Robustly transitive sets and heterodimensional cycles

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Abstract

It is known that non-hyperbolic robustly transitive sets Λ_φ have all a dominated splitting and contain generically periodic points of different indices. We show that for a C^1-dense open subset of diffeomorphisms φ the indices of periodic points in a robust transitive set Λ_φ form an interval in $\mathbb{N}$. We also proof that the homoclinic classes of two periodic points in Λ_φ are robustly equal. Finally, we describe what kind of homoclinic tangencies can appear in Λ_φ by analyzing the dominated splittings of Λ_φ.

1 Introduction

When a diffeomorphism ϕ is *hyperbolic*, i.e., it verifies the Axiom A, the Spectral Decomposition Theorem of Smale says that its limit set (set of non-wandering points) is the union of finitely many *basic pieces* satisfying nice properties: they are invariant, compact, *transitive* (there is a dense orbit), pairwise disjoint and isolated (each piece is the maximal invariant set in a neighborhood of itself). Moreover, by construction, a basic piece is the *homoclinic class* of a hyperbolic periodic point, i.e., the closure of the transverse intersections of its invariant manifolds.

Even if the dynamics is non-hyperbolic, the homoclinic classes of hyperbolic periodic points seem to be the natural elementary pieces of the dynamics, satisfying many of the properties of the basic sets of the Smale's theorem: invariance, compactness, transitivity and density of hyperbolic periodic points. Recent results in [BD$_2$], [Ar] and [CMP] show that, for $\mathcal{C}^1$-generic diffeomorphisms (i.e., those belonging to a residual subset of $\mathrm{Diff}^1(M)$) two homoclinic classes are either disjoint or equal and they are maximal transitive sets (i.e., every transitive set intersecting a homoclinic class is contained in it). Let us observe that, in general, the homoclinic classes fail to be hyperbolic, isolated and pairwise disjoint.

In [BDP] it is shown that, for $\mathcal{C}^1$-generic diffeomorphisms, a homoclinic class is either contained in the closure of an infinite set of sinks or sources, or satisfies some weak form of hyperbolicity (partial hyperbolicity or, at least, existence of a *dominated splitting*). The first situation (called the Newhouse phenomenon) can be locally generic (in the residual sense): there are open sets in $\mathrm{Diff}^r(M)$ where the diffeomorphisms with infinitely many sinks or sources are (locally) residual for the $\mathcal{C}^r$-topology, see [N] for $r \geq 2$ for surface diffeomorphisms, [PV] for $r \geq 2$ in higher dimensions, and [BD$_1$] for $r = 1$ in dimensions greater or equal than 3. Certainly, the Newhouse phenomenon

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exhibits very wild behavior and it is conjectured that (in some sense) diffeomorphisms satisfying this phenomenon are very rare (for instance, for generic parametrized families of diffeomorphisms, the Lebesgue measure of the parameters corresponding to diffeomorphisms satisfying the Newhouse phenomenon is zero), see [Pa].

We focus here on the opposite behavior, more precisely, on the so-called *robustly transitive sets* introduced in [DPU] as a non-hyperbolic generalization of the basic sets of the Spectral Decomposition of Smale: a robustly transitive set Λ of a diffeomorphism ϕ is a transitive set which is locally maximal in some neighbourhood U of it and such that, for every C^1-perturbation ψ of the diffeomorphism ϕ, the maximal invariant set of ψ in U is transitive. From the results in [M$_2$], [DPU] and [BDP] every robustly transitive set Λ admits a *dominated splitting,* say $T_\Lambda M = E_1 \oplus \cdots \oplus E_k$, and by [BD$_2$], $\mathcal{C}^1$-generically, it is a *homoclinic class.* An invariant set may admit more than one dominated splitting. The reason is that, in some cases, one can sum some bundles of a dominated splitting, obtaining a new dominated splitting with less bundles, or, conversely, split some bundle of the splitting in a dominated way. So it is natural to consider the *finest dominated splitting* of the set Λ (i.e., one of which it is not possible to split any bundle of the splitting to get a new dominated splitting).

In this paper we study the interplay between the dominated splittings (especially the finest one) of a robustly transitive set Λ and its dynamics, answering questions about the *indices* (dimension of the stable manifold) of the periodic points of Λ, the possible bifurcations (saddle-node and homoclinic tangencies) occurring in this set as well as its dynamical structure.

In order to present our results we need to give some precise definitions.

In what follows, M denotes a compact, closed Riemannian manifold and $\mathrm{Diff}^1(M)$ the space of $\mathcal{C}^1$-diffeomorphisms of M endowed with the usual topology.

Let Λ be a compact invariant set of a diffeomorphism ϕ. A ϕ_*-invariant splitting $T_\Lambda M = E \oplus F$ over Λ is *dominated* if the fibers of E and F have constant dimension and there exists a k such that for every $x \in \Lambda$ one has

$$||\phi_*^k|_{E(x)}|| \cdot ||\phi_*^{-k}|_{F(\phi^k(x))}|| < \frac{1}{2},$$

that is, ϕ_*^k expands the vectors in F uniformly more than the vectors in E. Then we say that F *dominates* E and write $E \prec F$.

An invariant bundle E over Λ is *uniformly contracting* if there exists a k such that for every $x \in \Lambda$ one has:

$$||\phi_*^k|_{E(x)}|| < \frac{1}{2}.$$

An invariant bundle is *uniformly expanding* if it is uniformly contracting for ϕ_*^{-1}.

Let $T_\Lambda M = E_1 \oplus E_2 \oplus \cdots \oplus E_m$ be a ϕ_*-invariant splitting over Λ such that the fibers of the bundles E_i have constant dimension. Denote by $E_i^j = \bigoplus_i^j E_k$. Observe that $E_1^{k-1} \oplus E_k^m$ is a splitting of $T_\Lambda M$ for all $k \in \{2, \ldots, m\}$. We say that $E_1 \oplus E_2 \oplus \cdots \oplus E_m$ is the *finest dominated splitting* of Λ if $E_1^{k-1} \oplus E_k^m$ is a dominated splitting for each $k \in \{2, \ldots, m\}$ and every E_k is *indecomposable* (i.e., it does not admit any (nontrivial) dominated splitting). See [BDP] for the existence and uniqueness of the finest dominated splitting.

Consider a set $V \subset M$ and a diffeomorphism $\varphi\colon M \to M$. We denote by $\Lambda_\varphi(V)$ the *maximal invariant set* of φ in V, i.e., $\Lambda_\varphi(V) = \bigcap_{i \in \mathbb{Z}} \varphi^i(V)$. Given an open set U of M the set $\Lambda_\varphi(U)$ is *robustly transitive* if $\Lambda_\psi(U)$ is equal to $\Lambda_\psi(\bar{U})$ and is *transitive* for all ψ in a $\mathcal{C}^1$-neighbourhood of

φ. We say that a ψ-invariant closed set K is transitive if there is some $x \in K$ having a positive dense orbit in the whole set K.

If a robustly transitive set $\Lambda_\phi(U)$ is not (uniformly) hyperbolic then, by a $\mathcal{C}^1$-small perturbation of ϕ, one can create non-hyperbolic periodic points, and thus hyperbolic periodic points with different indices in $\Lambda_\phi(U)$ (see [M$_2$]). Our first two results describe the possible indices of the periodic points of $\Lambda_\phi(U)$, in terms of the finest dominated splitting of $\Lambda_\phi(U)$:

Theorem A. *Let U be an open set of M and $\mathcal{M}(U)$ a $\mathcal{C}^1$-open set of $\mathrm{Diff}^1(M)$ such that $\Lambda_\varphi(U)$ is robustly transitive for every $\varphi \in \mathcal{M}(U)$. Then there is a dense open subset $\mathcal{N}(U)$ of $\mathcal{M}(U)$ such that, for every $\varphi \in \mathcal{N}(U)$, the set of indices of the hyperbolic periodic points of $\Lambda_\varphi(U)$ is an interval of integers (i.e., if P and Q are hyperbolic periodic points of indices p and q, $p \geq q$, of $\Lambda_\varphi(U)$, $\varphi \in \mathcal{N}(U)$, and $j \in [q,p]$, then $\Lambda_\varphi(U)$ has a hyperbolic periodic point of index j).*

In the next result we use the arguments in [M$_2$] to relate the uniform contraction or dilatation of the extremal bundles of the finest dominated splitting of a robustly transitive set with the indices of the periodic points of this set.

Theorem B. *Consider an open set U of a compact manifold M and an integer $q \in \mathbb{N}^*$. Let $\mathcal{U}$ be a $\mathcal{C}^1$-open set of $\mathrm{Diff}^1(M)$ such that for every $\phi \in \mathcal{U}$ the maximal invariant set $\Lambda_\phi(\bar{U})$ satisfies the following properties:*

1. *the set $\Lambda_\phi(\bar{U})$ is contained in U and admits a dominated splitting $E_\phi \oplus F_\phi$, $E_\phi \prec F_\phi$, with $\dim E_\phi(x) = q$,*
2. *the set $\Lambda_\phi(\bar{U})$ has no periodic points of index $k < q$.*

Then the bundle E_ϕ is uniformly contracting for every $\phi \in \mathcal{U}$.

We can summarize the two results above to get a characterization of the set of indices of the periodic points of the set $\Lambda_\phi(\bar{U})$, as follows.

Let U be an open set of M and φ a diffeomorphism such that $\Lambda_\varphi(U)$ is robustly transitive with a finest dominated splitting of the form $T_{\Lambda_\varphi(U)}M = E_1 \oplus \cdots \oplus E_{k(\varphi)}$, $E_i \prec E_{i+1}$. Denote by E^s the sum of all the uniformly contracting bundles of this splitting and let E_α be the first non-uniformly contracting bundle, i.e., $E^s = E_1 \oplus \cdots \oplus E_{\alpha-1}$. In the same way, denote by E^u the sum of all the uniformly expanding bundles of the splitting and let E_β be the last non-uniformly expanding bundle, i.e., $E^u = E_{\beta+1} \oplus \cdots \oplus E_{k(\varphi)}$. Let $\mathcal{U}$ be a $\mathcal{C}^1$-neighborhood of φ such that for every $\psi \in \mathcal{U}$ the set $\Lambda_\psi(U)$ has the same properties as $\Lambda_\varphi(U)$ (i.e., robustly transitive and the number $k(\psi)$ of bundles of the finest dominated splitting is equal to $k(\varphi)$) and the dimensions of bundles $E^s(\psi)$, $E_\alpha(\psi)$, $E_\beta(\psi)$ and $E^u(\psi)$, defined in the obvious way, are constant in $\mathcal{U}$ and equal to corresponding bundles for ϕ.

Corollary C. *With the notation above, there are a C^1-open and dense subset $\mathcal{V}$ of $\mathcal{U}$ and locally constant functions $i, j \colon \mathcal{V} \to \mathbb{N}^*$ such that*

$$i(\psi) \in [\dim(E^s), \dim(E^s) + \dim(E_\alpha)] \cap \mathbb{N}^*,$$
$$j(\psi) \in [\dim(E^u), \dim(E^u) + \dim(E_\beta)] \cap \mathbb{N}^*,$$

and, for every $\psi \in \mathcal{V}$, the set of indices of the hyperbolic periodic points of $\Lambda_\psi(U)$ is the interval $[i(\psi), \dim(M) - j(\psi)] \cap \mathbb{N}^*$.

The first known examples of non-hyperbolic robustly transitive sets had a one-dimensional central direction, see [M_1] and [Sh]. As a consequence, these examples do not present *homoclinic tangencies* (non-transverse homoclinic intersections between the invariant manifolds of some periodic point). Let us observe that if a periodic point has a homoclinic tangency then, after a perturbation of the diffeomorphism, one create a Hopf bifurcation (a periodic point whose derivative has a pair of conjugate nonreal eigenvalues of modulus one), see [YA] and [R], and hence points whose central direction has dimension at least two. Currently examples of robustly transitive sets having a central direction of dimension two or more are known, see [BD_1], [B] and [BV]. Moreover, in some cases these sets exhibit homoclinic tangencies, see [B] and [BV]. Our next result explains what kind of dominated splitting of a robustly transitive set prevents homoclinic bifurcations.

We say that a robustly transitive set $\Lambda_\varphi(U)$ is *$\mathcal{C}^1$-far from homoclinic tangencies* if there are no *homoclinic tangencies* in $\Lambda_\psi(U)$ for any ψ in a C^1-neighbourhood of φ.

Theorem D. *Given an open set U of M let $\mathcal{P}(U) \subset \mathrm{Diff}^1(M)$ be an open set of diffeomorphisms φ such that:*

1. *The set $\Lambda_\varphi(U)$ is robustly transitive and the minimum and the maximum of the indices of the hyperbolic periodic points of $\Lambda_\varphi(U)$ are constant in $\mathcal{P}(U)$. Denote these numbers by i_s and i_c, respectively.*
2. *The set $\Lambda_\varphi(U)$ is $\mathcal{C}^1$-far from homoclinic tangencies.*

Then there is an open and dense subset $\mathcal{O}(U)$ of $\mathcal{P}(U)$ such that, for every $\varphi \in \mathcal{O}(U)$, the set $\Lambda_\varphi(U)$ has a dominated splitting $T_{\Lambda_\varphi(U)} = E^s \oplus E_1 \oplus \cdots \oplus E_r \oplus E^u$, such that

- *E^s is uniformly contracting and has dimension $i_s \geq 1$,*
- *E^u is uniformly expanding and has dimension $\dim(M) - i_c \geq 1$,*
- *$r = i_c - i_s$ and the bundle E_i has dimension one and is not uniformly hyperbolic for every $i = 1, \ldots, r$.*

Actually, from the proof of this theorem we get somewhat more: given any robustly transitive set $\Lambda_\phi(U)$ the dimensions of the non-hyperbolic bundles of its finest dominated splitting determine, for diffeomorphisms in a C^1-neighbourhood of ϕ, the *ranks* of the homoclinic tangencies (that is, the index of the periodic point exhibiting the tangency) that can occur in $\Lambda_\psi(U)$. The precise statement of this result is in Section 6, see Theorem F.

Finally, for robustly transitive sets which are far from homoclinic tangencies, we prove that the *(relative) homoclinic classes* of two periodic points of this set are equal in a $\mathcal{C}^1$-robust way. More precisely, let P_φ be a hyperbolic periodic point of a diffeomorphism φ. We denote by H_{P_φ} the set of transverse intersections of the invariant manifolds of P_φ. Observe that the homoclinic class of P_φ is the closure of H_{P_φ}. Given an open set U, the *relative homoclinic class* of P_φ in U is the closure of the set $H_{P_\varphi}(U)$ of transverse homoclinic points of P_φ whose orbits are contained in U.

Theorem E. *Let U be an open set of M and $\mathcal{S}(U) \subset \mathrm{Diff}^1(M)$ an open set of diffeomorphisms φ such that the set $\Lambda_\varphi(U)$ is robustly transitive and there are no homoclinic tangencies (in the whole manifold) associated to periodic points of $\Lambda_\varphi(U)$.*

Consider any pair of hyperbolic periodic points P_φ and Q_φ of $\Lambda_\varphi(U)$ with indices p and q whose continuations are defined for every ψ in $\mathcal{S}(U)$. Then there is an open and dense subset $\mathcal{D}(U)$ of $\mathcal{S}(U)$ such that

$$\overline{H_{P_\psi}(U)} = \overline{H_{Q_\psi}(U)}$$

for every ψ in $\mathcal{D}(U)$.

Unfortunately, in the theorem above we cannot ensure that the relative homoclinic classes of P_ψ and Q_ψ are equal to $\Lambda_\psi(U)$, although by the results in [BD$_2$] this is true for a residual subset of $\mathcal{S}(U)$.

Let us now say a few words about the proofs of our results. One of the main tools is the notion of *heterodimensional cycle.* Given a diffeomorphism ϕ with two hyperbolic periodic points P_ϕ and Q_ϕ with different indices, say index$(P_\phi) >$ index(Q_ϕ), we say that ϕ has a *heterodimensional cycle associated to P_ϕ and Q_ϕ,* denoted by $\Gamma(\phi, P_\phi, Q_\phi)$, if $W^s(P_\phi)$ and $W^u(Q_\phi)$ have a (nontrivial) transverse intersection and $W^u(P_\phi)$ and $W^s(Q_\phi)$ have a quasi-transverse intersection along the orbit of some point x, i.e., $T_xW^u(P_\phi) + T_xW^s(Q_\phi)$ is a direct sum. Observe that in this case $\dim(M) - \dim(T_xW^u(P_\phi) + T_xW^s(Q_\phi))$ is equal to index$(P_\phi)-$ index(Q_ϕ), this number being the *codimension of the cycle.*

The proof of Theorem A has two main ingredients. The first is Theorem 3.1, which implies that, by unfolding a heterodimensional cycle associated to points of indices q and p as above, one gets hyperbolic periodic points of some index in between q and p (a priori, we do not know the index of such a point). The second ingredient of the proof is the Connecting Lemma of Hayashi (see Theorem 2.1 and [H]) which allows us to create (after a $\mathcal{C}^1$-perturbation) heterodimensional cycles associated to any pair of periodic points of a robustly transitive set.

Two other important tools are the constructions in [M$_2$] and in [BDP], specially the periodic linear systems with transitions of [BDP]. In this paper we need to introduce *transitions between points of different indices* in the same homoclinic class, generalizing the construction in [BDP], in which only transitions between points with the same index were considered.

Finally, to prove Theorem E, the main ingredient, besides the Connecting Lemma, is the proposition below concerning the structure of the homoclinic classes of hyperbolic points having a heterodimensional cycle.

We say that a hyperbolic periodic point R_ϕ is *$\mathcal{C}^1$-far from tangencies* if there is a $\mathcal{C}^1$-neighbourhood $\mathcal{W}$ of ϕ in $\mathrm{Diff}^1(M)$ such that every $\psi \in \mathcal{W}$ has no homoclinic tangencies associated to R_ψ. A heterodimensional cycle $\Gamma(\phi, P_\phi, Q_\phi)$ is *$\mathcal{C}^1$-far from homoclinic tangencies* if the points P_ϕ and Q_ϕ in the cycle are $\mathcal{C}^1$-far from homoclinic tangencies.

Finally, we say that two points x and y are *transitively related* for ϕ if there exists a transitive set of ϕ containing x and y. The points x and y are *transitively related in an open set U* if there exists a transitive set of ϕ contained in U that contains x and y.

Proposition 1.1. *Let U be an open set, φ a diffeomorphism and P_φ and Q_φ a pair of hyperbolic periodic points of φ of indices p and $q = p - 1$, respectively. Consider a neighbourhood $\mathcal{W} \subset$ $\mathrm{Diff}^1(M)$ of φ of diffeomorphisms ψ such that,*

- *the continuations of P_ψ and Q_ψ are defined and $\mathcal{C}^1$-far from tangencies,*
- *the points P_ψ and Q_ψ are transitively related in U.*

Then there is a $\mathcal{C}^1$-open subset $\mathcal{W}_\varphi$ of $\mathcal{W}$ whose closure contains φ such that the relative homoclinic classes of P_ψ and Q_ψ in U are equal for every $\psi \in \mathcal{W}_\varphi$.

[DR, Theorem A] guarantees that given any heterodimensional cycle $\Gamma(\phi, P_\phi, Q_\phi)$ of codimension one far from homoclinic tangencies, there is a $\mathcal{C}^1$-open set, whose closure contains ϕ, of diffeomorphisms φ such that P_φ and Q_φ are transitively related. Thus, for any diffeomorphism ϕ with a heterodimensional cycle which is far from homoclinic tangencies, there are diffeomorphisms φ arbitrarily close to ϕ satisfying the hypotheses of the proposition. The proof of Proposition 1.1 follows from the results in [DR] and the Connecting Lemma of Hayashi.

This paper is organized as follows. In Section 2 we get some results concerning heterodimensional cycles, robustly transitive sets and homoclinic classes using the Hayashi's Connecting Lemma. In Section 3 we prove Theorem A. For that we need to study the creation of periodic points in the unfolding of heterodimensional cycles (of any codimension). In Section 4 we prove Theorem B, for that we recall some folklore results concerning dominated splittings and remember and reformulate some results in [M_2]. In Sections 5 and 6 we study the interplay between the finest dominated splitting of a robustly transitive set and the creation of homoclinic tangencies inside this set. Finally, in Section 7 we prove the results concerning (relative) homoclinic classes.

2 Transitively related points

We begin the proofs of our results by recalling the Hayashi's Connecting Lemma and deducing some consequences from it.

2.1 Connecting lemma and transitively related points

Theorem 2.1. (Hayashi's Connecting Lemma, [H]) *Let P_φ and Q_φ be a pair of hyperbolic periodic points of a $\mathcal{C}^1$-diffeomorphism φ such that there are sequences of points x_n and of natural numbers k_n such that the sequences x_n and $\varphi^{k_n}(x_n)$ accumulate on $W^s_{loc}(Q_\varphi)$ and on $W^u_{loc}(P_\varphi)$, respectively.*

Then there is a diffeomorphism ψ arbitrarily $\mathcal{C}^1$-close to φ such that $W^s(Q_\psi)$ and $W^u(P_\psi)$ have a nonempty intersection.

Remark 2.2. *Every pair of hyperbolic periodic points P_φ and Q_φ which are transitively related satisfy the hypotheses of the Connecting Lemma (Theorem 2.1).*

Proof of the remark: Consider a transitive set Λ containing P_φ and Q_φ and a point x of Λ whose positive orbit is dense in Λ. Then there are sequences of natural numbers m_n and r_n, $m_n, r_n \to \infty$ as $n \to \infty$, such that $\varphi^{m_n}(x) \to P_\varphi$ and $\varphi^{r_n}(x) \to Q_\varphi$. Then it is immediate to get new sequences m'_n and r'_n, with $m'_n, r'_n \to \infty$, such that $\varphi^{m'_n}(x)$ and $\varphi^{r'_n}(x)$ converge to some point of $W^u_{loc}(P_\varphi)$ and of $W^s_{loc}(Q_\varphi)$, respectively. Taking subsequences, if necessary, we can assume that $r'_n = m'_n + k_n$ for some $k_n > 0$. Now it suffices to take $x_n = \varphi^{m'_n}(x)$ and consider the sequences x_n and k_n. □

2.2 Homoclinic relative classes and robustly transitive sets

By [BD$_2$, Theorem B] there is a residual subset of $\text{Diff}^1(M)$ consisting of diffeomorphisms such that the homoclinic classes of any pair of transitively related hyperbolic periodic points are equal. The proof of this result is based on the Hayashi's Connecting Lemma. Using the *relative version* of the connecting lemma we get a relative version of [BD$_2$, Theorem B] whose prove we omit here.

Theorem 2.3. (Relative version of [BD$_2$, Theorem B]). *Given an open set U of M there is a residual subset $\mathcal{G}(U)$ of $\text{Diff}^1(M)$ such that for every $\varphi \in \mathcal{G}(U)$ a pair of hyperbolic periodic points P_φ and Q_φ of φ, are transitively related in U if and only if the relative homoclinic class in U of P_φ and Q_φ are equal, i.e., $\overline{H_{P_\varphi}(U)} = \overline{H_{Q_\varphi}(U)}$.*

Let $\mathcal{A}(U)$ be an open set of $\text{Diff}^1(M)$ such that $\Lambda_\varphi(U)$ is robustly transitive for all φ. By the Pugh closing lemma (see [Pu]) and a Kupka-Smale argument, there is a residual subset $\mathcal{R}(U)$ of $\mathcal{A}(U)$ of diffeomorphisms φ such that the hyperbolic periodic points form a dense subset of $\Lambda_\varphi(U)$. Taking $\mathcal{T}(U) = \mathcal{G}(U) \cap \mathcal{R}(U)$, where $\mathcal{G}(U)$ and $\mathcal{R}(U)$ are as above we get the following:

Proposition 2.4. *Let U and $\mathcal{A}(U)$ be open sets of M and of $\text{Diff}^1(M)$, respectively, such that $\Lambda_\varphi(U)$ is robustly transitive for all $\varphi \in \mathcal{A}(U)$. Then there is a residual subset $\mathcal{T}_\mathcal{A}(U)$ of $\mathcal{A}(U)$ such that*

$$\overline{H_{P_\varphi}(U)} = \Lambda_\varphi(U)$$

for every $\varphi \in \mathcal{T}_\mathcal{A}(U)$ and every hyperbolic periodic point P_φ of $\Lambda_\varphi(U)$.

2.3 Heterodimensional cycles

We will use the following lemma which is a consequence of the Connecting Lemma and an argument of transversality:

Lemma 2.5. *Let P_φ and Q_φ be a pair of hyperbolic periodic points of a diffeomorphism φ of indices p and q, $p \geq q$. Suppose that P_ψ and Q_ψ are transitively related for every ψ in a neighbourhood $\mathcal{V}$ of φ. Then there is a dense subset $\mathcal{W}$ of $\mathcal{V}$ such that every ϕ in $\mathcal{W}$ has a heterodimensional cycle $\Gamma(\phi, P_\phi, Q_\phi)$ of codimension $(p - q)$.*

Proof: Consider any $\psi \in \mathcal{V}$, since P_ψ and Q_ψ are transitively related, by Remark 2.2, we can apply Theorem 2.1 to get ξ arbitrarily close to ψ (hence ξ is in $\mathcal{V}$) such that $W^s(P_\xi) \cap W^u(Q_\xi) \neq \emptyset$. Since

$$\dim(W^s(P_\xi)) + \dim(W^u(Q_\xi)) = p + (\dim(M) - q) \geq \dim(M),$$

we can assume that this intersection is transverse.

Since ξ belongs to $\mathcal{V}$ the points P_ξ and Q_ξ are transitively related. Thus, again by Remark 2.2, we can apply Theorem 2.1 to get ϕ arbitrarily close to ξ (ϕ in $\mathcal{V}$) such that $W^s(P_\phi)$ and $W^u(Q_\phi)$ have (non empty) transverse intersection and $W^u(P_\phi) \cap W^s(Q_\phi) \neq \emptyset$. After a new perturbation, if necessary, we can assume that the last intersection is quasi-transverse, obtaining a heterodimensional cycle $\Gamma(\phi, P_\phi, Q_\phi)$ of codimension $(p - q)$, ending the proof of the lemma. □

Let us state two remarks of the proof above that we will use in Section 7.

Remark 2.6. *Let P_φ and Q_φ be a pair of hyperbolic periodic points of a diffeomorphism φ of indices p and q, $p \geq q$. Suppose that P_ψ and Q_ψ are transitively related for every ψ in a neighbourhood $\mathcal{V}$ of φ. Then there is a dense and open subset $\mathcal{D}$ of $\mathcal{V}$ such that $W^s(P_\psi)$ and $W^u(Q_\psi)$ have a nontrivial transverse intersection for every ψ in $\mathcal{D}$.*

If in Theorem 2.3 one assumes that the points P_φ and Q_φ have the same index, one has the following stronger version of it:

Remark 2.7. *Let P_φ and Q_φ be a pair of hyperbolic periodic points of the same index of a diffeomorphism φ and U an open set containing the orbits of P_φ and Q_φ. Suppose that P_ψ and Q_ψ are transitively related for every ψ in a neighbourhood $\mathcal{V}$ of φ. Then there is a dense and open subset $\mathcal{O}$ of $\mathcal{V}$ such that, for every ψ in $\mathcal{O}$, the relative homoclinic classes of P_ψ and Q_ψ in U are equal.*

3 Proof of Theorem A: unfolding heterodimensional cycles

3.1 Transitions for heterodimensional cycles

We begin this section stating a somewhat technical result introducing the notion of *transition between periodic points of different indices.*

Theorem 3.1. *Let P and Q be two hyperbolic periodic points of a diffeomorphism φ of indices p and q, $p > q$, and periods $n(P)$ and $n(Q)$, respectively. Denote by M_P and M_Q the linear maps*

$$\varphi_*^{n(P)}(P) \colon T_P M \to T_P M \quad \textit{and} \quad \varphi_*^{n(Q)}(Q) \colon T_Q \to T_Q M.$$

Assume that there are dominated splittings

$$T_P M = E_1(P) \oplus E_2(P) \oplus E_3(P) \quad \textit{and} \quad T_Q M = E_1(Q) \oplus E_2(Q) \oplus E_3(Q),$$

with $\dim(E_1(P)) = \dim(E_1(Q)) = q$ and $\dim(E_3(P)) = \dim(E_3(Q)) = \dim(M) - p$, which are invariant by M_P and M_Q, respectively. Assume, in addition, that there is a heterodimensional cycle $\Gamma(\varphi, U, P, Q)$ in some open subset U of M.

Then, fixed any $\varepsilon > 0$, there are matrices T_0 and T_1 and $\delta > 0$ such that, for every n and $m \geq 0$, and every family matrices $(I_i), i = 0, \ldots, (n+m)+2$, δ-close to identity, there is a diffeomorphism ψ ε-C^1-close to φ having a periodic orbit R of period $n(R)$ such that the linear map $M_R = \psi_*^{n(R)}$ is conjugate to

$$I_{n+m+2} \circ T_1 \circ I_{n+m+1} \circ M_Q \circ I_{n+m} \circ \cdots \circ I_{n+2} \circ M_Q \circ I_{n+1} \circ T_0 \circ I_n \circ M_P \circ I_{n-1} \circ \cdots \circ I_1 \circ M_P \circ I_0.$$

Moreover, $n(R) = t_1 + t_2 + n \cdot n(P) + m \cdot n(Q)$, where t_1 and t_2 are constants depending only on the choice of T_0 and T_1.

The maps T_0 and T_1 are called *transitions* (from P to Q and from Q to P, respectively). These maps are a generalization of the transitions introduced in [BDP] for hyperbolic periodic points which are homoclinically related.

Theorem 3.1 is the main step in the proof of Theorem A. Taking appropriate n and m and assuming that $\text{index}(P) > \text{index}(Q) + 1$, using the theorem one gets that the index of R is in

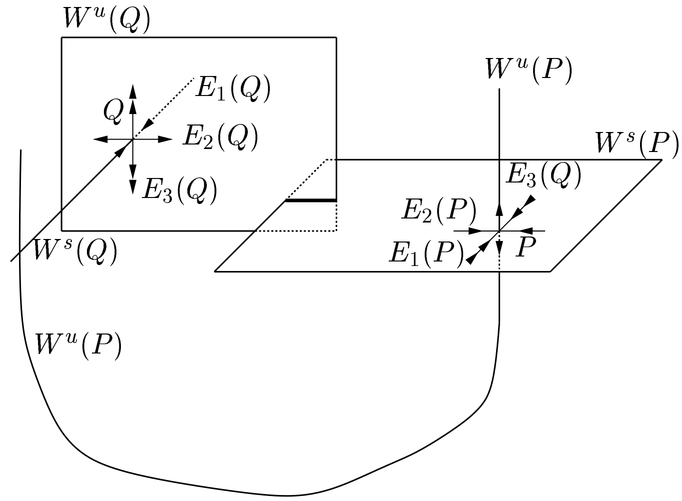

Figure 1: A heterodimensional cycle

between the indices of P and Q, see Corollary 3.6. This construction will also allow us to get points R corresponding to saddle-node bifurcations.

Proof: For simplicity let us assume that P and Q are fixed points. Notice that $E_1(Q)$ is the stable direction of Q, $E_1(P)$ is the strong stable direction of P, $E_3(Q)$ is the strong unstable direction of Q and $E_3(P)$ is the unstable direction of P.

We now perform a C^1-perturbation of the diffeomorphism φ to get appropriate linearizing coordinates of the cycle. The properties of this linearization are summarized in the next lemma:

Lemma 3.2. *Let φ be a diffeomorphism satisfying the hypotheses of Theorem 3.1. Then there is ϕ arbitrarily C^1-close to φ having a heterodimensional cycle $\Gamma(\phi, U, P, Q)$ such that:*

1. *There are smooth linearizing charts*

$$U_P,\, U_Q \simeq [-1,1]^q \times [-1,1]^{p-q} \times [-1,1]^{\dim(M)-p}$$

defined on neighbourhoods of P and Q where ϕ is a linear map such that, for every $x \in U_P \cap \phi^{-1}(U_P)$ or $x \in U_Q \cap \phi^{-1}(U_Q)$, one has:

   (a) *In these charts P and Q correspond to the point $\{0\}^{\dim(M)}$ and $\phi_*(P) = \varphi_*(P)$ and $\phi_*(Q) = \varphi_*(Q)$,*

   (b) *The foliation by q-planes parallel to $[-1,1]^q \times \{0\}^{p-q} \times \{0\}^{\dim(M)-p}$ (called* strong stable foliation, *$\mathcal{F}^s$) is locally invariant and corresponds to the smaller (in modulus) eigenvalues of the linear maps induced by ϕ in U_P and U_Q.*

   (c) *The foliation by $(p-q)$-planes parallel to $\{0\}^q \times [-1,1]^{p-q} \times \{0\}^{\dim(M)-p}$ (called* central foliation, *$\mathcal{F}^c$) is locally invariant.*

   (d) *The foliation by $(n-p)$-planes parallel to $\{0\}^q \times \{0\}^{p-q} \times [-1,1]^{\dim(M)-p}$ (called* strong unstable foliation, *$\mathcal{F}^u$) is locally invariant and corresponds to the bigger (in modulus) eigenvalues of the linear maps induced by ϕ in U_P and U_Q.*

2. *There are points* $X_0 \in (W^u(Q) \pitchfork W^s(P)) \cap U_Q$ *and* $Y_0 = \phi^{k_0}(X_0) \in U_P$, $k_0 > 0$, *such that, in these coordinates,* $X_0 \in \{0\}^q \times [-1,1]^{p-q} \times \{0\}^{\dim(M)-p}$ *(the* local center-unstable manifold *of* Q*, denoted by* $W^{cu}_{loc}(Q)$*) and* $Y_0 \in \{0\}^q \times [-1,1]^{p-q} \times \{0\}^{\dim(M)-p}$ *(the* local center-stable manifold *of* P*, denoted by* $W^{cs}_{loc}(P)$*).*

3. *There are points* $X_1 \in (W^s(Q) \cap W^u(P)) \cap U_P$ *and* $Y_1 = \phi^{k_1}(X_1) \in U_Q$, $k_1 > 0$, *such that, in these coordinates,* $X_1 \in \{0\}^q \times \{0\}^{p-q} \times [-1,1]^{\dim(M)-p}$ *(the* local unstable manifold *of* P, $W^u_{loc}(P)$*) and* $Y_1 \in [-1,1]^q \times \{0\}^{p-q} \times \{0\}^{\dim(M)-p}$ *(the* local stable manifold $W^s_{loc}(Q)$ *of* Q*).*

4. *There are small cubes* $C_0 \subset U_Q$ *and* $C_1 \subset U_P$ *centered at* X_0 *and* X_1*, respectively, such that*
   (a) $\phi^{k_0}(C_0) \subset U_P$ *and* $\phi^{k_1}(C_1) \subset U_Q$,
   (b) *the restrictions* $T_0 = \phi^{k_0}|_{C_0}$ *and* $T_1 = \phi^{k_1}|_{C_1}$ *are affine maps which preserve the strong stable, central and strong unstable foliations above.*

Proof: To get a point of the heteroclinic intersection $W^u(Q) \pitchfork W^s(P)$ in the central direction just observe that, generically, there are points X of such an intersection which are not in the strong unstable manifold of Q nor in the strong stable manifold of P. Thus, after an arbitrarily small perturbation of φ, we can assume that this is our case. Considering a point X with this property and using the domination, we have that the backward orbit of X approaches to the center-unstable manifold of Q. Similarly, the forward iterates of X approach to the center-stable manifold of P. Now by two local small $\mathcal{C}^1$-perturbations one gets the announced points X_0 and $Y_0 = \phi^{k_0}(X_0)$. Observe that we can perform these two perturbations without breaking the cycle (i.e., preserving the non-transverse intersection between $W^s(Q)$ and $W^u(P)$). Observe now that the points X_1 and $Y_1 = \phi^{k_1}(X_1)$ in the lemma are directly given by the intersection $W^s(Q) \cap W^u(P)$.

After a new perturbation, we can assume that ϕ is linear in small neighbourhoods of P and of Q and that ϕ^{k_0} and ϕ^{k_1} are both affine in small neighbourhoods of X_0 and X_1. The only difficulty is to see that these affine maps can be chosen preserving the foliations (strong stable, central and strong unstable). This fact follows along the lines of the proof of [BDP, Lemma 4.13] using the domination. Let us explain all that in details.

In our linearizing charts there are foliations $\mathcal{F}^{cs}$ (resp., $\mathcal{F}^{cu}$) tangent to the sum $E_1 \oplus E_2$ of the stable and central directions (resp., the sum $E_2 \oplus E_3$ of the central and unstable directions). By genericity, we can assume that the images by ϕ^{k_0} of the foliations $\mathcal{F}^s$, $\mathcal{F}^u$, $\mathcal{F}^c$, $\mathcal{F}^{cs}$ and $\mathcal{F}^{cu}$ are in general position. Now, one checks that the forward iterates of the images by ϕ^{k_0} of the leaves of $\mathcal{F}^{cu}$ become close to the center-unstable leaves in U_P. Replacing the initial k_0 by $k_0 + \ell$, for some big positive ℓ, and doing a small perturbation, one gets an invariant center-unstable foliation.

To get the invariance of the strong stable foliation we consider negative iterates of the foliations in the neighbourhood of Y_0. By the previous construction, the center-unstable foliation is preserved by negative iterations. So the negative iterates of the strong stable foliation are transverse to the center-unstable one. As above, the backwards iterates of the strong stable leaves approach to the leaves of the strong stable foliation in U_Q. So we can replace X_0 by some (large) negative iterate of it, say $-\ell'$, and perform a small perturbation (preserving the center unstable foliation) in such a way the transition map $\phi^{k_0+\ell+\ell'}$ from a neighbourhood of X_0 to a neighbourhood of Y_0 preserves the strong stable and center-unstable foliations.

To get the invariance of the strong unstable and center foliations (keeping the invariance of the strong stable one) one repeats all the arguments above inside the center-unstable foliation. We omit the details of this construction. This gives the transition T_0.

The transition T_1 is obtained using the same arguments, so we do not go into the details. The proof of the lemma is now complete. □

Definition 3.1. *Consider a* $\dim(M)$*-cube* $C = I^s \times I^c \times I^u$*, where* I^s *is a* q*-cube,* I^c *a* $(p-q)$*-cube and* I^u *a* $(\dim(M)-p)$*-cube, where there are defined coordinates* (x^s, x^c, x^u) *as above.*

A subset Δ *of* C *is* s*-complete if, for every* $Z = (z^s, z^c, z^u) \in \Delta$*, the horizontal* q*-cube* $I^s \times \{(z^c, z^u)\}$ *is contained in* Δ*. Similarly, a subset* Δ *of* C *is* u*-complete if, for every point* $Z \in \Delta$*, the vertical* $(\dim(M)-p)$*-cube* $\{z^s, z^c\} \times I^u$ *is contained in* Δ*.*

By shrinking, if necessary, the size of the neighbourhood U_Q in the strong unstable direction and taking an appropriate cube C_1 around X_1, we can assume that the image by T_1 of any u-complete disk Δ of C_1 (contained in a leaf of the strong unstable foliation) is a u-complete disk of U_Q.

For simplicity let us denote by A and B the restrictions of ϕ to U_Q and U_P, respectively.

Lemma 3.3. *There is* $\ell_0 \geq 0$ *such that:*

1. *Consider any* $Z \in W^u_{loc}(Q)$ *and any* s*-complete disk* Δ^s *of* C_0 *(contained in a leaf of the strong stable foliation) containing* Z*. Then the connected component of* $A^{-n}(\Delta^s) \cap U_Q$ *containing* $A^{-n}(Z)$ *is a* s*-complete disk in* U_Q *for all* $n \geq \ell_0$*.*

2. *Consider any* u*-complete disk* Δ^u *of* C_0 *(in a leaf of the strong unstable foliation). Then the intersection between* Δ^u *and* $T_0^{-1}(W^s_{loc}(P))$ *is a unique point* W*. Let* Δ^u_m *be the connected component of* $(B^m \circ T_0(\Delta^u)) \cap U_P$ *containing* $B^m \circ T_0(W)$*. Then* $\Delta^u_m \cap C_1$ *is a complete* u*-disk (in* C_1*) for every* $m \geq \ell_0$*.*

Proof: For instance to see the first item just observe that A^{-1} expands the s-direction and recall the A-invariance of the foliations. The second item follows using that B expands in the u-direction and the B-invariance of the foliations. □

We are now ready to end the proof of Theorem 3.1. Given $\varepsilon > 0$ there is an $\varepsilon/2$-perturbation ϕ of φ satisfying Lemmas 3.2 and 3.3. We will now obtain the final diffeomorphism considering a perturbation of ϕ obtained by composing the transition T_1 with a small translation T_v in the direction of a vector v parallel to the central direction (in U_Q). Let us now explain the details of this construction.

In our coordinates, $X_0 = (0^s, x_0^c, 0^u)$. Consider now the su-disk

$$\Delta = \left([-1,1]^q \times \{x_0^c\} \times [-1,1]^{\dim(M)-p}\right) \cap C_0.$$

With the terminology above, the disk Δ is u and s-complete in C_0.

Given n and m bigger than ℓ_0, ℓ_0 as in Lemma 3.3, let Δ^{-m} and Δ^n_0 be the connected components of $A^{-m}(\Delta) \cap U_Q$ containing $A^{-m}(X_0)$ and of $(B^n \circ T_0(\Delta)) \cap U_P$ containing $B^n(T_0(X_0))$, respectively. Let $\Delta^n_1 = \Delta^n_0 \cap C_1$. Write $\Delta^n = T_1(\Delta^n_1)$. By Lemma 3.3 and the observation before, Δ^{-m} and Δ^n are a s-complete and u-complete disks in U_Q and C_1, respectively.

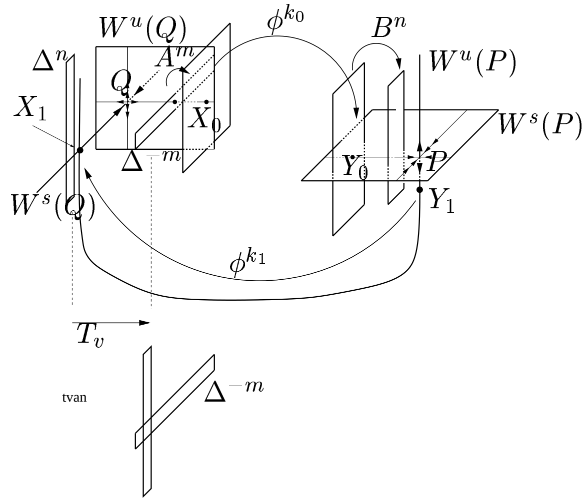

Figure 2: A periodic orbit

Observe that there is a unique vector v parallel to the central direction such that the intersection between $T_v(\Delta^n)$ and Δ^{-m} is not empty. Moreover, since these sets are both su-disks of U_Q, such an intersection is a sub-rectangle R intersecting *completely* Δ^{-m} in the u-direction and Δ^n in the s-direction. Here by a *complete intersection in the u-direction* we mean that, for every $Z \in R$, the leaf $F^u(Z)$ of the strong unstable foliation containing Z is such that the connected components of $F^u(Z) \cap R$ and of $\Delta^{-m} \cap F^u(Z)$ containing Z are equal. The definition of *complete intersection in the s-direction* is totally analogous (considering strong stable leaves).

Now a classical argument of hyperbolicity implies that the map $T = T_v \circ T_1 \circ B^n \circ T_0 \circ A^m$ has a fixed point W in Δ^{-m}. Observe that the derivative of T at W is $\tilde{T}_1 \circ B^n \circ \tilde{T}_0 \circ A^m$ (where $\tilde{T}_i$ is the linear part of the affine map T_i).

So it remains to see that the size of the translation T_v can be taken smaller than $\varepsilon/2$. For that first observe that the disks Δ^{-m} and Δ^n can be taken passing arbitrarily close to the heteroclinic intersection Y_1, for that it is enough to take n and m large enough. Thus there is n_0 such that the distance between Δ^{-m} and Δ^n is less than $\varepsilon/2$ for every n and m greater n_0. Fixed such an n_0 and replacing T_0 by $T_0 \circ A^{n_0}$ and T_1 by $T_1 \circ B^{n_0}$, we get that for every positive n and m there is a translation T_v, $v = v(n,m)$, such that the modulus of v is less than $\varepsilon/2$.

The diffeomorphism ψ in the statement of the theorem is obtained from ϕ by composing T_1 with T_v. By construction, ψ has a periodic point R of period $n_R = t_0 + t_1 + n + m$, where $t_0 = k_0 + n_0$ and $t_1 = k_1 + n_1$, such that

$$\psi_*^{n_R}(R) = \tilde{T}_1 \circ B^n \circ \tilde{T}_0 \circ A^m.$$

Observe that t_0 and t_1 depend exclusively on the transitions T_0 and T_1. The theorem now follows from the definition of A and B and the lemma below, that allows us to perform any small perturbation of the derivative of a diffeomorphism along the orbit of a periodic point in a dynamical way.

Lemma 3.4. **([F], [M$_2$])** *Consider a C^1-diffeomorphism φ and a φ-invariant finite set Σ. Let A be an ε-perturbation of φ_* along Σ (i.e., the linear maps $A(x)$ and $\varphi_*(x)$ are ε-close for all $x \in \Sigma$). Then for every neighbourhood U of Σ there is a diffeomorphism ϕ C^1-ε-close to φ such that*

- $\varphi(x) = \phi(x)$ *if* $x \in \Sigma$ *or if* $x \notin U$,
- $\phi_*(x) = A(x)$ *for all* $x \in \Sigma$.

The proof of Theorem 3.1 is now complete. □

We end this subsection by stating a lemma that follows from the proof of [BDP, Lemma 4.13]:

Lemma 3.5. *Let M_P and M_Q be linear maps as in the statement of Theorem 3.1. Suppose that M_P and M_Q preserve the dominated splittings $T_P M = E^1_P \oplus \cdots \oplus E^k_P$ and $T_Q M = E^1_Q \oplus \cdots \oplus E^k_Q$, where $\dim(E^i_P) = \dim(E^i_Q)$ for every i. Then one can choose the matrices T_0 and T_1 in Theorem 3.1 such that*

$$T_0(E^i_P) = E^i_Q \text{ and } T_1(E^i_Q) = E^i_P, \quad \text{for every } i \in \{1, \dots, k\}.$$

3.2 Periodic points in the unfolding of heterodimensional cycles

Using Lemma 3.4 we get the following two corollaries of Theorem 3.1. First we use the notation $\Gamma(\varphi, U, P, Q)$ to localize a cycle, that is, if we are only concerned with the intersection between the invariant manifolds of P and Q whose orbit is contained in U.

Corollary 3.6. *Consider a heterodimensional cycle $\Gamma(\varphi, U, P, Q)$ associated to the hyperbolic periodic points P and Q of indices p and q, where $p > q$, having positive real eigenvalues of multiplicity one. Then, for every integer $\ell \in [q, p]$, there is a diffeomorphism ϕ arbitrarily close to φ with a hyperbolic periodic point of index ℓ in $\Lambda_\phi(U)$.*

Proof: This corollary is trivial when $\ell = p$ or q. So let us fix some $\ell \in]q, p[$. Define the matrices M_P and M_Q as in the statement of Theorem 3.1 and denote by $\lambda^1_P, \cdots, \lambda^{\dim(M)}_P$ the eigenvalues of M_P, where $0 < \lambda^1_P < \cdots < \lambda^{\dim(M)}_P$, and by $\lambda^1_Q, \cdots, \lambda^{\dim(M)}_Q$ the eigenvalues of M_Q, where $0 < \lambda^1_Q < \cdots < \lambda^{\dim(M)}_Q$.

For each $i \in \{1, \dots, \dim(M)\}$ let $E^i(P)$ and $E^i(Q)$ the eigenspaces corresponding to λ^i_P and λ^i_Q, respectively. We now consider the invariant splittings (of M_P and M_Q) given by

$$\begin{array}{lll} E_1(P) = E^1(P) \oplus \cdots E^{\ell-1}(P), & E_2(P) = E^\ell(P), & E_3(P) = E^{\ell+1}(P) \oplus \cdots E^{\dim(M)}(P), \\ E_1(Q) = E^1(Q) \oplus \cdots E^{\ell-1}(Q), & E_2(Q) = E^\ell(Q), & E_3(Q) = E^{\ell+1}(Q) \oplus \cdots E^{\dim(M)}(Q). \end{array}$$

Observe that, by the hypotheses on the eigenvalues of P and Q, the splittings $E_1(R), E_2(R)$ and $E_3(R)$, $R = P, Q$, are dominated (for M_P and M_Q), thus they satisfy the hypotheses of Theorem 3.1.

Since $q < \ell < p$ we have that $\lambda^\ell_P < 1 < \lambda^\ell_Q$. Thus there are constants C and C', $0 < C < 1 < C'$, and arbitrarily big natural numbers n_0 and m_0 such that

$$(\lambda^{\ell-1}_P)^{n_0}\,(\lambda^{\ell-1}_Q)^{m_0} < C < (\lambda^{\ell}_P)^{n_0}\,(\lambda^{\ell}_Q)^{m_0} < C' < (\lambda^{\ell+1}_P)^{n_0}\,(\lambda^{\ell+1}_Q)^{m_0}.$$

Applying Theorem 3.1 to the matrices M_P and M_Q, $n = n_0$, $m = m_0$, and the matrices $I_0, \dots, I_{n+m+2}$ equal to the identity, we get transitions T_0 and T_1 and a diffeomorphism ϕ close to φ having a periodic point $R \in \Lambda_\phi(U)$ of period $n(R) \simeq n_0 + m_0$ such that $\phi^{n(R)}_*$ is conjugate to

$$M_R = T_1 \circ M^{m_0}_Q \circ T_0 \circ M^{n_0}_P.$$

By Lemma 3.5 we can suppose that T_0 and T_1 preserve the splittings $E_1 \oplus E_2 \oplus E_3$. Hence the ℓ^{th}-eigenvalue λ_R^ℓ of M_R is such that

$$\frac{C}{k_1} < |\lambda_R^\ell| < k_2\, C',$$

where k_1 is the product of the norms of T_0^{-1} and T_1^{-1} and k_2 is the product of the norms of T_0 and T_1. Observe that a priori we can not guarantee that this eigenvalue is positive (we do not know if the transitions preserve the orientation). Thus, taking n_0 and m_0 big enough, we can assume that $|\log(\lambda_R^\ell)|/(n_0+m_0)$ is arbitrarily close to zero.

Applying now Lemma 3.4 to the derivative of ϕ along the orbit of R, we can assume that the eigenvalues $\lambda_R^1, \dots, \lambda_R^{\dim(M)}$ of $\phi_*^{n(R)}(R)$ satisfy

$$0 < |\lambda_R^1| < \cdots < |\lambda_R^{\ell-1}| < 1 = |\lambda_R^\ell| < |\lambda_R^{\ell+1}| < \cdots < |\lambda_R^{\dim(M)}|. \tag{1}$$

After a final perturbation, we have that R has index ℓ, ending the proof of the corollary. $\square$

Finally, a minor modification of the proof of Corollary 3.6 gives the following:

Corollary 3.7. *Consider a heterodimensional cycle $\Gamma(\varphi, U, P, Q)$ satisfying the hypothesis of Theorem 3.1. Moreover suppose that there is a dominated splitting $F_1 \oplus \cdots \oplus F_i \oplus \cdots \oplus F_k$ over $\Lambda_\varphi(U)$ such that the moduli of the Jacobians of φ restricted to F_i along the orbits of Q and P are strictly bigger and less than one, respectively.*

Then there is a diffeomorphism ϕ arbitrarily C^1-close to φ with a hyperbolic periodic point $R \in \Lambda_\phi(U)$ such that the modulus of the Jacobian of $\phi^{n(R)}$ over F_i at R is equal to one.

Proof: Consider the dominated splittings

$$E_1 = F_1 \oplus \cdots \oplus F_{i-1}, \quad E_2 = F_i \quad E_3 = F_{i+1} \oplus \cdots \oplus F_k.$$

Just observe that by Lemma 3.5 we can choose the transitions T_i preserving the dominated splitting $E_1 \oplus E_2 \oplus E_3$. The result follows arguing as in Corollary 3.6. $\square$

3.3 End of the proof of Theorem A

We need the following lemma,

Lemma 3.8. ([BDP, Lemma 5.4]) *Let V be an open set of M and R_φ a hyperbolic periodic point of a diffeomorphism φ such that its relative homoclinic class in V, $\overline{H_{R_\varphi}(V)}$, is non trivial. Then there is a diffeomorphism ϕ arbitrarily $\mathcal{C}^1$-close to φ such that $\overline{H_{R_\phi}(V)}$ contains a hyperbolic periodic point of the same index of R_ϕ whose eigenvalues are all real, positive and of multiplicity one.*

Under the hypothesis of Theorem A, this lemma allows us to assume that, after perturbing the original diffeomorphism and replacing the initial points P_φ and Q_φ by other points of $\Lambda_\varphi(U)$ of the same index, we can assume that the points P_φ and Q_φ of $\Lambda_\varphi(U)$ have real positive eigenvalues of multiplicity one. To see why this is so just observe that, by Theorem 2.3, after a $\mathcal{C}^1$-perturbation of φ we can assume that $\overline{H_{P_\varphi}(U)} = \overline{H_{Q_\varphi}(U)} \subset \Lambda_\varphi(U)$, thus these two relative homoclinic classes are non-trivial. Hence we can now apply Lemma 3.8 to such homoclinic classes to get the periodic points (of indices p and q) in $\Lambda_\varphi(U)$ with real positive eigenvalues of multiplicity one. So we lose no generality assuming that the points P_φ and Q_φ in Theorem A have real positive eigenvalues of multiplicity one. Using Lemma 2.5 and Corollary 3.6 one gets:

Lemma 3.9. *Given $p > q$ and $\ell \in]q, p]$ let $\varphi \in \mathcal{M}(U)$ be a diffeomorphism with two hyperbolic periodic points P_φ and Q_φ in $\Lambda_\varphi(U)$ of indices p and q having positive real eigenvalues of multiplicity one. Then there is $\phi \in \mathcal{M}(U)$ arbitrarily C^1-close to φ having a hyperbolic periodic point of index ℓ in $\Lambda_\phi(U)$.*

Proof: By hypothesis, the continuations P_ϕ and Q_ϕ of P_φ and Q_φ are transitively related for every ϕ in a neighbourhood of φ in $\mathcal{M}(U)$ (just observe that set $\Lambda_\phi(U)$ is robustly transitive and P_ϕ and Q_ϕ belong to $\Lambda_\phi(U)$). Hence we can apply Lemma 2.5 to P_φ and Q_φ to create a heterodimensional cycle $\Gamma(\psi, U, P_\psi, Q_\psi)$ for some ψ arbitrarily close to φ. Corollary 3.6 now gives ϕ close to ψ (thus close to φ) with a periodic point of index ℓ in $\Lambda_\phi(U)$, ending the proof of the lemma. □

Given $\varphi \in \mathcal{M}(U)$ consider a neighbourhood $\mathcal{U}_\varphi$ of φ in $\mathcal{M}(U)$ such that every $\psi \in \mathcal{U}_\varphi$ has hyperbolic periodic points of indices q and p. Let $\mathcal{H}_j$ be the set of diffeomorphisms $\psi \in \mathcal{U}_\varphi$ having some hyperbolic periodic point of index j in $\Lambda_\varphi(U)$. Applying Lemma 3.9 finitely many times, one gets that the sets $\mathcal{H}_j$, $j \in [q, p]$, are dense in $\mathcal{U}_\varphi$.

Theorem A now follows by observing that, for every j, the set $\mathcal{H}_j$ is open. Now it is enough to consider the set $\cap_q^p \mathcal{H}_j$, which is a dense open subset of $\mathcal{U}_\varphi$. This ends the proof of Theorem A.

4 Hyperbolicity of the extremal bundles

In this section we prove Theorem B. For that, as in the hypotheses of this theorem, consider an open set U of a compact manifold M and $q \in \mathbb{N}^*$ and let $\mathcal{U}$ be a C^1-open set of $\mathrm{Diff}^1(M)$ such that for every diffeomorphism $\phi \in \mathcal{U}$ the set $\Lambda_\phi(\bar{U})$ has a dominated splitting $E_\phi \oplus F_\phi$ with $\dim(E_\phi(x)) = q$ for all $x \in \Lambda_\phi(\bar{U})$. Suppose that every $\phi \in \mathcal{U}$ has no periodic points of index $r < q$. Then we prove that the bundle E_ϕ is uniformly contracting for every $\phi \in \mathcal{U}$.

The proof of this result follows using the arguments in [M_2] after some small technical modifications. So here we will just sketch this proof, emphasizing the main modifications that we need to introduce.

The results in [M_2] are formulated in terms of *families of periodic sequences of linear maps.* It is considered the family obtained by taking all the diffeomorphism ϕ in an open set of $\mathrm{Diff}^1(M)$ and the restrictions of the derivatives of these diffeomorphisms to their periodic orbits. He consider perturbations of this system of linear maps without caring if such perturbations come from perturbations of the initial diffeomorphism. However, a Lemma of Franks' (see Lemma 3.4 above) allows one to perform dynamically the perturbation of the derivative: given a diffeomorphism φ and a periodic point x of φ, to each perturbation A of the derivative φ_* throughout the orbit of x corresponds a diffeomorphism ψ C^1-close to φ which preserves the φ-orbit of x and such that $A(z) = \psi_*(z)$ for all z in the φ-orbit of x.

We begin by recalling some results about dominated splittings, see next section. In Section 4.2 we recall the terminology about families of periodic linear systems and some results in [M_2]. Finally, in Section 4.3 we prove Theorem B.

4.1 Remarks on dominated splittings

In this subsection we state precisely some folklore results on dominated splittings. Before that let us observe that if $\Lambda_\varphi(U)$ is a robustly transitive, then, by definition, it is a φ-invariant compact subset of U which is the maximal φ-invariant set of $\bar{U}$. This implies that, for any neighbourhood V of $\Lambda_\varphi(U)$ and every diffeomorphism ϕ close to φ, the set $\Lambda_\phi(U)$ coincides with $\Lambda_\phi(\bar{U})$ and is contained in V. Thus $\Lambda_\phi(U)$ depends lower-semi-continuously on ϕ. We say that $\Lambda_\phi(U)$ is the *continuation* of $\Lambda_\varphi(U)$ for ϕ.

Lemma 4.1. *Let φ be a diffeomorphism and U an open subset of M such that $\Lambda_\varphi(U)$ coincides with $\Lambda_\varphi(\bar{U})$ and admits a dominated splitting $T_{\Lambda_\varphi(U)}M = E \oplus F$, $E \prec F$. Then, for every diffeomorphism ψ close enough to φ, there is a unique dominated splitting $E_\psi \oplus F_\psi$, $E_\psi \prec F_\psi$, defined over $\Lambda_\psi(U)$ such that $\dim(E_\psi) = \dim(E)$.*

The splitting $E_\psi \oplus F_\psi$ above is the *continuation* of $E \oplus F$. Moreover, the continuations E_ψ and F_ψ depend continuously on ψ. This lemma also holds for dominated splitting with an arbitrary number of bundles.

Proof: Let us just sketch the proof of the lemma. By the definition of domination, there is a strictly φ_*-invariant continuous cone field $\mathcal{C}^+$ defined over $\Lambda_\varphi(U)$ such that the bundle F is obtained as the intersection of the forward φ_*-iterates of the cones of $\mathcal{C}^+$. Similarly, there is a strictly (φ_*^{-1})-invariant continuous cone field $\mathcal{C}^-$ defined over $\Lambda_\varphi(U)$ such that the intersections of the backward iterates of $\mathcal{C}^-$ define E. These cone fields can be extended continuously to invariant cone fields $\mathcal{C}_0^+$ and $\mathcal{C}_0^-$ defined on a compact neighbourhood V of $\Lambda_\varphi(U)$.

Observe that every ψ close to φ let invariant the cone fields $\mathcal{C}_0^+$ and $\mathcal{C}_0^-$ and recall that $\Lambda_\psi(U) \subset V$. We now define the bundles E_ψ and F_ψ as the intersection of the (backward and forward, respectively) iterates by ψ_* of the cones of $\mathcal{C}_0^-$ and $\mathcal{C}_0^+$, respectively. By construction, the splitting $E_\psi \oplus F_\psi$ is dominated and satisfies $\dim E_\psi = \dim E$.

For the continuous dependence of the bundles E_ψ and F_ψ on the diffeomorphism ψ we refer the reader to [BDP, Lemma 1.4], for instance. This ends the sketch of the proof. □

Lemma 4.2. ([BDP, Lemma 1.4]). *Let ϕ be a diffeomorphism and Σ a ϕ-invariant set having a dominated splitting $E \oplus F$. Then this splitting can be extended (in a dominated way) to the closure of Σ.*

Remark 4.3. *Let φ be a diffeomorphism, K a transitive φ-invariant compact set, $T_K M = E_1 \oplus E_2 \oplus \cdots \oplus E_m$ the finest dominated splitting of φ over K, and $\Sigma \subset K$ a φ-invariant dense subset of K. Then the finest dominated splitting of φ over Σ is given by the restriction to Σ of the bundles E_i.*

Proof of the remark: We argue by contradiction, suppose that there is a dominated splitting over Σ which refines the splitting given by the restrictions to Σ of the bundles E_i. Then, by Lemma 4.2, such a splitting can be extended to the whole K, contradicting that the splitting $E_1 \oplus \cdots \oplus E_m$ is the finest one. □

Let us state a final result whose proof we omit.

Remark 4.4. *Let φ be a diffeomorphism and E a φ_*-invariant bundle defined on a φ-invariant compact set K_1. Consider any φ-invariant dense subset K_2 of K_1. Then we have the following:*

- *The bundle E is uniformly hyperbolic over K_1 if and only if its restriction to K_2 is uniformly hyperbolic.*

- *The diffeomorphism φ contracts (resp. expands) uniformly the volume in E over K_1 if and only if it contracts uniformly (resp. expands) the volume in E over K_2.*

4.2 Families of periodic sequences of linear maps and dominated splittings

We begin this section by recalling some definitions in [M_2].

Definition 4.1.

1. *A* periodic sequence of a linear maps *is a periodic map $\xi\colon \mathbb{Z} \to GL(N,\mathbb{R})$, $n \mapsto \xi_n$, such that the sequence of norms $\|\xi_n\|$ and $\|\xi_n^{-1}\|$ are uniformly bounded (independently of n). We denote this family by $\{\xi_n\}$.*

2. *A periodic sequence of linear maps $\{\xi_n\}$ of period n is called* contracting *if the product $\xi_{n-1} \circ \cdots \circ \xi_0$ is an uniform contraction, i.e., all its eigenvalues have modulus strictly less than 1.*

3. *A family $\Xi = \{\xi^{(\alpha)}\}_{\alpha\in\mathcal{A}}$ of periodic sequences of linear maps is* robustly contracting[1] *if there is $\varepsilon > 0$ such that any family $\Theta = \{\theta^{(\alpha)}\}_{\alpha\in\mathcal{A}}$ having the same period function $n(\alpha)$ and ε-close to Ξ (i.e., $\|\theta^\alpha_n - \xi^\alpha_n\| < \varepsilon$ for all $\alpha \in \mathcal{A}$ and $n \in \mathbb{Z}$) is contracting.*

The example of family of periodic sequence of linear maps that will be play a key role in the proof of Theorem B is obtained as follows. Let $\phi \in \mathcal{U}$, $\mathcal{U}$ as in Theorem B, and $\delta > 0$ such that every diffeomorphism ψ which is 2δ-C^1-close to ϕ belongs to $\mathcal{U}$. Now let $\mathcal{A}_\phi$ be the set of pairs $\alpha = (x,\psi)$ such that ψ is δ-close to ϕ and the ψ-orbit of x is contained in U and periodic. Consider now some trivialization of the bundles E_ψ (as in Theorem B) over the set of periodic points (by choosing an orthonormal basis of $E_\psi(x)$) and for each $\alpha = (x,\psi) \in \mathcal{A}_\phi$ define ξ^α by the restrictions of the differential ψ_* to $\{E_\psi(\psi^i(x))\}_{i\in\mathbb{Z}}$. We now have that $\Xi_\phi = \{\xi^\alpha\}_{\alpha\in\mathcal{A}_\phi}$ is a family of periodic sequences of linear maps.

Lemma 4.5. *The family Ξ_ϕ defined above is robustly contracting.*

Proof: The proof if by contradiction. Otherwise, there are $(x,\psi) \in \mathcal{A}_\phi$ and a linear map ν corresponding to a perturbation of the restriction of the differential of ψ to E_ψ along the periodic ψ-orbit of x, having an eigenvalue of modulus bigger or equal than one, i.e.,

$$\nu(\psi^{n(x)-1}(x)) \circ \cdots \circ \nu(x)\colon E_\psi(x) \to E_\psi(x)$$

has an eigenvalue λ such that $|\lambda| \geq 1$, where $n(x)$ is the ψ-period of x.

Using Lemma 3.4 we get a diffeomorphism ζ close to ψ, thus in $\mathcal{U}$, such that x is a periodic point of $\Lambda_\zeta(\bar{U})$ and

$$\zeta_*^{n(x)}(x) = \zeta_*(\zeta^{n(x)-1}) \circ \cdots \zeta_*(x) = \nu(\psi^{n(x)-1}(x)) \circ \cdots \circ \nu(x).$$

[1]This notion is called *uniformly contracting* in [M_2], but we rename it to avoid ambiguity with the now usually accepted notion of uniform hyperbolicity or uniform contraction.

Thus the restriction of $\zeta_*^{n(x)}(x)$ to $E_\zeta(x)$ has at most $(q-1)$ eigenvalues of modulus (strictly) less than one. On the other hand, by the domination $E_\psi \prec F_\psi$, the eigenvalues of the restriction of $\zeta_*^{n(x)}(x)$ to $F_\zeta(x)$ are all strictly bigger than one in modulus. This implies that there is a periodic point x in $\Lambda_\zeta(\bar{U})$ of index (strictly) less than q, contradicting the definition of $\mathcal{U}$. This contradiction ends the proof of the lemma. □

We now borrow the following lemma from [M$_2$].

Lemma 4.6. ([M$_2$, Lemma II.7]). *Let $\{\xi^{(\alpha)}, \alpha \in \mathcal{A}\}$ be a robustly contracting family of periodic sequences of isomorphisms of $\mathbb{R}^N$. Then there exist $K>0$, $0<\lambda<1$ and $m \in \mathbb{N}^*$ such that:*

a) if $\alpha \in \mathcal{A}$ and ξ^α has minimum period $n \geq m$, then

$$\prod_{j=0}^{k-1} \left\| \prod_{i=0}^{m-1} \xi^{(\alpha)}_{i+mj} \right\| \leq K\lambda^k,$$

where k is the integer part of n/m;

b) for all $\alpha \in \mathcal{A}$

$$\limsup_{n\to+\infty} \frac{1}{n} \sum_{j=0}^{n-1} \log\left(\left\| \prod_{i=0}^{m-1} \xi^{(\alpha)}_{i+mj} \right\| \right) < 0.$$

Applying Lemma 4.6 to the family Ξ_ϕ defined above we get the next proposition which is a reformulation of [M$_2$, Proposition II.1]:

Proposition 4.7. *Let $\phi \in \mathcal{U}$ ($\mathcal{U}$ as in Theorem B). Then there are a neighborhood $\mathcal{V}$ of ϕ and constants $K>0$, $m \in \mathbb{N}^*$ and $0<\lambda<1$ such that for every $g \in \mathcal{V}$ and every periodic point x of ψ whose orbit is contained in U one has:*

a) If x has minimum period $n \geq m$ then

$$\prod_{i=0}^{k-1} \left\| (\psi^m)_*(\psi^{mi}(x))|_{E_\psi(\psi^{mi}(x))} \right\| \leq K\lambda^k,$$

where k is the entire part of n/m.

b) Moreover,

$$\limsup_{r\to+\infty} \frac{1}{r} \sum_{i=0}^{r-1} \log\left(\left\| (\psi^m)_*(\psi^{mi}(x))|_{E_\psi(\psi^{mi}(x))} \right\| \right) < 0.$$

Theorem B will be a consequence of Proposition 4.7 and the Mañé's Ergodic Closing Lemma that we now recall, for completeness:

Theorem 4.8. (Ergodic Closing Lemma, [M$_2$, Theorem A]). *Consider a diffeomorphism ϕ defined on a compact manifold. Then there is a ϕ-invariant set $\Sigma(\phi)$ (named set of* well closable points*of ϕ) such that:*

1. *The set $\Sigma(\phi)$ has total measure (i.e. $\mu(\Sigma(\phi)) = 1$ for every ϕ-invariant probability measure μ).*

2. *For every $x \in \Sigma(\phi)$ and $\varepsilon > 0$ there is a diffeomorphism ψ, which is ε-close to ϕ in the C^1-topology, such that x is periodic for ψ and the distance $\operatorname{dist}(\phi^i(x), \psi^i(x)) < \varepsilon$ for all $i \in [0, n(x,\psi)]$, where $n(x,\psi)$ is the period of x for ψ.*

4.3 End of the proof of Theorem B

The proof of the theorem now follows almost exactly as the proof of [M$_2$, Theorem B], see pages 520-524. Let us recall the main steps of this proof and point out the changes we need to introduce.

Proof: Let $\phi \in \mathcal{U}$. By compactness of the set $\Lambda_\phi(\bar{U})$, as in [M$_2$] to get the uniform contraction of the bundle E_ϕ it is enough to see that

$$\liminf_{n\to+\infty} ||\phi_*^n|_{E_\phi(x)}|| = 0.$$

We argue by contradiction. If ϕ_* is not uniformly contracting on E_ϕ over $\Lambda_\phi(\bar{U})$ then there are a constant $\kappa > 0$, point $x \in \Lambda_\phi(\bar{U})$ and $n_0 \in \mathbb{N}$ such that

$$\|\phi_*^n|_{E_\phi(x)}\| > \kappa > 0$$

for every $n > n_0$. We now choose a sequence j_n, $j_n \to +\infty$, such that the sequence of probabilities μ_n defined by

$$\mu_n = \frac{1}{j_n}\sum_{i=0}^{j_n-1} \delta(\phi^{mi}(x))$$

converges (in the weak topology) to a probability μ, where $\delta(z)$ is the Dirac measure at the point z and m is as in Proposition 4.7.

Let $\varphi^\phi = \log ||\phi_*^m|_{E_\phi}||$. By Lemma 4.1 the bundle E_ϕ is continuous on $\Lambda_\phi(\bar{U})$, so φ^ϕ is continuous on $\Lambda_\phi(\bar{U})$. By the choice of x, one has $\int \varphi^\phi \, d\mu_n \geq 0$ for every large enough n, so $\int \varphi^\phi \, d\mu \geq 0$. Using the Birkhoff's Theorem and the Ergodic Closing Lemma we get a point $p \in \Lambda_\phi(\bar{U}) \cap \Sigma(\phi)$ such that

$$\lim_{n\to+\infty} \frac{1}{j_n}\sum_{i=0}^{j_n-1} \log \|\phi_*^m|_{E_\phi(\phi^{mi}(p))}\| \geq 0.$$

By item (b) of Proposition 4.7, the point p is not periodic. Now, by Theorem 4.8, there is ψ arbitrarily C^1-close to ϕ (so $\psi \in \mathcal{V} \subset \mathcal{U}$, $\mathcal{V}$ as in Proposition 4.7) such that p is a periodic point of ψ of period $n(p)$ and the distance $\operatorname{dist}(\phi^i(p), \psi^i(p))$ is less than an arbitrarily small $\varepsilon > 0$ for every $i \in [0, n(p)]$. Observe that since p is not periodic for ϕ, the period $n(p)$ goes to infinity as ε goes to zero, i.e., ψ tends to ϕ.

Since the fibers $E_\psi(y)$ vary continuously with (y,ψ), recall Lemma 4.1, the function

$$\varphi^\psi(y) = \log ||\psi_*^m|_{E_\psi(y)}||$$

is continuous. Now for λ as in Proposition 4.7 take λ_0 and $n_0 \in \mathbb{N}^*$ such that $\lambda < \lambda_0 < 1$ and for every $n \geq n_0$ one has

$$\frac{1}{n}\sum_{i=0}^{n-1} \varphi^\phi(\phi^{mi}(p)) \geq \frac{1}{2}\log(\lambda_0).$$

We can also assume that $K\lambda^n < \lambda_0^n$ for every $n \geq n_0$. So if ψ is close enough to ϕ then

$$|\varphi^\psi(\psi^i(p)) - \varphi^\phi(\phi^i(p))| < |\frac{1}{2}\log(\lambda_0)|$$

for every $i \in [0, n(p)]$. Moreover, the entire part k of $n(p)/m$ is greater than n_0. Thus

$$\frac{1}{k}\sum_{i=0}^{k-1} \varphi^\psi(\psi^{mi}(p)) \geq \log(\lambda_0) > \frac{1}{k}\log(K\lambda^k),$$

contradicting item (a) of Proposition 4.7. This contradiction ends the proof of Theorem B. □

5 Proof of Theorem D

5.1 Perturbation of the derivative at periodic points

In this section we recall some results from [BDP]. These results are formulated in terms of *families of periodic linear systems,* that is, considering the differential of the diffeomorphism as an abstract linear cocycle over the set $\Lambda_\varphi(U)$ and perturbations of this cocycle, without caring if such perturbations come from perturbations of the diffeomorphism. However, as in Section 4, Lemma 3.4 allows us to perform dynamically the final abstract cocycle. Let us explain these results in a detailed way.

Given a diffeomorphism φ and a hyperbolic periodic point P_φ of φ of index p denote by Σ_{P_φ} the subset of $\overline{H(P_\varphi, U)}$ of hyperbolic periodic points R of index p homoclinically related to P_φ, i.e., $W^s(R) \pitchfork W^u(P_\varphi) \neq \emptyset$ and $W^u(R) \pitchfork W^s(P_\varphi) \neq \emptyset$. Observe that in our setting we can assume that Σ_{P_φ} is not trivial (different to the orbit of P_φ).

As above, given $x \in \Sigma_{P_\varphi}$ denote by M_x the matrix $M_x = \varphi_*^{n(x)}(x)\colon T_xM \to T_xM$, where $n(x)$ is the period of x. The first important property formalized in [BDP] is that the matrices M_x corresponding to different points of Σ_{P_φ} (the derivatives of $\varphi^{n(x)}$ at these points x) can be *essentially* multiplied how many times as one wants, and the resulting product corresponds to a matrix of the system at some different point. More precisely,

Lemma 5.1. *Let φ be a diffeomorphism and P_φ a hyperbolic periodic point of φ. Consider any pair of periodic points of x and y of φ in Σ_{P_φ} and $\varepsilon > 0$. Suppose that M_x and M_y let invariant dominated splittings*

$$T_xM = E_x^1 \oplus \cdots \oplus E_x^k, \quad E_i(x) \prec E_{i+1}(x), \quad \textit{and} \quad T_yM = E_y^1 \oplus \cdots \oplus E_y^k, \quad E_i(y) \prec E_{i+1}(y),$$

such that $\dim(E_x^i) = \dim(E_y^i)$ for every i. Then there is $\delta \in]0, \varepsilon[$ satisfying the following property:

Given any pair of δ-perturbations $\tilde{M}_x$ and $\tilde{M}_y$ of M_x and M_y, respectively, $\tilde{M}_x\colon T_xM \to T_xM$ and $\tilde{M}_y\colon T_yM \to T_yM$, there are linear maps

$$T_1\colon T_xM \to T_yM \quad \textit{and} \quad T_2\colon T_yM \to T_xM$$

preserving the dominated splittings above (i.e., $T_1(E_x^i) = E_y^i$ and $T_2(E_y^i) = E_x^i$ for every i) such that, for any $n \geq 0$ and $m \geq 0$, there are a periodic point $z \in \Sigma_{P_\varphi}$ and an ε-perturbation of φ_* along the orbit of z,

$$A^i\colon T_{\varphi^i(z)}M \to T_{\varphi^{i+1}(z)}M, \quad i = 0, \ldots, n(z) - 1,$$

such that

$$\tilde{M}_z = A^{n(z)-1} \circ \cdots \circ A^0 \colon T_zM \to T_zM$$

is conjugate to the product $T_2 \circ M_y^m \circ T_1 \circ M_x^n$.

Remark 5.2. *In fact, in* [BDP] *it is shown that Lemma 5.1 holds for any finite number of orbits* $x_1, \ldots, x_k$ *of* Σ_{P_φ}. *This allows us to get linear maps* $T_i \colon T_{x_i}M \to T_{x_{i+1}}M$ *preserving a dominated splitting such that, for every* $n_1, \ldots, n_k$, *there are a point* $z \in \Sigma_{P_\varphi}$ *and perturbations* A^i *of the derivative of* φ_* *at* $\varphi^i(z)$ *such that* $\tilde{M}_z = A^{n(z)-1} \circ \cdots \circ A^0$ *is conjugate to* $T_k \circ M_{x_k}^{n_k} \circ \cdots \circ T_2 \circ M_{x_2}^{n_2} \circ T_1 \circ M_{x_1}^{n_1}$.

The maps T_i correspond to the so called *transitions*, recall also Theorem 3.1. The fact that the transitions can be taken preserving a dominated splitting has been proved in [BDP, Lemma 4.13]. This property is the basis of the proof of the following result:

Lemma 5.3. *Let* $E_1 \oplus \cdots \oplus E_m$, $E_i \prec E_{i+1}$, *be the finest dominated splitting of* TM *over* Σ_{P_φ} *of* φ_*. *Then, for every* $\varepsilon > 0$, *there are a dense subset* Σ_ε *of* Σ_{P_φ} *and an* ε*-perturbation* A_ε *of* φ_* *preserving the splitting* $E_1 \oplus \cdots \oplus E_m$ *such that, for every* $R \in \Sigma_\varepsilon$, *the restriction of the linear maps*

$$M_{A_\varepsilon}(R) = A_\varepsilon(\varphi^{n(R)-1}(x)) \circ \cdots \circ A_\varepsilon(\varphi(x)) \circ A_\varepsilon(x)$$

to $E_i(R)$ *is a homothety.*

Moreover, if there are $i \in \{1, \ldots, m\}$ *and* $Q \in \Sigma_{P_\varphi}$ *such that the modulus of the Jacobian of the restriction of* $\varphi_*^{n(Q)}$ *to* $E_i(Q)$ *is one then* R *can be taken such that the restriction of* $M_{A_\varepsilon}(R)$ *to* $E_i(R)$ *is identity.*

This lemma is a consequence of [BDP, Propositions 2.4 and 2.5]. To see that these propositions can be applied in our context, one just needs to observe that the restriction of φ_* to each bundle E_i (over Σ_{P_φ}) defines a periodic linear system with transitions. For that it is enough to recall that the transitions of φ_* can be chosen preserving the bundles E_j of the dominated splitting (see [BDP, Section 4]).

Given a hyperbolic linear map A of an Euclidean space (i.e., without eigenvalues of modulus equal to 1) the *index of* A is the number of eigenvalues of A of modulus less than 1, counted with multiplicity.

Lemma 5.4. ([BDP, Lemma 4.16]) *Given* $\varepsilon > 0$ *there exist* $x \in \Sigma_{P_\varphi}$ *and an* ε*-perturbation of* φ_* *along the orbit of* x *such that the corresponding matrix* M_x *has index* p, $p = index(P_\varphi)$, *and all the eigenvalues of* M_x *are real, positive and with multiplicity* 1.

5.2 Tangencies and codimension one heterodimensional cycles

The existence of non-real eigenvalues in the *central direction* of the saddles in a (codimension one) heterodimensional cycle produces homoclinic tangencies. That is formalized in the following result we export from [DR].

Let A be a linear map of an n-dimensional Euclidean space E, we say that a non-real eigenvalue $\lambda \in (\mathbb{C} \setminus \mathbb{R})$ of A has *rank* ℓ if there are $(\ell - 1)$ eigenvalues (counted with multiplicity) of A of modulus strictly less than $|\lambda|$ and $(n - \ell - 1)$ eigenvalues of modulus strictly bigger than $|\lambda|$. A periodic point P of a diffeomorphism φ has a *non-real eigenvalue of rank* ℓ if its derivative $\varphi_*^{n(P)}(P)$ has a a non-real eigenvalue of rank ℓ.

Lemma 5.5. *Let $\Gamma(\phi, U, R^1_\phi, R^2_\phi)$ be a codimension one heterodimensional cycle associated to hyperbolic periodic points of indices $(r+1)$ and r. Suppose that R^1_ϕ (resp. R^2_ϕ) has a non-real eigenvalue of rank r (resp. $r+1$). Then there is ψ arbitrarily close to ϕ with a homoclinic tangency associated to R^2_ψ (resp. R^1_ψ) in $\Lambda_\psi(U)$.*

Proof: Just observe that if R^1_ϕ has a non-real eigenvalue of rank r then the unstable manifold of R^2_ϕ spiralizes around $W^u(R^1_\phi)$. Now unfolding the cycle $\Gamma(\phi, U, R^1_\phi, R^2_\phi)$ one gets a homoclinic tangency associated to the continuation of R^2_ϕ. See [DR, Section 8.1] for details. □

5.3 Proof of Theorem D

Consider $\varphi \in \mathcal{P}(U)$ and its finest dominated splitting $E_1(\varphi) \oplus \cdots \oplus E_{m(\varphi)}(\varphi)$ over $\Lambda_\varphi(U)$. By Lemma 4.1, the continuation of this splitting over $\Lambda_\phi(U)$ is uniquely defined for every ϕ close to φ. Denote such a continuation by $E_1(\phi) \oplus \cdots \oplus E_{m(\varphi)}(\phi)$. By Lemma 4.1, the the number $m(\varphi)$ of bundles of the finest dominated splitting of $\Lambda_\varphi(U)$ is lower semi-continuous, thus locally constant in an open and dense subset $\mathcal{P}_1(U)$ of $\mathcal{P}(U)$. Moreover, the dimensions of the bundles of the finest dominated splitting are also locally constant in $\mathcal{P}_1(U)$. So there is an open and dense subset $\mathcal{O}(U)$ of $\mathcal{P}(U)$ where $m(\varphi)$ and the dimensions of the bundles of the finest dominated splitting are continuous functions. This set $\mathcal{O}(U)$ is the open and dense subset of $\mathcal{P}(U)$ announced in Theorem D.

Observe that it is enough to prove the theorem for a connected component of $\mathcal{O}(U)$. So from now on we restrict our attention to a fixed connected component $\mathcal{O}_0$ of $\mathcal{O}(U)$.

Given $\varphi \in \mathcal{O}_0$ consider the finest dominated splitting of $\Lambda_\varphi(U)$, say $T_{\Lambda_\varphi(U)}M = E_1(\varphi) \oplus E_2(\varphi) \oplus \cdots \oplus E_{m(\varphi)}(\varphi)$, as the dimensions and the number of bundles of the splitting do not depend on $\varphi \in \mathcal{O}_0$ from now on we will omit such a (in)dependence.

Let us now introduce some notations. For simplicity write $p = i_c$ and $q = i_s$ (the maximum and minimum indices of the hyperbolic periodic points of $\Lambda_\varphi(U)$). Given i and j in $\{1, \ldots, m\}$, with $i < j$, let

$$E_i^j = E_i \oplus E_{i+1} \oplus \cdots \oplus E_j.$$

Denote by d_i and d_i^j the dimensions of E_i and E_i^j, respectively, thus $d_i^j = \sum_{k=i}^j d_k$. We define i_q and i_p by the relations

$$d_1^{i_q - 1} < q \le d_1^{i_q} \quad \text{and} \quad d_1^{i_p - 1} < p \le d_1^{i_p}.$$

To prove Theorem D it is enough to see the following:

(A) $d_1^{i_q} = q$ and $d_{i_p+1}^m = \dim(M) - p$,

(B) $d_j = 1$ and the bundle E_j is not uniformly hyperbolic for all $j \in \{i_q + 1, \ldots, i_p\}$,

(C) $E_1^{i_q}$ and $E_{i_p+1}^m$ are uniformly contracting and expanding, respectively.

In Lemmas 5.6, 5.7 and 5.8 we will prove these items.

Lemma 5.6. (Proof of (A)). *$d_1^{i_q} = q$ and $d_{i_p+1}^m = \dim(M) - p$.*

Proof: Let us prove the first part of the lemma. The proof is by contradiction, assume that $d_1^{i_q} > q$, then, by definition of $d_1^{i_q}$, one has

$$d_1^{i_q-1} < q < q+1 \leq d_1^{i_q} = d_1^{i_q-1} + d_{i_q},$$

thus

$$d_{i_q} \geq 2. \tag{2}$$

By Proposition 2.4 and the definition of $\mathcal{O}_0$, there is a diffeomorphism $\varphi \in \mathcal{O}_0$ with a hyperbolic periodic point Q_φ of index q such that Σ_{Q_φ} is dense in $\Lambda_\varphi(U)$. By Remark 4.3, the finest dominated splitting of φ over Σ_{Q_φ} is the restriction to Σ_{Q_φ} of the bundles E_i.

By equation (2), E_{i_q} is undecomposable and has dimension d_{i_q} equal or bigger than 2. Applying Lemma 5.3 to the set Σ_{Q_φ} and the bundle E_{i_q}, we get $R_\varphi \in \Sigma_{Q_\varphi}$ of period $n(R_\varphi)$ and a perturbation A of φ_* throughout the φ-orbit of R_φ such that

$$M_A(R_\varphi)) = A(\varphi^{n(R_\varphi)-1}(R_\varphi)) \circ \cdots \circ A(\varphi(R_\varphi)) \circ A(R_\varphi)$$

is a homothety in $E_{i_q}(R_\varphi)$. We observe that the perturbation A of φ_* can be obtained (and we do so) such that its restrictions to the bundles $E_k(R_\varphi)$, $k \neq i_q$, coincide with φ_*. Thus, since all points of Σ_{Q_φ} have index q, one has that, for every $T_\varphi \in \Sigma_{Q_\varphi}$, the bundles $E_j(T_\varphi)$, $j > i_q$, correspond to expanding eigenvalues of $\varphi_*^{n(T_\varphi)}$. Hence the number of contracting eigenvalues of $M_A(R_\varphi)$ is at most $d_1^{i_q}$.

First, if the ratio of this homothety (the restriction of $M_A(R_\varphi)$ to $E_{i_q}(R_\varphi)$) is bigger or equal than one, using Lemma 3.4, one gets ϕ close to φ ($\phi \in \mathcal{O}_0$) with a hyperbolic periodic point $R_\phi \in \Lambda_\phi(U)$ having at most $d_1^{i_q-1}$ contracting eigenvalues. By hypothesis, $d_1^{i_q-1} < q$, thus the index of R_ϕ is strictly inferior than q, contradicting the definition of q (minimality of the index of the points of $\Lambda_\phi(U)$, $\phi \in \mathcal{P}(U)$).

So we can assume that the ratio of the homothety $M_A(R_\varphi)|_{E_{i_q}(R_\varphi)}$ is less than one. As the restriction of $\varphi_*^{n(R_\varphi)}$ to each $E_i(R_\varphi)$, $i > i_q$, has expanding eigenvalues, the index of R_ϕ is exactly $d_1^{i_q}$. Now, the definition of p implies that $d_1^{i_q} \leq p$.

Write $\ell = d_1^{i_q} \leq p$. Since all the eigenvalues of the restriction of $\phi_*^{n(R_\varphi)} = M_A(R_\varphi)$ to $E_{i_q}(R_\phi)$ are equal and $\dim(E_{i_q}(R_\phi)) \geq 2$, using again Lemma 3.4, one gets a diffeomorphisms υ close to ϕ such that R_υ has index ℓ and $\upsilon_*^{n(R_\upsilon)}(R_\upsilon)$ has a contracting non-real eigenvalue of rank $(\ell - 1)$.

By Theorem A, since $q \leq \ell - 1$, there is a diffeomorphism ζ close to υ with a periodic point $S_\zeta \in \Lambda_\zeta(U)$ of index $(\ell - 1)$. Using Lemma 2.5, we obtain η close to ζ with a codimension one heterodimensional cycle in U associated to R_η and S_η, say $\Gamma(\eta, U, R_\eta, S_\eta)$. Since η can be taken arbitrarily close to υ we can assume that R_η has index ℓ and a non-real eigenvalue of rank $\ell - 1$ and that S_η has index $(\ell - 1)$. Finally, by Lemma 5.5 there is a diffeomorphism $\xi \in \mathcal{O}_0$ arbitrarily close to η with a homoclinic tangency in $\Lambda_\xi(U)$ associated to the point S_ξ of index $(\ell - 1)$, contradicting the definition of $\mathcal{P}(U)$. This ends the proof of the first assertion in the lemma.

Using the same arguments one gets that $d_{i_p+1}^m = (\dim(M) - p)$, so we omit this proof. □

Lemma 5.7. (Proof of (B)). *The bundle E_i is one dimensional and non-uniformly hyperbolic for all $i \in \{i_q + 1, \ldots, i_p\}$.*

Proof: Given $k \in \{i_q+1, \cdots, i_p\}$ let $\ell = d_1^k = \dim(E_1^k)$. Observe that by, Lemma 5.6, $q < \ell \leq p$.

The bundle E_k is not uniformly hyperbolic: We argue by contradiction. Otherwise, since E_k is undecomposable, it would be either uniformly contracting or expanding. In the first case, using the domination of the splitting, one has that every periodic point of $\Lambda_\varphi(U)$ has index bigger or equal than $\ell > q$, contradicting the definition of q. In the second case, again by the domination of the splitting, every periodic point of $\Lambda_\varphi(U)$ has index strictly less than $\ell \leq p$, contradicting the definition of p.

The bundle E_k is one-dimensional: The proof is by contradiction, assuming that $\dim(E_k) = d_k \geq 2$. By Theorem A and Proposition 2.4, there is $\varphi \in \mathcal{O}_0$ having a hyperbolic periodic point $R_\varphi \in \Lambda_\varphi(U)$ of index ℓ such that Σ_{R_φ} is dense in $\Lambda_\varphi(U)$. By Lemma 5.3, there are a perturbation A of φ_* and a point $S_\varphi \in \Sigma_{R_\varphi}$ such that the restriction of $M_A(S_\varphi)$ to $E_k(S_\varphi)$ is a homothety. Moreover, as above we can take A such that its restrictions to the bundles $E_i(S_\varphi)$, $i \neq k$, coincide with the one of φ_*.

Suppose, for instance, that the ratio of such a homothety is bigger than one. From $S_\varphi \in \Sigma_{R_\varphi}$ and the definition of Σ_{R_φ}, the restrictions of $\varphi_*^{n(S_\varphi)}$ to the bundles $E_i(S_\varphi)$, $i > k$, have only expanding eigenvalues. Thus the matrix $M_A(S_\varphi)$ has exactly $r = d_1^{k-1}$ contracting eigenvalues, where

$$q \leq d_1^{i_q} \leq d_1^{k-1} = r \leq d_1^{i_p-1} < d_1^{i_p} = p \quad \text{and} \quad r < r + d_k \leq r + 2 \leq p.$$

Using Lemma 3.4, we get $\phi \in \mathcal{O}_0$ with a hyperbolic periodic point $S_\phi \in \Lambda_\phi(U)$ of index r such that the restriction of ϕ_* to $E_k(S_\phi)$ is equal to A. After a new perturbation, if necessary, we can assume that $\phi_*^{n(S_\phi)}(S_\phi)$ has a expanding non-real eigenvalue of rank $(r+1)$.

As in the proof of Lemma 5.6, by Theorem A and Lemma 2.5, there is $\psi \in \mathcal{O}_0$ close to ϕ with a periodic point $T_\psi \in \Lambda_\psi(U)$ of index $(r+1) < p$ and a heterodimensional cycle $\Gamma(\psi, U, T_\psi, S_\psi)$, where S_ψ has index r and a (expanding) non-real eigenvalue of rank $(r+1)$. Finally, by Lemma 5.5, there is $\xi \in \mathcal{O}_0$ close to ψ with a homoclinic tangency associated to T_ξ, contradicting the definition of $\mathcal{O}_0$. This ends the proof of the lemma in this case. If the homothety given by the restriction of $M_A(S_\varphi)$ to E_k has ratio less than one the proof follows similarly.

□

Lemma 5.8. (Proof of (C)). *The bundles $E_1^{i_q}$ and $E_{i_p+1}^m$ are uniformly volume contracting and volume expanding, respectively.*

Proof: This lemma follows from Theorem B. To see, for instance, that $E = E_1^{i_q}$ is uniformly contracting just observe that the set $\mathcal{O}_0$ and the dominated splitting $E_1^{i_q} \oplus E_{i_q+1}^m$ satisfy the hypotheses of Theorem B, recall that, by Lemma 5.6, $q = d_1^{i_q} = \dim(E_1^{i_q})$.

The uniform expansion of $E_{i_p+1}^m$ follows analogously. This completes the proof of the lemma and of the theorem. □

6 Homoclinic tangencies

We now analyze the dimensions of the bundles of finest dominated splitting of a robust transitive set to deduce the different types of homoclinic bifurcations that that this set may exhibit.

We consider an open set U of M and an open set of diffeomorphisms $\mathcal{N}(U)$ such that, for every $\varphi \in \mathcal{N}(U)$, the set $\Lambda_\varphi(U)$ is robustly transitive and

- the maximum and the minimum of the indices of the periodic points of $\Lambda_\varphi(U)$ are constant, equal to p and q, respectively,
- the dimensions of the bundles of the finest dominated splitting of $\Lambda_\varphi(U)$ do not depend on $\varphi \in \mathcal{N}(U)$.

Let us observe that in this section we do not assume that there are no homoclinic tangencies in $\Lambda_\varphi(U)$, as in the previous section.

We use the notation introduced in Section 5.3 for the dimensions of the bundles of the finest dominated splitting. Recall that, with this notation and by definition, $q \leq d_1^{i_q}$ and $p \leq d_1^{i_p}$.

We say that a robustly transitive set $\Lambda_\varphi(U)$ has a *homoclinic tangency of rank* r if there is a periodic point $R_\varphi \in \Lambda_\varphi(U)$ of index r having a homoclinic tangency and such a point of tangency belongs to $\Lambda_\varphi(U)$.

Theorem F. *Let U, $\mathcal{N}(U)$, p and q as above. Consider any $\varphi \in \mathcal{N}(U)$.*

- *If $d_1^{i_q} > q$ then there is ϕ arbitrarily close to φ such that $\Lambda_\phi(U)$ has a homoclinic tangency of rank $(d_1^{i_q} - 1)$.*
- *If $d_1^{i_p} > p$ then there is ϕ arbitrarily close to φ such that $\Lambda_\phi(U)$ has a homoclinic tangency of rank $(d_1^{i_p-1} + 1)$.*
- *If $d_j \geq 2$ for some $j \in \{i_q + 1, \dots, i_p\}$ then, for every $k \in [d_1^{j-1} + 1, d_1^j)$, there is ϕ arbitrarily close to φ such that $\Lambda_\phi(U)$ has a homoclinic tangency of rank k.*

This theorem is a generalization of the result [DPU, Corollary G] for three dimensional robustly transitive sets, which says that the existence of an undecomposable bundle of dimension strictly bigger than one leads to the creation of homoclinic tangencies in a (non-hyperbolic) robustly transitive set.

The proof of Theorem F follows from a small modification of the the proofs of Lemmas 5.6 and 5.7 and involves heterodimensional cycles.

Denote by $\mathcal{T}_k(U)$, $k = 1, \dots, \dim(M) - 1$, the subset of $\mathcal{N}(U)$ of diffeomorphisms ϕ such that $\Lambda_\phi(U)$ has a homoclinic tangency of rank k. Theorem F now follows from the next two lemmas.

Lemma 6.1. *Under the hypothesis of Theorem F, we have the following*

- *If $d_1^{i_q} > q$ then $\mathcal{T}_{d_1^{i_q}-1}(U)$ is dense in $\mathcal{N}(U)$.*
- *If $d_1^{i_p} > p$ then $\mathcal{T}_{d_1^{i_p-1}+1}(U)$ is dense in $\mathcal{N}(U)$.*

Proof: First, observe that, by definition, if $d_1^{i_q} > q$ (resp. $d_1^{i_p} > p$) then $d_{i_q} > 1$ (resp. $d_{i_p} > 1$).

To prove the first part of the lemma it is enough to see that if $\varphi \in \mathcal{N}(U)$ and $d_1^{i_q} > q$ then there is υ arbitrarily close to φ such that $\Lambda_\upsilon(U)$ has a homoclinic tangency of rank $(d_1^{i_q} - 1)$. Let us recall that in the proof of Lemma 5.6, under the assumption that $\ell = d_1^{i_q} > q$, we got υ close to φ having a hyperbolic periodic point $R_v \in \Lambda_\nu(U)$ of index ℓ with a non-real eigenvalue of rank $(\ell - 1)$.

Since $q \leq \ell - 1 < p$, by Theorem A and Lemma 2.5, after a $\mathcal{C}^1$-perturbation of v, we can assume that v has a periodic point S_v of index $(\ell - 1)$ and a (codimension one) heterodimensional cycle $\Gamma(v, U, R_v, S_v)$ (R_v of index ℓ with a non-real eigenvalue of rank $(\ell - 1)$). By Lemma 5.5 there is ξ close to v with a homoclinic tangency associated to S_ξ. This ends the first part of the lemma.

The second part of the lemma follows similarly. □

Lemma 6.2. *Under the hypotheses of Theorem F, suppose that* $d_j \geq 2$, $j \in \{i_q + 1, \ldots, i_p - 1\}$. *Then, for every* $k \in [d_1^{j-1} + 1, d_1^j)$, *the set* $\mathcal{T}_k(U)$ *is dense in* $\mathcal{N}(U)$.

Proof: As in the previous lemma, given any $\varphi \in \mathcal{N}(U)$ with $d_j \geq 2$ and $k \in [d_1^{j-1} + 1, d_1^j)$ we will obtain ϕ arbitrarily close to φ such that $\Lambda_\varphi(U)$ has a homoclinic tangency or rank k. By Theorem A and since

$$q \leq d_1^{j-1} < d_1^j \leq d_1^{i_p - 1} < p,$$

after perturbing φ, we can assume that φ has a pair of hyperbolic periodic points $S_\varphi, T_\varphi \in \Lambda_\varphi(U)$ of indices d_1^j and d_1^{j-1}, respectively.

By Lemma 2.5, there is ψ close to φ with a heterodimensional cycle $\Gamma(\psi, U, S_\psi, T_\psi)$. Observe that the modulus of the restriction of the Jacobian of $\psi_*^{n(T_\psi)}$ to $E_j(T_\psi)$ is bigger than one and the modulus of the restriction of the Jacobian of $\psi_*^{n(S_\psi)}$ to $E_j(S_\psi)$ is less than one. By Corollary 3.7, unfolding this cycle, we get ϕ close to φ with a hyperbolic periodic point $R_\phi \in \Lambda_\phi(U)$ with index r, $r \in [d_1^{j-1}, d_1^j]$, such that the modulus of the Jacobian of $\phi_*^{n(R_\phi)}$ to $E_j(R_\phi)$ is exactly one.

By Proposition 2.4, after a perturbation of ϕ, we can assume that $\Sigma_{R_\phi}(\phi)$ is dense in $\Lambda_\phi(U)$. Since $E_j(R_\phi)$ is undecomposable and has dimension equal or bigger than 2, arguing exactly as in the proof of Lemma 5.7, but now applying the final part of Lemma 5.3, we get ξ (arbitrarily close to ϕ) with a periodic point $A_\xi \in \Lambda_\xi(U)$ such that the restriction of $\xi_*^{n(A_\xi)}$ to $E_j(A_\xi)$ is the identity.

Take now any $k \in [d_1^{j-1} + 1, d_1^j)$, after a perturbation of ξ we can assume that the index of A_ξ is $k - 1$, and that $\xi_*^{n(A_\xi)}(A_\xi)$ has an expanding non-real eigenvalue of rank k. Again, by Theorem A, we can assume there is a periodic point $B_\xi \in \Lambda_\xi(U)$ of index k, where $k > q$. Finally, by Lemma 2.5, there is η close to ξ with a codimension one cycle $\Gamma(\eta, U, B_\eta, A_\eta)$, A_η of index $(k - 1)$ and with an expanding non-real eigenvalue of rank k and B_η of index k. Now the lemma follows from Lemma 5.5.
□

7 Proof of Theorem E

As we have mentioned in the introduction, Theorem E follows from Proposition 1.1. So before proving the proposition let us deduce the theorem from it.

Recall that U and $\mathcal{S}(U)$ are open sets of M and $\text{Diff}^1(M)$ such that for every diffeomorphism $\varphi \in \mathcal{S}(U)$ the set $\Lambda_\varphi(U)$ is robustly transitive and has no homoclinic tangencies (in the whole manifold). By Theorem D, there is an open and dense subset $\mathcal{I}(U)$ of $\mathcal{S}(U)$, such that if φ belongs to $\mathcal{I}(U)$ and $\Lambda_\varphi(U)$ contains periodic points on indices q and p, $q < p$, then $\Lambda_\varphi(U)$ contains points of every index in between q and p. So it is enough to prove the theorem for the subset $\mathcal{I}(U)$ of $\mathcal{S}(U)$.

Consider the maps $i^+, i^- : \mathcal{I}(U) \to \mathbb{N}^*$ that associate to each $\varphi \in \mathcal{I}(U)$ the maximum and the minimum of the indices of the hyperbolic periodic points of $\Lambda_\varphi(U)$, respectively. These two functions are semi-continuous, so they are continuous in an open and dense subset $\mathcal{I}_0(U)$ of $\mathcal{I}(U)$. Now it is enough to fix a connected component $\mathcal{I}_0$ of $\mathcal{I}(U)$ where i^+ and i^- are both constant and to prove the theorem for this set. Suppose that $i^+(\varphi) = p$ and $i^-(\varphi) = q$ for all $\varphi \in \mathcal{I}_0$, $q \le p$.

Let us assume that $q < p$ (the case $q = p$ follows from Remark 2.7, so we omit it). Let Q_φ and P_φ be points of indices q and p of $\Lambda_\varphi(U)$. For notational simplicity let us assume that their continuations are defined in the whole $\mathcal{I}_0$. Since P_φ and Q_φ are transitively related in $\mathcal{I}_0$, by Remark 2.6, there is an open and dense subset $\mathcal{I}_1$ of $\mathcal{I}_0$ such that $W^s(P_\phi)$ and $W^u(Q_\phi)$ have nonempty transverse intersection for all $\phi \in \mathcal{I}_1$. So it is enough to prove the theorem for $\mathcal{I}_1$.

For each $j \ge 0$ with $q + j \le p$, let $\mathcal{A}(j)$ be the subset of $\mathcal{I}_1$ of diffeomorphisms ψ such that $\Lambda_\psi(U)$ contains hyperbolic periodic points $R^0_\psi, R^1_\psi, \dots, R^j_\psi$ such that

- $\mathrm{index}(R^i_\psi) = q + i$,
- $\overline{H_{R^0_\varphi}(U)} = \overline{H_{R^1_\varphi}(U)} = \cdots = \overline{H_{R^j_\varphi}(U)}$ for every φ in a neighbourhood of ψ

To end the proof of Theorem E it is enough to see the following.

Lemma 7.1. *The set $\mathcal{A}(j)$ is open and dense in $\mathcal{I}_1$ for every $j \in (0, r]$, $r = p - q$.*

Before proving this lemma let us end the proof of the theorem.

Observe that by Lemma 7.1, $\mathcal{A}(r)$ is open and dense in $\mathcal{I}_1$ and for every ψ in $\mathcal{A}(r)$ there are hyperbolic periodic points R^0_ψ and R^r_ψ of $\Lambda_\psi(U)$ of indices q and $q + r = p$ such that

$$\overline{H_{R^r_\psi}(U)} = \overline{H_{R^0_\psi}(U)}.$$

As above, for notational simplicity, let us assume that the continuations of R^0_ψ and R^r_ψ are defined in the whole $\mathcal{A}(r)$. The points Q_ψ and R^0_ψ have index q and are transitively related in $\mathcal{A}(r)$. Thus, by Remark 2.7, there is an open and dense subset $\mathcal{D}_1$ of $\mathcal{A}(r)$ of diffeomorphisms ζ such that the relative homoclinic classes of Q_ψ and R^0_ψ in U are equal. Similarly, there is an open and dense subset $\mathcal{D}_2$ of $\mathcal{A}(r)$ of diffeomorphisms ζ such that the relative homoclinic classes of P_ψ and R^r_ψ in U are equal. Thus, for all $\zeta \in \mathcal{D}_1 \cap \mathcal{D}_2$, one has that

$$\overline{H_{P_\zeta}(U)} = \overline{H_{R^r_\zeta}(U)} = \overline{H_{R^0_\zeta}(U)} = \overline{H_{Q_\zeta}(U)}.$$

Since $\mathcal{D}_1 \cap \mathcal{D}_2$ is open and dense in $\mathcal{A}(r)$, thus in $\mathcal{I}_1$, this ends the proof of the theorem.

Proof of the lemma: The proof of Lemma 7.1 is by induction. To see that $\mathcal{A}(1)$ is open and dense in $\mathcal{I}_1$ it suffices to see that given any $\phi \in \mathcal{I}_1$ there is an open subset $\mathcal{A}_\phi$ of $\mathcal{I}_1$ such that

- ϕ belongs to the closure of $\mathcal{A}_\phi$,
- for every $\psi \in \mathcal{A}_\phi$ there is a hyperbolic periodic point $R^1_\psi \in \Lambda_\psi(U)$ of index $(q+1)$ such that $\overline{H_{Q_\psi}(U)} = \overline{H_{R^1_\psi}(U)}$ (here we take $R^0_\psi = Q_\psi$).

Since ϕ is in $\mathcal{I}_1$ there is a periodic point $R^1_\phi \in \Lambda_\phi(U)$ of index $(q+1)$. Observe that Q_ϕ and R^1_ϕ are transitively related and $\text{index}(Q_\psi)+1 = \text{index}(R^1_\psi)$. Thus, by Lemma 2.5, after a perturbation of ϕ, we can assume that ϕ has a (codimension one) cycle $\Gamma(\phi, U, R^1_\phi, Q_\phi)$. By hypothesis, this cycle is far from homoclinic tangencies. Thus, by Proposition 1.1, there is an open set $\mathcal{B}_\phi$, whose closure contains ϕ, such that $\overline{H_{Q_\zeta}(U)} = \overline{H_{R^1_\zeta}(U)}$ for all $\zeta \in B_\phi$. The first inductive step follows taking $\mathcal{A}_\phi = \mathcal{B}_\phi \cap \mathcal{I}_1$.

Suppose now defined inductively the open and dense subsets $\mathcal{A}(1), \mathcal{A}(2), \dots, \mathcal{A}(j-1)$, $q+j \le p$, of $\mathcal{I}_1$ satisfying the properties above. Then the set

$$\mathcal{A}'(j-1) = \mathcal{A}(1) \cap \cdots \cap \mathcal{A}(j-1)$$

is open and dense in $\mathcal{I}_1$. Now it is enough to get an open and dense subset $\mathcal{A}(j)$ of $\mathcal{A}'(j-1)$ with the announced properties. For that we argue exactly as in the step $j=1$.

Consider any $\phi \in \mathcal{A}'(j-1)$. Since $\phi \in \mathcal{I}_1$ the set $\Lambda_\phi(U)$ contains a hyperbolic periodic point R^j_ϕ of index $(q+j)$. As in the first step of the induction, using Lemma 2.5, we can assume (after a perturbation of ϕ) that ϕ has a (codimension one) cycle $\Gamma(\phi, U, R^j_\phi, R^{j-1}_\phi)$, where R^{j-1}_ϕ is the point of index $(q+j-1)$ in the inductive step $(j-1)$. By hypothesis, this cycle is far from homoclinic tangencies. Thus, by Proposition 1.1, there is an open set $\mathcal{B}_\phi \subset \mathcal{A}'(j-1)$ containing ϕ in its closure such that

$$\overline{H_{R^{j-1}_\zeta}(U)} = \overline{H_{R^j_\zeta}(U)}$$

for all $\zeta \in \mathcal{B}_\phi$. Since $\mathcal{B}_\phi \subset \mathcal{A}'(j-1)$, we have

$$\overline{H_{R^0_\zeta}(U)} = \overline{H_{R^1_\zeta}(U)} = \overline{H_{R^{j-1}_\zeta}(U)} = \overline{H_{R^j_\zeta}(U)}$$

for all $\zeta \in \mathcal{B}_\phi$, ending the proof of the lemma. □

7.1 Proof of Proposition 1.1

Suppose now that (as in the hypotheses of Proposition 1.1) the indices of P_φ and Q_φ are p and q with $p=(q+1)$. By [BDP, Lemma 5.4], we can assume that the robustly transitive set $\Lambda_\varphi(U)$ contains a pair of hyperbolic periodic points of indices q and $p+1$ having only real eigenvalues with multiplicity one and different modulos. For notational simplicity, let us assume that Q_φ and P_φ verify these hypotheses. In particular, these points verify the hypotheses of Corollary 3.6. By (1) in the proof of the corollary, after a small perturbation, we can assume that φ has a saddle-node periodic point (a point with an eigenvalue equal to one) with q contracting eigenvalues and $(\dim(M)-q-1)$ expanding eigenvalues. After a new perturbation, by unfolding the saddle-node, we can assume that φ has a pair of periodic points A_φ and B_φ of indices p and q, respectively, such that there is a curve γ whose extremes are A_φ and B_φ and whose interior is contained in $W^s(A_\varphi) \pitchfork W^u(B_\varphi)$. By Remark 2.7, we can assume that there is an open set $\mathcal{V}$ containing φ in its closure such that $\overline{H_{P_\psi}(U)} = \overline{H_{A_\psi}(U)}$ and $\overline{H_{Q_\psi}(U)} = \overline{H_{B_\psi}(U)}$ for all ψ in $\mathcal{V}$.

By Remark 2.6, there is a sequence of diffeomorphisms φ_k, $\varphi_k \to \varphi$ in the $\mathcal{C}^1$-topology, such that φ_k has a codimension one heterodimensional cycle $\Gamma(\varphi_k, U, A_{\varphi_k}, B_{\varphi_k})$. By construction, these cycles are *connected* ones, i.e., $W^s(A_{\varphi_k}) \pitchfork W^u(B_{\varphi_k})$ has a periodic connected component whose extremes are contained in the orbits of A_{φ_k} and B_{φ_k} (here the connected component is the continuation of the curve γ above).

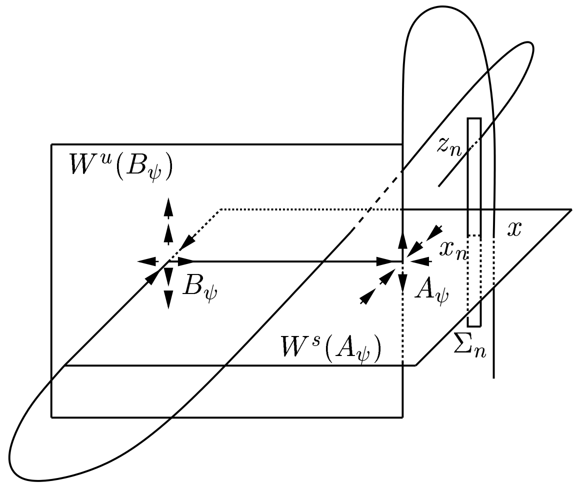

Figure 3: Homoclinic points

The proposition now follows directly from [DR]. For completeness let us state these results.

Lemma 7.2. *Let ζ be a $\mathcal{C}^1$-diffeomorphism with a codimension one connected heterodimensional cycle $\Gamma(\zeta, U, A_\zeta, B_\zeta)$ as above. Then given any $\mathcal{C}^1$-neighbourhood $\mathcal{A}$ of ζ there is a $\mathcal{C}^1$-open subset $\mathcal{U}(\zeta)$ of $\mathcal{A}$ such that $\overline{H_{A_\psi}(U)} = \overline{H_{B_\psi}(U)}$ for every $\psi \in \mathcal{U}(\zeta)$.*

By the lemma, for each φ_k as above there is an open set $\mathcal{U}(\varphi_k) \subset \mathcal{V}$ containing φ_k in its closure, such that, for every $\psi \in \mathcal{U}(\varphi_k)$, one has $\overline{H_{A_\psi}(U)} = \overline{H_{B_\psi}(U)}$. Since $\psi \in \mathcal{V}$, we have that $\overline{H_{A_\psi}(U)} = \overline{H_{P_\psi}(U)}$ and $\overline{H_{Q_\psi}(U)} = \overline{H_{B_\psi}(U)}$. Proposition 1.1 now follows taking $\mathcal{W}_\varphi = \bigcup_k \mathcal{U}(\varphi_k)$.

Proof of the lemma: Observe that the cycle $\Gamma(\zeta, U, A_\zeta, B_\zeta)$ is connected and far from homoclinic tangencies. In [DR], see the comments after Theorem A, it is proved that given any neighbourhood $\mathcal{U}$ of ζ there is an open subset $\mathcal{U}_0$ of $\mathcal{U}$ such that every $\psi \in \mathcal{U}_0$ has a transitive set Λ_ψ containing A_ψ and B_ψ such that $\Lambda_\psi \subset \overline{H(B_\psi)}$. The main step to prove this result is the fact we will state below.

Let us first observe that, by construction, there is a multiplicity one contracting eigenvalue $\lambda_c \in \mathbb{R}$ of the derivative of ζ at A_ζ such that $1 > |\lambda_c| > |\lambda|$ for every contracting eigenvalue λ of A_ζ different from λ_c (see condition (CE) in [DR, Section 3.1]). Thus for every ψ close to ζ the (codimension one) strong stable foliation $\mathcal{F}^s_\psi$ of $W^s(A_\psi)$ is defined. Similarly, we have that the (codimension one) strong unstable foliation $\mathcal{F}^u_\psi$ of $W^u(A_\psi)$ is defined. Now the lemma will follow form the following fact.

Fact: *Let $\mathcal{A}$ be as in Lemma 7.2.*

- *Let $u = (\dim(M) - p)$ be the dimension of the unstable bundle of A_ζ. There is an open subset $\mathcal{A}_0$ of $\mathcal{A}$ of diffeomorphisms ψ such that $W^s(B_\psi)$ meets transversely every $(u+1)$-disk Σ transverse to $\mathcal{F}^s_\psi$.*
- *Let s be the dimension of the stable bundle of A_ζ. There is an open subset $\mathcal{A}_0$ of $\mathcal{A}$ of diffeomorphisms ψ such that $W^u(A_\psi)$ meets transversely every $(s+1)$-disk Σ transverse to $\mathcal{F}^u_\psi$.*

This fact is a non-technical reformulation of [DR, Proposition 3.6 (b)]. Let us observe that (due to the context) in [DR] this proposition is stated for parametrized families of diffeomorphisms unfolding a connected cycle corresponding to a first bifurcation. But, as mentioned in [DR, Section 6], it holds in a much more general setting (including the case under consideration).

To see, for instance, that $\overline{H_{A_\psi}(U)}$ is contained in $\overline{H_{B_\psi}(U)}$ we use the first part of the fact. Take any x in $H_{A_\psi}(U)$. By the cycle configuration $W^u(A_\psi)$ is contained in the closure of $W^u(B_\psi)$, thus there is a sequence $x_n \to x$ with $x_n \in W^s(A_\psi) \pitchfork W^u(B_\psi)$ for all n. Associate to each x_n we have a $(u+1)$-disk Σ_n of diameter less than $1/n$ which is contained in $W^u(B_\psi)$ and transverse to $W^s(A_\psi)$ at x_n (see figure). The fact implies that for each n there is $z_n \in W^s(B_\psi) \pitchfork \Sigma_n$. By construction $z_n \in H_{A_\psi}$ (in fact one can take $z_n \in H_{B_\psi}(U)$) and $\lim z_n = \lim x_n = x$.

The inclusion $\overline{H_{B_\varphi}(U)} \subset \overline{H_{B_\varphi}(U)}$ follows similarly using the second part of the fact. This ends the sketch of the proof of the lemma. □

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