widetilde{f}_\varepsilon) < \frac{\varepsilon}{2}$ and arbitrarily large numbers $i_+, i_- \geqslant 0$ such that $\widetilde{f}^{i_\pm}_\varepsilon(L^{\mathrm{u}}_{\pm})$ transversely intersects $S_{\pm}$ at some point $Z_{i_\pm}$. The points $Z_{i_\pm}$ can be chosen converging to some point of $\gamma_\pm$. Item (3) of the proposition follows taking $Z^{\pm}_{\varepsilon} = \widetilde{f}^{-(N_1+i_\pm)}_\varepsilon(Z_{i_\ell})$. Note that, arguing as before, we can assume that the argument of the complex eigenvalue of $D\widetilde{f}_\varepsilon(Q)$ is irrational.

Note that by Remark 4.4, the transitions of $\widetilde{f}_\varepsilon$ and f are the same, therefore $\widetilde{f}_\varepsilon \in \mathcal{H}^r_{\mathrm{BH,h}}(M)$ and $\Gamma_{P,Q,X,Y}(\widetilde{f}_\varepsilon) = \Gamma_{P,Q,X,Y}(f)$. Moreover, we observe that all perturbations that we will perform in what follows will keep this property.

To prove item (2), consider small disjoint closed subdiscs $\widetilde{L}^{\mathrm{u}}_{\pm} = \widetilde{L}^{\mathrm{u}}_{\pm}(\varepsilon) \subset L^{\mathrm{u}}_{\pm}$ as follows (see Figure 3): Let $\ell \in \{+,-\}$, there are numbers $i_\ell = i_\ell(\varepsilon)$ with

- $\widetilde{f}^{j}_\varepsilon(\widetilde{L}^{\mathrm{u}}_\ell) \subset U_P$ for all $j \in \{0,\dots,i_\ell\}$,
- $\widetilde{f}^{j}_\varepsilon(\widetilde{L}^{\mathrm{u}}_\ell) \cap U_Y = \emptyset$ for all $j \in \{0,\dots,i_\ell - 1\}$ and $\widetilde{f}^{i_\ell}_\varepsilon(\widetilde{L}^{\mathrm{u}}_\ell) \subset U_Y$,
- $\widetilde{f}^{i_\ell}_\varepsilon(\widetilde{L}^{\mathrm{u}}_\ell) \pitchfork S_\ell$ at the point Z_{i_ℓ},
- the family
$$\mathcal{L}^{\mathrm{u}} \stackrel{\mathrm{def}}{=} \big\{\widetilde{f}^{j}_\varepsilon(\widetilde{L}^{\mathrm{u}}_\ell)\colon j \in \{0,\dots,i_\ell + N_2\},\ \ell \in \{+,-\}\big\}$$
consists of pairwise disjoint sets,
- $\widetilde{f}^{i_\ell+N_2}_\varepsilon(\widetilde{L}^{\mathrm{u}}_\ell) \subset U_{\widetilde{Y}},\ \ \ell = +,-.$

By the definition of the transition $\mathfrak{T}_2$ and the choice of the neighbourhoods $U_Q, U_Y, U_{\widetilde{Y}}$, the subfamily of $\mathcal{L}^{\mathrm{u}}$ given by

$$\mathcal{L}^{\mathrm{u}}_0 \stackrel{\mathrm{def}}{=} \mathcal{L}^{\mathrm{u}} \setminus \{\widetilde{f}^{i_\ell+N_2}_\varepsilon(\widetilde{L}^{\mathrm{u}}_\ell)\}$$

is disjoint from U_P.

To prove item (2) of the proposition, consider the homoclinic points of Q

$$\widehat{Z}_\ell \stackrel{\mathrm{def}}{=} \widetilde{f}^{N_2}_\varepsilon(Z_{i_\ell}) \in W^{\mathrm{s}}_{\mathrm{loc}}(Q,\widetilde{f}_\varepsilon) \pitchfork W^{\mathrm{u}}(Q,\widetilde{f}_\varepsilon), \quad \ell = +,-$$

and take arbitrarily small

$$0 < \rho < \frac{\varepsilon}{2(\|f\|_r + \varepsilon)} \tag{4.2}$$

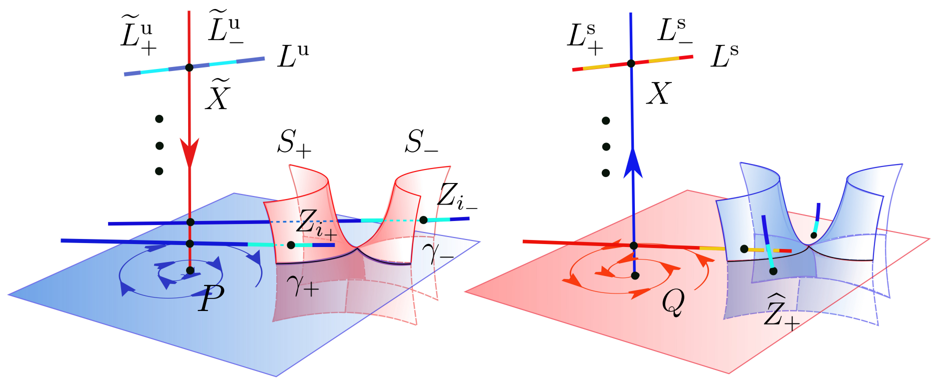

FIGURE 3. The discs $L^{\mathrm{u}}_{\pm}$ in $W^{\mathrm{u}}(Q,f)$ and $L^{\mathrm{s}}_{\pm}$ in $W^{\mathrm{s}}(P,f)$.

such that

$$B(\widetilde{Y},2\rho)\cup B(\widehat{Z}_+,2\rho)\cup B(\widehat{Z}_-,2\rho)\subset U_{\widetilde{Y}}$$

$$B(\widehat{Z}_+,2\rho),\ B(\widehat{Z}_-,2\rho),\ B(\widetilde{Y},2\rho) \text{ are pairwise disjoint.}$$

Arguing as above, considering $\widetilde{f}_\varepsilon^{-1}$ and applying item (a) of Lemma 4.1 to Q, the disc L^{s}, and the points $\widehat{Z}_\pm$ we get disjoint closed subdiscs $L^{\mathrm{s}}_\pm = L^{\mathrm{s}}_\pm(\rho)$ of L^{s} satisfying the following conditions (see Figure 3): Let K be the C^r norm of the map b_ρ in (3.1), there are numbers $k_\pm(\rho)=k_\pm$ such that:

- the family of sets
$$\{\widetilde{f}_\varepsilon^{-i}(L^{\mathrm{s}}_+) \colon i=0,\dots,k_+\}\cup\{\widetilde{f}_\varepsilon^{-j}(L^{\mathrm{s}}_-) \colon j=0,\dots,k_-\}$$
is pairwise disjoint,
- $\widetilde{f}_\varepsilon^{-i}(L^{\mathrm{s}}_\pm)\subset\Big(U_Q\setminus B\big(\widehat{Z}_\pm,\frac{\rho^{r+1}}{K^3}\big)\Big)$ for every $i\in\{0,\dots,k_\pm-1\}$,
- $\widetilde{f}_\varepsilon^{-k_\pm}(L^{\mathrm{s}}_\pm)\subset B\big(\widehat{Z}_\pm,\frac{\rho^{r+1}}{K^3}\big)$.

Let $X_\pm$ be the closest point of $\widehat{Z}_\pm$ in $\widetilde{f}_\varepsilon^{-k_\pm}(L^{\mathrm{s}}_\pm)$ and define the vector

$$\mathbf{w}_\pm \overset{\text{def}}{=} X_\pm-\widehat{Z}_\pm,\qquad \|\mathbf{w}_\pm\|\leqslant\frac{\rho^{r+1}}{K^3}. \tag{4.3}$$

Using the function Π_ρ in (3.2), consider the perturbation of the identity given by

$$\vartheta_{\pm,\rho}(\widehat{Z}_\pm+W)\overset{\text{def}}{=}\widehat{Z}_\pm+W+\Pi_\rho(W)\,\mathbf{w}_\pm,\quad \text{if}\quad \widehat{Z}_\pm+W\in B(\widehat{Z}_\pm,2\rho)$$

and the identity otherwise. Since $\|\mathbf{w}_\pm\|\leqslant\frac{\rho^{r+1}}{K^3}$ and $\|\Pi^\theta_\rho\|_r\leqslant K^3\,\rho^{-r}$ (recall (3.3)), it holds

$$\|\vartheta_{\pm,\rho}-\mathrm{id}\|_r\leqslant\|\Pi_\rho\|_r\cdot\|\mathbf{w}_\pm\|<\rho.$$

Finally, consider the perturbation of $\widetilde{f}_\varepsilon$ defined by

$$f_\varepsilon=f_{\varepsilon,\rho}\overset{\text{def}}{=}\vartheta_{\pm,\rho}\circ\widetilde{f}_\varepsilon.$$

Recalling the choice of ρ in (4.2), we get

$$d(f, f_\varepsilon)_r \leqslant d(f, \widetilde{f}_\varepsilon)_r + d(\widetilde{f}_\varepsilon, f_\varepsilon)_r \leqslant \frac{\varepsilon}{2} + \rho \|\widetilde{f}_\varepsilon\|_r < \frac{\varepsilon}{2} + \rho(\|f\|_r + \varepsilon) < \varepsilon.$$

By construction, $f_{\varepsilon,\rho}$ coincides with $\widetilde{f}_\varepsilon$ outside $\widetilde{f}_\varepsilon^{-1}\big(B(\widehat{Z}_+, 2\rho) \cup B(\widehat{Z}_-, 2\rho)\big)$ and by (4.3) we have that the points X, $X_{1,\varepsilon}$, and $X_{2,\varepsilon}$ with

$$X_{1,\varepsilon} \stackrel{\text{def}}{=} f_\varepsilon^{k_+}(X_+), \quad X_{2,\varepsilon} \stackrel{\text{def}}{=} f_\varepsilon^{k_-}(X_-) \in f_{\varepsilon,\rho}^{-N_1}\big(W^{\mathrm{s}}_{\mathrm{loc}}(P, f_{\varepsilon,\rho})\big) \cap W^{\mathrm{u}}(Q, f_{\varepsilon,\rho})$$

are quasi-transverse heteroclinic points of $f_{\varepsilon,\rho}$ with different orbits.

Thus, $f_{\varepsilon,\rho} \in \mathcal{H}^r_{\mathrm{BH,h}}(M)$ and satisfies items (1)-(3) in the proposition.

We now study the elliptic case when $f \in \mathcal{H}^r_{\mathrm{BH,e^+}}(M)$. We apply the variation of Lemma 4.1 in Remark 4.3 and observe that an arbitrarily small modification of the angle provides two transverse intersections (for the same iterate). The rest of the proof is identical to the hyperbolic case.

Finally, note that in our construction the transitions are preserved, thus if $f \in \mathcal{H}^r_{\mathrm{BH},*}(M)$ then $f_{\varepsilon,\rho} \in \mathcal{H}^r_{\mathrm{BH},*}(M)$, $* = \mathrm{h}, \mathrm{e}^+$. The proof of the proposition is now complete. □

4.3. **Transitions for the new heteroclinic points.** Given $f \in \mathcal{H}^r_{\mathrm{BH},*}(M)$, $* = \mathrm{h}, \mathrm{e}^+$, and small $\varepsilon > 0$, consider its perturbation $f_\varepsilon \in \mathcal{H}^r_{\mathrm{BH},*}(M)$ and the heteroclinic points $X_{1,\varepsilon}$ and $X_{2,\varepsilon}$ given by Proposition 4.1. Take disjoint neighbourhoods $U_{1,\varepsilon}$ and $U_{2,\varepsilon}$ of $X_{1,\varepsilon}$ and $X_{2,\varepsilon}$ contained in U_X where the transition map $\mathfrak{T}_1$ in (2.3) is defined. By shrinking these neighbourhoods, we can take a small neighbourhood $U_{0,\varepsilon} \subset U_X$ of $X_{0,\varepsilon} = X$ disjoint from $U_{1,\varepsilon}$ and $U_{2,\varepsilon}$. We write

$$X_{i,\varepsilon} \stackrel{\text{def}}{=} X + Z_{i,\varepsilon}, \quad Z_{i,\varepsilon} \stackrel{\text{def}}{=} (x_{i,\varepsilon}, y_{i,\varepsilon}, z_{i,\varepsilon}), \quad i = 0, 1, 2. \tag{4.4}$$

Note that $Z_{1,\varepsilon}, Z_{2,\varepsilon} \to \mathbf{0}$ as $\varepsilon \to 0$.

Denote by $\mathfrak{T}_{1,i,\varepsilon}$ the restriction of $\mathfrak{T}_1$ to $U_{i,\varepsilon}$,

$$\mathfrak{T}_{1,i,\varepsilon} : U_{i,\varepsilon} \to U_{\widetilde{X}}, \quad \mathfrak{T}_{1,i,\varepsilon}(Z) = f_\varepsilon^{N_1}(Z) = f^{N_1}(Z), \quad i = 0, 1, 2. \tag{4.5}$$

Hence, recalling (2.3),

$$\widetilde{X}_{i,\varepsilon} \stackrel{\text{def}}{=} \mathfrak{T}_{1,i,\varepsilon}(X_{i,\varepsilon}) = \widetilde{X} + A(Z_{i,\varepsilon}) + \widetilde{H}(Z_{i,\varepsilon}),$$

Using equation (4.4) and that $X_{i,\varepsilon} \in f_\varepsilon^{-N_1}\big(W^{\mathrm{s}}_{\mathrm{loc}}(P, f_\varepsilon)\big) \cap U_X$ we get

$$\widetilde{X}_{i,\varepsilon} \stackrel{\text{def}}{=} \widetilde{X} + \widetilde{Z}_{i,\varepsilon} \in W^{\mathrm{s}}_{\mathrm{loc}}(P, f_\varepsilon), \quad \text{where} \quad \widetilde{Z}_{i,\varepsilon} = A(Z_{i,\varepsilon}) + \widetilde{H}(Z_{i,\varepsilon}). \tag{4.6}$$

Note that $\widetilde{X}, \widetilde{X}_{i,\varepsilon} \in W^{\mathrm{s}}_{\mathrm{loc}}(P, f_\varepsilon)$ and that (in local coordinates) $W^{\mathrm{s}}_{\mathrm{loc}}(P, f_\varepsilon)$ is contained in $\{(x, 0, 0)\} \subset U_P$. Hence

$$\widetilde{Z}_{i,\varepsilon} = (\tilde{x}_{i,\varepsilon}, 0, 0). \tag{4.7}$$

Then the map $\mathfrak{T}_{1,i,\varepsilon}$ can be written as follows: for $X_{i,\varepsilon} + W \in U_{i,\varepsilon}$ we have

$$\mathfrak{T}_{1,i,\varepsilon}(X_{i,\varepsilon} + W) = \widetilde{X}_{i,\varepsilon} + A(W) + \widetilde{H}^i_\varepsilon(W), \tag{4.8}$$

where $\widetilde{H}^i_\varepsilon \colon \mathbb{R}^3 \to \mathbb{R}^3$ is defined by

$$\widetilde{H}^i_\varepsilon(W) \stackrel{\text{def}}{=} \widetilde{H}(Z_{i,\varepsilon} + W) - \widetilde{H}(Z_{i,\varepsilon}). \tag{4.9}$$

Remark 4.5. Let $\widetilde{H}^i_\varepsilon = (\widetilde{H}^i_{\varepsilon,1}, \widetilde{H}^i_{\varepsilon,2}, \widetilde{H}^i_{\varepsilon,3})$. Then

$$\widetilde{H}^i_{\varepsilon,1}(\mathbf{0}) = \widetilde{H}^i_{\varepsilon,2}(\mathbf{0}) = \widetilde{H}^i_{\varepsilon,3}(\mathbf{0}) = 0.$$

Although the maps $\widetilde{H}^i_{\varepsilon,j}$ do not satisfy the same "flat conditions" at $\mathbf{0}$ satisfied by the terms $\widetilde{H}_i$ of $\mathfrak{T}_1$, see (2.3), the following convergence property holds:

$$\frac{\partial}{\partial x}\widetilde{H}^i_{\varepsilon,k}(\mathbf{0}) \to 0, \quad \frac{\partial}{\partial y}\widetilde{H}^i_{\varepsilon,k}(\mathbf{0}) \to 0, \quad \frac{\partial}{\partial z}\widetilde{H}^i_{\varepsilon,k}(\mathbf{0}) \to 0, \quad \varepsilon \to 0, \quad k = 1, 2, 3.$$

4.4. **Parameterisations of unstable discs throughout the points $X_{i,\varepsilon}$.** Take an unitary vector $\mathbf{v}_{i,\varepsilon} \in T_{X_{i,\varepsilon}} W^{\mathrm{u}}(Q, f_\varepsilon)$. For small $\delta > 0$, consider the parameterised segment of the local unstable manifold of Q containing $X_{i,\varepsilon}$ in $U_{i,\varepsilon}$ obtained considering its Taylor expansion,

$$L^{\mathrm{u}}_{i,\varepsilon}(\delta) \stackrel{\mathrm{def}}{=} \big\{ X_{i,\varepsilon} + t\, \mathbf{v}_{i,\varepsilon} + \widetilde{\rho}_{i,\varepsilon}(t) : |t| < \delta \big\} \subset W^{\mathrm{u}}(Q, f_\varepsilon), \tag{4.10}$$

here $\widetilde{\rho}_{i,\varepsilon}$ is an C^r map satisfying

$$\widetilde{\rho}_{i,\varepsilon}(0) = \frac{d}{dt}\widetilde{\rho}_{i,\varepsilon}(0) = \mathbf{0}. \tag{4.11}$$

Remark 4.6. By the λ-lemma we have that $\mathbf{v}_{i,\varepsilon} \to \mathbf{e}_2 = (0,1,0)$ and $\|\widetilde{\rho}_{i,\varepsilon}\|_r \to 0$ as $\varepsilon \to 0$. By a C^r perturbation of f_ε, we can assume that $\mathbf{v}_{i,\varepsilon} = \mathbf{e}_2$. To see this, let $\pi_{i,\varepsilon}$ be the plane generated by $\mathbf{v}_{i,\varepsilon}$ and $\mathbf{e}_2$ in $T_{X_{i,\varepsilon}} M$ and $\alpha_{i,\varepsilon}$ the smallest angle (modulus 2π) of the rotation map taking $\mathbf{v}_{i,\varepsilon}$ into $\mathbf{e}_2$. Note that $\alpha_{i,\varepsilon} \to 0$ as $\varepsilon \to 0$. Performing a rotation-like C^r perturbation as in Section 3.3 at $X_{i,\varepsilon}$ around of the orthogonal direction to $\Pi_{i,\varepsilon}$, we get a diffeomorphism $O(\alpha_{i,\varepsilon})$ C^r close to f_ε (that we continue to call f_ε) such that

$$L^{\mathrm{u}}_{i,\varepsilon}(\delta) = \big\{ X_{i,\varepsilon} + t\, \mathbf{e}_2 + \widetilde{\rho}_{i,\varepsilon}(t) : |t| < \delta \big\} \subset W^{\mathrm{u}}(Q, f_\varepsilon), \quad i = 1, 2. \tag{4.12}$$

With a slight abuse of notation, the higher order terms $\widetilde{\rho}_{i,\varepsilon}$ in (4.12) are denoted as the ones in (4.10). As the latter are obtained as "small rotations" of the terms in (4.10), they satisfy the flat conditions in (4.11).

5. BLENDER-HORSESHOES AND CENTER-UNSTABLE HÉNON-LIKE FAMILIES

In this section, we introduce blender-horseshoes and their main properties (Section 5.1) and explain how they may lead to robust tangencies (Section 5.2). We also state their occurrence in center-unstable Hénon-like families (Section 5.3). Finally, we study the geometry of the unstable manifolds of these blenders (Lemma 5.5). All blenders considered in this paper are blender-horseshoes, thus if there is no misunderstanding in some cases we will refer to them simply as blenders.

5.1. **Blender-horseshoes.** We refrain to give a precise definition of a blender-horseshoe (for details see [8, 13]), instead we will focus on their relevant properties. We also restrict our discussion to our three dimensional context. A *blender-horseshoe* is a locally maximal hyperbolic set Γ_f of a diffeomorphism $f : M \to M$ that is conjugate to the complete full shift on two symbols and satisfies a geometrical condition stated in Lemmas 5.1 and 5.2. We now go to the details.

There is an open neighbourhood Δ of Γ such that

$$\Gamma_f = \bigcap_{i\in\mathbb{Z}} f^i(\,\overline{\Delta}\,) \subset \Delta$$

The set Γ_f is also partially hyperbolic: there is a dominated splitting with one-dimensional bundles $E^{\mathrm{s}} \oplus E^{\mathrm{cu}} \oplus E^{\mathrm{uu}}$ of $T_{\Gamma_f}M$ such that $E^{\mathrm{u}} \stackrel{\text{def}}{=} E^{\mathrm{cu}} \oplus E^{\mathrm{uu}}$ and E^{s} are the unstable and stable bundles of Γ, respectively. The bundle E^{uu} is the strong unstable direction. We consider a Df-invariant cone fields $\mathcal{C}^{\mathrm{uu}}$ and C^{u} around E^{uu} and E^{u}, a Df^{-1}-invariant cone field $\mathcal{C}^{\mathrm{ss}}$ around E^{ss}, and a center unstable cone field C^{cu} around E^{cu}. The latter is not Df invariant, but the norm of the vectors in C^{cu} are uniformly expanded by Df.

As a hyperbolic set, the blender-horseshoe Γ_f has a continuation Γ_g for every g sufficiently close. The important fact is that for diffeomorphisms nearby these continuations are also blender-horseshoes. Blender-horseshoes can be also defined for endomorphisms, see [17, Definition 2.7], with the following reformulation of the continuation property: every map (diffeomorphism or endomorphism) close to an endomorphism with a blender-horseshoe also has a blender-horseshoe with the same reference domain.

A key ingredient of a blender-horseshoe is its *superposition region.* To describe it define the local stable manifold of Γ_f by

$$W^{\mathrm{s}}_{\mathrm{loc}}(\Gamma_f, f) \stackrel{\text{def}}{=} \big\{\, x \in M : \; f^i(x) \in \overline{\Delta} \;\; \text{for every} \;\; i \geqslant 0 \,\big\} \tag{5.1}$$

and observe that the set Γ_f has two fixed points P^+ and P^-, called the *reference fixed points of the blender*. One defines "large" one-dimensional uu-discs (contained in U) tangent to $\mathcal{C}^{\mathrm{uu}}$ *at the right and at the left of* $W^{\mathrm{s}}_{\mathrm{loc}}(P^{\pm}, f)$. The set of such discs at the right of $W^{\mathrm{s}}_{\mathrm{loc}}(P^-, f)$ and at the left of $W^{\mathrm{s}}_{\mathrm{loc}}(P^+, f)$ form the *superposition region* denoted by $\mathcal{D}_f$. We say that these uu-discs are *in-between*.

The next two lemmas (see for instance [5, Lemma 3.13] and [17, Lemma 2.5]) state an intersection property for discs in the superposition region of the blender that will play a key role to get robust heterodimensional cycles.

Lemma 5.1 (The superposition region)**.** *Let Γ_f be a blender-horseshoe of f and D compact disc whose interior contains a disc in the superposition region of Γ_f. Then for every C^1-neighbourhood $\mathcal{U}$ of D there is a C^1-neighbourhood*[6] *$\mathcal{V}$ of f such that every compact disc in $\mathcal{U}$ contains a disc in the superposition region of Γ_g for every $g \in \mathcal{V}$.*

For further explanation of this lemma see Remark 5.4. See also Lemma 5.4 for a "quantitative version" of this result.

Lemma 5.2. *Let Γ_f be a blender-horseshoe of f and D a compact disc containing a disc the superposition region of Γ_f. Then $W^{\mathrm{s}}_{\mathrm{loc}}(\Gamma_f, f) \cap D \neq \emptyset$ and this intersection is quasi-transverse.*

[6]Let $r > 1$ and $f \in \mathrm{Diff}^r(M)$. Any C^1-neighbourhood of f contains a C^r neighbourhood of f.

This lemma follows from the fact that the image of any disc D in the superposition region contains a disc in the superposition region. Arguing inductively, it follows that any disc in the superposition region contains a point whose forward orbit is contained in Δ. The lemma now follows from the characterisation of $W^{\mathrm{s}}_{\mathrm{loc}}(\Gamma_f, f)$ in (5.1). This construction also guarantees that the obtained intersection between D and $W^{\mathrm{s}}_{\mathrm{loc}}(\Gamma_f, f)$ is quasi-transverse: the disks in the superposition region are tangent to a strong unstable cone field, $W^{\mathrm{s}}_{\mathrm{loc}}(\Gamma_f, f)$ is tangent to a stable cone field, and these cone fields have no common directions.

5.2. **Blender-horseshoes: tubes and folding manifolds.** We now analise when the local stable manifold $W^{\mathrm{s}}_{\mathrm{loc}}(\Gamma_f, f)$ of a blender-horseshoe Γ_f has a (robust) tangency with a surface S "passing throughout its domain Δ". In [8] it is proved that occurrence when S is a *folding manifold.* Motivated by this fact, we introduce the notion of u-*tubes* and prove that they generate folding manifolds after iterations. In this way, Proposition 5.2 provides a mechanism guaranteeing the robust tangencies in Theorem 1.1. We now go into the details of this construction that follows the ideas in [8, Section 4].

5.2.1. *Folding manifolds and tubes.* Throughout this section, we consider a diffeomorphism f with a blender-horseshoe Γ_f with domain Δ and reference fixed points P^- and P^+. Next definition is an extension of [8, Definition 4.2].

Definition 5.1 (Strips, tubes, and folding manifolds)**.** Consider a surface with boundary S of the form

$$S = \bigcup_{t\in[0,1]} D_t \subset \Delta,$$

where $(D_t)_{t\in[0,1]}$ is a family of uu-discs depending continuously on t. We say that S is a

- u-*strip* if the family of discs $(D_t)_{t\in[0,1]}$ is pairwise disjoint and S is tangent to the unstable cone field C^{u},
- uu-*tube* if the family of discs $(D_t)_{t\in[0,1)}$ is pairwise disjoint and $D_0 = D_1$.
- a *folding surface* if the family of discs $(D_t)_{t\in[0,1]}$ is pairwise disjoint, D_0 and D_1 both intersect $W^{\mathrm{s}}_{\mathrm{loc}}(P^-, f)$ (or both intersect $W^{\mathrm{s}}_{\mathrm{loc}}(P^+, f)$), and D_t is in-between P^- and P^+ for every $t \in (0,1)$.

A strip or a tube S is *in-between* if every D_t is in-between P^+ and P^-. Note that a folding surface cannot be in-between.

In what follows, we will use the letters S, T, and F to denote strips, tubes, and folding manifolds, respectively. The main result of this section is the following:

Proposition 5.2. *Let Γ_f be a blender-horseshoe of a diffeomorphism f and T a* uu-*tube in-between. Then $W^{\mathrm{s}}_{\mathrm{loc}}(\Gamma_f, f)$ and T have a tangency point.*

An immediate consequence of this proposition is the following version of [8, Corollary 4.11] where folding manifolds are replaced by uu-tubes:

Corollary 5.1. *Let Γ_f be a blender-horseshoe of f and T a* uu*-tube in-between contained in the unstable manifold of a saddle R_f of index two. Then there is a C^r neighbourhood $\mathcal{V}_f$ of f such that $W^{\mathrm{s}}_{\mathrm{loc}}(\Gamma_g, g)$ and $W^{\mathrm{u}}(R_g, g)$ have a tangency point for every $g \in \mathcal{V}_f$.*

Proposition 5.2 will follow from [8, Proposition 4.4]: any folding manifold has a tangency with $W^{\mathrm{s}}_{\mathrm{loc}}(\Gamma_f, f)$. The main ingredient of the proof in [8] is the fact that the image of any folding manifold contains a folding manifold, see [8, Lemma 4.5]. We reformulate that lemma in our context. For that we need to analise the central width of iterations of u-tubes and u-strips. We refer to see [13, Section 2] for a detailed analysis of iterations of strips. We now go into the details.

First, a *central curve* is a curve tangent to the center unstable cone field $\mathcal{C}^{\mathrm{cu}}$. The *central width* of a u-strip $S = \cup_{t\in[0,1]} D_t$ is defined by

$$\mathrm{w}(S) = \inf\{\mathrm{length}(\ell) \colon \ell \subset S \text{ is a central curve joining } D_0 \text{ and } D_1\}.$$

Note that there is $\kappa = \kappa(\Gamma_f) > 0$ such that $\mathrm{w}(S) \leqslant \kappa$ for every u-strip in-between.

To define the *central width of a* uu*-tube* T in-between (denoted with a slight abuse of notation also by $\mathrm{w}(T)$), we note that $(\Delta \setminus T)$ has two connected components, one of them is disjoint from P^- and P^+. We denote this component by Δ_T and let

$$\mathrm{w}(T) = \sup\{\mathrm{w}(S) \colon S \text{ is a u-strip contained in } \Delta_T\}.$$

Note that the width of any uu-tube in-between is bounded by the constant κ above.

Lemma 5.3. *Let Γ_f be a blender-horseshoe. Then there is $\lambda > 1$ such that for every* uu*-tube T in-between one of the following possibilities holds true:*

(a) *$f(T)$ has tangency with either $W^{\mathrm{s}}_{\mathrm{loc}}(P^-, f)$ or with $W^{\mathrm{s}}_{\mathrm{loc}}(P^+, f)$,*
(b) *$f(T)$ contains a folding manifold,*
(c) *$f(T)$ contains a* uu*-tube T' in-between with $\mathrm{w}(T') \geqslant \lambda\, \mathrm{w}(T)$.*

Proposition 5.2 easily follows from this lemma. Observe that, by the comments above, in cases (a) and (b) we get a tangency between T and $W^{\mathrm{s}}_{\mathrm{loc}}(\Gamma_f, f)$ and we are done. Otherwise, we let $T_0 = T$ and get a new tube T_1 in-between such that $T_1 \subset f(T_0)$ and $\mathrm{w}(T_1) \geqslant \lambda\, \mathrm{w}(T_0)$. We can now apply Lemma 5.3 to T_1 and argue recurrently. But case (c) cannot occur infinitely many consecutive times: in such a case we get a sequence of tubes (T_n) in-between with

$$T_n \subset f(T_{n-1}) \quad \text{and} \quad \mathrm{w}(T_n) \geqslant \lambda\, \mathrm{w}(T_{n-1}) \geqslant \lambda^n\, \mathrm{w}(T_0).$$

Since the widths of the tubes T_n are bounded by κ there is a first step of the recurrent construction when we fall in cases (a) or (b). We now prove the lemma.

5.2.2. *Proof of Lemma 5.3.* We have the following version of Lemma 5.3 for u-strips imported from [8, Lemma 4.5] (see also the proof of [13, Proposition 2.3]).

Lemma 5.4. *Consider a blender-horseshoe Γ_f of f. Then there is $\lambda > 1$ such that for every* u*-strip S in-between then one of the two possibilities holds:*

(i) *$f(S)$ intersects transversely either $W^{\mathrm{s}}_{\mathrm{loc}}(P^-, f)$ or $W^{\mathrm{s}}_{\mathrm{loc}}(P^+, f)$,*
(ii) *$f(S)$ contains a* u*-strip S' in-between with $\mathrm{w}(S') \geqslant \lambda\, \mathrm{w}(S)$.*

The main ingredient of the proof of this lemma is in Remark 5.4.

Remark 5.3 (Iterations of u-strips in-between)**.** Lemma 5.4 implies that the orbit of any u-strip in-between transversely meets either $W^s_{\mathrm{loc}}(P^-, f)$ or $W^s_{\mathrm{loc}}(P^+, f)$. More precisely, let $S_0 = S$ and assume that $f(S_0)$ intersects transversely neither $W^{\mathrm{s}}_{\mathrm{loc}}(P^-, f)$ nor $W^{\mathrm{s}}_{\mathrm{loc}}(P^+, f)$. In that case, we get a new strip S_1 in-between such that $S_1 \subset f(S_0)$ and $\mathrm{w}(S_1) \geqslant \lambda\, \mathrm{w}(S_0)$. We can now apply Lemma 5.4 to S_1 and argue recurrently. As above, this possibility cannot occurs infinitely many consecutive times.

We are now ready to prove the lemma. Given any $\varepsilon > 0$, associated to the uu-tube $T_0 = T$ there is an internal strip $S_0 \subset \Delta_{T_0}$, with $\mathrm{w}(S_0) \geqslant \mathrm{w}(T_0) - \varepsilon$. Note that this strip is in-between. Consider now $f(T_0)$ and assume that cases (a) and (b) in Lemma 5.3 do not hold. This implies that the image of $f(S_0)$ satisfies (ii) in Lemma 5.4. We now see that and in that case there is u-tube $T_1 \subset f(T_0)$ such that $S_1 \subset \Delta_{T_1}$. Since this holds for every $\varepsilon > 0$ item (c) in the lemma follows.

We now explain how the tube T_1 is obtained. We will refer to [13, Section 2] for details. A blender-horseshoe has a Markov partition associated to two disjoint "subrectangles" $\Delta_{\mathbb{A}}, \Delta_{\mathbb{B}} \subset \Delta$ such that

$$\Gamma_f = \bigcap_{i \in \mathbb{Z}} f^i(\Delta_{\mathbb{A}} \cup \Delta_{\mathbb{B}}).$$

Moreover, the map that associates to each $x \in \Gamma_f$ the sequence $(\xi_i(x))_{\in \mathbb{Z}} \in \{\mathbb{A}, \mathbb{B}\}^{\mathbb{Z}}$ defined by $f^i(x) \in \Delta_{\xi_i(x)}$ is a conjugation between Γ_f and the complete shift on the symbols $\mathbb{A}$, $\mathbb{B}$. In particular, the reference fixed points of the blender satisfy $P^- \in \Delta_{\mathbb{A}}$ and $P^+ \in \Delta_{\mathbb{B}}$. In what follows, we denote by $f_{\mathbb{E}}$ the restriction of f to $\mathbb{E}$, $\mathbb{E} \in \{\mathbb{A}, \mathbb{B}\}$. Given a set X we let $X_{\mathbb{E}} \stackrel{\mathrm{def}}{=} X \cap \Delta_{\mathbb{E}}$.

Next remark explains the mechanism guaranteeing Lemma 5.1 and is a consequence of the definition of a blender-horseshoe.

Remark 5.4 (Iterations of uu-discs ans u-strip)**.** There is $\lambda > 1$ with the following property: Consider a uu-disc D and a u-strip S in-between.

- Then either $f(D_{\mathbb{A}})$ or $f(D_{\mathbb{B}})$ contains a uu-disc in-between.
- Suppose that $f(S_{\mathbb{E}})$ is a u-strip in-between, then $\mathrm{w} f(S_{\mathbb{E}}) \geqslant \lambda \mathrm{w}(S_{\mathbb{E}})$.

By Remark 5.4, for a given $t \in [0,1]$ either $f(D_{t,\mathbb{A}})$ or $f(D_{t,\mathbb{B}})$ is a uu-disc in-between. We have the following three cases:

(A) $f(D_{t,\mathbb{A}})$ is a uu-disc in-between for every $t \in [0,1]$,
(B) $f(D_{t,\mathbb{B}})$ is a uu-disv in-between for every $t \in [0,1]$,
(C) Cases (A) and (B) do not hold.

In case (A), by Remark 5.4 we have that $T_1 = f(T_{t,\mathbb{A}})$ is a uu-tube such that $\mathrm{w}(T_1) \geqslant \lambda \mathrm{w}(T)$. Analogously, in case (B) $T_1 = f(T_{t,\mathbb{B}})$ is a uu-tube satisfying $\mathrm{w}(T_1) \geqslant \lambda \mathrm{w}(T)$. In both cases, item (3) in the lemma holds. Thus it remains to consider case (iii).

In case (C), there is $c \in [0,1)$ such that $f(D_{c,\mathbb{A}})$ is a uu-disc in-between but $f(D_{c,\mathbb{B}})$ is not a uu-disc in-between (this possibility includes the case where

$f(D_{c,\mathbb{B}})$ is not a uu-disc). After changing the parameterisation of the tube we can take $c=0$. Recall that $D_0=D_1$. Let

$$t_1 \stackrel{\text{def}}{=} \sup\{t\in[0,1]\colon f(D_{s,\mathbb{A}}) \text{ is in-between for all } s\in[0,t]\},$$
$$t_2 \stackrel{\text{def}}{=} \inf\{t\in[0,1]\colon f(D_{s,\mathbb{A}}) \text{ is in-between for all } s\in[t,1]\}.$$

Note that $t_1=1$ if and only if $t_2=0$. Moreover, if this does not hold then $t_1\leqslant t_2$. Thus, a priory, there are the following cases, (i) $t_1=1$ and $t_2=0$, (ii) $t_1=t_2\in(0,1)$, and (iii) $0<t_1<t_2<1$. Case (i) implies that we are in case (A) above, a contradiction. Thus it can be discarded.

Note that, by continuity, if $t_1<1$ then $f(D_{t_1,\mathbb{A}})\cap W^s_{\mathrm{loc}}(P^-,f)\neq\emptyset$. Similarly, if $t_2>0$ then $f(D_{t_2,\mathbb{A}})\cap W^s_{\mathrm{loc}}(P^-,f)\neq\emptyset$. Note that the local stable manifolds are tangent to the stable cone field and the uu-discs are tangent to the strong unstable cone field, hence the previous intersections are quasi-transverse.

In case (iii), consider the interval $[t_2-1,t_1]$ and let $\widehat{D}_t=D_t$ if $t\in[0,t_1]$ and $\widehat{D}_t=D_{t+1}$ if $t\in[t_2-1,1]$. By construction,

$$S \stackrel{\text{def}}{=} \bigcup_{t\in[t_2-1,t_1]} f(\widehat{D}_{t,\mathbb{A}})$$

is a folding manifold, thus item (b) in the lemma holds.

Finally, in case (ii) we have that $T_1=\bigcup_{t\in[0,1]} f(D_{t,\mathbb{A}})$ is tangent to $W^s_{\mathrm{loc}}(P^-,f)$ and we are in case (a) in the lemma. This completes the proof of the lemma. □

5.3. **Blender-horseshoes in the center-unstable Hénon-like family.** Let us start by defining the Hénon-like families of endomorphisms that we will consider.

5.3.1. *Center-unstable Hénon-like families of endomorphisms.* We consider the parameterised families of Hénon-like endomorphisms $E_{\xi,\mu,\bar\varsigma}$, $G_{\xi,\mu,\bar\eta}\colon\mathbb{R}^3\to\mathbb{R}^3$, defined by

$$\begin{aligned} E_{\xi,\mu,\bar\varsigma}(x,y,z) &\stackrel{\text{def}}{=} (\xi\,x+\varsigma_1\,y,\ \mu+\varsigma_2\,y^2+\varsigma_3\,x^2+\varsigma_4\,x\,y,\ \varsigma_5\,y),\\ G_{\xi,\mu,\bar\eta}(x,y,z) &\stackrel{\text{def}}{=} (y,\mu+y^2+\eta_1\,y\,z+\eta_2\,z^2,\xi\,z+y), \end{aligned} \tag{5.2}$$

where $\xi>1,\mu\in\mathbb{R}$, $\bar\varsigma\stackrel{\text{def}}{=}(\varsigma_1,\varsigma_2,\varsigma_3,\varsigma_4,\varsigma_5)\in\mathbb{R}^5$, and $\bar\eta=(\eta_1,\eta_2)\in\mathbb{R}^2$. These families are called *center-unstable Hénon-like.*

These families are conjugate, this allows us to translate properties from one family to the other. More precisely:

Remark 5.5. Consider the families of endomorphisms

$$\widehat{E}_{\xi,\bar\varsigma},\ \widehat{G}_{\xi,\bar\eta}\colon\mathbb{R}^4\to\mathbb{R}^4$$

defined by

$$\begin{aligned} (\mu,x,y,z)&\mapsto\widehat{E}_{\xi,\bar\varsigma}(\mu,x,y,x)\stackrel{\text{def}}{=}\big(\mu,E_{\xi,\mu,\bar\varsigma}(x,y,z)\big),\\ (\mu,x,y,z)&\mapsto\widehat{G}_{\xi,\bar\eta}(\mu,x,y,z)\stackrel{\text{def}}{=}\big(\mu,G_{\xi,\mu,\bar\eta}(x,y,z)\big). \end{aligned} \tag{5.3}$$

Consider the map

$$\bar\eta(\bar\varsigma)=(\eta_1(\bar\varsigma),\eta_2(\bar\varsigma))=(\varsigma_1^2\varsigma_3\varsigma_2^{-1},\varsigma_1\varsigma_4\varsigma_2^{-1}). \tag{5.4}$$

Suppose that $\bar\varsigma$ is such that

$$\varsigma_1\varsigma_2\varsigma_5 \neq 0,$$

then $\widehat{E}_{\xi,\bar\varsigma}$ and $\widehat{G}_{\xi,\bar\eta(\bar\varsigma)}$ are conjugate:

$$\widehat\Theta_{\bar\varsigma}^{-1} \circ \widehat{E}_{\xi,\bar\varsigma} \circ \widehat\Theta_{\bar\varsigma} = \widehat{G}_{\xi,\bar\eta(\bar\varsigma)}.$$

where

$$\begin{aligned} &\widehat\Theta_{\bar\varsigma}\colon \mathbb{R}^4 \to \mathbb{R}^4, \quad \widehat\Theta_{\bar\varsigma}(\mu,x,y,z) = \big(\varsigma_2^{-1}\mu, \Theta_{\bar\varsigma}(x,y,x)\big),\\ &\Theta_{\bar\varsigma}\colon \mathbb{R}^3 \to \mathbb{R}^3, \quad \Theta_{\bar\varsigma}(x,y,z) = (\varsigma_2^{-1}\varsigma_1 z, \varsigma_2^{-1} y, \varsigma_2^{-1}\varsigma_5 x). \end{aligned} \tag{5.5}$$

5.3.2. *Occurrence of blender-horseshoes.* Consider the set

$$\Delta \stackrel{\text{def}}{=} [-4,4]^2 \times [-40,22]. \tag{5.6}$$

Theorem 5.6 (Theorem 1 in [17])**.** *There is $\varepsilon_{\mathrm{BH}} > 0$ such that for every*

$$(\xi,\mu,\bar\eta) \in \mathcal{O}_{\mathrm{BH}} \stackrel{\text{def}}{=} (1.18, 1.19) \times (-10,-9) \times (-\varepsilon_{\mathrm{BH}}, \varepsilon_{\mathrm{BH}})^2$$

the endomorphism $G_{\xi,\mu,\bar\eta}$ has a blender-horseshoe $\Lambda_{\xi,\mu,\bar\eta}$ with domain of reference Δ such that

$$\Lambda_{\xi,\mu,\bar\eta} = \bigcap_{i\in\mathbf{Z}} G^i_{\xi,\mu,\bar\eta}(\Delta) \subset \mathrm{interior}(\Delta).$$

As a consequence, every diffeomorphism or endomorphism sufficiently C^1 close to $G_{\xi,\mu,\bar\eta}$ has a blender-horseshoe in Δ.

Remark 5.7. By Remark 5.5, the map $E_{\xi,\mu,\bar\varsigma}$, with $\bar\varsigma = (\varsigma_1,\dots,\varsigma_5)$, has blender-horseshoes if $\varsigma_1\varsigma_2\varsigma_5 \neq 0$ and

$$(\xi, \mu, \varsigma_1^2\varsigma_3\varsigma_2^{-1}, \varsigma_1\varsigma_4\varsigma_2^{-1}) \in (1.18, 1.19) \times (-10,-9) \times (-\varepsilon_{\mathrm{BH}}, \varepsilon_{\mathrm{BH}})^2.$$

Let us say a few words about the blenders in Theorem 5.6. Let $P^{\pm}_{\xi,\mu,\bar\eta}$ be the *reference fixed points* of the blender $\Lambda_{\xi,\mu,\bar\eta}$ of $G_{\xi,\mu,\bar\eta}$ in in Δ. The fixed points of $G_{\xi,\mu} \stackrel{\text{def}}{=} G_{\xi,\mu,\bar 0}$ [7] in Δ, satisfy

$$P^-_{\xi,\mu} = (p^-_{\xi,\mu}, p^-_{\xi,\mu}, \widetilde{p}^-_{\xi,\mu}), \qquad P^+_{\xi,\mu} = (p^+_{\xi,\mu}, p^+_{\xi,\mu}, \widetilde{p}^+_{\xi,\mu}) \tag{5.7}$$

with

$$\begin{aligned} -2.7 < p^-_{\xi,\mu} < -2.5, &\quad 13 < \widetilde{p}^-_{\xi,\mu} < 15,\\ 3.5 < p^+_{\xi,\mu} < 3.71, &\quad -20.6 < \widetilde{p}^+_{\xi,\mu} < -18.4. \end{aligned}$$

For the map $G_{\xi,\mu}$, the strong unstable cone field is given by

$$\mathcal{C}^{\mathrm{uu}}(Z) \stackrel{\text{def}}{=} \big\{ (u,v,w) \in \mathbb{R}^3 \,:\, \sqrt{u^2+w^2} < \tfrac{1}{2}|v| \big\}, \tag{5.8}$$

see [17, Lemma 3.10]. We also have that

$$W^{\mathrm{s}}_{\mathrm{loc}}(P^{\pm}_{\xi,\mu}, G_{\xi,\mu}) = \big\{ (p^{\pm}_{\xi,\mu} + t, p^{\pm}_{\xi,\mu}, \widetilde{p}^{\pm}_{\xi,\mu}) :\ -4 - p^{\pm}_{\xi,\mu} \leqslant t \leqslant 4 - p^{\pm}_{\xi,\mu} \big\}.$$

[7] With the notation in [17], say $P^-_{\xi,\mu} = P_{\xi,\mu}$ and $P^+_{\xi,\mu} = Q_{\xi,\mu}$

The superposition region of the blender $\Lambda_{\xi,\mu} \overset{\text{def}}{=} \Lambda_{\xi,\mu,\bar{0}}$ consists of (large) discs tangent to the cone field $\mathcal{C}^{\mathrm{uu}}$ which are at the right of $W^{\mathrm{s}}_{\mathrm{loc}}(P^{-}_{\xi,\mu}, G_{\xi,\mu})$ and at the left of $W^{\mathrm{s}}_{\mathrm{loc}}(P^{+}_{\xi,\mu}, G_{\xi,\mu})$. These observations imply the following:

Remark 5.8 (A disc in the superposition region)**.** Consider the disc

$$L \overset{\text{def}}{=} \big\{ (0, y, 0) : |y| < 4 \big\} \subset \Delta \tag{5.9}$$

in the superposition region the blender $\Lambda_{\xi,\mu,\bar{\eta}}$ of $G_{\xi,\mu,\bar{\eta}}$ for every $(\xi, \mu, \bar{\eta}) \in \mathcal{O}_{\mathrm{BH}}$. Then, by Lemma 5.1, for every diffeomorphism F close enough to $G_{\xi,\mu,\bar{\eta}}$ every disc sufficiently close to L is in the superposition region of the blender Λ_F.

Lemma 5.5. *The unstable manifold $W^{\mathrm{u}}\big(P^{+}_{\xi,\mu,\bar{\eta}}, G_{\xi,\mu,\bar{\eta}}\big)$ is unbounded in the y- and z-directions.*

Proof. We will show that $W^{\mathrm{u}}\big(P^{+}_{\xi,\mu}, G_{\xi,\mu}\big)$ contains the set

$$G_{\xi,\mu}(\Pi^{+}_{\xi,\mu}), \quad \text{where} \quad \Pi^{+}_{\xi,\mu} \overset{\text{def}}{=} \big\{ (x, y, z) \colon y \geqslant p^{+}_{\xi,\mu} \big\} \subset \mathbb{R}^3.$$

The general case follows studying 2-dimensional projections of $W^{\mathrm{u}}\big(P^{+}_{\xi,\mu,\bar{\eta}}, G_{\xi,\mu,\bar{\eta}}\big)$.

Consider the plane $\Pi_{\xi,\mu} \overset{\text{def}}{=} \{ y = p^{+}_{\xi,\mu} \} \subset \mathbb{R}^3$ and note that

$$G_{\xi,\mu}(\Pi_{\xi,\mu}) = \mathcal{L}_{\xi,\mu} \overset{\text{def}}{=} \big\{ (p^{+}_{\xi,\mu}, p^{+}_{\xi,\mu}, \widetilde{p}^{+}_{\xi,\mu} + t) : t \in \mathbb{R} \big\} \subset \Pi_{\xi,\mu}.$$

It is easy to see that $\mathcal{L}_{\xi,\mu}$ is $G_{\xi.\mu}$-invariant and that the restriction of $G_{\xi.\mu}$ to $\mathcal{L}_{\xi,\mu}$ is an expanding linear map (with expansion factor $\xi > 1$). Consider the cone field

$$\mathcal{C}^{\mathrm{u}}(Z) \overset{\text{def}}{=} \big\{ (u, v, w) \in \mathbb{R}^3 : |u| < \tfrac{1}{2}\sqrt{v^2 + w^2} \big\}.$$

By [17, Claims 3.12 and 3.14], this cone field is $DG_{\xi,\mu}$-invariant and uniformly expanding for every $Z = (x, y, z)$ with $|y| > \sqrt{5}$. A simple calculation implies that for every $Z \in G_{\xi,\mu}(\Pi^{+}_{\xi,\mu})$ the following holds

$$G_{\xi,\mu}(\Pi^{+}_{\xi,\mu}) \subset \Pi^{+}_{\xi,\mu} \qquad \text{and} \qquad T_Z\big(G_{\xi,\mu}(\Pi^{+}_{\xi,\mu})\big) \subset \mathcal{C}^{\mathrm{u}}(Z).$$

These properties and the expanding property of $\mathcal{C}^{\mathrm{u}}$ imply the lemma. □

Notation. We will consider blenders in $\mathbb{R}^3$ and in the ambient manifold M. We will denote the first ones by Λ and the second ones by Υ.

6. The renormalisation scheme

In this section, we outline the renormalisation scheme in [16], see Proposition 6.5. For that we embed any $f \in \mathcal{H}^r_{\mathrm{BH}}(M)$ in a bifurcating eight-parameter family

$$\mathbb{R}^8 \ni \bar{v} \to f_{\bar{v}} \in \mathrm{Diff}^r(M) \quad \text{with} \quad f_{\mathbf{0}} = f$$

and construct a renormalisation scheme for f consisting of:

- a sequence of local charts Ψ_k from $\mathbb{R}^3$ to U_Q,
- a sequence of reparameterisations $\mathbb{R} \ni \mu \mapsto \bar{v}^{\xi}_k(\mu) \in \mathbb{R}^8$ with $\xi > 1$ and $\bar{v}^{\xi}_k(\mu) \to \mathbf{0}$ on compact sets,

- sequences n_k, $m_k \in \mathbb{N}$ such that the "return maps" $f^{N_2+m_k+N_1+n_k}_{\bar{\upsilon}^\xi_k(\mu)}$, defined on a neighbourhood of the heterodimensional tangency $\widetilde{Y}$, satisfies
$$\Psi_k^{-1} \circ f^{N_2+m_k+N_1+n_k}_{\bar{\upsilon}^\xi_k(\mu)} \circ \Psi_k \to E_{\xi,\mu,\bar{\varsigma}}$$
where the convergence is C^r, $E_{\xi,\mu,\bar{\varsigma}}$ is defined (5.2), and $\bar{\varsigma} = \bar{\varsigma}(\xi, f)$ satisfies (2.9).

This section is organised as follows. We define the unfolding family $(f_{\bar{\upsilon}})_{\bar{\upsilon}\in\mathbb{R}^8}$ in Section 6.1 and review the renormalisation scheme in Section 6.2. The convergence of the scheme is stated in Section 6.3.

6.1. **The unfolding family.** Given $f \in \mathcal{H}^r_{\mathrm{BH}}(M)$ and
$$\bar{\upsilon} = (\bar{\mu}, \bar{\nu}, \alpha, \beta) \in \mathbb{R}^3 \times \mathbb{R}^3 \times \mathbb{R} \times \mathbb{R} = \mathbb{R}^8,$$
we consider a (smooth) family $(f_{\bar{\upsilon}})_{\bar{\upsilon}\in\mathbb{R}^8}$ in $\mathrm{Diff}^r(M)$ with $f_{\mathbf{0}} = f$, $f_{\bar{\upsilon}}(Q) = Q$, and $f_{\bar{\upsilon}}(P) = P$ such that

- the parameter $\bar{\mu}$ unfolds the heterodimensional tangency $\widetilde{Y}$,
- the parameter $\bar{\nu}$ unfolds the quasi-transverse intersection $\widetilde{X}$, and
- the parameters α and β modify the arguments of $Df(P)$ and $Df(Q)$.

Recall the translation-like perturbations $T_{W,\bar{\omega},\varrho}$ and rotation-like perturbations $R^*_{\alpha,\theta,\kappa}$ in (3.4) and (3.5), respectively. For $\bar{\upsilon} = (\bar{\mu}, \bar{\nu}, \alpha, \beta)$ and small $\rho > 0$ consider the perturbation of the identity defined by
$$\Omega_{\bar{\upsilon},\rho}(Z) = \Omega_{(\bar{\mu},\bar{\nu},\alpha,\beta),\rho}(Z) = \begin{cases} R^x_{\alpha,\theta_1,\kappa_1}(Z), & \text{if } Z \in U_P, \\ T_{\widetilde{X},\bar{\nu},\rho}(Z), & \text{if } Z \in V_{\widetilde{X}}, \\ R^y_{\beta,\theta_2,\kappa_2}(Z), & \text{if } Z \in U_Q, \\ T_{\widetilde{Y},\bar{\mu},\rho}(Z), & \text{if } Z \in V_{\widetilde{Y}}, \\ \mathrm{id}(Z), & \text{if } Z \notin V_P \cup V_Q \cup V_{\widetilde{X}} \cup V_{\widetilde{Y}}, \end{cases}$$
where $V_P, V_Q, V_{\widetilde{X}}$, and $V_{\widetilde{Y}}$ are small neighbourhoods of $P, Q, \widetilde{X}$, and $\widetilde{Y}$ contained in $U_P, U_Q, U_{\widetilde{X}}$, and $U_{\widetilde{Y}}$, respectively, recall Section 2. The numbers θ_1, $\theta_2 > 1$, κ_1, $\kappa_2 > 0$ are chosen such that
$$\widetilde{X}, Y \in [-\kappa_1^{-1}, \kappa_1^{-1}]^3, \qquad \widetilde{Y}, X \in [-\kappa_2^{-1}, \kappa_2^{-1}]^3,$$
$$[-\theta_1\kappa_1^{-1}, \theta_1\kappa_1^{-1}]^3 \subset U_P, \qquad [-\theta_2\kappa_2^{-1}, \theta_2\kappa_2^{-1}]^3 \subset U_Q.$$
Finally, we let
$$f_{\bar{\upsilon},\rho} = \Omega_{\bar{\upsilon},\rho} \circ f. \tag{6.1}$$

Remark 6.1 (The parameter ρ)**.** Above we emphasise the role of the parameter ρ related to the size of the translation-like perturbation of f.

Remark 6.2 (Support of the rotation-like part of $\Omega_{\bar{\upsilon},\rho}$)**.** Consider the linear maps

$$f_{\lambda,\sigma,\varphi} = \begin{pmatrix} \lambda & 0 & \\ 0 & \sigma\sin 2\pi\varphi & \sigma\cos 2\pi\varphi \\ 0 & -\sigma\cos 2\pi\varphi & \sigma\sin 2\pi\varphi \end{pmatrix},$$

$$\widetilde{f}_{\lambda,\sigma,\varphi} = \begin{pmatrix} \sigma\sin 2\pi\varphi & 0 & \sigma\cos 2\pi\varphi \\ 0 & \lambda & 0 \\ -\sigma\cos 2\pi\varphi & 0 & \sigma\sin 2\pi\varphi \end{pmatrix}.$$

With this notation, the restriction of f to U_P is the map $f_{\lambda_P,\sigma_P,\varphi_P}$, recall Section 2.1.1. Note that if $Z \in f^{-1}\big([-\kappa_1,\kappa_1]^3\big) \cap [-\kappa_1,\kappa_1]^3$ then

$$\Omega_{\bar{\upsilon},\rho} \circ f(Z) = R^x_{\alpha,\theta_1,\kappa_1} \circ f(Z) = R^x_{\alpha,\theta_1,\kappa_1} \circ f_{\lambda_P,\sigma_P,\varphi_P} = f_{\lambda_P,\sigma_P,\varphi_P+\alpha}(Z).$$

If $Z \in f^{-1}\big([-\theta_1\kappa_1^{-1},\theta_1\kappa_1^{-1}]^3\big)^c \cap U_P$ then

$$\Omega_{\bar{\upsilon},\rho} \circ f(Z) = R^x_{\alpha,\theta_1,\kappa_1} \circ f(Z) = f(Z).$$

Similarly, the restriction of f to U_Q is $\widetilde{f}_{\lambda_Q,\sigma_Q,\varphi_Q}$ and analogous conditions hold.

Remark 6.3 (Support of the translation-like part of $\Omega_{\bar{\upsilon},\rho}$)**.** Note that

- $\Omega_{\bar{\upsilon},\rho} \circ f(Z) = T_{\widetilde{X},\bar{\nu},\rho} \circ f(Z) = f(Z)$ for every $Z \in M \setminus f^{-1}\big(B(\widetilde{X},2\rho)\big)$,
- $\Omega_{\bar{\upsilon},\rho} \circ f(Z) = T_{\widetilde{X},\bar{\nu},\rho} \circ f(f^{-1}(\widetilde{X})) = \widetilde{X} + \bar{\nu}$.

Analogously, we have that

- $\Omega_{\bar{\upsilon},\rho} \circ f(Z) = T_{\widetilde{y},\bar{\mu},\rho} \circ f(Z) = f(Z)$ for every $Z \in M \setminus f^{-1}\big(B(\widetilde{Y},2\rho)\big)$,
- $\Omega_{\bar{\upsilon},\rho} \circ f(Z) = T_{\widetilde{Y},\bar{\mu},\rho} \circ f(f^{-1}(\widetilde{Y})) = \widetilde{Y} + \bar{\mu}$.

6.2. **The renormalisation scheme.** We now summarise the ingredients of the renormalisation scheme: Sojourn times and adjusting arguments (Section 6.2.1), reparameterisations (Section 6.2.2), and changes of coordinates (Section 6.2.3).

6.2.1. *Sojourn times and adjusting arguments.* Fix $\xi > 1$ and consider

$$\tau \stackrel{\text{def}}{=} \frac{\gamma_3(a_3-a_2)}{\sqrt{2}}, \quad \text{where } \gamma_3 \text{ is as in (2.4) and } a_2, a_3 \text{ are as in (2.6).}$$

Note that $\tau > 0$, see (2.7). By [16, Lemma 6.1], associated to $\tau^{-1}\xi$, there is a residual subset $\mathcal{R} = \mathcal{R}_{\tau^{-1}\xi}$ of $(0,1) \times (1,\infty)$ consisting of pairs (σ,λ) having a sequence of *sojourn times* $\mathbf{s}_k = (m_k,n_k)$ in $\mathbb{N}^2$, *adapted to* $\tau^{-1}\xi$ satisfying

$$\lim_{k\to\infty} \sigma^{m_k}\lambda^{n_k} = \tau^{-1}\xi \tag{6.2}$$

where m_k and n_k are related by the inequality

$$m_k < \eta\, n_k + \tilde{\eta} + 1, \quad \eta \stackrel{\text{def}}{=} \frac{\log(\lambda^{-1})}{\log(\sigma)} \quad \text{and} \quad \tilde{\eta} \stackrel{\text{def}}{=} \frac{\log(\tau\,\xi^{-1})}{\log(\sigma)}.$$

Our hypotheses allow us to consider $(\sigma_P,\lambda_Q) \in \mathcal{R}$ having a sequence of sojourn times $\mathbf{s}_k = (m_k,n_k)$ adapted to $\tau^{-1}\,\xi$, see [14, Lemma 5.1] and [16, Lemma 6.1]. The spectral condition in (2.2) provides a constant $C > 0$ such that

$$\lambda_P^{\frac{m_k}{2}}\sigma_P^{m_k}\sigma_Q^{n_k} < C\left((\lambda_P^{\frac{1}{2}}\sigma_P)^{\eta}\sigma_Q\right)^{n_k} \to 0 \tag{6.3}$$

Associated to the sequence $(\mathbf{s}_k)$ there is a sequence $\Theta_k = (\zeta_{m_k}, \vartheta_{n_k}) \in \mathbb{R}^2$, with $\Theta_k \to (0,0)$, of *adjusting arguments* leading to the following argument maps

$$\begin{aligned} \alpha_k(\theta) &\stackrel{\text{def}}{=} \frac{1}{2\pi m_k}\left(\frac{\pi}{4} - 2\pi m_k\theta + 2\pi[m_k\theta] + \zeta_{m_k}\right); \\ \beta_k(\omega) &\stackrel{\text{def}}{=} \frac{1}{2\pi n_k}\left(\frac{\pi}{2} - 2\pi n_k\omega + 2\pi[n_k\omega] + \vartheta_{n_k}\right), \end{aligned} \tag{6.4}$$

where $[x]$ denotes the integer part of $x \in \mathbb{R}$. The sequence (ϑ_{n_k}) is chosen such that $\varphi_Q + \beta_k$ is irrational (here φ_Q is the argument in item **(A)** in Section 2.1.1). There are no further restrictions on the definition of (ζ_{m_k}).

6.2.2. *Reparameterisations.* Associated to the sequences $(\mathbf{s}_k)$ and (Θ_k) we define the sequence of reparameterisations $\bar{\upsilon} = \bar{\upsilon}_k^{\xi}$ of the family $f_{\bar{\upsilon},\rho}$ in (6.1) by:

$$\bar{\upsilon}_k^{\xi}\colon \mathbb{R} \to \mathbb{R}^8, \quad \bar{\upsilon}_k^{\xi}(\mu) \stackrel{\text{def}}{=} \left(\bar{\mu}_k^{\xi}(\mu), \bar{\nu}_k^{\xi}, \alpha_k^{\xi}(\varphi_P), \beta_k^{\xi}(\varphi_Q)\right) \in \mathbb{R}^3 \times \mathbb{R}^3 \times \mathbb{R} \times \mathbb{R},$$

where (for simplicity, in what follows we eliminate the dependence[8] of the coordinate maps of $\bar{\upsilon}_k^{\xi}$ on ξ):

- $\bar{\mu}_k : \mathbb{R} \to \mathbb{R}^3$ is defined by

$$\bar{\mu}_k(\mu) \stackrel{\text{def}}{=} (-\lambda_P^{m_k} a_1, \sigma_Q^{-n_k} + \sigma_Q^{-2n_k}\sigma_P^{-2m_k}\mu - \lambda_P^{m_k} b_1, -\lambda_P^{m_k} c_1), \tag{6.5}$$

where a_1, b_1, c_1 are as in (2.6). Note that $\bar{\mu}_k(\mu) \to (0,0,0)$ as $k \to \infty$.

- To define $\bar{\nu}_k \in \mathbb{R}^3$ consider first

$$\varphi_{P,k} \stackrel{\text{def}}{=} \varphi_P + \alpha_k(\varphi_P) \quad \text{and} \quad \varphi_{Q,k} \stackrel{\text{def}}{=} \varphi_Q + \beta_k(\varphi_Q)$$

and the sequences

$$\begin{aligned} \tilde{\mathfrak{c}}_k &\stackrel{\text{def}}{=} \cos\left(2\pi m_k(\varphi_{P,k})\right), \quad \tilde{\mathfrak{s}}_k \stackrel{\text{def}}{=} \sin\left(2\pi m_k(\varphi_{P,k})\right), \\ \mathfrak{c}_k &\stackrel{\text{def}}{=} \cos\left(2\pi n_k(\varphi_{Q,k})\right), \quad \mathfrak{s}_k \stackrel{\text{def}}{=} \sin\left(2\pi n_k(\varphi_{Q,k})\right). \end{aligned} \tag{6.6}$$

Remark 6.4. By the definition of $\alpha_k(\varphi_P)$ and $\beta_k(\varphi_Q)$ in (6.4), it follows that $\mathfrak{c}_k \to 0$, $\mathfrak{s}_k \to 1$, and $\tilde{\mathfrak{c}}_k, \tilde{\mathfrak{s}}_k \to 1/\sqrt{2}$.

Recalling the coordinated maps $\widetilde{H}_2$ and $\widetilde{H}_3$ of $\widetilde{H}$ in (2.3), we let

$$\begin{aligned} \tilde{\rho}_{2,k} &\stackrel{\text{def}}{=} \frac{1}{2}\frac{\partial^2}{\partial x^2}\widetilde{H}_2(\mathbf{0})(\mathfrak{c}_k - \mathfrak{s}_k)^2 + \frac{1}{2}\frac{\partial^2}{\partial z^2}\widetilde{H}_2(\mathbf{0})(\mathfrak{s}_k + \mathfrak{c}_k)^2, \\ \tilde{\rho}_{3,k} &\stackrel{\text{def}}{=} \frac{1}{2}\frac{\partial^2}{\partial x^2}\widetilde{H}_3(\mathbf{0})(\mathfrak{s}_k - \mathfrak{c}_k)^2 + \frac{1}{2}\frac{\partial^2}{\partial z^2}\widetilde{H}_3(\mathbf{0})(\mathfrak{s}_k + \mathfrak{c}_k)^2. \end{aligned} \tag{6.7}$$

Finally, we let

$$\begin{aligned} \bar{\nu}_k \stackrel{\text{def}}{=} \Big(&-\lambda_Q^{n_k}\left(\alpha_1\left(\mathfrak{c}_k - \mathfrak{s}_k\right) + \alpha_3\left(\mathfrak{s}_k + \mathfrak{c}_k\right)\right), \\ &\sigma_P^{-m_k}(\tilde{\mathfrak{c}}_k + \tilde{\mathfrak{s}}_k) - \lambda_Q^{2n_k}\tilde{\rho}_{2,k}, \\ &\sigma_P^{-m_k}(\tilde{\mathfrak{c}}_k - \tilde{\mathfrak{s}}_k) - \lambda_Q^{n_k}\gamma_3\left(\mathfrak{c}_k + \mathfrak{s}_k\right) - \lambda_Q^{2n_k}\tilde{\rho}_{3,k}\Big), \end{aligned} \tag{6.8}$$

where $\alpha_1, \alpha_3, \gamma_3$ are as in (2.4). Note that $\bar{\nu}_k \to \mathbf{0}$ as $k \to \infty$.

[8]This dependence is given by the choice in (6.2).

6.2.3. *Change of coordinates.* Using the local coordinates in U_Q, we consider the sequence of maps $\Psi_k\colon U_k \to U_Q = [-a_Q, a_Q]^3$ defined by

$$\begin{aligned}\Psi_k(x,y,z) \stackrel{\text{def}}{=} (1 + \sigma_P^{-m_k}\sigma_Q^{-n_k}\, x, \\ \sigma_Q^{-n_k} + \sigma_P^{-2m_k}\sigma_Q{}^{-2n_k}\, y, 1 + \sigma_P^{-m_k}\sigma_Q{}^{-n_k}\, z),\end{aligned} \tag{6.9}$$

where U_k is the "cube" of $\mathbb{R}^3$ such that $\Psi_k(U_k) = U_Q$. Recall that $\widetilde{Y} = (1,0,1)$ and note that for any compact set $K \subset \mathbb{R}^3$ it holds $\Psi_k(K) \to \{\widetilde{Y}\}$ as $k \to \infty$.

6.3. **Convergence of the renormalisation scheme.** Fixed $\xi > 1$, small $\rho > 0$, and $f \in \mathcal{H}^r_{\mathrm{BH}}(M)$, consider the renormalisation scheme above and the sequence of one-parameter family of maps

$$\begin{aligned}&\mathbb{R} \ni \mu \to \mathcal{R}_{\bar{v}_k^\xi(\mu),\rho}(f) \in \mathrm{Diff}^r(M) \\ &\mathcal{R}_{\bar{v}_k^\xi(\mu),\rho}(f) \stackrel{\text{def}}{=} f^{N_2}_{\bar{v}_k^\xi(\mu),\rho} \circ f^{m_k}_{\bar{v}_k^\xi(\mu),\rho} \circ f^{N_1}_{\bar{v}_k^\xi(\mu),\rho} \circ f^{n_k}_{\bar{v}_k^\xi(\mu),\rho},\end{aligned} \tag{6.10}$$

called *renormalised sequence of f*. Here we are emphasising the roles of ξ and ρ.

Proposition 6.5 (Theorem 1,[16])**.** *Fix $\xi > 1$, small $\rho > 0$, and $\mu \in \mathbb{R}$. Given any $f \in \mathcal{H}^r_{\mathrm{BH}}(M)$ the sequence of maps*

$$\Psi_k^{-1} \circ \mathcal{R}_{\bar{v}_k^\xi(\mu),\rho}(f) \circ \Psi_k\colon U_k \to \mathbb{R}^3, \quad k \in \mathbb{N},$$

converges, on compact sets of $\mathbb{R}^3$ and in the C^r topology, to the endomorphism $E_{\xi,\mu,\bar{\varsigma}}$ in (5.2)*, where $\bar{\varsigma} = \bar{\varsigma}(\xi, f)$ is as in* (2.8)*.*

Notation 6.6 (The parameters ρ and ξ)**.** When the role of ρ is not relevant it will omitted, writing $f_{\bar{v}}$ and $\mathcal{R}_{\bar{v}_k^\xi(\mu)}$ instead of $f_{\bar{v},\rho}$ and $\mathcal{R}_{\bar{v}_k^\xi(\mu),\rho}$. Similarly with $\xi > 1$.

Remark 6.7. Recall the perturbation f_ε of f in Proposition 4.1. The renormalisation scheme of f associated to X and Y can be applied to f_ε and it is preserved. In this way, we get the one-parameter family of diffeomorphisms $f_{\varepsilon,\bar{v}_k(\mu),\rho}$.

7. Interplay between blenders and heteroclinic points

In Section 7.1, see Proposition 7.2, we state the occurrence of blender-horseshoes in the renormalisation scheme for diffeomorphisms $f \in \mathcal{H}^r_{\mathrm{BH}}(M)$ and their perturbations $f_\varepsilon \in \mathcal{H}^r_{\mathrm{BH}}(M)$ given by Proposition 4.1.

Note that f_ε has additional heteroclinic points $X_{1,\varepsilon}, X_{2,\varepsilon}$. In Section 7.2, we see how these intersections are unfolded without modifying the blenders given by the renormalisation scheme.

Before going into the details, recall the definitions of the transitions $\mathfrak{T}_{1,i,\varepsilon}$ associated to $X_{i,\varepsilon}$ and their domains $U_{i,\varepsilon}$ in (4.5) and consider the neighbourhoods

$$\widetilde{U}_{i,\varepsilon} \stackrel{\text{def}}{=} \mathfrak{T}_{1,i,\varepsilon}(U_{i,\varepsilon}) \quad \text{of} \quad \widetilde{X}_{i,\varepsilon} = \mathfrak{T}_{1,i,\varepsilon}(X_{i,\varepsilon}), \quad i = 0,1,2.$$

Take sufficiently small $\rho = \rho(\varepsilon) > 0$ such that

$$B(\widetilde{X}, 2\,\rho) \subset \widetilde{U}_{0,\varepsilon}, \quad B(\widetilde{X}_{1,\varepsilon}, 2\,\rho) \subset \widetilde{U}_{1,\varepsilon}, \quad B(\widetilde{X}_{2,\varepsilon}, 2\,\rho) \subset \widetilde{U}_{2,\varepsilon}. \tag{7.1}$$

In particular, these three balls are disjoint. We can now consider the renormalisation scheme $\mathcal{R}_{\bar{v}_k^\xi(\mu),\rho}(f_\varepsilon)$ and observe that by the choice of ρ the renormalisation preserves the heteroclinic points $X_{1,\varepsilon}$ and $X_{2,\varepsilon}$. Note that as the transitions of f are not modified, recalling (2.8), we have

$$\bar{\varsigma} = \bar{\varsigma}(\xi, f) = \bar{\varsigma}(\xi, f_\varepsilon).$$

7.1. **Blenders in the renormalisation scheme.** Recall the definitions of Ψ_k and of U_k in (6.9) and of $\mathcal{R}_{\bar{v}_k^\xi(\mu),\rho}(f_\varepsilon)$ in (6.10). Define the maps

$$\widehat{\Psi}_k\colon \mathbb{R}\times U_k \to \mathbb{R}\times U_Q, \qquad \mathcal{R}_{\bar{v}_k^\xi(\mu),\rho}(f_\varepsilon)\colon \mathbb{R}\times M \to \mathbb{R}\times M$$

by

$$\widehat{\Psi}_k(\mu, X) \stackrel{\text{def}}{=} \big(\mu, \Psi_k(X)\big),$$
$$\widehat{\mathcal{R}}_{k,\rho}(f_\varepsilon)(\mu, X) \stackrel{\text{def}}{=} \big(\mu, \mathcal{R}_{\bar{v}_k^\xi(\mu),\rho}(f_\varepsilon)(X)\big)$$

and consider (with slight abuse of notation on the domain of definitions) the maps Φ_k and $\widehat{\Phi}_k$ defined by

$$\begin{aligned} X \in \mathbb{R}^3 \ &\mapsto\ \Phi_k(X) \stackrel{\text{def}}{=} \Psi_k \circ \Theta_{\bar{\varsigma}}(X) \in U_Q, \\ (\mu, X) \in \mathbb{R}\times\mathbb{R}^3 \ &\mapsto\ \widehat{\Phi}_k(\mu, X) \stackrel{\text{def}}{=} \widehat{\Psi}_k \circ \widehat{\Theta}_{\bar{\varsigma}}(\mu, X) \\ &\qquad\qquad = \big(\varsigma_2^{-1}\mu, \Phi_k(X)\big) \in \mathbb{R}\times U_Q, \end{aligned} \tag{7.2}$$

where $\Theta_{\bar{\varsigma}}$ and $\widehat{\Theta}_{\bar{\varsigma}}$ are the conjugations in (5.5).

The following explicit form of the maps Φ_k^{-1} will be used in Section 9.1.

Remark 7.1. Note that for $(1+x, y, 1+z) \in U_Q$ close to $\widetilde{Y} = (1,0,1)$ we have that

$$(\tilde{x}, \tilde{y}, \tilde{z}) \stackrel{\text{def}}{=} \Phi_k^{-1}(1+x, y, 1+z), \qquad \begin{cases} \tilde{x} = \sigma_Q^{n_k}\,\sigma_P^{m_k}\,\varsigma_2\,\varsigma_5^{-1}\,z, \\ \tilde{y} = \varsigma_2\,\sigma_Q^{2n_k}\,\sigma_P^{2m_k}\,(y - \sigma_Q^{-n_k}), \\ \tilde{z} = \sigma_Q^{n_k}\,\sigma_P^{m_k}\,\varsigma_2\,\varsigma_1^{-1}\,x. \end{cases}$$

Finally, consider the sequence of maps $\mathfrak{R}_{k,\rho}(f_\varepsilon)\colon \mathbb{R}\times\mathbb{R}^3 \to \mathbb{R}\times\mathbb{R}^3$

$$(\mu, X) \ \mapsto\ \mathfrak{R}_{k,\rho}(f_\varepsilon)(\mu, X) \stackrel{\text{def}}{=} \widehat{\Phi}_k^{-1} \circ \widehat{\mathcal{R}}_{k,\rho}(f_\varepsilon) \circ \widehat{\Phi}_k(\mu, X).$$

Finally, note that for each fixed μ the projection of $\mathfrak{R}_{k,\rho}(f_\varepsilon)(\mu,\cdot)$ in the "second coordinate" $\mathbb{R}^3$ is exactly the map $\mathcal{R}_{\bar{v}_k^\xi(\mu),\rho}(f_\varepsilon)$.

Recalling the definition of $\bar{\eta}(\bar{\varsigma})$ in (5.4) and of $\bar{\varsigma}(\xi, f) = \bar{\varsigma}(\xi, f_\varepsilon)$ in (2.8) we define (with slight abuse of notation) the map

$$\bar{\eta}(\xi, f_\varepsilon) \stackrel{\text{def}}{=} \bar{\eta}(\bar{\varsigma}(\xi, f_\varepsilon)) = \bar{\eta}(\xi, f).$$

Recalling Proposition 6.5, Theorem 5.6 (and the definition of the set $\mathcal{O}_{\mathrm{BH}}$ there), Remark 5.5, and the definition of $\widehat{G}_{\xi,\bar{\eta}}$ in (5.3) we get the following:

Proposition 7.2. *Consider $f \in \mathcal{H}^r_{\mathrm{BH}}(M)$, $\xi > 1$, small $\varepsilon > 0$, and f_ε as in Proposition 4.1. There is $\rho(\varepsilon)$ such that the sequence of maps $(\mathfrak{R}_{k,\rho(\varepsilon)}(f_\varepsilon))$ converges in the C^r topology and on compact sets of $\mathbb{R}^4$ to $\widehat{G}_{\xi,\bar\eta(\xi,f)}$.*

As a consequence, for every k large enough the map $\mathcal{R}_{\bar\upsilon^\xi_k(\mu),\rho(\varepsilon)}(f_\varepsilon)$ has a blender-horseshoe $\Lambda_{\bar\upsilon^\xi_k(\mu),\rho(\varepsilon)}$ in $\Delta = [-4,4]^2 \times [-40,22]$.

We now describe more precisely the blenders $\Lambda_{\bar\upsilon^\xi_k(\mu),\rho(\varepsilon)}$. For $\mu \in (-10,-9)$, large k, and small ε, consider the diffeomorphism

$$f_{\varepsilon,\bar\upsilon_k(\mu),\rho(\varepsilon)} \stackrel{\mathrm{def}}{=} \Omega_{\bar\upsilon_k(\mu),\rho(\varepsilon)} \circ f_\varepsilon,$$

where $\Omega_{\bar\upsilon_k(\mu),\rho(\varepsilon)}$ is defined as in (6.1). Proposition 7.2 implies that $f_{\varepsilon,\bar\upsilon_k(\mu),\rho(\varepsilon)}$ has a blender-horseshoe defined as follows. Let

$$\begin{aligned}
\Delta(\mu) &\stackrel{\mathrm{def}}{=} \{\mu\} \times \Delta,\\
\Delta_k(\mu) &\stackrel{\mathrm{def}}{=} \Delta_k = \Phi_k(\Delta),\\
\widehat{\Delta}_k(\mu) &\stackrel{\mathrm{def}}{=} \widehat{\Phi}_k\big(\Delta(\mu)\big) = \big(\varsigma_2^{-1}\,\mu, \Phi_k(\Delta)\big) = \big(\varsigma_2^{-1}\,\mu, \Delta_k\big).
\end{aligned} \tag{7.3}$$

Recalling that $\mathcal{R}_{\bar\upsilon^\xi_k(\mu),\rho(\varepsilon)}(f_\varepsilon) = f^{N_2+m_k+N_1+n_k}_{\varepsilon,\bar\upsilon_k(\mu),\rho(\varepsilon)}$, see (6.10), we have that

$$\begin{aligned}
\Upsilon_{f_{\varepsilon,\bar\upsilon_k(\mu),\rho(\varepsilon)}} &\stackrel{\mathrm{def}}{=} \bigcap_{\ell\in\mathbb{Z}} \big(\mathcal{R}_{\bar\upsilon^\xi_k(\mu),\rho(\varepsilon)}(f_\varepsilon)\big)^\ell(\Delta_k(\mu))\\
&= \bigcap_{\ell\in\mathbb{Z}} f^{(N_2+m_k+N_1+n_k)\,\ell}_{\varepsilon,\bar\upsilon_k(\mu),\rho(\varepsilon)}(\Delta_k).
\end{aligned} \tag{7.4}$$

is a blender-horseshoe of $f^{N_2+m_k+N_1+n_k}_{\varepsilon,\bar\upsilon_k(\mu),\rho(\varepsilon)}$. Note that, by construction,

$$\Lambda_{\bar\upsilon^\xi_k(\mu),\rho(\varepsilon)} = \Phi_k^{-1}\big(\Upsilon_{f_{\varepsilon,\bar\upsilon_k(\mu),\rho(\varepsilon)}}\big). \tag{7.5}$$

Notation 7.3. The reference saddles $P^\pm_{\bar\upsilon^\xi_k(\mu),\rho(\varepsilon)}$ of $\Lambda_{\bar\upsilon^\xi_k(\mu),\rho(\varepsilon)}$ are the continuations of the saddles $P^\pm_{\xi,\mu,\bar\eta}$ of the blender of $G_{\xi,\mu,\bar\eta}$ in (5.7).

Remark 7.4. Consider $\big(\xi,\mu,\bar\eta(\xi,f)\big) \in \mathcal{O}_{\mathrm{BH}}$ and write $\bar\eta = \bar\eta(\xi,f)$. Recall the definition of the disc $L \subset \mathbb{R}^3$ in the superposition region of the blender of $G_{\xi,\mu,\bar\eta}$, see Remark 5.8. The second part of that remark and the C^r convergence

$$\Phi_k^{-1} \circ \mathcal{R}_{\bar\upsilon^\xi_k(\mu),\rho(\varepsilon)}(f_\varepsilon) \circ \Phi_k \to G_{\xi,\mu,\bar\eta}$$

on compact subsets of $\mathbb{R}^3$ imply that for every large k the set

$$\Phi_k^{-1} \circ \mathcal{R}_{\bar\upsilon^\xi_k(\mu),\rho(\varepsilon)}(f_\varepsilon) \circ \Phi_k(L)$$

contains a disc in the superposition region of the blender $\Lambda_{\bar\upsilon^\xi_k(\mu),\rho(\varepsilon)}$.

7.2. **Unfolding the heteroclinic points** $X_{i,\varepsilon}$**.** By construction, we have that $X_{1,\varepsilon}$ and $X_{2,\varepsilon}$ are quasi-transverse heteroclinic points of $f_{\varepsilon,\bar{\upsilon}_k(\mu),\rho(\varepsilon)}$ and (recalling the definitions in (4.6))

$$\widetilde{X}_{i,\varepsilon} \in W^{\mathrm{s}}_{\mathrm{loc}}(P, f_{\varepsilon,\bar{\upsilon}_k(\mu),\rho(\varepsilon)}) \cap W^{\mathrm{u}}(Q, f_{\varepsilon,\bar{\upsilon}_k(\mu),\rho(\varepsilon)}).$$

Remark 7.5. The choices of ε and $\rho(\varepsilon)$ imply that the closure of the orbits of $\widetilde{X}_{i,\varepsilon}$ and the orbit of the blender $\Upsilon_{f_{\varepsilon,\bar{\upsilon}_k(\mu),\rho(\varepsilon)}}$ are disjoint. Moreover, the orbit of the blender is also disjoint from the neighbourhoods $U_{i,\varepsilon}$ of $X_{i,\varepsilon}$ (and thus from the neighbourhoods $\widetilde{U}_{i,\varepsilon}$ of $\widetilde{X}_{i,\varepsilon}$).

We now consider a "local unfolding of the heteroclinic point $\widetilde{X}_{1,\varepsilon}$ independent of the renormalisation process": this unfolding is given by a perturbation whose support is disjoint from $B\big(\widetilde{X}, \rho(\varepsilon)\big)$ and $B\big(\widetilde{Y}, \rho(\varepsilon)\big)$. For that, consider a family of local perturbations of f_ε given by

$$g_{\varepsilon,\bar{\upsilon}_k(\mu),\rho(\varepsilon)} = \theta_{\varepsilon,k} \circ f_{\varepsilon,\bar{\upsilon}_k(\mu),\rho(\varepsilon)}, \tag{7.6}$$

where $\theta_{\varepsilon,k}$ is a C^r perturbation of identity supported on $B\big(\widetilde{X}_{1,\varepsilon}, 2\rho(\varepsilon)\big) \subset \widetilde{U}_{1,\varepsilon}$ satisfying $\lim_{k\to\infty} d_r(\theta_{\varepsilon,k}, \mathrm{id}) = 0$. To define $\theta_{\varepsilon,k}$, recall the definitions of the bump function Π_δ in (3.2) and the sequences m_k, n_k in Section 6.2.1, $\tilde{\mathfrak{c}}_k$ and $\tilde{\mathfrak{s}}_k$ in (6.6), and $\bar{\nu}_k$ in (6.8), and consider the sequence of vectors

$$\bar{\tau}_k \stackrel{\text{def}}{=} \big(0, \sigma_P^{-m_k}(\tilde{\mathfrak{c}}_k + \tilde{\mathfrak{s}}_k), \sigma_P^{-m_k}(\tilde{\mathfrak{c}}_k - \tilde{\mathfrak{s}}_k)\big) \in \mathbb{R}^3, \qquad \bar{\tau}_k \to \mathbf{0}. \tag{7.7}$$

The map $\theta_{\varepsilon,k} \colon M \to M$ is defined by:

$$\begin{aligned} &\theta_{\varepsilon,k}\big(Z\big) \stackrel{\text{def}}{=} Z + \Pi_{\rho(\varepsilon)}(W)\,\bar{\tau}_k, \quad \text{if} \quad Z = \widetilde{X}_{1,\varepsilon} + W \in B\big(\widetilde{X}_{1,\varepsilon}, 2\rho(\varepsilon)\big),\\ &\theta_{\varepsilon,k}(Z) \stackrel{\text{def}}{=} Z, \quad \text{if} \quad Z \notin B\big(\widetilde{X}_{1,\varepsilon}, 2\rho(\varepsilon)\big). \end{aligned} \tag{7.8}$$

Recalling that $\|\Pi_{\rho(\varepsilon)}\|_r \leqslant (\|b\|_r)^3\,\rho(\varepsilon)^{-r}$, see (3.3), and that $|\tilde{\mathfrak{c}}_k \pm \tilde{\mathfrak{s}}_k| \leqslant 2$, see (6.6), we get

$$d_r(\theta_{\varepsilon,k}, \mathrm{id}) \leqslant 2\,(\|b\|_r)^3\,\rho(\varepsilon)^{-1}\,\sigma_P^{-m_k} \to 0, \qquad k \to \infty.$$

Remark 7.6. By definition of $\theta_{\varepsilon,k}$, for every $W \notin f_\varepsilon^{-1}\big(B(\widetilde{X}_{1,\varepsilon}, 2\rho(\varepsilon)\big)$ it holds that

$$g_{\varepsilon,\bar{\upsilon}_k(\mu),\rho(\varepsilon)}(W) = \theta_{\varepsilon,k} \circ f_{\varepsilon,\bar{\upsilon}_k(\mu),\rho(\varepsilon)}(W) = f_{\varepsilon,\bar{\upsilon}_k(\mu),\rho(\varepsilon)}(W).$$

As a consequence, if $W \notin f_\varepsilon^{-1}\big(B(\widetilde{X}_{1,\varepsilon}, 2\delta)\big)$ then

$$\mathcal{R}_{\bar{\upsilon}^{\xi}_k(\mu),\rho(\varepsilon)}(f_\varepsilon)(W) = \mathcal{R}_{\bar{\upsilon}^{\xi}_k(\mu),\rho(\varepsilon)}(g_{\varepsilon,\bar{\upsilon}_k(\mu),\rho(\varepsilon)})(W).$$

Hence, by Remark 7.5, the maps $g_{\varepsilon,\bar{\upsilon}_k(\mu),\rho(\varepsilon)}$ and $f_{\varepsilon,\bar{\upsilon}_k(\mu),\rho(\varepsilon)}$ have the common blender $\Upsilon_{f_{\varepsilon,\bar{\upsilon}_k(\mu),\rho(\varepsilon)}}$ defined in (7.4).

Moreover, by construction, the curve $L^{\mathrm{u}}_{2,\varepsilon}(\rho(\varepsilon))$ in (4.12) containing $X_{2,\varepsilon}$ is contained in $W^{\mathrm{u}}(Q, g_{\varepsilon,\bar{\upsilon}_k(\mu),\rho(\varepsilon)})$.

8. Orbits and itineraries associated to the renormalisation

This is a preparatory section to the proof of Theorem 1.1. We study admissible points and returns: we select a set of points whose itineraries for the diffeomorphisms $g_{\varepsilon,\bar{v}_k(\mu),\rho(\varepsilon)}$ in (7.6) are associated to the renormalisation scheme, see (8.1).

Recall the charts $\Phi_k = \Phi_{k,\bar{\varsigma}} = \Psi_k \circ \Theta_{\bar{\varsigma}} \colon U_k \subset \mathbb{R}^3 \to U_Q$ in (7.2) and that for every compact set $K \subset \mathbb{R}^3$ it holds $\Phi_k(K) \to \widetilde{Y}$ as $k \to \infty$.

Recall the definitions of the neighbourhoods U_X, $U_{\widetilde{X}}$, U_Y, and $U_{\widetilde{Y}}$ of the heteroclinic points of the cycle in Section 2.1.2, and of the balls $B(\widetilde{X}, 2\rho(\varepsilon)) \subset U_{\widetilde{X}}$ and $B(\widetilde{Y}, 2\rho(\varepsilon)) \subset U_{\widetilde{Y}}$ in (7.1).

Consider the subset $U_{\varepsilon,\bar{v}_k(\mu),\rho(\varepsilon)}$ of $B(\widetilde{Y}, 2\rho(\varepsilon)) \subset \Phi_k(U_k) = U_Q$ of points having the following itinerary for $f_{\varepsilon,\bar{v}_k(\mu),\rho(\varepsilon)}$: a point $w \in U_{\varepsilon,\bar{v}_k(\mu),\rho(\varepsilon)}$ if

$$w \in f_{\varepsilon,\bar{v}_k(\mu),\rho(\varepsilon)}^{-(N_2+m_k+N_1+n_k)}\big(B(\widetilde{Y}, 2\rho(\varepsilon))\big) \cap B(\widetilde{Y}, 2\rho(\varepsilon))$$

and it satisfies (see Figure 4):

$$\begin{aligned}
f^{i}_{\varepsilon,\bar{v}_k(\mu),\rho(\varepsilon)}(w) &\in U_Q, \quad \text{for every} \quad 0 \leqslant i \leqslant n_k,\\
f^{n_k}_{\varepsilon,\bar{v}_k(\mu),\rho(\varepsilon)}(w) &\in U_X,\\
f^{N_1+n_k}_{\varepsilon,\bar{v}_k(\mu),\rho(\varepsilon)}(w) &\in B(\widetilde{X}, 2\rho(\varepsilon)),\\
f^{j+N_1+n_k}_{\varepsilon,\bar{v}_k(\mu),\rho(\varepsilon)}(w) &\in U_P, \quad \text{for every} \quad 0 \leqslant j \leqslant m_k,\\
f^{m_k+N_1+n_k}_{\varepsilon,\bar{v}_k(\mu),\rho(\varepsilon)}(w) &\in U_Y, \text{ and}\\
f^{N_2+m_k+N_1+n_k}_{\varepsilon,\bar{v}_k(\mu),\rho(\varepsilon)}(w) &\in B(\widetilde{Y}, 2\rho(\varepsilon)).
\end{aligned} \tag{8.1}$$

We say that the points in $U_{\varepsilon,\bar{v}_k(\mu),\rho(\varepsilon)}$ are $(\varepsilon, \bar{v}_k(\mu), \rho(\varepsilon))$-*admissible* and that $n_k + N_1 + m_k + N_2$ is the $(\varepsilon, \bar{v}_k(\mu), \rho(\varepsilon))$-*admissible return.* We now define the maps

$$\begin{aligned}
&F_{\varepsilon,\bar{v}_k(\mu),\rho(\varepsilon)} \colon \Phi_k^{-1}(U_{\varepsilon,\bar{v}_k(\mu),\rho(\varepsilon)}) \to \mathbb{R}^3,\\
&F_{\varepsilon,\bar{v}_k(\mu),\rho(\varepsilon)} \stackrel{\text{def}}{=} \Phi_k^{-1} \circ f^{N_2+m_k+N_1+n_k}_{\varepsilon,\bar{v}_k(\mu),\rho(\varepsilon)} \circ \Phi_k.
\end{aligned}$$

Recall that $g_{\varepsilon,\bar{v}_k(\mu),\rho(\varepsilon)} = \theta_{\varepsilon,k} \circ f_{\varepsilon,\bar{v}_k(\mu),\rho(\varepsilon)}$ and that for the points in $U_{\varepsilon,\bar{v}_k(\mu),\rho(\varepsilon)}$ the map $\theta_{\varepsilon,k}$ is the identity, so $g_{\varepsilon,\bar{v}_k(\mu),\rho(\varepsilon)} = f_{\varepsilon,\bar{v}_k(\mu),\rho(\varepsilon)}$ for points in $U_{\varepsilon,\bar{v}_k(\mu),\rho(\varepsilon)}$.

Remark 8.1. Every point of the blender is $(\varepsilon, \bar{v}_k(\mu), \rho(\varepsilon))$-admissible:

$$\begin{aligned}
\Upsilon_{\varepsilon,\bar{v}_k(\mu),\rho(\varepsilon)} &\subset \bigcap_{j\in\mathbb{Z}} f^{j\,(N_2+m_k+N_1+n_k)}_{\varepsilon,\bar{v}_k(\mu),\rho(\varepsilon)}(U_{\varepsilon,\bar{v}_k(\mu),\rho(\varepsilon)})\\
&= \bigcap_{j\in\mathbb{Z}} g^{j\,(N_2+m_k+N_1+n_k)}_{\varepsilon,\bar{v}_k(\mu),\rho(\varepsilon)}(U_{\varepsilon,\bar{v}_k(\mu),\rho(\varepsilon)}).
\end{aligned}$$

Notation 8.2. In what follows, if there is no possibility of misunderstanding, we will simply write:

- $f_{\varepsilon,k,\mu}$ and $g_{\varepsilon,k,\mu}$ in the places of $f_{\varepsilon,\bar{v}_k(\mu),\rho(\varepsilon)}$ and $g_{\varepsilon,\bar{v}_k(\mu),\rho(\varepsilon)}$,
- $\Lambda_{\varepsilon,k,\mu}$ and $\Upsilon_{\varepsilon,k,\mu}$ in the place of $\Lambda_{\bar{v}_k(\mu),\rho(\varepsilon)}$ and $\Upsilon_{f_{\varepsilon,\bar{v}_k(\mu),\rho(\varepsilon)}}$,
- $\mathcal{R}_{\varepsilon,k,\mu}$ in the place of $\mathcal{R}_{\bar{v}_k(\mu),\rho(\varepsilon)}$.

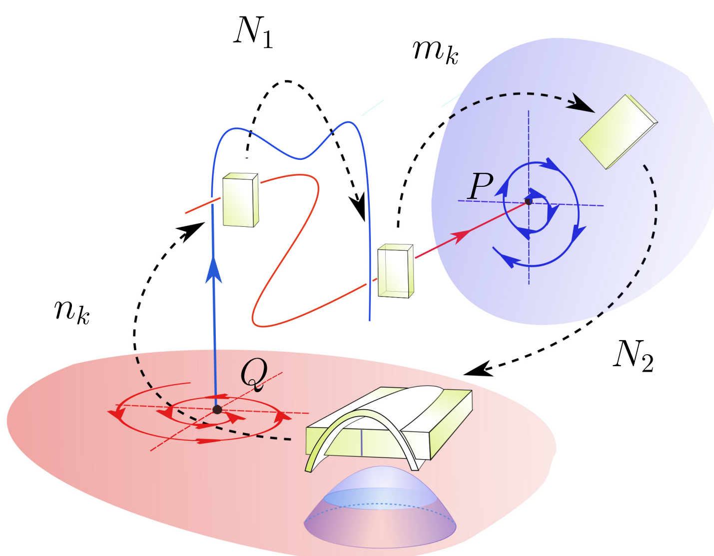

FIGURE 4. The points in the blender are admissible points

9. PROOF OF THEOREM 1.1: INTERSECTIONS BETWEEN THE TWO-DIMENSIONAL MANIFOLDS

Throughout this and the next section, we will assume that $f \in \mathcal{H}^r_{\mathrm{BH}}(M)$, $r \geqslant 2$, and consider the perturbations $g_{\varepsilon,k,\mu}$ of f. By Proposition 7.2 the set $\Upsilon_{\varepsilon,k,\mu}$ in Remark 8.1 is a blender of $g_{\varepsilon,k,\mu}$.

We prove that the unstable manifolds of the blenders $\Upsilon_{\varepsilon,k,\mu}$ of $g_{\varepsilon,k,\mu}$ transversely intersects the stable manifold of the saddle Q.

Proposition 9.1. *For every small $\varepsilon > 0$, large k, and $\mu \in (-10,-9)$ it holds*

$$W^{\mathrm{u}}\big(\Upsilon_{g_{\varepsilon,k,\mu}}, g_{\varepsilon,k,\mu}\big) \pitchfork W^{\mathrm{s}}\big(Q_{g_{\varepsilon,k,\mu}}, g_{\varepsilon,k,\mu}\big) \neq \emptyset.$$

The proof of this proposition is inspired by [23, Proposition 1, Chapter 6.4]. Here there are additional difficulties due to the heterodimensional nature of the bifurcation. A comparison between the two settings is done in Section 9.1.1.

Notation 9.2. Recall definitions of f_*, F_*, Υ_*, and U_*, with $* = (\varepsilon, \bar{\upsilon}_k(\mu), \rho(\varepsilon))$, in Section 8. If there is no possible misunderstanding, in what follows ε and $\rho(\varepsilon)$ will be omitted, simply writing $f_{\bar{\upsilon}_k(\mu)}$, $F_{\bar{\upsilon}_k(\mu)}$, $\Upsilon_{\bar{\upsilon}_k(\mu)}$, and $U_{\bar{\upsilon}_k(\mu)}$.

By Proposition 7.2, the sequence of maps $F_{\bar{\upsilon}_k(\mu)} = F_{\varepsilon,\bar{\upsilon}_k(\mu),\rho(\varepsilon)}$ converges (on compacta) to the family of endomorphisms $G_{\xi,\mu,\bar{\eta}}$ for some fixed ξ and $\bar{\eta}$. Hence we write G_μ in the place of $G_{\xi,\mu,\bar{\eta}}$.

This section is organised as follows. In Section 9.1, we introduce an auxiliary one-dimensional foliation $\mathcal{D}$ in U_P which subfoliates the reference domain of the blenders. The main property of this foliation is that the strong unstable foliation of the blender approaches to $\mathcal{D}$. We translate the foliation $\mathcal{D}$ to $U_{\widetilde{Y}}$ by $f^{N_2}_{\bar{v}_k(\mu)}$ and thereafter by Φ_k^{-1} to $\mathbb{R}^3$, obtaining in this way a foliation $\widetilde{\mathcal{D}}_{\bar{v}_k(\mu)}$. In Lemma 9.1, we see that the leaves of $\widetilde{\mathcal{D}}_{\bar{v}_k(\mu)}$ converge to parabolas when $k \to \infty$. In Section 9.2, using the foliation $\mathcal{D}$, for the admissible points and returns in Section 8, we study the expansion of vectors and how the angles change, see Lemma 9.2. In Section 9.4, we translate these estimates for the map $F_{\bar{v}_k(\mu)}$. In Section 9.3, we study the separatrices of the saddles of the blenders nearby heterodimensional tangencies. Finally, in Section 9.5 we conclude the proof of Proposition 9.1.

9.1. **The auxiliary one dimensional foliation.** We start with some preliminary constructions. We consider first auxiliary foliations $\mathcal{F}^{\mathrm{u}}_{R,\bar{v}_k(\mu)}$, $R = P, Q$, defined on the neighbourhoods U_P and U_Q as the natural extensions of the invariant local manifolds of the saddles in the cycle. The leaf $\mathcal{F}^{\mathrm{u}}_{P,\bar{v}_k(\mu)}(A)$ of the point $A \in U_P$ of the foliation $\mathcal{F}^{\mathrm{u}}_{P,\bar{v}_k(\mu)}$ is the intersection of the set U_P and the plane parallel to the coordinate plane yz containing A. Similarly, the leaf $\mathcal{F}^{\mathrm{s}}_{P,\bar{v}_k(\mu)}(A)$ of $\mathcal{F}^{\mathrm{s}}_{P,\bar{v}_k(\mu)}$ is the intersection of U_P and the straight line parallel to the axis x containing A. Thus these leaves are "parallel" to $W^{\mathrm{u}}_{\mathrm{loc}}(P)$ and $W^{\mathrm{s}}_{\mathrm{loc}}(P)$, respectively. The leaves of the foliations $\mathcal{F}^{\mathrm{u}}_{Q,\bar{v}_k(\mu)}$ and $\mathcal{F}^{\mathrm{s}}_{Q,\bar{v}_k(\mu)}$ are defined similarly and are "parallel" to $W^{\mathrm{u}}_{\mathrm{loc}}(Q)$ and $W^{\mathrm{s}}_{\mathrm{loc}}(Q)$. Note that the foliations $\mathcal{F}^{*}_{P,Q,\bar{v}_k(\mu)}$, $* = \mathrm{s}, \mathrm{u}$, do not depend on $\bar{v}_k(\mu)$.

For $* = \mathrm{s}, \mathrm{u}$, we "transport" the foliations $\mathcal{F}^{*}_{Q,\bar{v}_k(\mu)}$ from $U_X \subset U_Q$ to $U_{\widetilde{X}} \subset U_P$ by the transition $f^{N_1}_{\bar{v}_k(\mu)}$ and continue denoting the resulting foliations by $\mathcal{F}^{*}_{Q,\bar{v}_k(\mu)}$. Similarly, we "transport" the foliations $\mathcal{F}^{*}_{P,\bar{v}_k(\mu)}$ from $U_Y \subset U_P$ to $U_{\widetilde{Y}} \subset U_Q$ by $f^{N_2}_{\bar{v}_k(\mu)}$ and continue denoting these foliations by $\mathcal{F}^{*}_{P,\bar{v}_k(\mu)}$. Note that these extensions do depend on $\bar{v}_k(\mu)$.

We now consider an auxiliary one-dimensional foliation $\mathcal{D}$ in U_P. For that consider the family of curves

$$\ell_{(s,a)} \stackrel{\text{def}}{=} \big\{ (s, a, -a) + (0, t, t) \,:\, t \in \mathbb{R} \big\} \cap U_P, \quad s, a \in \mathbb{R}.$$

and define the *diagonal foliation* of $U_P = [-a_P, a_P]^3$ by

$$\mathcal{D} \stackrel{\text{def}}{=} \big\{ \ell_{(s,a)} \,:\, a, s \in [-a_P, a_P] \big\}. \tag{9.1}$$

Note that $\mathcal{D}$ "subfoliates" the leaves of $\mathcal{F}^{\mathrm{u}}_{P,\bar{v}_k(\mu)}$ in U_P, see Figure 5.

Consider the domain $\Delta_k(\mu)$ of the blender $\Upsilon_{\bar{v}_k(\mu)}$ in (7.3). By [16, Lemma 3], for every large k, the coordinates $(x, 1+y, 1+z) \in U_P$ of the points in $f^{m_k+N_1+n_k}_{\bar{v}_k(\mu)}\big(\Delta_k(\mu)\big)$ are close to $Y = (0,1,1)$. Moreover, they have Landau symbols

$$x = O(\lambda_P{}^{m_k}), \quad y = O(\sigma_P^{-m_k}\, \sigma_Q{}^{-n_k}), \quad z = O(\sigma_P^{-m_k}\, \sigma_Q{}^{-n_k}).$$

These conditions and equation (6.3) imply that

$$f^{m_k+N_1+n_k}_{\bar{v}_k(\mu)}(\Delta_k(\mu)) \subset \bigcup_{s \in J_k} \big\{ \ell_{(s,a)} \,:\, a \in [-a_P, a_P] \big\} \tag{9.2}$$

where

$$J_k \stackrel{\text{def}}{=} [\,-\sigma_Q^{-2n_k}\,\sigma_P^{-2m_k} a_P\,,\,\sigma_Q^{-2n_k}\,\sigma_P^{-2m_k} a_P\,]. \tag{9.3}$$

As above, we consider the intersection of the leaves of $\mathcal{D}$ with U_Y and "transport" them by $f^{N_2}_{\bar{v}_k(\mu)}$, obtaining the following foliation of $U_{\widetilde{Y}}$ (see Figure 5):

$$\mathcal{D}_{\bar{v}_k(\mu)} \stackrel{\text{def}}{=} \big\{\,\ell_{(s,a,\bar{v}_k(\mu))} \stackrel{\text{def}}{=} f^{N_2}_{\bar{v}_k(\mu)}(\ell_{(s,a)} \cap U_Y)\,:\,a,s \in [-a_P,a_P]\,\big\}. \tag{9.4}$$

Similarly, we let

$$\widetilde{\mathcal{D}}_{\bar{v}_k(\mu)} \stackrel{\text{def}}{=} \big\{\,\widetilde{\ell}_{(s,a,\bar{v}_k(\mu))} \stackrel{\text{def}}{=} \Phi_k^{-1}(\ell_{(s,a,\bar{v}_k(\mu))})\,:\,a,s \in [-a_P,a_P]\,\big\}. \tag{9.5}$$

9.1.1. *Comparison of the homoclinic and heteroclinic settings.* Our heterodimensional analysis is inspired by the one in [23, Chapter 6.4] for homoclinic tangencies. Let us highlight some key differences and similarities. For that recall that [23] considers a surface diffeomorphism with saddle R having a homoclinic tangency Z. There are associated auxiliary local stable and unstable foliations $\mathcal{W}^*_\mu$, $* = \mathrm{s},\mathrm{u}$, defined on a neighbourhood of Z, here μ refers to a "renormalisation" parameter unfolding the tangency. In [23] the renormalisation scheme converges to the quadratic family $\varphi_\mu(x,y) = (y, y^2+\mu)$.

The construction in [23] implicitly uses the fact that (for suitable parameters) the family φ_μ has a fixed point whose unstable manifold has an "infinite" separatrix. Here we have a property with the same flavour stated in Lemma 5.5.

The foliations $\mathcal{W}^{\mathrm{s}}_\mu$ and $\mathcal{W}^{\mathrm{u}}_\mu$ converge (in the charts of the corresponding renormalisation scheme) to foliations whose leaves are horizontal lines and parabolas of the form $(x, x^2+\mu)$, respectively. Here, the foliation $\mathcal{D}_{\bar{v}_k(\mu)}$ plays the role of the unstable foliation $\mathcal{W}^{\mathrm{u}}_\mu$. Lemma 9.1 states a convergence property of the foliation $\mathcal{D}_{\bar{v}_k(\mu)}$ (involving some projections). The stable foliation $\mathcal{F}^{\mathrm{s}}_{Q,\bar{v}_k(\mu)}$ defined above is similar to $\mathcal{W}^{\mathrm{s}}_\mu$.

Both foliations $\mathcal{W}^{\mathrm{s}}_\mu$ and $\mathcal{W}^{\mathrm{u}}_\mu$ foliate a neighbourhood of the tangency Z. Here, we have that the foliation $\mathcal{D}_{\bar{v}_k(\mu)}$ covers a neighbourhood of the heterodimensional tangency $\widetilde{Y}$ and therefore of the reference domain $\Delta_k(\mu)$ of the blender $\Upsilon_{\bar{v}_k(\mu)}$, see (9.2). There is a similar assertion for $\mathcal{F}^{\mathrm{s}}_{Q,\bar{v}_k(\mu)}$.

In [23], the unstable leaves of the limit thick horseshoes approach to parabola of the limit unstable foliation and its projection along stable leaves "covers" several fundamental domains of the local unstable manifold of the saddle R. Here, we have a similar property: the leaves of the strong unstable foliation of the blenders $\Upsilon_{\bar{v}_k(\mu)}$ are close to the leaves of $\mathcal{D}_{\bar{v}_k(\mu)}$. Due to the lack of domination of our setting, it is not possible to get a similar covering property. Instead, we prove that "projections" of the leaves of $\mathcal{D}_{\bar{v}_k(\mu)}$ covers a fixed proportion of a fixed fundamental domain of $W^{\mathrm{u}}_{\mathrm{loc}}(Q)$. This will be enough to see that the blenders are involved in the robust cycles with the saddle Q and are homoclinically related to the saddle P.

9.1.2. *Convergence to parabolas.* We now go to the details of our construction. Consider:

- the projection $\pi_{1,2}\colon \mathbb{R}^3 \to \mathbb{R}^2$ given by $\pi_{1,2}(x,y,z) = (x,y)$;

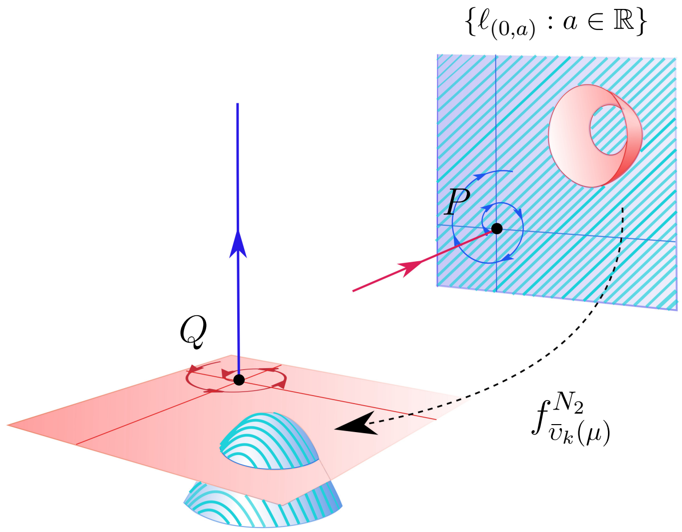

FIGURE 5. The diagonal foliation on the leaf $\mathcal{F}^{\mathrm{u}}(P)$.

- the maps $\alpha,\ \beta : \mathbb{R} \to \mathbb{R}$ defined by

$$\begin{aligned} \alpha(a) &\stackrel{\text{def}}{=} \sqrt{2}\,\beta_2\,(b_2 - b_3)\,a, \\ \beta(\mu, s, a) &\stackrel{\text{def}}{=} \mu + b_1\,\varsigma_2\,s + (b_2 + b_3 - b_4)\,\varsigma_2\,a^2, \end{aligned} \tag{9.6}$$

  where b_1, b_2, b_3, b_4 are parameters associated to the heterodimensional tangency in (2.6) and ς_2 is as in the definition of the Hénon-like maps in (5.2);
- the family of curves

$$\bar{\ell}_{(s,a,\bar{\upsilon}_k(\mu))}(t) \stackrel{\text{def}}{=} \pi_{1,2}\Big(\widetilde{\ell}_{(s,a,\bar{\upsilon}_k(\mu))}(t)\Big) = \pi_{1,2}\Big(\Phi_k^{-1}\big(\ell_{(s,a,\bar{\upsilon}_k(\mu))}(t)\big)\Big); \tag{9.7}$$

- the re-scaling maps $\widehat{s}_k, \widehat{a}_k, \widehat{t}_k \colon [-a_P, a_P] \to \mathbb{R}$, given by

$$\begin{aligned} \widehat{s}_k(s) &\stackrel{\text{def}}{=} \sigma_Q^{-2n_k}\,\sigma_P^{-2m_k}\,s, \\ \widehat{a}_k(a) &\stackrel{\text{def}}{=} \sigma_Q^{-n_k}\,\sigma_P^{-m_k}\,a, \\ \widehat{t}_k(t) &\stackrel{\text{def}}{=} \frac{\sqrt{2}}{\beta_2(b_2 + b_3 + b_4)}\,\sigma_Q^{-2n_k}\,\sigma_P^{-2m_k}\,t. \end{aligned} \tag{9.8}$$

Noting that $\widehat{s}_k(s) \in J_k$ for every $s \in [-a_P, a_P]$, we can define the curves

$$\widehat{\ell}_{(s,a,\bar{\upsilon}_k(\mu))}(t) \stackrel{\text{def}}{=} \bar{\ell}_{(\,\widehat{s}_k(s)\,,\,\widehat{a}_k(a)\,,\,\bar{\upsilon}_k(\mu)\,)}(\,\widehat{t}_k(t)\,), \quad (s,a) \in [-a_P, a_P]$$

and the sets

$$\widehat{L}_{(s,a,\bar{\upsilon}_k(\mu))} \stackrel{\text{def}}{=} \{\,\widehat{\ell}_{(s,a,\bar{\upsilon}_k(\mu))}(t)\ :\ t \in [-a_P, a_P]\,\}.$$

Lemma 9.1. *The sequence of sets $\widehat{L}_{(s,a,\bar{\upsilon}_k(\mu))}$ converges to the parabola*

$$y = x^2 + \alpha(a)\,x + \beta(a, s, \mu), \tag{9.9}$$

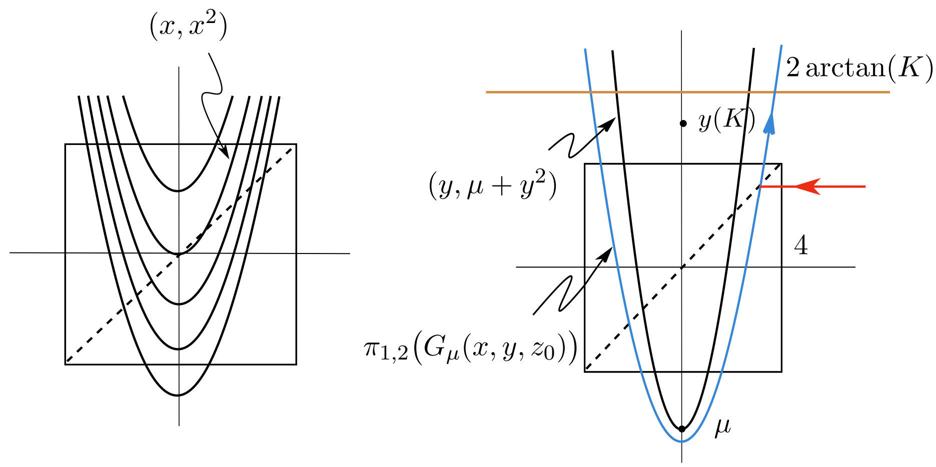

FIGURE 6. The limit family of parabolas of the sets $\widehat{L}_{(s,0,\bar{\mu}_k(\mu))}$ (left). Angles between parabolas and horizontal lines (right)

when $k \to \infty$. The convergence is C^r uniform on compact sets of $\mathbb{R}^2$.

Next remark will be used in Section 9.3. It relates the parabolas in Lemma 9.1, hence the foliation $\widetilde{\mathcal{D}}_{\bar{v}_k(\mu)}$, and the Hénon-like maps. It will play a key role in the arguments for controlling the size of unstable sets of blenders.

Remark 9.3. Recall the reference domain $\Delta = [-4, 4]^2 \times [-40, 22]$ in (5.6) of the blender of G_μ and that

$$\pi_{1,2}(G_\mu(x, y, z)) = (y, \mu + y^2 + \kappa\, y\, z + \eta\, z^2).$$

Hence for each fixed z_0 we have that $\pi_{1,2}(G_\mu(x, y, z_0))$ is a curve of the family in (9.9). Therefore given any $K > 0$ there is $y(K)$ such that for every $z_0 \in [-40, 22]$ and every $y \geqslant y(K)$ the angle between the curve $\pi_{1,2}(G_\mu(x, y, z_0))$ and the parallel to the x-axis at the point $(0, y)$ are strictly bigger than $2\arctan(K)$, see Figure 6.

Proof of Lemma 9.1. We first give an explicit calculation of the coordinates of the curves $\ell_{(s,a,\bar{v}_k(\mu))}$ in (9.4). Consider the parameterisation of $\ell_{(s,a,\bar{v}_k(\mu))} \subset U_{\widetilde{Y}}$ given by

$$\begin{aligned}\ell_{(s,a,\bar{v}_k(\mu))}(t) &\stackrel{\text{def}}{=} f^{N_2}_{\bar{v}_k(\mu)}\big(s, 1 + t + a, 1 + t - a\big)\\ &\stackrel{\text{def}}{=} \Big(\ell^1_{(s,a,\bar{v}_k(\mu))}(t), \ell^2_{(s,a,\bar{v}_k(\mu))}(t), \ell^3_{(s,a,\bar{v}_k(\mu))}(t)\Big).\end{aligned}$$

Recalling the expression $f^{N_2}_{\bar{v}_k(\mu)}$ in (2.5), Remark 6.3, and (6.5), we get

$$\begin{aligned}
\ell^1_{(s,a,\bar{v}_k(\mu))}(t) &= 1 + a_1 s + (a_2 - a_3)a + (a_2 + a_3)t \\
&\quad + H_1(s, t+a, t-a) - \lambda_P^{m_k}\, a_1, \\
\ell^2_{(s,a,\bar{v}_k(\mu))}(t) &= b_1 s + (b_2 + b_3 - b_4)\, a^2 + (b_2 + b_3 + b_4)\, t^2 + 2(b_2 - b_3)\, a\, t \\
&\quad + H_2(s, t+a, t-a) + \sigma_Q^{-n_k} + \sigma_Q^{-2n_k}\, \sigma_P^{-2m_k}\, \mu - \lambda_P^{m_k}\, b_1, \\
\ell^3_{(s,a,\bar{v}_k(\mu))}(t) &= 1 + c_1 s + (c_2 + c_3)t + H_3(s, t+a, t-a) - \lambda_P^{m_k}\, c_1.
\end{aligned}$$

Write

$$\overline{\ell}_{(s,a,\bar{v}_k(\mu))}(t) \stackrel{\text{def}}{=} \Big(\overline{\ell}^1_{(s,a,\bar{v}_k(\mu))}(t), \overline{\ell}^2_{(s,a,\bar{v}_k(\mu))}(t)\Big).$$

From the definitions of $\overline{\ell}_{(s,a,\bar{v}_k(\mu))}(t)$ in (9.7) and of Φ_k^{-1} in Remark 7.1, we get

$$\begin{aligned}
\overline{\ell}^1_{(s,a,\bar{v}_k(\mu))}(t) &= c_1\, \varsigma_2\, \varsigma_5^{-1}\, \sigma_Q^{n_k}\, \sigma_P^{m_k} s + \frac{\beta_2(b_2 + b_3 + b_4)}{\sqrt{2}}\, \sigma_Q^{n_k}\, \sigma_P^{m_k}\, t \\
&\quad + \varsigma_2\, \varsigma_5^{-1}\, \sigma_Q^{n_k}\, \sigma_P^{m_k} H_3(s, t+a, t-a) - \lambda_P^{m_k}\, \sigma_Q^{n_k}\, \sigma_P^{m_k}\, \varsigma_2\, \varsigma_5^{-1}\, c_1, \\
\overline{\ell}^2_{(s,a,\bar{v}_k(\mu))}(t) &= b_1\, \varsigma_2\, \sigma_Q^{2n_k}\, \sigma_P^{2m_k} s + \varsigma_2\, (b_2 + b_3 - b_4)\, \sigma_Q^{2n_k}\, \sigma_P^{2m_k} a^2 \\
&\quad + \frac{\beta_2^2(b_2 + b_3 + b_4)^2}{2} \sigma_Q^{2n_k}\, \sigma_P^{2m_k}\, t^2 + 2(b_2 - b_3)\, \varsigma_2\, \sigma_Q^{2n_k}\, \sigma_P^{2m_k}\, a\, t \\
&\quad + \varsigma_2\, \sigma_Q^{2n_k}\, \sigma_P^{2m_k}\, H_2(s, t+a, t-a) + \mu - \lambda_P^{m_k}\, \sigma_Q^{2n_k}\, \sigma_P^{2m_k}\, \varsigma_2\, b_1.
\end{aligned}$$

Recalling the re-scaling maps $\widehat{s}_k, \widehat{a}_k, \widehat{t}_k$, in (9.8) and performing the corresponding substitutions, a straightforward calculation implies that

$$\widehat{\ell}_{(s,a,\bar{v}_k(\mu))}(t) \ \to (t, t^2 + \alpha(a)\, t + \beta(\mu, s, a)),$$

where α and β are as in (9.6). This ends the proof of the lemma. □

9.2. **Estimates of angles and expansion for admissible iterations.** We prove the lemma below, which is version of [23, Claim 1 in Chapter 6.4] in our context. To state this lemma consider the projection $\pi_{2,3} \colon \mathbb{R}^3 \to \mathbb{R}^2$ defined by $\pi_{2,3}(x, y, z) = (y, z)$ and recall the definitions of the foliation $\mathcal{D}_{\bar{v}_k(\mu)}$ of $U_{\widetilde{Y}}$ in (9.4) and of the set $U_{\bar{v}_k(\mu)}$ of points with an admissible itinerary as in (8.1).

Lemma 9.2. *There is a constant $C > 0$ such that for every $\mu \in (-10, -9)$ and every sufficiently large k the following holds:*

Consider $K > 0$, a $\bar{v}_k(\mu)$-admissible point $w \in U_{\bar{v}_k(\mu)}$, and a vector $v \in T_w M$ such that

$$\text{angle}\big(v, \mathcal{F}^{\mathrm{s}}_{Q,\bar{v}_k(\mu)}\big) \geqslant K\, \sigma_Q^{-n_k} \sigma_P^{-m_k}. \tag{9.10}$$

Then

(1) $\|\pi_{2,3}\big(D_w(f^{N_2+m_k+N_1+n_k}_{\bar{v}_k(\mu)})(v)\big)\| \geqslant C\, K\, \|v\|$,
(2) $\text{angle}\big(D_w(f^{N_2+m_k+N_1+n_k}_{\bar{v}_k(\mu)})(v), \mathcal{F}^{\mathrm{u}}_{P,\bar{v}_k(\mu)}\big) = O(\lambda_P^{m_k} \sigma_Q^{n_k}) \to 0$ *as* $k \to \infty$*, and*
(3) $\text{angle}\big(D_w(f^{N_2+m_k+N_1+n_k}_{\bar{v}_k(\mu)})(v), \mathcal{D}_{\bar{v}_k(\mu)}\big) \to 0,$ *as* $k \to \infty$.

We begin with some preliminary estimates. For $R = P, Q$, consider the coordinate vector fields $\left\{\frac{\partial}{\partial x_R}, \quad \frac{\partial}{\partial y_R}, \quad \frac{\partial}{\partial z_R}\right\}$, defined on the neighbourhood U_R and tangent to the corresponding foliations $\mathcal{F}^*_{R,\bar{v}_k(\mu)}$, $* = \mathrm{u}, \mathrm{s}$. The derivatives of the transition maps $\mathfrak{T}_1 = f^{N_1}$ in (2.4) and $\mathfrak{T}_2 = f^{N_2}$ in (2.6) satisfy

$$
\begin{aligned}
D\mathfrak{T}_1\Big(\frac{\partial}{\partial x_Q}\Big) &= \alpha_1 \frac{\partial}{\partial x_P},\\
D\mathfrak{T}_1\Big(\frac{\partial}{\partial y_Q}\Big) &= \alpha_2 \frac{\partial}{\partial x_P} + \beta_2 \frac{\partial}{\partial y_P},\\
D\mathfrak{T}_1\Big(\frac{\partial}{\partial z_Q}\Big) &= \alpha_3 \frac{\partial}{\partial x_P} + \gamma_3 \frac{\partial}{\partial z_P},\\
D\mathfrak{T}_2\Big(\frac{\partial}{\partial x_P}\Big) &= a_1 \frac{\partial}{\partial x_Q} + b_1 \frac{\partial}{\partial y_Q} + c_1 \frac{\partial}{\partial z_Q},\\
D\mathfrak{T}_2\Big(\frac{\partial}{\partial y_P}\Big) &= a_2 \frac{\partial}{\partial x_Q} + c_2 \frac{\partial}{\partial z_Q},\\
D\mathfrak{T}_2\Big(\frac{\partial}{\partial z_P}\Big) &= a_3 \frac{\partial}{\partial x_Q} + c_3 \frac{\partial}{\partial z_Q}.
\end{aligned}
\tag{9.11}
$$

Given a point $w \in U_{\bar{v}_k(\mu)}$ and a vector

$$
v = v_x \frac{\partial}{\partial x_Q} + v_y \frac{\partial}{\partial y_Q} + v_z \frac{\partial}{\partial z_Q}, \tag{9.12}
$$

we give the explicit expression of $D_w(f^{N_2+m_k+N_1+n_k}_{\bar{v}_k(\mu)})(v)$. It follows the sketch of the step by step calculations of this derivative, for details see [16, Section 7].

Recalling the linearising coordinates of f at Q in (2.1) and the definitions of $\mathfrak{c}_k, \mathfrak{s}_k$ in (6.6), we have

$$
D_w(f^{n_k}_{\bar{v}_k(\mu)})(v) = \lambda_Q^{n_k}(\mathfrak{c}_k - \mathfrak{s}_k)\, v_x \frac{\partial}{\partial x_Q} + \sigma_Q^{n_k}\, v_y \frac{\partial}{\partial y_Q} + \lambda_Q^{n_k}(\mathfrak{c}_k + \mathfrak{s}_k)\, v_z \frac{\partial}{\partial z_Q}.
$$

Write

$$
D_w(f^{N_1+n_k}_{\bar{v}_k(\mu)})(v) = v_{x,\bar{v}_k(\mu)} \frac{\partial}{\partial x_P} + v_{x,\bar{v}_k(\mu)} \frac{\partial}{\partial y_P} + v_{z,\bar{v}_k(\mu)} \frac{\partial}{\partial z_P}.
$$

Using (9.11) we have that

$$
\begin{aligned}
v_{x,\bar{v}_k(\mu)} &= \lambda_Q^{n_k}(\mathfrak{c}_k - \mathfrak{s}_k)\, v_x\, \alpha_1 + \sigma_Q^{n_k}\, v_y\, \alpha_2 + \lambda_Q^{n_k}(\mathfrak{c}_k - \mathfrak{s}_k)\, v_z\, \alpha_3,\\
v_{y,\bar{v}_k(\mu)} &= \sigma_Q^{n_k}\, v_y\, \beta_2,\\
v_{z,\bar{v}_k(\mu)} &= \lambda_Q^{n_k}(\mathfrak{c}_k + \mathfrak{s}_k)\, v_z\, \gamma_3.
\end{aligned}
\tag{9.13}
$$

Write

$$
D_w(f^{m_k+N_1+n_k}_{\bar{v}_k(\mu)})(v) = \widetilde{v}_{x,\bar{v}_k(\mu)} \frac{\partial}{\partial x_P} + \widetilde{v}_{y,\bar{v}_k(\mu)} \frac{\partial}{\partial y_P} + \widetilde{v}_{z,\bar{v}_k(\mu)} \frac{\partial}{\partial z_P}.
$$

Using (9.13), the linearising coordinates of f at P in (2.1), and the definitions of $\tilde{\mathfrak{c}}_k, \tilde{\mathfrak{s}}_k$ in (6.6), we have that

$$
\begin{aligned}
&\widetilde{v}_{x,\bar{v}_k(\mu)} = \lambda_P^{m_k}\big(\lambda_Q^{n_k}(\mathfrak{c}_k-\mathfrak{s}_k)\, v_x\,\alpha_1 + \sigma_Q^{n_k}\, v_y\,\alpha_2 + \lambda_Q^{n_k}(\mathfrak{c}_k-\mathfrak{s}_k)\, v_z\,\alpha_3\big),\\
&\widetilde{v}_{y,\bar{v}_k(\mu)} = \sigma_P^{m_k}\,\sigma_Q^{n_k}\,\tilde{\mathfrak{c}}_k\, v_y\,\beta_2 - \sigma_P^{m_k}\,\lambda_Q^{n_k}\,\tilde{\mathfrak{s}}_k\,(\mathfrak{c}_k+\mathfrak{s}_k)\, v_z\,\gamma_3,\\
&\widetilde{v}_{z,\bar{v}_k(\mu)} = \sigma_P^{m_k}\,\sigma_Q^{n_k}\,\tilde{\mathfrak{s}}_k\, v_y\,\beta_2 + \sigma_P^{m_k}\,\lambda_Q^{n_k}\,\tilde{\mathfrak{c}}_k\,(\mathfrak{c}_k+\mathfrak{s}_k)\, v_z\,\gamma_3.
\end{aligned} \tag{9.14}
$$

Finally, we write

$$
D_w(f_{\bar{v}_k(\mu)}^{N_2+m_k+N_1+n_k})(v) = \widehat{v}_{x,\bar{v}_k(\mu)}\,\frac{\partial}{\partial x_Q} + \widehat{v}_{x,\bar{v}_k(\mu)}\,\frac{\partial}{\partial y_Q} + \widehat{v}_{z,\bar{v}_k(\mu)}\,\frac{\partial}{\partial z_Q}.
$$

Using (9.11) and (9.14) we get

$$
\begin{aligned}
\widehat{v}_{x,\bar{v}_k(\mu)} &= a_1\,\widetilde{v}_{x,\bar{v}_k(\mu)} + a_2\,\widetilde{v}_{y,\bar{v}_k(\mu)} + a_3\,\widetilde{v}_{z,\bar{v}_k(\mu)},\\
\widehat{v}_{y,\bar{v}_k(\mu)} &= b_1\,\widetilde{v}_{x,\bar{v}_k(\mu)},\\
\widehat{v}_{z,\bar{v}_k(\mu)} &= c_1\,\widetilde{v}_{x,\bar{v}_k(\mu)} + c_2\,\widetilde{v}_{y,\bar{v}_k(\mu)} + c_3\,\widetilde{v}_{z,\bar{v}_k(\mu)}.
\end{aligned} \tag{9.15}
$$

Proof of Lemma 9.2. Take an unitary vector v, write it as in (9.12), and let $\alpha_{k,\mu} \stackrel{\text{def}}{=} \operatorname{angle}\big(v, \mathcal{F}^{\mathrm{s}}_{Q,\bar{v}_k(\mu)}\big)$. Note that by hypothesis (9.10)

$$
|v_y| = \sin(\alpha_{k,\mu}) \geqslant \sin(K\,\sigma_Q^{-n_k}\,\sigma_P^{-m_k}) \approx K\,\sigma_Q^{-n_k}\sigma_P^{-m_k}. \tag{9.16}
$$

To prove item (1), recall the definitions of $\widehat{v}_{z,\bar{v}_k(\mu)}$ in (9.15) and of $\widetilde{v}_{x,\bar{v}_k(\mu)}$, $\widetilde{v}_{y,\bar{v}_k(\mu)}$, and $\widetilde{v}_{z,\bar{v}_k(\mu)}$ in (9.14) and note that

$$
\frac{\widehat{v}_{z,\bar{v}_k(\mu)}}{\sigma_P^{m_k}\,\sigma_Q^{n_k}} = \frac{c_1\,\widetilde{v}_{x,\bar{v}_k(\mu)}}{\sigma_P^{m_k}\,\sigma_Q^{n_k}} + \frac{c_2\,\widetilde{v}_{y,\bar{v}_k(\mu)}}{\sigma_P^{m_k}\,\sigma_Q^{n_k}} + \frac{c_3\,\widetilde{v}_{z,\bar{v}_k(\mu)}}{\sigma_P^{m_k}\,\sigma_Q^{n_k}}.
$$

Recalling that $\sigma_P, \sigma_Q > 1$, $\sigma_P^{m_k}\,\lambda_Q^{n_k} \to \tau^{-1}\xi$ (see (6.2)) and $\tilde{\mathfrak{c}}_k$, $\tilde{\mathfrak{s}}_k \to 1/\sqrt{2}$, $\mathfrak{c}_k \to 0$, $\mathfrak{s}_k \to 1$ (see Remark 6.4), we get

$$
\frac{\widetilde{v}_{x,\bar{v}_k(\mu)}}{\sigma_P^{m_k}\,\sigma_Q^{n_k}} \to 0, \qquad \frac{\widetilde{v}_{y,\bar{v}_k(\mu)}}{\sigma_P^{m_k}\,\sigma_Q^{n_k}} \to \frac{\beta_2\,v_y}{\sqrt{2}}, \qquad \frac{\widetilde{v}_{z,\bar{v}_k(\mu)}}{\sigma_P^{m_k}\,\sigma_Q^{n_k}} \to \frac{\beta_2\,v_y}{\sqrt{2}}.
$$

Thus, for every large enough k, we get

$$
|\widehat{v}_{z,\bar{v}_k(\mu)}| \approx \sqrt{2}\,\sigma_P^{m_k}\,\sigma_Q^{n_k}\,|c_2+c_3|\,|\beta_2|\,|v_y|. \tag{9.17}
$$

Note that $c_2+c_3 \neq 0$, see (2.7) and (2.6), and $\beta_2 \neq 0$, see (2.4). Finally, from (9.16) and (9.17) it follows

$$
\begin{aligned}
\|\pi_{2,3}(D_w(f_{\bar{v}_k(\mu)}^{N_2+m_k+N_1+n_k})(v))\| &\geqslant |\widehat{v}_{z,\bar{v}_k(\mu)}|\\
&\approx \sqrt{2}\,\sigma_P^{m_k}\,\sigma_Q^{n_k}\,|c_2+c_3|\,|\beta_2|\,|v_y|\\
&\geqslant C\,K,
\end{aligned} \tag{9.18}
$$

where $C = \sqrt{2}\,|\beta_2|\,|c_2+c_3|$. This proves the first item in the lemma.

To prove item (2), let $\beta_{k,\mu}$ be the angle between the vector $D_w(f^{m_k+N_1+n_k}_{\bar{v}_k(\mu)})(v)$ and the unstable foliation $\mathcal{F}^u_{P,\bar{v}_k(\mu)}$ (tangent to $\frac{\partial}{\partial y_P}$ and $\frac{\partial}{\partial z_P}$). Then

$$\sin(\beta_{k,\mu}) = \frac{|\widetilde{v}_{x,\bar{v}_k(\mu)}|}{\|D_w(f^{m_k+N_1+n_k}_{\bar{v}_k(\mu)})(v)\|} \leqslant \frac{|\widetilde{v}_{x,\bar{v}_k(\mu)}|}{\|\pi_{2,3}(D_w(f^{m_k+N_1+n_k}_{\bar{v}_k(\mu)})(v))\|} \lesssim \frac{\lambda_P^{m_k}\sigma_Q^{n_k}}{C\,K},$$

where in the last inequality we use (9.18) and that $\widetilde{v}_{x,\bar{v}_k(\mu)} = O(\lambda_P^{m_k}\sigma_Q^{n_k})$, see (9.14). Note that by (6.3)

$$\lambda_P^{m_k}\sigma_Q^{n_k} \leqslant \lambda_P^{m_k}\sigma_Q^{2n_k}\sigma_P^{2m_k} \to 0.$$

Therefore,

$$\beta_{k,\mu} \approx \sin(\beta_{k,\mu}) = O(\lambda_P^{m_k}\sigma_Q^{n_k}) \to 0,$$

proving the second item in the lemma.

To prove the last item in the lemma, recall again the definitions of $\widetilde{v}_{y,\bar{v}_k(\mu)}$ and $\widetilde{v}_{z,\bar{v}_k(\mu)}$ in (9.14) and note that

$$\begin{aligned} \frac{|\widetilde{v}_{y,\bar{v}_k(\mu)}|}{|\widetilde{v}_{z,\bar{v}_k(\mu)}|} &= \frac{|\sigma_P^{m_k}\,\sigma_Q^{n_k}\,\tilde{\mathfrak{c}}_k\,v_y\,\beta_2 - \sigma_P^{m_k}\,\lambda_Q^{n_k}\,\tilde{\mathfrak{s}}_k\,(\mathfrak{c}_k+\mathfrak{s}_k)\,v_z\,\gamma_3|}{|\sigma_P^{m_k}\,\sigma_Q^{n_k}\,\tilde{\mathfrak{s}}_k\,v_y\,\beta_2 + \sigma_P^{m_k}\,\lambda_Q^{n_k}\,\tilde{\mathfrak{c}}_k\,(\mathfrak{c}_k+\mathfrak{s}_k)\,v_z\,\gamma_3|} \\ &= \frac{\left|\tilde{\mathfrak{c}}_k\,v_y\,\beta_2 - \frac{\sigma_P^{m_k}\,\lambda_Q^{n_k}\,\tilde{\mathfrak{s}}_k\,(\mathfrak{c}_k+\mathfrak{s}_k)\,v_z\,\gamma_3}{\sigma_P^{m_k}\,\sigma_Q^{n_k}}\right|}{\left|\tilde{\mathfrak{s}}_k\,v_y\,\beta_2 + \frac{\sigma_P^{m_k}\,\lambda_Q^{n_k}\,\tilde{\mathfrak{c}}_k\,(\mathfrak{c}_k+\mathfrak{s}_k)\,v_z\,\gamma_3}{\sigma_P^{m_k}\,\sigma_Q^{n_k}}\right|} \to 1, \end{aligned} \tag{9.19}$$

where for the limit we use again that $\tilde{\mathfrak{c}}_k,\ \tilde{\mathfrak{s}}_k \to 1/\sqrt{2}$ and $\mathfrak{c}_k \to 0$, $\mathfrak{s}_k \to 1$.

This implies that the angle between $D_w(f^{m_k+N_1+n_k}_{\bar{v}_k(\mu)})(v)$ and the diagonal foliation $\mathcal{D}$ in (9.1) tends to 0 as k goes to infinity. Hence, by the definition of $\mathcal{D}_{\bar{v}_k(\mu)}$, the angle between $D_w(f^{N_2+m_k+N_1+n_k}_{\bar{v}_k(\mu)})(v)$ and $\mathcal{D}_{\bar{v}_k(\mu)}$ also tends to 0 as k goes to infinity. This ends the proof of the lemma. □

9.3. **Separatrices of the saddles of blenders nearby heterodimensional tangencies.** We now study the stable and strong unstable separatrices of the reference saddle $P^+_{\bar{v}_k(\mu)}$ of the blenders $\Lambda_{\bar{v}_k(\mu)}$ of $F_{\bar{v}_k(\mu)}$ in $\mathbb{R}^3$, recall Notation 7.3. Our goal are the angular estimates in Lemma 9.3.

Let us go to the details. Consider first blenders for the endomorphisms G_μ. Take $\mu \in (-10,-9)$ and the blender $\Lambda_\mu = \Lambda_{\xi,\mu,\bar{\eta}}$ of G_μ and its reference fixed point $P^+_\mu = (p^+_\mu, p^+_\mu, \widetilde{p}^+_\mu)$ in (5.7).

Denote by σ^{uu}_μ the separatrix[9] of $W^{\mathrm{uu}}(P^+_\mu)$ contained in $\{y \geqslant p^+_\mu\}$ and by σ^{s}_μ the separatrix of $W^{\mathrm{s}}(P^+_\mu)$ contained in $\{x \geqslant p^+_\mu,\ y = p^+_\mu,\ z = \widetilde{p}^+_\mu\}$. We consider the curve $\widetilde{\sigma}^{\mathrm{s}}_\mu \overset{\mathrm{def}}{=} \{P^+_\mu\} \cup \sigma^{\mathrm{s}}_\mu$. We now introduce the ingredients of our construction which are depicted in Figure 7.

[9]That is, one of the connected components of $W^{\mathrm{uu}}(P^+_\mu) \setminus \{P^+_\mu\}$

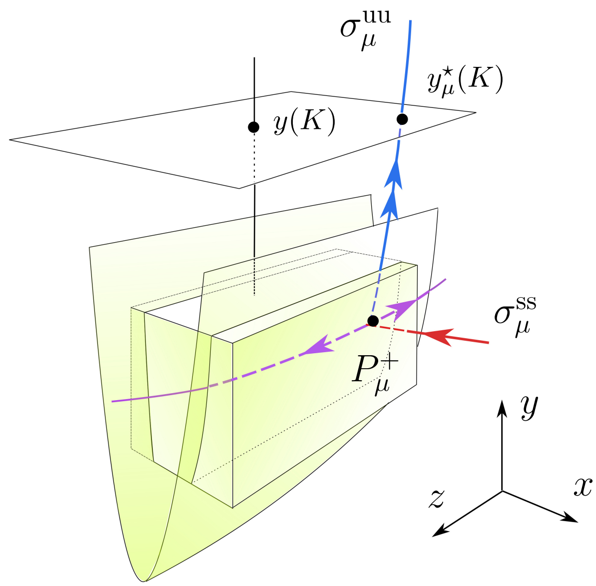

FIGURE 7. Separatrices of the reference saddles of the blender

• *Strong unstable separatrices.* Given $K > 0$ define $y(K)$ as in Remark 9.3 and consider the point

$$y^\star_\mu(K) \stackrel{\text{def}}{=} \sigma^{\mathrm{uu}}_\mu \cap \{y = y(K)\}.$$

Let $\sigma^{\mathrm{uu}}_{K,\mu}$ be the segment of σ^{uu}_μ bounded by $y^\star_\mu(K)$ and P^+_μ and consider its fundamental domain

$$\widetilde{\sigma}^{\mathrm{uu}}_{K,\mu} \stackrel{\text{def}}{=} \sigma^{\mathrm{uu}}_{K,\mu} \setminus G^{-1}_\mu(\sigma^{\mathrm{uu}}_{K,\mu}).$$

After increasing K, if necessary, we can assume that the angles between the curve $\pi_{1,2}(\widetilde{\sigma}^{\mathrm{uu}}_{K,\mu})$ and the lines parallel to the x-axis are strictly bigger than $\arctan(K)$.

For $\mu \in (-10,-9)$ and large k, we consider the continuations of the objects defined above for G_μ:

- the separatrices $\sigma^{\mathrm{uu}}_{\bar{v}_k(\mu)}$ of $W^{\mathrm{uu}}(P^+_{\bar{v}_k(\mu)})$,
- the points $y^\star_{\bar{v}_k(\mu)}(K) \stackrel{\text{def}}{=} \sigma^{\mathrm{uu}}_{\bar{v}_k(\mu)} \cap \{y = y(K)\}$,
- the curves $\sigma^{\mathrm{uu}}_{K,\bar{v}_k(\mu)}$ of $\sigma^{\mathrm{uu}}_{\bar{v}_k(\mu)}$ bounded by $y^\star_{\bar{v}_k(\mu)}(K)$ and $P^+_{\bar{v}_k(\mu)}$, and
- the fundamental domain

$$\widetilde{\sigma}^{\mathrm{uu}}_{K,\bar{v}_k(\mu)} = \sigma^{\mathrm{uu}}_{K,\bar{v}_k(\mu)} \setminus F^{-1}_{\bar{v}_k(\mu)}(\sigma^{\mathrm{uu}}_{K,\bar{v}_k(\mu)}).$$

By Remark 9.3 and continuity, for every $\mu \in (-10,-9)$ and large k, the angles between the curve $\pi_{1,2}(\widetilde{\sigma}^{\mathrm{uu}}_{K,\bar{v}_k(\mu)})$ and the lines parallel to the x-axis are strictly bigger than $\arctan(K)$.

• *Stable separatrices.* For large k, define the continuations $\widetilde{\sigma}^{\mathrm{s}}_{\bar{v}_k(\mu)}$ of $\widetilde{\sigma}^{\mathrm{s}}_{\mu}$ for $F_{\bar{v}_k(\mu)}$ contained in a separatrix of $W^{\mathrm{s}}(P^{+}_{\bar{v}_k(\mu)})$ and whose boundary contains $P^{+}_{\bar{v}_k(\mu)}$.

• *Angles between the separatrices.* Note that, by equation (5.8), the tangent space of $W^{\mathrm{uu}}(P^{+}_{\mu})$ at P^{+}_{μ} is contained in a cone field transverse to the horizontal line (parallel to the x-axis). In particular, equation (5.8) implies that for every $\mu \in (-10,-9)$ the angle between $\sigma^{\mathrm{uu}}_{\mu}$ and $\widetilde{\sigma}^{\mathrm{s}}_{\mu}$ at P^{+}_{μ} is bounded from below by $\pi/7$. Thus, for every k large enough, the angle between $\sigma^{\mathrm{uu}}_{\bar{v}_k(\mu)}$ and $\widetilde{\sigma}^{\mathrm{s}}_{\bar{v}_k(\mu)}$ at $P^{+}_{\bar{v}_k(\mu)}$ is bigger than $\pi/8$.

For the next lemma recall the definition of the curves $\widetilde{\ell}_{(s,a,\bar{v}_k(\mu))}$ in (9.5) and of the parameter interval J_k in (9.3).

Lemma 9.3. *For every $K>0$ there is $k_0=k_0(K)\geqslant 1$ such that for every $\mu\in(-10,-9)$ and every $k\geqslant k_0$ the angles between*

- *the lines parallel to the x-axis in $\mathbb{R}^2$ and the curves $\pi_{1,2}(\widetilde{\sigma}^{\mathrm{uu}}_{K,\bar{v}_k(\mu)})$ are at least* $\arctan(K)$,
- *the curves $\big(\widetilde{\ell}_{(s,a,\bar{v}_k(\mu))}\big)_{a\in[-a_P,a_P],\, s\in J_k}$ and $\pi_{1,2}(\widetilde{\sigma}^{\mathrm{s}}_{\bar{v}_k(\mu)})$ are at least $\frac{\pi}{8}$,*
- *the lines parallel to the x-axis in $\mathbb{R}^2$ and $\pi_{1,2}(\widetilde{\sigma}^{\mathrm{s}}_{\bar{v}_k(\mu)})$ are at most K^{-1},*
- *the curves $\big(\widetilde{\ell}_{(s,a,\bar{v}_k(\mu))}\big)_{a\in[-a_P,a_P],\, s\in J_k}$ and the curves $\pi_{1,2}(\widetilde{\sigma}^{\mathrm{uu}}_{K,\bar{v}_k(\mu)})$ are at most K^{-1}.*

Proof. The first three items of the lemma follow from the discussion before the lemma. The last item follows from Lemma 9.1 and Remark 9.3. □

9.4. **Estimates of angles and expansion for iterations of $F_{\bar{v}_k(\mu)}$.** By the formula of Φ_k in Remark 7.1, we have that $D_w\Phi_k$ is a diagonal matrix that does not depend on the point w. Hence we will omit this dependence. Thus, after identifying the tangent spaces in $U_{\widetilde{Y}}$ with $\mathbb{R}^3$, we have that

$$\begin{aligned}||D_wF_{\bar{v}_k(\mu)}(v)|| &= ||D\Phi_k^{-1}\big(D_{\Phi_k(w)}f^{N_2+m_k+N_1+n_k}_{\bar{v}_k(\mu)}(D\Phi_k)(v)\big)||\\ &= ||D_{\Phi_k(w)}f^{N_2+m_k+N_1+n_k}_{\bar{v}_k(\mu)}(v)||.\end{aligned}$$

Observe also that the angles in Lemma 9.3 are taken with respect the coordinates in the xy-plane in $\mathbb{R}^3$. To get these angles in $U_{\widetilde{Y}}$ we need to replace each angle α by $\arctan(\sigma_P^{-m_k}\sigma_Q^{-n_k}\tan(\alpha))$.

We have the following consequence of Lemma 9.2. For that recall also the definition of the foliation $\widetilde{\mathcal{D}}_{\bar{v}_k(\mu)}$ in 9.5:

Lemma 9.4. *There is a constant $C>0$ such that for every $\mu\in(-10,-9)$ and every sufficiently large k the following holds:*

Take $K>0$, a point $w\in\Phi_k^{-1}(U_{\bar{v}_k(\mu)})$, and a unitary vector $v=(v_1,v_2,v_3)\in\mathbb{R}^3$ such that

$$|v_2|\geqslant\arctan(K). \tag{9.20}$$

Then

(1) $\|\pi_{1,2}\big(D_w F_{\bar{v}_k(\mu)}(v)\big)\| \geqslant C\,K\,\|v\|$,
(2) $\mathrm{angle}\big(D_w F_{\bar{v}_k(\mu)}(v), \Phi_k^{-1}(\mathcal{F}^{\mathrm{u}}_{P,\bar{v}_k(\mu)})\big) = \arctan\big(O(\lambda_P^{m_k}\sigma_Q^{2n_k}\sigma_P^{m_k})\big) \to 0$ *as* $k \to \infty$, *and*
(3) $\mathrm{angle}\big(D_w F_{\bar{v}_k(\mu)}(v), \widetilde{\mathcal{D}}_{\bar{v}_k(\mu)}\big) \to 0, \quad$ *as* $k \to \infty$.

Proof. The first two items of the lemma are direct translations of the corresponding items of Lemma 9.2. For the convergence to zero in the second item we use (6.3). The third item follows from (9.19) and the definition of Φ_k. □

9.5. **End of the proof of Proposition 9.1.** Using the local coordinates in U_Q, fixed small $\delta > 0$ consider the neighbourhood

$$U_X(\delta) \stackrel{\mathrm{def}}{=} [-\delta,\delta] \times [1-\delta, 1+\delta] \times [-\delta,\delta]$$

of the heteroclinic point $X = (0,1,0)$ and the two-disc (see Figure 8)

$$S(\delta) \stackrel{\mathrm{def}}{=} [-\delta,\delta] \times \{1-\delta\} \times [-\delta,\delta] \subset \partial U_X(\delta).$$

For large $k > 0$ and $\mu \in (-10,-9)$ define the set

$$\widehat{S}_{\bar{v}_k(\mu)}(\delta) \stackrel{\mathrm{def}}{=} C\Big(f^{-n_k}_{\bar{v}_k(\mu)}(0,1-\delta,0), f^{-n_k}_{\bar{v}_k(\mu)}(S(\delta)) \cap U_Q\Big),$$

recall that $C(x,A)$ is the connected component of the set A containing the point x. We finally let

$$\widetilde{S}_{\bar{v}_k(\mu)}(\delta) \stackrel{\mathrm{def}}{=} \widehat{S}_{\bar{v}_k(\mu)}(\delta) \cap U_{\widetilde{Y}} \quad \text{and} \quad S_{\bar{v}_k(\mu)}(\delta) \stackrel{\mathrm{def}}{=} \Phi_k^{-1}(\widetilde{S}_{\bar{v}_k(\mu)}(\delta)) \subset \mathbb{R}^3.$$

Lemma 9.5. *For every* $\delta > 0$ *there is* k_0 *such that for every* $k \geqslant k_0$ *and every* $\mu \in (-10,-9)$ *it holds*

$$W^{\mathrm{uu}}(P^+_{\bar{v}_k(\mu)}, F_{\bar{v}_k(\mu)}) \pitchfork S_{\bar{v}_k(\mu)}(\delta) \neq \emptyset.$$

Proof. Let $C > 0$ be the constant in Lemma 9.4 and take large $K > 0$ with $C\,K = \tau > 1$. Applying Lemma 9.3 to K, we get k_0 such that the angles between the lines parallel to the x-axis in $\mathbb{R}^2$ and $\pi_{1,2}(\widetilde{\sigma}^{\mathrm{uu}}_{K,\bar{v}_k(\mu)})$ are at least $\arctan(K)$. Thus, after increasing k_0, we get the angular condition (9.20) in Lemma 9.4. Hence

$$\mathrm{lenght}\big(F_{\bar{v}_k(\mu)}(\widetilde{\sigma}^{\mathrm{uu}}_{K,\bar{v}_k(\mu)})\big) \geqslant \tau\,\mathrm{lenght}\big(\widetilde{\sigma}^{\mathrm{uu}}_{K,\bar{v}_k(\mu)}\big).$$

The curve $F_{\bar{v}_k(\mu)}(\widetilde{\sigma}^{\mathrm{uu}}_{K,\bar{v}_k(\mu)})$ is contained in the strong unstable separatrix $\sigma^{\mathrm{uu}}_{\bar{v}_k(\mu)}$ of $W^{\mathrm{uu}}(P^+_{\bar{v}_k(\mu)})$. By construction, the curve $F_{\bar{v}_k(\mu)}(\widetilde{\sigma}^{\mathrm{uu}}_{K,\bar{v}_k(\mu)})$ also satisfies the angular condition (9.20). The proof now follows inductively: the curves $F_k^n(\widetilde{\sigma}^{\mathrm{uu}}_{K,\bar{v}_k(\mu)})$ are contained in $\sigma^{\mathrm{uu}}_{\bar{v}_k(\mu)}$ and their lengths grow exponentially. This implies that this separatrix transversally intersects the two-disc $S_{\bar{v}_k(\mu)}(\delta)$. □

Note that, for every sufficiently large k, there are defined the continuations $Z^{\pm}_{\bar{v}_k(\mu)} = Z^{\pm}_{\varepsilon,\bar{v}_k(\mu)}$ of the transverse homoclinic point $Z^{\pm}_{\varepsilon}$ of Q in Proposition 4.1. These points are transverse homoclinic points of Q for $f_{\bar{v}_k(\mu)}$ and (in the coordinates in U_Q) are of the form

$$Z^{\pm}_{\bar{v}_k(\mu)} = (0, 1 \pm \zeta^{\pm}_{\bar{v}_k(\mu)}, 0), \qquad \zeta^{\pm}_{\bar{v}_k(\mu)} \in (0,\delta).$$

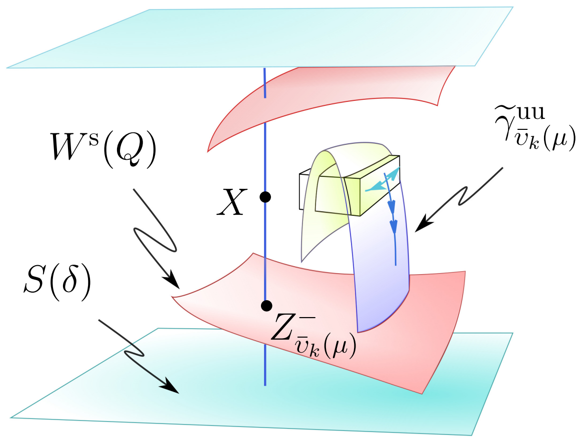

FIGURE 8. Two-dimensional connection between the blender and the saddle Q.

For each k and μ, we can also consider a disc $W^{\pm}_{\bar{v}_k(\mu)}$ in $W^{\mathrm{s}}(Q, f_{\bar{v}_k(\mu)})$ centred at $Z^{\pm}_{\bar{v}_k(\mu)}$ which has uniform size and is uniformly transverse to $W^{\mathrm{u}}_{\mathrm{loc}}(Q, f_{\bar{v}_k(\mu)})$, meaning that the angles between $W^{\pm}_{\bar{v}_k(\mu)}$ and $W^{\mathrm{u}}_{\mathrm{loc}}(Q, f_{\bar{v}_k(\mu)})$ at $Z^{\pm}_{\bar{v}_k(\mu)}$ are uniformly bounded from below.

We are now ready to conclude the proof of Proposition 9.1. Let $\gamma^{\mathrm{uu}}_{\bar{v}_k(\mu)}$ be the segment of $\sigma^{\mathrm{uu}}_{\bar{v}_k(\mu)}$ joining $P^{+}_{\bar{v}_k(\mu)}$ and $S_{\bar{v}_k(\mu)}(\delta)$ and consider (see Figure 8)

$$\widetilde{\gamma}^{\mathrm{uu}}_{\bar{v}_k(\mu)} \stackrel{\mathrm{def}}{=} f^{n_k}_{\bar{v}_k(\mu)}(\Phi_k(\gamma^{\mathrm{uu}}_{\bar{v}_k(\mu)})).$$

Consider the domain $\Delta_k(\mu)$ of the blender $\Upsilon_{\bar{v}_k(\mu)}$ in (7.3). The calculations in [16, Step A, eq. (35)] imply that for every large k the coordinates $(x, 1+y, z) \in U_Q$ of the points in $f^{n_k}_{\bar{v}_k(\mu)}(\Delta_k(\mu))$ are close to $X = (0,1,0)$ and they have Landau symbols

$$x = O({\lambda_Q}^{n_k}), \quad y = O(\sigma_P^{-2m_k}\, {\sigma_Q}^{-n_k}), \quad z = O({\lambda_Q}^{n_k}).$$

Hence $f^{n_k}_{\bar{v}_k(\mu)}(P^{+}_{\bar{v}_k(\mu)})$ converges to X as $k \to \infty$. It follows now from Lemma 9.5 that the curves $\widetilde{\gamma}^{\mathrm{uu}}_{\bar{v}_k(\mu)}$ accumulate to the disc

$$\{0\} \times [1-\delta, 1] \times \{0\} \subset W^{\mathrm{u}}_{\mathrm{loc}}(Q, f^{n_k}_{\bar{v}_k(\mu)}).$$

This implies that for sufficiently large k the curve $\widetilde{\gamma}^{\mathrm{uu}}_{\bar{v}_k(\mu)}$ transversely intersects the disc $W^-_{\varepsilon,\bar{v}_k(\mu)} \subset W^{\mathrm{s}}_{\mathrm{loc}}(Q, f_{\bar{v}_k(\mu)})$. As the curve $\widetilde{\gamma}^{\mathrm{uu}}_{\bar{v}_k(\mu)}$ is contained in the unstable manifold of the blender this ends the proof of Proposition 9.1.

10. Proof of Theorem 1.1: Intersections between one-dimensional manifolds

We now prove that the stable manifold the blender $\Upsilon_{\varepsilon,k,\mu}$ of $g_{\varepsilon,k,\mu}$ robustly intersects the unstable manifold of the saddle Q.

Proposition 10.1. *For every small $\varepsilon > 0$, large k, and $\mu \in (-10, -9)$, there is a C^r neighbourhood $\mathcal{V}_{\varepsilon,k,\mu}$ of $g_{\varepsilon,k,\mu}$ consisting of diffeomorphisms g such that*

$$W^{\mathrm{s}}\big(\Upsilon_g, g\big) \cap W^{\mathrm{u}}\big(Q_g, g\big) \neq \emptyset,$$

where Υ_g and Q_g are the continuations of $\Upsilon_{\varepsilon,k,\mu}$ and Q. Moreover, this intersection can be chosen quasi-transverse.

By Lemmas 5.1 and 5.2 this proposition follows from the next result:

Lemma 10.1. *For every small $\varepsilon > 0$, large k, and $\mu \in (-10, -9)$, the unstable manifold $W^{\mathrm{u}}(Q, g_{\varepsilon,k,\mu})$ contains a uu-disc in the superposition region of $\Upsilon_{\varepsilon,k,\mu}$.*

10.1. **Proof of Lemma 10.1.** We import some ingredients from Section 5.3. Consider the disc L in (5.9). By Remark 7.4, for sufficiently large k the set

$$\Phi_k^{-1} \circ \mathcal{R}_{\varepsilon,k,\mu}(g_{\varepsilon,k,\mu}) \circ \Phi_k\big(L\big) \subset \mathbb{R}^3$$

contains a disc in the superposition region of the blender $\Lambda_{\varepsilon,k,\mu}$ in (7.5).

Take small $\delta = \delta(\varepsilon) > 0$ and the segment $L^{\mathrm{u}}_{1,\varepsilon} \stackrel{\mathrm{def}}{=} L^{\mathrm{u}}_{1,\varepsilon}(\delta) \subset W^{\mathrm{u}}(Q, f_\varepsilon)$ in (4.12). By the definition of $g_{\varepsilon,k,\mu}$, this disc is contained in $W^{\mathrm{u}}(Q, g_{\varepsilon,k,\mu})$. Thus

$$J^{\mathrm{u}}_{\varepsilon,k,\mu} \stackrel{\mathrm{def}}{=} g^{N_1}_{\varepsilon,k,\mu}\big(L^{\mathrm{u}}_{1,\varepsilon}\big) \subset W^{\mathrm{u}}(Q, g_{\varepsilon,k,\mu}). \tag{10.1}$$

Note that the transition $g^{N_1}_{\varepsilon,k,\mu}$ does not depend on μ, thus in what follows we will omit the dependence on μ of the sets $J^{\mathrm{u}}_{\varepsilon,k,\mu}$ writing just $J^{\mathrm{u}}_{\varepsilon,k}$. We will prove that there is a compact subdisc $\widehat{J}^{\mathrm{u}}_{\varepsilon,k}$ of $J^{\mathrm{u}}_{\varepsilon,k}$ such that the C^r distance between the discs

$$\Phi_k^{-1} \circ g^{N_2+m_k}_{\varepsilon,k,\mu}(\widehat{J}^{\mathrm{u}}_{\varepsilon,k}) \quad \text{and} \quad \Phi_k^{-1} \circ \mathcal{R}_{\varepsilon,k,\mu}(g_{\varepsilon,k,\mu}) \circ \Phi_k(L)$$

goes to zero as $k \to \infty$. As the latter set contains a disc in the superposition region of the blender, this implies the lemma (see Figure 9). We now go to the details of the proof.

Recall the definition of $\Phi_k = \Psi_k \circ \Theta_{\bar{\varsigma}}$ in (7.2) and consider the parameterisation of $\Phi_k(L)$ given by

$$\gamma_k : [-4, 4] \to M, \quad \gamma_k(t) \stackrel{\mathrm{def}}{=} \Psi_k \circ \Theta_{\bar{\varsigma}}(0, t, 0) = \Psi_k(0, \varsigma_2^{-1} t, 0).$$

We will provide a parameterisation $\gamma_{\varepsilon,k} : [-4, 4] \to M$ of $\widehat{J}^{\mathrm{u}}_{\varepsilon,k}$ such that

$$\lim_{k\to\infty} \big\| \big(\Phi_k^{-1} \circ g^{N_2+m_k+N_1+n_k}_{\varepsilon,k,\mu} \circ \gamma_k - \Phi_k^{-1} \circ g^{N_2+m_k}_{\varepsilon,k,\mu} \circ \gamma_{\varepsilon,k}\big)|_{[-4,4]} \big\|_r = 0. \tag{10.2}$$

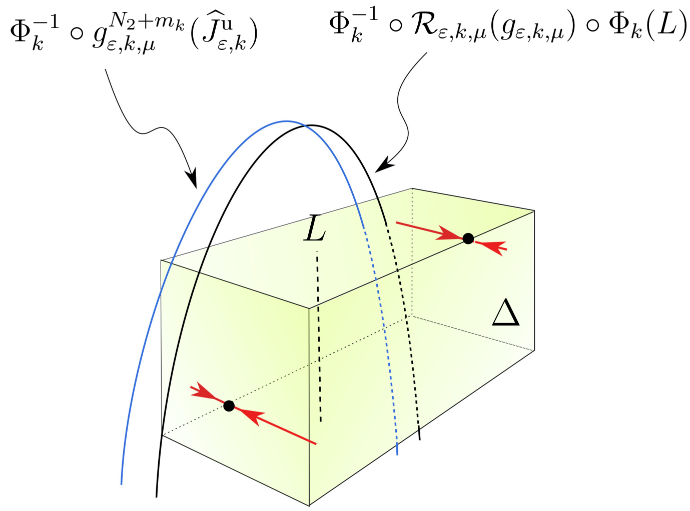

FIGURE 9. One-dimensional connection between the blender and the saddle Q.

As the set $\Psi_k^{-1} \circ g_{\varepsilon,k,\mu}^{N_2+m_k+N_1+n_k} \circ \gamma_k([-4,4])$ contains a disc in the superposition region of the blenders (see Remark 7.4) this proves Lemma 10.1.

To get equation (10.2) we observe that

$$\begin{aligned}&\big\| \Phi_k^{-1} \circ g_{\varepsilon,k,\mu}^{N_2+m_k+N_1+n_k} \circ \gamma_k - \Phi_k^{-1} \circ g_{\varepsilon,k,\mu}^{N_2+m_k} \circ \gamma_{\varepsilon,k} \big\|_r = \\ &\quad \big\| \Theta_{\bar\varsigma}^{-1} \circ \Psi_k^{-1} \circ g_{\varepsilon,k,\mu}^{N_2+m_k+N_1+n_k} \circ \gamma_k - \Theta_{\bar\varsigma}^{-1} \circ \Psi_k^{-1} \circ g_{\varepsilon,k,\mu}^{N_2+m_k} \circ \gamma_{\varepsilon,k} \big\|_r \leqslant \\ &\quad C \big\| \Psi_k^{-1} \circ g_{\varepsilon,k,\mu}^{N_2+m_k+N_1+n_k} \circ \gamma_k - \Psi_k^{-1} \circ g_{\varepsilon,k,\mu}^{N_2+m_k} \circ \gamma_{\varepsilon,k} \big\|_r,\end{aligned}$$

where C is an upper bound of the C^r norm of $\Theta_{\bar\varsigma}^{-1}$ in the cube Δ. Thus to conclude the proof of the proposition it is enough to prove the following:

Lemma 10.2. *There is a parameterisation $\gamma_{\varepsilon,k} : [-4,4] \to M$ of $\widehat{J}^{\mathrm{u}}_{\varepsilon,k}$ such that*

$$\lim_{k\to\infty} \big\| \big(\Psi_k^{-1} \circ g_{\varepsilon,k,\mu}^{N_2+m_k+N_1+n_k} \circ \gamma_k - \Psi_k^{-1} \circ g_{\varepsilon,k,\mu}^{N_2+m_k} \circ \gamma_{\varepsilon,k}\big)|_{[-4,4]} \big\|_r = 0.$$

We begin by giving the explicit parameterisation of $\gamma_{\varepsilon,k}$ of $\widehat{J}^{\mathrm{u}}_{\varepsilon,k}$.

10.1.1. *The parameterisations* $\gamma_{\varepsilon,k}$. We first write the segment $J^{\mathrm{u}}_{\varepsilon,k}$ in (10.1) in local coordinates. Recalling the definitions of $L^{\mathrm{u}}_{1,\varepsilon}$ in (4.10), $\mathfrak{T}_{1,1,\varepsilon} = f_\varepsilon^{N_1}|_{U_{1,\varepsilon}}$ in (4.8), and $\theta_{\varepsilon,k}$ in (7.8) (where $\bar\tau_k$ is defined), we have

$$\begin{aligned}J^{\mathrm{u}}_{\varepsilon,k} = \big\{ &\widetilde{X}_{1,\varepsilon} + A\big(t\,\mathbf{e}_2 + \bar\rho_{1,\varepsilon}(t)\big) + \widetilde{H}^1_\varepsilon\big(t\,\mathbf{e}_2 + \bar\rho_{1,\varepsilon}(t)\big) + \\ &+ \Pi_\delta\big(\, A\big(t\,\mathbf{e}_2 + \bar\rho_{1,\varepsilon}(t)\big) + \widetilde{H}^1_\varepsilon\big(t\,\mathbf{e}_2 + \bar\rho_{1,\varepsilon}(t)\big)\,\big)\bar\tau_k : |t| < \delta = \delta(\varepsilon)\big\}.\end{aligned}$$

Using (4.11), (4.9), $\bar\rho_{1,\varepsilon}(0) = 0$, and Remark 4.5, we get $\widehat{\delta} \in (0,\delta)$ such that for every $|t| < \widehat{\delta}$ it holds

$$A\big(t\,\mathbf{e}_2 + \bar\rho_{1,\varepsilon}(t)\big) + \widetilde{H}^1_\varepsilon\big(t\,\mathbf{e}_2 + \bar\rho_{1,\varepsilon}(t)\big) \in B(\mathbf{0},\delta).$$

As the map Π_δ is igual to 1 in $B(\mathbf{0},\delta)$ (recall (3.2)), we can consider the subdisc

$$\widetilde{J}^{\mathrm{u}}_{\varepsilon,k} \stackrel{\mathrm{def}}{=} \big\{ \widetilde{X}_{1,\varepsilon} + A\big(t\,\mathbf{e}_2 + \bar\rho_{1,\varepsilon}(t)\big) + \widetilde{H}^1_\varepsilon\big(t\,\mathbf{e}_2 + \bar\rho_{1,\varepsilon}(t)\big) + \bar\tau_k : |t| \leqslant \widehat{\delta}\,\big\} \subset J^{\mathrm{u}}_{\varepsilon,k}.$$

For convenience, we write $\widetilde{J}^{\mathrm{u}}_{\varepsilon,k}$ in the following compact form:

$$\widetilde{J}^{\mathrm{u}}_{\varepsilon,k} = \big\{ \widetilde{X}_{1,\varepsilon} + t\,A(\mathbf{v}_{1,\varepsilon}) + \widetilde{\rho}_{1,\varepsilon}(t) + \bar\tau_k : |t| \leqslant \widehat{\delta}\,\big\},$$

where

$$\begin{aligned}\mathbf{v}_{1,\varepsilon} &\stackrel{\mathrm{def}}{=} \mathbf{e}_2 + A^{-1} D\widetilde{H}^1_\varepsilon(\mathbf{0})\,\mathbf{e}_2,\\ \widetilde{\rho}_{1,\varepsilon}(t) &\stackrel{\mathrm{def}}{=} A\big(\bar\rho_{1,\varepsilon}(t)\big) + \widetilde{H}^1_\varepsilon\big(t\,\mathbf{e}_2 + \bar\rho_{1,\varepsilon}(t)\big) - t\,D\widetilde{H}^1_\varepsilon(\mathbf{0})\,\mathbf{e}_2.\end{aligned}$$

Note that, by (4.11), we have that

$$\frac{d}{dt}(\widetilde{\rho}_{1,\varepsilon})(0) = \widetilde{\rho}_{1,\varepsilon}(0) = \mathbf{0}.$$

We now consider the subdisc $\widehat{J}^{\mathrm{u}}_{\varepsilon,k}$ of $\widetilde{J}^{\mathrm{u}}_{\varepsilon,k}$ obtained by rescaling the parameter t by the factor $\sigma_P^{-2m_k}\sigma_Q{}^{-n_k}\,|\,\varsigma_2^{-1}| \ll \frac{\widehat{\delta}}{4} < 1$ as follows:

$$\widehat{J}^{\mathrm{u}}_{\varepsilon,k} \stackrel{\mathrm{def}}{=} \big\{ \widetilde{X}_{1,\varepsilon} + \sigma_P^{-2m_k}\sigma_Q{}^{-n_k}\,\varsigma_2^{-1}\,t\,A(\mathbf{v}_{1,\varepsilon}) + \widehat{\rho}_{1,\varepsilon,k}(t) + \bar\tau_k : |t| \leqslant 4\big\},$$

where

$$\widehat{\rho}_{1,\varepsilon,k}(t) \stackrel{\mathrm{def}}{=} \widetilde{\rho}_{1,\varepsilon}\big(\sigma_P^{-2m_k}\sigma_Q{}^{-n_k}\,\varsigma_2^{-1}\,t\big).$$

To rewrite the set $\widehat{J}^{\mathrm{u}}_{\varepsilon,k}$ in a compact form, let

$$\begin{aligned}(\widetilde{w}^{1,\varepsilon}_1, \widetilde{w}^{1,\varepsilon}_2, \widetilde{w}^{1,\varepsilon}_3) &\stackrel{\mathrm{def}}{=} D\widetilde{H}^1_\varepsilon(\mathbf{0})\,\mathbf{e}_2,\\ \widehat{\rho}^{\ell}_{1,\varepsilon,k}(t) &\stackrel{\mathrm{def}}{=} \widetilde{\rho}^{\ell}_{1,\varepsilon}\big(\sigma_P^{-2m_k}\sigma_Q{}^{-n_k}\,\varsigma_2^{-1}\,t\big), \qquad \ell = 1,2,3,\end{aligned} \tag{10.3}$$

where $\widetilde{\rho}^{\ell}_{1,\varepsilon}$ is the ℓ-th coordinate of $\widetilde{\rho}_{1,\varepsilon}$.

Remark 10.2. Remark 4.5 implies that $(\widetilde{w}^{1,\varepsilon}_1, \widetilde{w}^{1,\varepsilon}_2, \widetilde{w}^{1,\varepsilon}_3) \to (0,0,0)$ as $\varepsilon \to 0$.

Recalling that $\widetilde{X}_{1,\varepsilon} = (1 + \widetilde{x}_{1,\varepsilon}, 0, 0)$, see (4.6) and (4.7), and the definitions of A in (2.4) and of $\bar\tau_k$ in (7.7), we can write

$$\widehat{J}^{\mathrm{u}}_{\varepsilon,k} = \big\{ \big(1 + \widetilde{x}_{1,\varepsilon} + x_{\varepsilon,k}(t), y_{\varepsilon,k}(t), z_{\varepsilon,k}(t)\big) : |t| \leqslant 4\big\},$$

where

$$\begin{aligned}x_{\varepsilon,k}(t) &\stackrel{\mathrm{def}}{=} \sigma_P^{-2m_k}\sigma_Q{}^{-n_k}\,\big(\alpha_2 + \widetilde{w}^{1,\varepsilon}_1\big)\,\varsigma_2^{-1}\,t + \widehat{\rho}^1_{1,\varepsilon,k}(t),\\ y_{\varepsilon,k}(t) &\stackrel{\mathrm{def}}{=} \sigma_P^{-2m_k}\sigma_Q{}^{-n_k}\,\big(\beta_2 + \widetilde{w}^{1,\varepsilon}_2\big)\,\varsigma_2^{-1}\,t + \widehat{\rho}^2_{1,\varepsilon,k}(t)\\ &\qquad + \sigma_P^{-m_k}(\tilde{\mathfrak{c}}_k + \tilde{\mathfrak{s}}_k),\\ z_{\varepsilon,k}(t) &\stackrel{\mathrm{def}}{=} \sigma_P^{-2m_k}\sigma_Q{}^{-n_k}\,\widetilde{w}^{1,\varepsilon}_3\,\varsigma_2^{-1}\,t + \widehat{\rho}^3_{1,\varepsilon,k}(t) + \sigma_P^{-m_k}(\tilde{\mathfrak{c}}_k - \tilde{\mathfrak{s}}_k).\end{aligned} \tag{10.4}$$

The announced parameterisation of $\widehat{J}^{\mathrm{u}}_{\varepsilon,k}$ is given by

$$\gamma_{\varepsilon,k} : [-4,4] \to M, \quad \gamma_{\varepsilon,k}(t) = (1 + \widetilde{x}_{1,\varepsilon} + x_{\varepsilon,k}(t), y_{\varepsilon,k}(t), z_{\varepsilon,k}(t)). \tag{10.5}$$

10.1.2. *End of the proof of Lemma 10.2 (thus of Proposition 10.1).* We calculate separately the two terms in the lemma. This involves some explicit calculations in the renormalisation scheme borrowed from [16] which are stated in Section 13.

• *The term* $\Psi_k^{-1}\circ g_{\varepsilon,k,\mu}^{N_2+m_k+N_1+n_k}\big(\gamma_k(t)\big)$. Write

$$(\overline{x}_{\varepsilon,k,\mu}(t),\overline{y}_{\varepsilon,k,\mu}(t),\overline{z}_{\varepsilon,k,\mu}(t)) \stackrel{\text{def}}{=} \Psi_k^{-1}\circ g_{\varepsilon,k,\mu}^{N_2+m_k+N_1+n_k}\big(\gamma_k(t)\big) = \Psi_k^{-1}\circ g_{\varepsilon,k,\mu}^{N_2+m_k+N_1+n_k}\circ \Psi_k(\Theta_{\bar\varsigma}(0,t,0)). \tag{10.6}$$

Using the formula in (13.1), replacing $f_{\bar{\upsilon}_k(\mu),\rho}$ by $g_{\varepsilon,k,\mu}=\theta_{\varepsilon,k}\circ f_{\varepsilon,\bar{\upsilon}_k(\mu),\rho(\varepsilon)}$ (this leads to a dependence on ε of the next expressions), and considering the curve $\Theta_{\bar\varsigma}(0,t,0)=(0,\varsigma_2^{-1}t,0)$ (see (5.5)), from equation (13.3) we get

$$\overline{x}_{\varepsilon,k,\mu}(t) = \Big(a_1\,\lambda_P^{m_k}\,\sigma_P^{-m_k}\,\alpha_2 + \big(\tilde{\mathfrak{c}}_k\,a_2+\tilde{\mathfrak{s}}_k\,a_3\big)\,\beta_2\Big)\varsigma_2^{-1}\,t + {\sigma_P}^{m_k}\,\sigma_Q^{n_k}\,\mathrm{hot}^x_{\varepsilon,k,\mu}(t),$$

$$\overline{y}_{\varepsilon,k,\mu}(t) = \varsigma_2^{-1}\,\mu + b_1\,\lambda_P^{m_k}\,{\sigma_Q}^{n_k}\,\alpha_2\,\varsigma_2^{-1}\,t + \Big(\tilde{\mathfrak{c}}_k^2\,b_2+\tilde{\mathfrak{s}}_k^2\,b_3+\tilde{\mathfrak{c}}_k\,\tilde{\mathfrak{s}}_k\,b_4\Big)\,\beta_2^2\,\varsigma_2^{-2}\,t^2 + {}$$
$$+\,{\sigma_P}^{2m_k}\,\sigma_Q^{2n_k}\,\mathrm{hot}^y_{\varepsilon,k,\mu}(t),$$

$$\overline{z}_{\varepsilon,k,\mu}(t) = \Big(c_1\,\lambda_P^{m_k}\,\sigma_P^{-m_k}\,\alpha_2 + \big(\tilde{\mathfrak{c}}_k\,c_2+\tilde{\mathfrak{s}}_k\,c_3\big)\,\beta_2\Big)\,\varsigma_2^{-1}\,t + {\sigma_P}^{m_k}\,\sigma_Q^{n_k}\,\mathrm{hot}^z_{\varepsilon,k,\mu}(t),$$

where $\mathrm{hot}^*_{\varepsilon,k,\mu}(t)$, $*=x,y,z$, are high order terms[10].

Remark 10.3 (Lemma 8.3 in [16])**.** The terms

$${\sigma_P}^{m_k}\,\sigma_Q^{n_k}\,\mathrm{hot}^x_{\varepsilon,k,\mu}(t),\quad {\sigma_P}^{2m_k}\,\sigma_Q^{2n_k}\,\mathrm{hot}^y_{\varepsilon,k,\mu}(t),\quad {\sigma_P}^{m_k}\,\sigma_Q^{n_k}\,\mathrm{hot}^z_{\varepsilon,k,\mu}(t),$$

go to zero in the C^r topology as k goes to infinity.

• *The term* $\Psi_k^{-1}\circ g_{\varepsilon,k,\mu}^{N_2+m_k}\big(\gamma_{\varepsilon,k}(t)\big)$.

Recalling the parameterisation of $\gamma_{\varepsilon,k}(t)$ in (10.5), write

$$(\widetilde{x}_{\varepsilon,k,\mu}(t),\widetilde{y}_{\varepsilon,k,\mu}(t),\widetilde{z}_{\varepsilon,k,\mu}(t)) \stackrel{\text{def}}{=} \Psi_k^{-1}\circ g_{\varepsilon,k,\mu}^{N_2+m_k}\big(\gamma_{\varepsilon,k}(t)\big) = \Psi_k^{-1}\circ g_{\varepsilon,k,\mu}^{N_2+m_k}\big(1+\widetilde{x}_{1,\varepsilon}+x_{\varepsilon,k}(t),y_{\varepsilon,k}(t),z_{\varepsilon,k}(t)\big). \tag{10.7}$$

Using equation (10.4) and the linearity of $g_{\varepsilon,k,\mu}$ in U_P, we get

$$g_{\varepsilon,k,\mu}^{m_k}\big(1+\widetilde{x}_{1,\varepsilon}+x_{\varepsilon,k}(t),y_{\varepsilon,k}(t),z_{\varepsilon,k}(t)\big) \stackrel{\text{def}}{=} \big(\widehat{x}_{\varepsilon,k}(t),1+\widehat{y}_{\varepsilon,k}(t),1+\widehat{z}_{\varepsilon,k}(t)\big),$$

where

$$\widehat{x}_{\varepsilon,k}(t) = {\lambda_P}^{m_k}\,(1+\widetilde{x}_{1,\varepsilon}) + {\lambda_P}^{m_k}\sigma_P^{-2m_k}{\sigma_Q}^{-n_k}\,\big(\alpha_2+\widetilde{w}_1^{1,\varepsilon}\big)\,\varsigma_2^{-1}\,t$$
$$+\,{\lambda_P}^{m_k}\widehat{\rho}^1_{1,\varepsilon,k}(t),$$
$$\widehat{y}_{\varepsilon,k}(t) = \big(\tilde{\mathfrak{c}}_k(\beta_2+\widetilde{w}_2^{1,\varepsilon}) - \widetilde{w}_3^{1,\varepsilon}\,\tilde{\mathfrak{s}}_k\big)\,\sigma_P^{-m_k}\,{\sigma_Q}^{-n_k}\,\varsigma_2^{-1}\,t + {\sigma_P}^{m_k}\,u_{\varepsilon,k}(t),$$
$$\widehat{z}_{\varepsilon,k}(t) = \big(\tilde{\mathfrak{s}}_k(\beta_2+\widetilde{w}_2^{1,\varepsilon}) + \widetilde{w}_3^{1,\varepsilon}\,\tilde{\mathfrak{c}}_k\big)\,\sigma_P^{-m_k}\,{\sigma_Q}^{-n_k}\,\varsigma_2^{-1}\,t + {\sigma_P}^{m_k}\,v_{\varepsilon,k}(t),$$

[10] In [16] these high order terms are denoted by $\mathrm{h.o.t.}^*$, $\mathrm{h.o.t.}^{**}$, $\mathrm{h.o.t.}^{***}$.

with

$$u_{\varepsilon,k}(t) \stackrel{\text{def}}{=} \tilde{\mathfrak{c}}_k\, \widehat{\rho}^2_{1,\varepsilon,k}(t) - \tilde{\mathfrak{s}}_k\, \widehat{\rho}^3_{1,\varepsilon,k}(t),$$
(10.8)
$$v_{\varepsilon,k}(t) \stackrel{\text{def}}{=} \tilde{\mathfrak{s}}_k\, \widehat{\rho}^2_{1,\varepsilon,k}(t) + \tilde{\mathfrak{c}}_k\, \widehat{\rho}^3_{1,\varepsilon,k}(t).$$

Recalling the expressions of $\widehat{\rho}^2_{1,\varepsilon,k}(t)$, $\widehat{\rho}^3_{1,\varepsilon,k}(t)$ in (10.3), for $i = 2, 3$ we get

(10.9) $$O(u_{\varepsilon,k}(t)) = O(v_{\varepsilon,k}(t)) = O(\widehat{\rho}^i_{1,\varepsilon,k}(t)) = O(\sigma_P^{-4m} {\sigma_Q}^{-2n_k}).$$

Thus the Landau symbols of $\widehat{x}_{\varepsilon,k}(t), \widehat{y}_{\varepsilon,k}(t)$, and $\widehat{z}_{\varepsilon,k}(t)$ are of the form

$$\widehat{x}_{\varepsilon,k}(t) = O({\lambda_P}^{m_k}), \qquad \widehat{y}_{\varepsilon,k}(t) = O(\sigma_P^{-m_k} {\sigma_Q}^{-n_k}) = \widehat{z}_{\varepsilon,k}(t).$$

Note that

$$(\widetilde{x}_{\varepsilon,k,\mu}(t), \widetilde{y}_{\varepsilon,k,\mu}(t), \widetilde{z}_{\varepsilon,k,\mu}(t)) = \Psi_k^{-1} \circ g^{N_2}_{\varepsilon,k,\mu}\big(\widehat{x}_{\varepsilon,k}(t), 1 + \widehat{y}_{\varepsilon,k}(t), 1 + \widehat{z}_{\varepsilon,k}(t)\big).$$

Recalling the definitions of Ψ_k in (6.9) and of $f^{N_2}_{\bar{v}_k(\mu)}$ (see Remark 6.3) and that $g^{N_2}_{\varepsilon,k,\mu} = f^{N_2}_{\varepsilon,\bar{v}_k(\mu),\rho(\varepsilon)} = f^{N_2}_{\bar{v}_k(\mu)}$ in the neighbourhood $f^{-N_2}_{\bar{v}_k(\mu)}(B(\widetilde{Y}, \rho(\varepsilon)))$ of Y that we are considering, we get

$$\begin{aligned}
\widetilde{x}_{\varepsilon,k,\mu}(t) &= a_1\, {\lambda_P}^{m_k} \sigma_P^{m_k} {\sigma_Q}^{n_k}\, \widetilde{x}_{1,\varepsilon} \\
&\quad + \Big(a_1\, {\lambda_P}^{m_k} {\sigma_P}^{-m_k}\, (\alpha_2 + \widetilde{w}_1^{1,\varepsilon}) + a_2\big(\tilde{\mathfrak{c}}_k\, (\beta_2 + \widetilde{w}_2^{1,\varepsilon}) - \widetilde{w}_3^{1,\varepsilon}\, \tilde{\mathfrak{s}}_k\big) \\
&\quad + a_3\big(\tilde{\mathfrak{s}}_k\, (\beta_2 + \widetilde{w}_2^{1,\varepsilon}) + \widetilde{w}_3^{1,\varepsilon}\, \tilde{\mathfrak{c}}_k\big)\Big)\ \varsigma_2^{-1}\, t + \mathrm{Hot}^x_{\varepsilon,k,\mu}(t); \\
\widetilde{y}_{\varepsilon,k,\mu}(t) &= \varsigma_2^{-1}\, \mu + b_1\, {\lambda_P}^{m_k} \sigma_P^{2m_k} {\sigma_Q}^{2n_k}\, \widetilde{x}_{1,\varepsilon} + b_1\, {\lambda_P}^{m_k} \sigma_Q^{n_k}\, (\alpha_2 + \widetilde{w}_1^{1,\varepsilon})\, \varsigma_2^{-1}\, t \\
&\quad + \Big(b_2\, \big(\tilde{\mathfrak{c}}_k(\beta_2 + \widetilde{w}_2^{1,\varepsilon}) - \widetilde{w}_3^{1,\varepsilon}\, \tilde{\mathfrak{s}}_k\big)^2 + b_3\, \big(\tilde{\mathfrak{s}}_k(\beta_2 + \widetilde{w}_2^{1,\varepsilon}) + \widetilde{w}_3^{1,\varepsilon}\, \tilde{\mathfrak{c}}_k\big)^2 \\
&\quad + b_4\, \big(\tilde{\mathfrak{c}}_k(\beta_2 + \widetilde{w}_2^{1,\varepsilon}) - \widetilde{w}_3^{1,\varepsilon}\, \tilde{\mathfrak{s}}_k\big)\big(\tilde{\mathfrak{s}}_k(\beta_2 + \widetilde{w}_2^{1,\varepsilon}) + \widetilde{w}_3^{1,\varepsilon}\, \tilde{\mathfrak{c}}_k\big)\Big)\, \varsigma_2^{-2}\, t^2 \\
&\quad + \mathrm{Hot}^y_{\varepsilon,k,\mu}(t); \\
\widetilde{z}_{\varepsilon,k,\mu}(t) &= c_1\, {\lambda_P}^{m_k} \sigma_P^{m_k} {\sigma_Q}^{n_k}\, \widetilde{x}_{1,\varepsilon} \\
&\quad + \Big(c_1\, {\lambda_P}^{m_k} {\sigma_P}^{-m_k}\, (\alpha_2 + \widetilde{w}_1^{1,\varepsilon}) + c_2\big(\tilde{\mathfrak{c}}_k\, (\beta_2 + \widetilde{w}_2^{1,\varepsilon}) - \widetilde{w}_3^{1,\varepsilon}\, \tilde{\mathfrak{s}}_k\big) \\
&\quad + c_3\big(\tilde{\mathfrak{s}}_k\, (\beta_2 + \widetilde{w}_2^{1,\varepsilon}) + \widetilde{w}_3^{1,\varepsilon}\, \tilde{\mathfrak{c}}_k\big)\Big)\ \varsigma_2^{-1}\, t + \mathrm{Hot}^z_{\varepsilon,k,\mu}(t),
\end{aligned}$$

where $\mathrm{Hot}^*_{\varepsilon,k,\mu}(t)$, $* = x, y, z$, are high order terms. Their explicit expressions can be found in Section 13.2.

• *Comparing the terms in Lemma 10.2* We are now ready to estimate the difference between the coordinates of the points in (10.6) and (10.7). For that writing

$$\delta^w_{\varepsilon,k,\mu}(t) \stackrel{\text{def}}{=} \widetilde{w}_{\varepsilon,k,\mu}(t) - \overline{w}_{\varepsilon,k,\mu}(t), \quad w = x, y, z,$$

we obtain

$$\begin{aligned}
\delta^x_{\varepsilon,k,\mu}(t) &= a_1\, {\lambda_P}^{m_k} \sigma_P^{m_k} {\sigma_Q}^{n_k}\, \widetilde{x}_{1,\varepsilon} + \Big(a_1\, {\lambda_P}^{m_k} {\sigma_P}^{-m_k}\, \widetilde{w}_1^{1,\varepsilon} + a_2\big(\tilde{\mathfrak{c}}_k\, \widetilde{w}_2^{1,\varepsilon} - \tilde{\mathfrak{s}}_k\, \widetilde{w}_3^{1,\varepsilon}\big) \\
&\quad + a_3\big(\tilde{\mathfrak{s}}_k\, \widetilde{w}_2^{1,\varepsilon} + \tilde{\mathfrak{c}}_k\, \widetilde{w}_3^{1,\varepsilon}\big)\Big)\ \varsigma_2^{-1}\, t + \mathrm{Hot}^x_{\varepsilon,k,\mu}(t) - \sigma_P^{m_k}\, \sigma_Q^{n_k} \mathrm{hot}^x_k(t);
\end{aligned}$$

$$\begin{aligned}\delta^y_{\varepsilon,k,\mu}(t) &= b_1\,{\lambda_P}^{m_k}\sigma_P^{2m_k}{\sigma_Q}^{2n_k}\,\widetilde{x}_{1,\varepsilon} + b_1\,{\lambda_P}^{m_k}\sigma_Q^{n_k}\,\widetilde{w}_1^{1,\varepsilon}\,\varsigma_2^{-1}\,t\\ &+\Big(b_2\left[2\,\beta_2\,\widetilde{w}_2^{1,\varepsilon}\,\tilde{\mathfrak{c}}_k^2+(\widetilde{w}_2^{1,\varepsilon})^2\,\tilde{\mathfrak{c}}_k^2-2\,\tilde{\mathfrak{c}}_k\,\tilde{\mathfrak{s}}_k\,\widetilde{w}_3^{1,\varepsilon}(\beta_2+\widetilde{w}_2^{1,\varepsilon})+(\widetilde{w}_3^{1,\varepsilon})^2\,\tilde{\mathfrak{s}}_k^2\right]\\ &+b_3\left[2\,\beta_2\,\widetilde{w}_2^{1,\varepsilon}\,\tilde{\mathfrak{s}}_k^2+(\widetilde{w}_2^{1,\varepsilon})^2\,\tilde{\mathfrak{s}}_k^2+2\,\tilde{\mathfrak{c}}_k\,\tilde{\mathfrak{s}}_k\,\widetilde{w}_3^{1,\varepsilon}(\beta_2+\widetilde{w}_2^{1,\varepsilon})+(\widetilde{w}_3^{1,\varepsilon})^2\,\tilde{\mathfrak{c}}_k^2\right]\\ &+b_4\left[2\,\beta_2\,\widetilde{w}_2^{1,\varepsilon}\,\tilde{\mathfrak{s}}_k\,\tilde{\mathfrak{c}}_k+(\widetilde{w}_2^{1,\varepsilon})^2\,\tilde{\mathfrak{s}}_k\,\tilde{\mathfrak{c}}_k\right.\\ &\left.+(\tilde{\mathfrak{c}}_k^2-\tilde{\mathfrak{s}}_k^2)\,\widetilde{w}_3^{1,\varepsilon}(\beta_2+\widetilde{w}_2^{1,\varepsilon})-(\widetilde{w}_3^{1,\varepsilon})^2\tilde{\mathfrak{s}}_k\,\tilde{\mathfrak{c}}_k\right]\Big)\,\varsigma_2^{-2}\,t^2\\ &+\mathrm{Hot}^y_{\varepsilon,k,\mu}(t)-\sigma_P^{2m_k}{\sigma_Q}^{2n_k}\,\mathrm{hot}^y_k(t);\end{aligned}$$

$$\begin{aligned}\delta^z_{\varepsilon,k,\mu}(t) &= c_1\,{\lambda_P}^{m_k}\sigma_P^{m_k}{\sigma_Q}^{n_k}\,\widetilde{x}_{1,\varepsilon}+\Big(c_1\,{\lambda_P}^{m_k}{\sigma_P}^{-m_k}\,\widetilde{w}_1^{1,\varepsilon}+c_2\big(\tilde{\mathfrak{c}}_k\,\widetilde{w}_2^{1,\varepsilon}-\tilde{\mathfrak{s}}_k\,\widetilde{w}_3^{1,\varepsilon}\big)\\ &+c_3\big(\tilde{\mathfrak{s}}_k\,\widetilde{w}_2^{1,\varepsilon}+\tilde{\mathfrak{c}}_k\,\widetilde{w}_3^{1,\varepsilon}\big)\Big)\,\varsigma_2^{-1}\,t+\mathrm{Hot}^z_{\varepsilon,k,\mu}(t)-\sigma_P^{m_k}\,\sigma_Q^{n_k}\mathrm{hot}^z_k(t).\end{aligned}$$

We now prove that $\left\|\delta^x_{\varepsilon,k,\mu}\,|_{[-4,4]}\right\|_r\to 0$. The proofs of $\left\|\delta^*_{\varepsilon,k,\mu}\,|_{[-4,4]}\right\|_r\to 0$, $*=y,z$, are similar and hence omitted. For that write

$$\delta^x_{\varepsilon,k,\mu}(t)=\mathcal{A}_{\varepsilon,k,\mu}(t)+\mathrm{Hot}^x_{\varepsilon,k,\mu}(t)-\sigma_P^{m_k}\,\sigma_Q^{n_k}\mathrm{hot}^x_k(t),$$

where $\mathcal{A}_{\varepsilon,k,\mu}(t)$ denotes the affine part of $\delta^x_{\varepsilon,k,\mu}(t)$.

Claim 10.4.

(1) $\lim_{k\to\infty}\left\|\,\mathcal{A}_{\varepsilon,k,\mu}\,|_{[-4,4]}\right\|_r=0$,
(2) $\lim_{k\to\infty}\left\|\sigma_P^{m_k}\,\sigma_Q^{n_k}\,\mathrm{hot}^x_{\varepsilon,k,\mu}\,|_{[-4,4]}\right\|_r=0$,
(3) $\lim_{k\to\infty}\left\|\mathrm{Hot}^x_{\varepsilon,k,\mu}\,|_{[-4,4]}\right\|_r=0$.

Clearly, this claim implies Lemma 10.2.

Proof of Claim 10.4. For the first item recall that by (6.3) we get $\lambda_P^{m_k}\sigma_P^{2m_k}{\sigma_Q}^{2n_k}\to 0$ and that by Remark 10.2 the norm of $(\widetilde{w}_1^{1,\varepsilon},\widetilde{w}_2^{1,\varepsilon},\widetilde{w}_3^{1,\varepsilon})$ is small. The second item was stated in Remark 10.3. The proof of last item is postponed to Section 13.2.1. □

The proof of Proposition 10.1 is now complete.

11. Proof of Theorem 1.1: homoclinic relations

We now prove that blender $\Upsilon_{\varepsilon,k,\mu}$ and the saddle P are homoclinically related.

Proposition 11.1. *For every small $\varepsilon>0$, large k, and $\mu\in(-10,-9)$ there is g arbitrarily C^r close to $g_{\varepsilon,k,\mu}$ such that*

$$W^{\mathrm{u}}\big(\Upsilon_g,g\big)\pitchfork W^{\mathrm{s}}\big(P,g\big)\neq\emptyset\quad\textit{and}\quad W^{\mathrm{s}}\big(\Upsilon_g,g\big)\pitchfork W^{\mathrm{u}}\big(P,g\big)\neq\emptyset.$$

Proof. Note that Propositions 9.1 and 10.1 hold for perturbations g of $g_{\varepsilon,k,\mu}$. Recall the definition of the quasi-transverse heteroclinic point $X_{2,\varepsilon}$ in Proposition 4.1 and observe that this intersection point is preserved by the local perturbations we have considered. Therefore $X_{2,\varepsilon}$ is also a heteroclinic point of $g_{\varepsilon,k,\mu}$. Note also that after a perturbation g of $g_{\varepsilon,k,\mu}$ (that does not destroy the heteroclinic points)

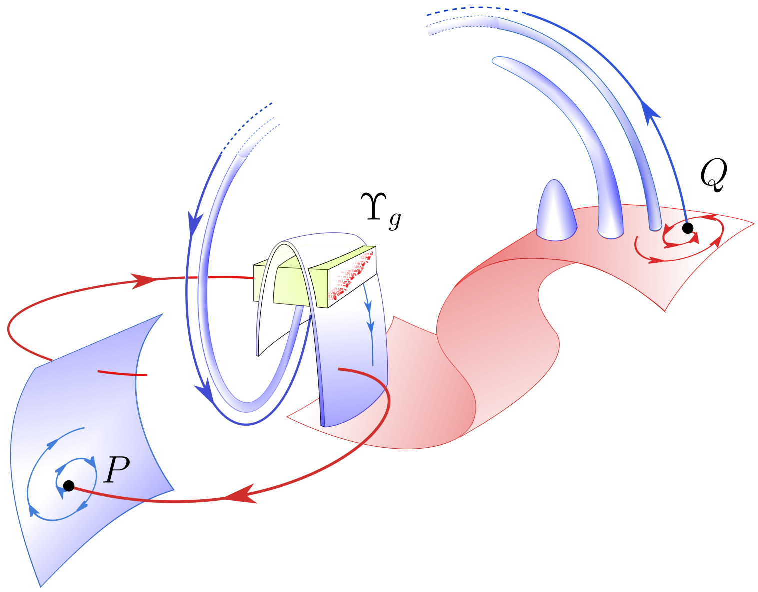

FIGURE 10. Robust homoclinic tangencies

we can assume that the argument of $Dg(Q)$ is irrational. Thus the hypotheses of Lemma 4.1 hold.

By Proposition 9.1, $W^{\mathrm{u}}\big(\Upsilon_g, g\big) \pitchfork W^{\mathrm{s}}\big(Q, g\big) \neq \emptyset$. Thus, after a perturbation, that we continue denoting by g and does not change the argument of the complex eigenvalue of $Dg(Q)$, we can assume that there is a disc $S \subset W^{\mathrm{u}}(\Upsilon_g, g)$ that transversely intersects $W^{\mathrm{s}}_{\mathrm{loc}}(Q, g)$ along a curve γ with nontrivial radial projection. Lemma 4.1 now implies that $W^{\mathrm{s}}(P, g)$ transversally intersects $S \subset W^{\mathrm{u}}(\Upsilon_g, g)$, proving the first item of the proposition. Note also that g has a robust cycle associated to Q and Υ_g.

To prove the second intersection, note that $W^{\mathrm{u}}(P, g)$ and $W^{\mathrm{s}}_{\mathrm{loc}}(Q, g)$ interesects transversely along a curve with a nontrivial radial projection. Note also that by Proposition 10.1, $W^{\mathrm{s}}(\Upsilon_g, g)$ and $W^{\mathrm{u}}\big(Q, g\big)$ intersects quasi-transversely at some point Z. Consider a curve $D \subset W^{\mathrm{s}}(\Upsilon_g, g)$ and containing the point Z in its interior. Arguing as above and applying Lemma 4.1, we have that the negative iterates of D (contained in $W^{\mathrm{s}}(\Upsilon_g, g)$ transversally intersects $W^{\mathrm{s}}(P, g)$. This completes the proof of the proposition. □

12. PROOF OF THEOREM 1.1: HOMOCLINIC TANGENCIES

In this section, we consider perturbations $g_{\varepsilon,k,\mu}$ of the diffeomorphisms f in $\mathcal{H}^r_{\mathrm{BH,e^+}}(M)$. Next proposition implies the part of Theorem 1.1 about homoclinic tangencies.

Proposition 12.1. *For every small $\varepsilon > 0$, large k, and $\mu \in (-10, -9)$ there is g arbitrarily C^r close to $g_{\varepsilon,k,\mu}$ with a C^r robust homoclinic tangency associated to blender-horseshoe Υ_g.*

Proof. Note that close to the original heterodimensional tangency, the manifold $W^{\mathrm{u}}\big(P, g_{\varepsilon,k,\mu}\big)$ intersect $W^{\mathrm{s}}\big(Q, g_{\varepsilon,k,\mu}\big)$ in closed curve denoted by $C_{\varepsilon,k,\mu}$. Let $S_{\varepsilon,k,\mu}$ be the two-dimensional compact disc contained in $W^{\mathrm{u}}\big(P, g_{\varepsilon,k,\mu}\big)$ bounded by $C_{\varepsilon,k,\mu}$. By the λ-lemma, the forward iterates $g^i_{\varepsilon,k,\mu}(S_{\varepsilon,k,\mu})$ of $S_{\varepsilon,k,\mu}$ accumulated to the unstable manifold of Q. By Lemma 10.1, the unstable manifold of Q contains a disc in the superposition region of the blender $\Upsilon_{\varepsilon,k,\mu}$. Thus there are infinitely many iterates of $S_{\varepsilon,k,\mu}$ containing uu-tubes in the superposition region of the blender, see Figure 10 and recall Definition 5.1. Corollary 5.1 implies that the manifold $W^{\mathrm{u}}\big(P, g_{\varepsilon,k,\mu}\big)$ and $W^{\mathrm{s}}_{\mathrm{loc}}\big(\Upsilon_{\varepsilon,k,\mu}, g_{\varepsilon,k,\mu}\big)$ have a C^r robust tangency. The proposition follows noting that by Proposition 11.1 the point P and the blender $\Upsilon_{\varepsilon,k,\mu}$ are homoclinically related. □

13. Calculations in the renormalisation scheme

We collect some calculations from [16] that we used in the previous sections.

13.1. **The renormalisation formula.** First, recall the perturbations $f_{\bar{\upsilon},\rho}$ of $f \in \mathcal{H}^r_{\mathrm{BH}}(M^3)$ in (6.1). For that, we borrow from [16, Section 7.3] the explicit formula for compositions of the form

$$\Psi_k^{-1} \circ f^{N_2+m_k+N_1+n_k}_{\bar{\upsilon}_k(\mu),\rho} \circ \Psi_k(x,y,z) \stackrel{\mathrm{def}}{=} (\check{x}_{k,\mu,\rho}, \check{y}_{k,\mu,\rho}, \check{z}_{k,\mu,\rho}). \tag{13.1}$$

Here, the iterates corresponding to n_k occurs in U_Q, the N_1 iterates correspond to the transition $\mathfrak{T}_1$, the iterates corresponding to m_k occurs in U_P, and the N_2 iterates correspond to the transition $\mathfrak{T}_2$.

Consider the heteroclinic points X, Y in the cycle (see Section 2.1.2) and write

$$\begin{aligned} \mathbf{x}_{k,\mu,\rho}(x,y,z) &\stackrel{\mathrm{def}}{=} f^{n_k}_{\bar{\upsilon}_k(\mu),\rho} \circ \Psi_k(x,y,z) - X, \\ \widehat{\mathbf{x}}_{k,\mu,\rho}(x,y,z) &\stackrel{\mathrm{def}}{=} f^{m_k+N_1+n_k}_{\bar{\upsilon}_k(\mu),\rho} \circ \Psi_k(x,y,z) - Y. \end{aligned} \tag{13.2}$$

Using the notation in Section 6, the composition in (13.1) reads as follows:

$$
\begin{aligned}
\breve{x}_{k,\mu,\rho} &= a_1\,\lambda_P^{m_k}\,{\lambda_Q}^{n_k}\,\big(\alpha_1\,(\mathfrak{c}_k\,x-\mathfrak{s}_k\,z)+\alpha_3\,(\mathfrak{s}_k\,x+\mathfrak{c}_k\,z)\big)\\
&\quad+a_1\,\lambda_P^{m_k}\,\sigma_P^{-m_k}\,\alpha_2\,y+\big(\tilde{\mathfrak{c}}_k\,a_2+\tilde{\mathfrak{s}}_k\,a_3\big)\,\beta_2\,y\\
&\quad+\sigma_P^{m_k}\,\lambda_Q^{n_k}\,\gamma_3\,\big(\tilde{\mathfrak{c}}_k\,a_3-\tilde{\mathfrak{s}}_k\,a_2\big)\,(\mathfrak{s}_k\,x+\mathfrak{c}_k\,z)\\
&\quad+{\sigma_P}^{m_k}\,\sigma_Q^{n_k}\,\mathrm{hot}^x_{k,\mu,\rho},\\
\breve{y}_{k,\mu,\rho} &= \mu+b_1\,\lambda_P^{m_k}\,{\sigma_Q}^{n_k}\,\alpha_2\,y\\
&\quad+b_1\,{\lambda_P}^{m_k}\,{\sigma_P}^{m_k}\,\lambda_Q^{n_k}\,\sigma_Q^{n_k}\,\big(\alpha_1\,(\mathfrak{c}_k\,x-\mathfrak{s}_k\,z)\alpha_3\,(\mathfrak{s}_k\,x+\mathfrak{c}_k\,z)\big)\\
&\quad+\Big(\tilde{\mathfrak{c}}_k^2\,b_2+\tilde{\mathfrak{s}}_k^2\,b_3+\tilde{\mathfrak{c}}_k\,\tilde{\mathfrak{s}}_k\,b_4\Big)\,\beta_2^2\,y^2\\
&\quad+\sigma_P^{2m_k}\,\lambda_Q^{2n_k}\Big(\tilde{\mathfrak{s}}_k^2\,b_2+\tilde{\mathfrak{c}}_k^2\,b_3-\tilde{\mathfrak{c}}_k\,\tilde{\mathfrak{s}}_k\,b_4\Big)\,\gamma_3^2\,(\mathfrak{s}_k\,x+\mathfrak{c}_k\,z)^2\\
&\quad+\sigma_P^{m_k}\,\lambda_Q^{n_k}\,\Big(2\tilde{\mathfrak{c}}_k\,\tilde{\mathfrak{s}}_k\,(b_3-b_2)(\tilde{\mathfrak{c}}_k^2\,-\tilde{\mathfrak{s}}_k^2)\,b_4\Big)\beta_2\,\gamma_3(\mathfrak{s}_k\,x\,y+\mathfrak{c}_k\,y\,z)\\
&\quad+{\sigma_P}^{2m_k}\,\sigma_Q^{2n_k}\,\mathrm{hot}^y_{k,\mu,\rho},\\
\breve{z}_{k,\mu,\rho} &= c_1\,\lambda_P^{m_k}\,{\lambda_Q}^{n_k}\,\big(\alpha_1\,(\mathfrak{c}_k\,x-\mathfrak{s}_k\,z)+\alpha_3\,(\mathfrak{s}_k\,x+\mathfrak{c}_k\,z)\big)\\
&\quad+c_1\,\lambda_P^{m_k}\,\sigma_P^{-m_k}\,\alpha_2\,y+\big(\tilde{\mathfrak{c}}_k\,c_2+\tilde{\mathfrak{s}}_k\,c_3\big)\,\beta_2\,y\\
&\quad+\sigma_P^{m_k}\,\lambda_Q^{n_k}\,\gamma_3\,\big(\tilde{\mathfrak{c}}_k\,c_3-\tilde{\mathfrak{s}}_k\,c_2\big)\,(\mathfrak{s}_k\,x+\mathfrak{c}_k\,z)\\
&\quad+{\sigma_P}^{m_k}\,\sigma_Q^{n_k}\,\mathrm{hot}^z_{k,\mu,\rho},
\end{aligned}
\tag{13.3}
$$

where $\mathrm{hot}^*_{k,\mu,\rho}=\mathrm{hot}^*_{k,\mu,\rho}(x,y,z)$, $*=x,y,z$, are higher order terms whose Laundau's symbols satisfy the following conditions (see [16, Lemma 8.3]). Write

$$\widehat{H}_2(\mathbf{x}_{k,\mu,\rho})\stackrel{\mathrm{def}}{=}\widetilde{H}_2(\mathbf{x}_{k,\mu,\rho})-\lambda_Q^{2n_k}\,\tilde{\rho}_{2,k},$$

where $\tilde{\rho}_{2,k}$ is defined in (6.7). Then

(i) $O\big(\mathrm{hot}^x_{k,\mu,\rho}(x,y,z)\big)=O\big({\lambda_P}^{m_k}\,\widetilde{H}_1(\mathbf{x}_{k,\mu,\rho})\big)+O\big(\sigma_P^{m_k}\,\widehat{H}_2(\mathbf{x}_{k,\mu,\rho})\big)$ $+O\big(H_1(\widehat{\mathbf{x}}_{k,\mu,\rho})\big)$,

(ii) $O\big(\mathrm{hot}^y_{k,\mu,\rho}(x,y,z)\big)=O({\lambda_P}^{m_k})+O\big(\widetilde{H}_1(\mathbf{x}_{k,\mu,\rho})\big)+O\big(\big(\sigma_P^{m_k}\,\widehat{H}_2(\mathbf{x}_{k,\mu,\rho})\big)^2\big)$ $+O\big(\sigma_Q^{-n_k}\,\widehat{H}_2(\mathbf{x}_{k,\mu,\rho})\big)+O\big(H_2(\widehat{\mathbf{x}}_{k,\mu,\rho})\big)$,

(iii) $O\big(\mathrm{hot}^z_{k,\mu,\rho}(x,y,z)\big)=O\big({\lambda_P}^{m_k}\widetilde{H}_1(\mathbf{x}_{k,\mu,\rho})\big)+O\big(\sigma_P^{m_k}\,\widehat{H}_2(\mathbf{x}_{k,\mu,\rho})\big)$ $+O\big(H_3(\widehat{\mathbf{x}}_{k,\mu,\rho})\big)$.

13.2. **The high order terms of $\Psi_k^{-1}\circ g^{N_2+m_k}_{\varepsilon,k,\mu}\big(\gamma_{\varepsilon,k}(t)\big)$.** Recall the definition of $\gamma_{\varepsilon,k}$ in (10.5). We now provide an explicit expression of the high order terms $\mathrm{Hot}^*_{\varepsilon,k,\mu}(t)$ in $\Psi_k^{-1}\circ g^{N_2+m_k}_{\varepsilon,k,\mu}\big(\gamma_{\varepsilon,k}(t)\big)$. Define $\mathbf{x}_{\varepsilon,k,\mu}$ and $\widehat{\mathbf{x}}_{\varepsilon,k,\mu}$ as in (13.2), where $f_{\bar{v}_k(\mu)}$ is replaced by $f_{\varepsilon,\bar{v}_k(\mu)}$ (this is why the subscript ε appears). Write

$$\mathbf{w}_{\varepsilon,k,\mu}(t)\stackrel{\mathrm{def}}{=}\mathbf{x}_{\varepsilon,k,\mu}\circ\Theta_{\bar{\varsigma}}(0,t,0),\qquad \widehat{\mathbf{w}}_{\varepsilon,k,\mu}(t)\stackrel{\mathrm{def}}{=}\widehat{\mathbf{x}}_{\varepsilon,k,\mu}\circ\Theta_{\bar{\varsigma}}(0,t,0),$$

where $\Theta_{\bar{\varsigma}}$ is an in (5.5). We have

$$\begin{aligned}
\mathrm{Hot}^{x}_{\varepsilon,k,\mu}(t) &= a_1\,{\lambda_P}^{m_k}\sigma_P^{m_k}{\sigma_Q}^{n_k}\widehat{\rho}^{1}_{\varepsilon,k}(t) + a_2\,\sigma_P^{2m_k}{\sigma_Q}^{n_k}u_{\varepsilon,k}(t)\\
&\quad + a_3\,{\sigma_P}^{2m_k}{\sigma_Q}^{n_k}v_{\varepsilon,k}(t) + \sigma_P^{m_k}{\sigma_Q}^{n_k}H_1\big(\widehat{\mathbf{w}}_{\varepsilon,k,\mu}(t)\big);\\
\mathrm{Hot}^{y}_{\varepsilon,k,\mu}(t) &= b_1\,{\lambda_P}^{m_k}\sigma_P^{2m_k}{\sigma_Q}^{2n_k}\widehat{\rho}^{1}_{\varepsilon,k}(t)\\
&\quad + \sigma_P^{2m_k}{\sigma_Q}^{n_k}\Big(2b_2\big(\tilde{\mathfrak{c}}_k(\alpha_2+\widetilde{w}_2^{1,\varepsilon}) - \widetilde{w}_3^{1,\varepsilon}\tilde{\mathfrak{s}}_k\big)\\
&\qquad\qquad + b_4\big(\tilde{\mathfrak{s}}_k(\alpha_2+\widetilde{w}_2^{1,\varepsilon}) + \widetilde{w}_3^{1,\varepsilon}\,\tilde{\mathfrak{c}}_k\big)\Big)\varsigma_2^{-1}\,t\,u_{\varepsilon,k}(t)\\
&\quad + \sigma_P^{2m_k}{\sigma_Q}^{n_k}\Big(2b_3\big(\tilde{\mathfrak{s}}_k(\alpha_2+\widetilde{w}_2^{1,\varepsilon}) + \widetilde{w}_3^{1,\varepsilon}\tilde{\mathfrak{c}}_k\big)\\
&\qquad\qquad + b_4\big(\tilde{\mathfrak{c}}_k(\alpha_2+\widetilde{w}_2^{1,\varepsilon}) - \widetilde{w}_3^{1,\varepsilon}\,\tilde{\mathfrak{s}}_k\big)\Big)\varsigma_2^{-1}\,t\,v_{\varepsilon,k}(t)\\
&\quad + \sigma_P^{4m_k}{\sigma_Q}^{2n_k}\Big(b_2\,\big(u_{\varepsilon,k}(t)\big)^2 + b_3\big(v_{\varepsilon,k}(t)\big)^2\\
&\qquad\qquad + b_4\,u_{\varepsilon,k}(t)\,v_{\varepsilon,k}(t)\Big) + \sigma_P^{2m_k}{\sigma_Q}^{2n_k}H_2\big(\widehat{\mathbf{w}}_{\varepsilon,k,\mu}(t)\big);\\
\mathrm{Hot}^{z}_{\varepsilon,k,\mu}(t) &= c_1\,{\lambda_P}^{m_k}\sigma_P^{m_k}{\sigma_Q}^{n_k}\widehat{\rho}^{1}_{\varepsilon,k}(t) + c_2\,\sigma_P^{2m_k}{\sigma_Q}^{n_k}u_{\varepsilon,k}(t)\\
&\quad + c_3\,{\sigma_P}^{2m_k}{\sigma_Q}^{n_k}v_{\varepsilon,k}(t) + \sigma_P^{m_k}{\sigma_Q}^{n_k}H_3\big(\widehat{\mathbf{w}}_{\varepsilon,k,\mu}(t)\big).
\end{aligned}$$

13.2.1. *Proof of item (3) in Claim 10.4.* We claim that

$$\left\|\left(\mathrm{Hot}^{x}_{\varepsilon,k,\mu} - \sigma_P^{m_k}{\sigma_Q}^{n_k}H_1\circ\widehat{\mathbf{w}}_{\varepsilon,k,\mu}\right)|_{[-4,4]}\right\|_r \to 0.$$

For this just note that

- $u_{\varepsilon,k}(t)$, $v_{\varepsilon,k}(t)$ and $\widehat{\rho}^{\ell}_{\varepsilon,k}(t)$, $\ell = 2,3$, have the same symbol of Landau $O(\sigma_P^{-4m_k}\,\sigma_Q^{-2n_k})$, see (10.8) and (10.9),
- $\widehat{\rho}^{1}_{\varepsilon,k}$ is bounded and (6.3).

Finally, the convergence

$$\lim_{k\to\infty}\left\|\sigma_P^{m_k}{\sigma_Q}^{n_k}H_1\circ\widehat{\mathbf{w}}_{\varepsilon,k,\mu}|_{[-4,4]}\right\|_r = 0$$

follows exactly as in [16, Claim 8.4].

References

[1] Asaoka, Masayuki, *Hyperbolic sets exhibiting C^1-persistent homoclinic tangency for higher dimensions*, Proc. Amer. Math. Soc., 136 (2008), pp. 677–686.

[2] Barrientos, Pablo G. and Ki, Yuri and Raibekas, Artem *Symbolic blender-horseshoes and applications*, Nonlinearity, 27 (2014), pp. 2805–2839.

[3] Barrientos, Pablo G. and Pérez, Sebastián A., *Robust Heteroclinic Tangencies*, Bull. Braz. Math. Soc. (N.S.), 51(2020), pp. 1041–1056.

[4] Barrientos, Pablo G. and Raibekas, Artem, *Robust tangencies of large codimension*, Nonlinearity, 30 (2017), pp. 4369–4409.

[5] Bochi, Jairo and Bonatti, Christian and Díaz, Lorenzo J., *Robust criterion for the existence of nonhyperbolic ergodic measures*, Comm. Math. Phys., 344 (2016), pp. 751–795.

[6] Bonatti, Christian, *Towards a global view of dynamical systems, for the C^1-topology*, Ergodic Theory Dynam. Systems, 31(2011), pp. 959–993.

[7] BONATTI, CHRISTIAN AND DÍAZ, LORENZO J., *Robust heterodimensional cycles and C^1-generic dynamics*, Journal of the Institute of Mathematics of Jussieu, 7(2008), pp. 469–525.
[8] BONATTI, CHRISTIAN AND DÍAZ, LORENZO J., *Abundance of C^1-robust homoclinic tangencies*, Trans. Amer. Math. Soc., 364(2012), pp. 5111–5148.
[9] BONATTI, CHRISTIAN AND DÍAZ, LORENZO J. AND KIRIKI, SHIN, *Stabilization of heterodimensional cycles*, Nonlinearity, 25(2012), pp. 931.
[10] BONATTI, CH. AND CROVISIER, S. AND DÍAZ, L. J. AND GOURMELON N., *Internal perturbations of homoclinic classes: non-domination, cycles, and self-replication*, Ergodic Theory Dynam. Systems, 33 (2013), pp. 739–776.
[11] CROVISIER, SYLVAIN AND PUJALS, ENRIQUE R., *Essential hyperbolicity and homoclinic bifurcations: a dichotomy phenomenon/mechanism for diffeomorphisms*, Invent. Math., 201 (2015), pp. 385–517.
[12] CROVISIER, SYLVAIN AND SAMBARINO, MARTIN AND YANG, DAWEI, *Partial hyperbolicity and homoclinic tangencies*, J. Eur. Math. Soc. (JEMS), 17(2015), pp. 1–49.
[13] DÍAZ, LORENZO J. AND GELFERT, KATRIN AND SANTIAGO, BRUNO, *Weak$*$ and entropy approximation of nonhyperbolic measures: a geometrical approach*, arXiv:1804.05913, To appear in Camb. Phil. Soc.,
[14] DÍAZ, LORENZO J. AND KIRIKI, SHIN AND SHINOHARA, KATSUTOSHI, *Blenders in centre unstable Hénon-like families: with an application to heterodimensional bifurcations*, Nonlinearity, 27(2014), pp. 353–378.
[15] L . J. DÍAZ AND A. NOGUEIRA AND E. R. PUJALS, *Heterodimensional tangencies*, Nonlinearity, 19(2006) pp. 2543–2566.
[16] DÍAZ, LORENZO J. AND PÉREZ, SEBASTIAN A., *Hénon-like families and blender-horseshoes at nontransverse heterodimensional cycles*, Internat. J. Bifur. Chaos Appl. Sci. Engrg., 29(2019), pp. 1930006, 22.
[17] DÍAZ, LORENZO J. AND PÉREZ, SEBASTIÁN A., *Blender-horseshoes in center-unstable Hénon-like families*, New trends in one-dimensional dynamics, Springer Proc. Math. Stat., 285(2019), pp. 137–163.
[18] KIRIKI, SHIN AND SOMA, TERUHIKO, *C^2-robust heterodimensional tangencies*, Nonlinearity, 25(2012), pp. 3277–3299.
[19] MOREIRA, CARLOS GUSTAVO, *There are no C^1-stable intersections of regular Cantor sets*, Acta Mathematica, 206(2011), pp. 311–323.
[20] NEWHOUSE, SHELDON E., *The abundance of wild hyperbolic sets and non-smooth stable sets for diffeomorphisms*, Publications Mathématiques de l'Institut des Hautes Études Scientifiques, 50(1),pp. 101–151.
[21] NEWHOUSE, SHELDON E., *Nondensity of Axiom A (a) on $\mathbb{S}^2$*, Global analysis, 1(1970), pp. 191–202.
[22] PALIS, JACOB. *A global view of dynamics and a conjecture on the denseness of finitude of attractors*, Astérisque, 261(2000), pp. 339–351.
[23] PALIS, JACOB AND TAKENS, FLORIS, *Hyperbolicity and sensitive chaotic dynamics at homoclinic bifurcations*, Cambridge Studies in Advanced Mathematics, 35(1993),
[24] PALIS, J. AND TAKENS, F., *Hyperbolicity and the creation of homoclinic orbits*, Ann. of Math. (2), 125(1987), pp. 337–374.
[25] PALIS, J. AND VIANA, M., *High dimension diffeomorphisms displaying infinitely many periodic attractors* Ann. of Math. (2), 140(1994), pp. 207–250.
[26] PUJALS, ENRIQUE R. AND SAMBARINO, MARTÍN, *Homoclinic tangencies and hyperbolicity for surface diffeomorphisms*, Ann. of Math. (2), 151(2000), pp. 961–1023.
[27] ROMERO,NEPTALI, *Persistence of homoclinic tangencies in higher dimensions*, Ergodic Theory Dynam. Systems, 15(1995), pp. 735–757.

DEPARTAMENTO DE MATEMÁTICA PUC-RIO, MARQUÊS DE SÃO VICENTE 225, GÁVEA, RIO DE JANEIRO 225453-900, BRAZIL
Email address: lodiaz@mat.puc-rio.br

INSTITUTO DE MATEMÁTICAS, PONTIFICIA UNIVERSIDAD CATÓLICA DE VALPARAÍSO, BLANCO VIEL 596, CERRO BARÓN, VALPARAÍSO, CHILE.
Email address: sebastian.perez.o@pucv.cl