Generic Symplectic Diffeomorphisms and Partial Hyperbolicity

Workshop on Symplectic Dynamics
Institute for Advanced Study (Princeton)

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Plan of the talk

- Recall basic definitions.
- State a theorem on partial hyperbolicity of generic symplectic diffeomorphisms.
- Discuss consequences and further developments of that theorem (including the ergodicity result).
- Compare with a cousin theorem for volume-preserving diffeomorphisms.
- Sketch the main ideas of the proof, and so expain:
  - why symplectic is more difficult than volume-preserving;
  - the probabilistic method for constructing the perturbations.

Lyapunov exponents, Oseledets splitting

$f: M \to M$ diffeomorphism of a compact manifold of dimension d.
By the **Oseledets theorem**, there exists a full probability set $R \subset M$ such that for every (*regular point*) $x \in R$ there is a (*Oseledets*) splitting

$$T_x M = E^1(x) \oplus \cdots \oplus E^{k(x)}(x), \quad (\text{each} \neq \{0\})$$

and numbers (*Lyapunov exponents*) $\lambda_1(x) > \cdots > \lambda_{k(x)}(x)$ such that

$$\frac{1}{n} \log \|Df^n(x) \cdot v\| \xrightarrow[n \to \pm\infty]{} \lambda_i(x) \quad \forall v \in E^i(x) \setminus \{0\}.$$

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The *zipped Oseledets splitting* is obtained by summing together all spaces with exponents of the same sign:

$$T_x M = E^+(x) \oplus E^0(x) \oplus E^-(x).$$

($E^*(x) = \{0\}$ now allowed, of course.)

Lyapunov exponents, Oseledets splitting (continued)

The multiplicity of each Lyapunov exponent $\hat{\lambda}_j(x)$ is $\dim E^j(x)$ (by definition).
Indicate the Lyapunov exponents repeated according to multiplicity by:

$$\lambda_1(x) \geq \lambda_2(x) \geq \cdots \geq \lambda_d(x), \qquad (d = \dim M).$$

If f preserves a symplectic form ω on M then (d is even and) the exponents are symmetric:

$$\lambda_1 = -\lambda_d, \quad \lambda_2 = -\lambda_{d-1}, \quad \ldots \quad , \quad \lambda_{\frac{d}{2}} = -\lambda_{\frac{d}{2}+1}.$$

In particular, if $T_x M = E^+(x) \oplus E^0(x) \oplus E^-(x)$ is the zipped Oseledets splitting then

$$\dim E^+(x) = \dim E^-(x), \quad \dim E^0(x) = \text{even}.$$

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1. A splitting $T_\Lambda M = E^u \oplus E^s$ is called *uniformly hyperbolic* if E^s is uniformly contracted and E^u is uniformly expanded (i.e. contracted in the past): there are constants $C > 0$, $\tau > 1$ such that

$$\forall n \geq 0, \ \begin{cases} \|Df^n(x) \cdot v^s\| \leq C\tau^{-n}\|v^s\| & \forall v^s \in E^s(x) \smallsetminus \{0\} \\ \|Df^{-n}(x) \cdot v^u\| \leq C\tau^{-n}\|v^u\| & \forall v^u \in E^u(x) \smallsetminus \{0\} \end{cases}$$

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Remark: These bundles are automatically uniformly continuous, and thus extend (with the same hyperbolicity properties) to the closure $\overline{\Lambda}$.

Domination: the weakest uniform form of hyperbolicity

If E, F are Df-invariant subbundles of $T_\Lambda M$ then we say that E *dominates* F (in symbols, $E \succ F$), if there are constants $c > 0$, $\tau > 1$ such that for all unit vectors $\vec{e} \in E(x)$, $\vec{f} \in F(x)$ and all $n \geq 0$

$$\frac{\|Df^n(x)\cdot\vec{e}\|}{\|Df^n(x)\cdot\vec{f}\|} > c\tau^n.$$

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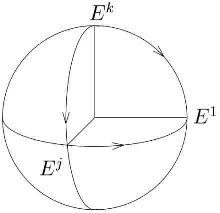
Dominated splitting: $T_\Lambda M = E^1 \oplus \cdots \oplus E^k$ with $E_1 \succ E_2 \succ \cdots \succ E_k$.
"Morse–Smale-like" dynamics on projective space:



Domination is a.k.a. (in ODE theory) as *exponential separation*. It dates back to Perron (rediscovered by Mañé.)

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Moreover, the 2nd alternative occurs for a positive μ-measure set of points $x \in M$ if and only if $f \in \mathscr{R}$ is Anosov.

Rem.: A weaker version (with no PH) was proved earlier in [B., Viana '05].

Discussion: $\dim M = 2$

If $\dim M = 2$ then the 3rd alternative in the theorem (partial hyperbolicity with 3 bundles) is impossible, so we get:

Corollary (B. '02)

C^1-generic area-preserving diffeomorphisms are either Anosov or have zero Lyapunov exponents almost everywhere.

Rem.: The proof of this result appeared much before, and relied heavily on Mañé's ideas.

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One can break M (minus a zero set) invariantly:

$$M = Z \sqcup \bigsqcup \Lambda_n \text{ mod } 0 \text{ where } \begin{cases} Z = \{\text{all } \lambda_i = 0\}, \\ \Lambda_n = \text{partially hyperbolic sets} \end{cases}$$

- Each Λ_n (or its closure $\overline{\Lambda}_n$) has of course its hyperbolicity constants c_n, τ_n.
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- However, these constants become weaker and weaker as n grows.

Please note that this is much stronger than what's is given by Oseledets theorem, which gives no uniformity along the orbits.

Also note that if $f \in \mathscr{R}$ is *ergodic* then the situation is much simpler...

The case of globally partially hyperbolic maps

Let $\mathrm{PH}^1_\omega(M)$ be the subset of $\mathrm{Diff}^1_\omega(M)$ formed by the diffeos that have a partially hyperbolic splitting over the whole tangent bundle.

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Proof of the theorem: Just combine the previous theorem with this:

Theorem (Corollary of Dolgopyat–Wilkinson)

Generic $f \in \mathrm{PH}^1_\omega(M)$ are (accessible and) weakly ergodic (i.e., almost every point has a dense orbit).

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Nice thing: The zero exponents in the center give a *nonuniform version of Burns–Wilkinson's center bunching.*

Ergodicity

Indeed many of Burns–Wilkinson's arguments work with *nonuniform center bunching*.

Putting these together with other [non obvious!] arguments, we get:

Theorem (Avila, B., Wilkinson '09)

The generic $f \in \mathrm{PH}^1_\omega(M)$ is ergodic.

(**Curiosity:** This paper was published before its ancestors [BW'10] and [B'10].)

This gives a C^1-generic, symplectic version of the Pugh–Shub ergodicity conjecture.

Proofs?

Now let's give an idea of the proof of the theorem stated by Mañé:

Theorem

For every f in a residual (dense G_δ) subset $\mathscr{R}$ of $\mathrm{Diff}^1_\omega(M)$, the following properties hold: For μ-a.e. $x \in M$, the zipped Oseledets splitting $T_\Lambda M = E^+ \oplus E^0 \oplus E^-$ on $\Lambda = \mathrm{orb}(x) = \{f^n(x);\ n \in \mathbb{Z}\}$ is:

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For the "moreover" part, we show that for C^2 (and hence C^1-generic) diffeos, hyperbolic sets have either zero or full measure. [B., Viana '04].

Back to basics: domination, and the symplectic case

In fact, to prove the result above, one "only" needs to show the following:

Theorem (Main Theorem)

If f is generic in $\mathrm{Diff}^1_\omega(M)$ then for almost every $x \in M$, the Oseledets splitting along the orbit of x is either trivial or dominated.

A Df-invariant splitting $T_\Lambda M = E^1 \oplus \cdots \oplus E^k$ is called

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Then, to conclude the proof of Mañé's statement, one has to use that for symplectic maps, "domination implies partially hyperbolicity". [B., Viana '04].

Rem.: Actually we obtain more information than in Mañé's statement, since also get domination between different exponents of the same sign.

Comparison with the volume-preserving case

The "Main Theorem" just stated is also true replacing "symplectic" by "volume-preserving":

Theorem (B. Viana '05)

If f is generic in $\mathrm{Diff}^1_{vol}(M)$ then for almost every $x \in M$, the Oseledets splitting along the orbit of x is either trivial or dominated.

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Rem.: There are some recent "global" improvements of this result: Avila–B. (ArXiv 2010), Jana Rodriguez–Hertz (in preparation), but that's another story...

Another reduction

Integrated summed Lyapunov exponent:

$$L_p(f) = \int_M (\lambda_1 + \cdots + \lambda_p)\, d\mu$$
$$= \int \Big(\lim_{n\to\infty} \log \| \wedge^p (Df^n) \| \Big)\, d\mu.$$

Easy fact: $L_p : \mathrm{Diff}^1_\omega(M) \to \mathbb{R}$ is upper-semicontinuous, and thus continuous on a residual subset.

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Let $N = d/2$.

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If f is a point of continuity of L_1, … and L_N then the Oseledets splitting is trivial or dominated along almost every orbit.

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Strategy of the proof: Suppose that one can detect non-domination of the Oseledets splitting on a positive measure set. Then produce a perturbation of f for which some L_p drops.

Setup for the proof

Fix f, $p \in \{1, \ldots, N = d/2\}$. Assume that the following set has positive measure:

$$\Sigma_p = \{x \in M;\ x \text{ regular non periodic with } \lambda_p(x) > \lambda_{p+1}(x)\}$$

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Another "zipped" Oseledets splitting:

$$T_{\Sigma_p} M = E^u \oplus E^c \oplus E^s \quad \text{where} \begin{cases} \dim E^u = \dim E^s = p \\ \dim E^c = 2(N-p) \end{cases}$$

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Denote $E^{uc} = E^u \oplus E^c$ etc. Then

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Main steps (sketchy)

- If a point in Σ_p "sees" non-domination $E^u \not\succ E^{cs}$ (for example, if $\sphericalangle(E^u, E^{cs})$ is small) then we can find a perturbation g of f that sends a vector from E^u_f to E^{cs}_f.

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- Do that in the *middle* of a long segment of orbit $\{x, \dots, f^n x\}$. Then one gets

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Main steps (sketchy)

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  - Something $= \frac{1}{2}(\lambda_p(f,x) + \lambda_{p+1}(f,x))$ (which is POSITIVE).
  - In fact we should obtain the inequality not only for x [a zero measure set is useless], but for most z around x in the support of the perturbation.

Main steps (continued)

Example: $\dim M = 2$

$$Df^n(x) = \begin{pmatrix} 2 & 0 \\ 0 & 1/2 \end{pmatrix}^k \mathrm{Id}^m \begin{pmatrix} 2 & 0 \\ 0 & 1/2 \end{pmatrix}^k, \qquad k \simeq n/2 \gg m \gg 1.$$

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Back to the steps of the general proof:

- Around a segment of orbit $\{x, \dots, f^n x\}$ that sees nondomination we find a thin and long *tower* $U \sqcup f(U) \sqcup \cdots \sqcup f^n(U)$, and find a perturbation supported in the tower so that the "finite-time" summed exponent drops (as explained above).
- Cover the a ("large") positive measure set of the manifold with these towers.
- This causes a significant drop of $L_p = \int (\lambda_1 + \cdots + \lambda_p)$, as we wanted.

The 4 types of non-dominance

Assume $x \in \Sigma_p$ and the segment of orbit $\{x, \ldots, f^m x\}$ is very long ($m \gg 1$) and "sees" non-domination $E^u \not> E^{cs}$; more precisely:

$$\frac{\|Df^m(x)|E^{cs}(x)\|}{\mathfrak{m}(Df^m(x)|E^u(x))} \geq \frac{1}{2}. \qquad (\mathfrak{m}(L) = \|L^{-1}\|^{-1}).$$

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Case I: small angle. There is a point $y \in \{x, \dots, f^m x\}$ such that $\sphericalangle(E^{cs}(y), E^u(y)) \ll 1$.
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How to perturb it: Compose with a single small rotation.

Case II: inverted behaviors. There are unit vectors $v^{cs} \in E^{cs}(x)$, $v^u \in E^u(x)$ such that $\|Df^m(x) \cdot v^{cs}\| \gg \|Df^m(x) \cdot v^u\|$. (Replace if necessary the whole segment of orbit by a subsegment.)
How to perturb it: Use two small rotations, one at the beginning and the other at the end.

The 4 types of non-dominance (continued)

Case III: Identity on a symplectic ($\omega \neq 0$) plane.
There is a long subsegment of orbit such the following holds: There is a (2-dim) plane spanned by a vector in E^u and a vector in E^s such that the restriction of Df to this plane "looks like the identity" (or more precisely, becomes an isometry after a bounded change of coordinates).

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How to perturb it: This case is essentially 2-dimensional. Use several "nested" rotations, as explained in a previous example...

The 4 types of non-dominance (continued)

Case IV: Expansion on a null ($\omega \equiv 0$) plane.
There is a (2-dim) plane P spanned by a vector in E^u and a vector in E^c which is **uniformly expanded** and **conformal** (along the segment of orbit). That is, after a bounded change of coordinates we have

$$Df(f^i x)|P = \begin{pmatrix} \tau_i & 0 \\ 0 & \tau_i \end{pmatrix}, \qquad \tau_i > c > 1.$$

The plane is necessarily null ($\omega \equiv 0$).

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However, this perturbation is not symplectic!

(In the **volume-preserving** case this idea would work; there are only 3 cases to be considered there.)

4-dimensional problem

There are standard symplectic coordinates $p_1, \ldots, p_N$, $q_1, \ldots, q_N$ (so $\omega = \sum_i dp_i \wedge dq_i$) such that

$$P = \Big\langle \underbrace{\frac{\partial}{\partial p_1}}_{\in E^u}, \underbrace{\frac{\partial}{\partial p_2}}_{\in E^c} \Big\rangle,$$

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Nested rotations don't work! The problem is that hyperbolicity of Df quickly distorts the domain where the perturbation should be supported.

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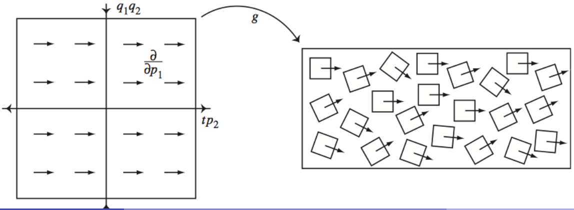
- Look the image $g(D)$ and the image v_1 of the field $v_0 = \frac{\partial}{\partial p_1}$ by Dg. Using Vitali Lemma, cover most of $g(D)$ by many disjoint tiny boxes D_i that look like (basically after a change of scale) the original box:



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- Then **THE RANDOM VARIABLES Θ_0 AND Θ_1 ARE (approximately) INDEPENDENT AND IDENTICALLY DISTRIBUTED!**
- Continuing this process we obtain a **RANDOM WALK** $S_n = \Theta_0 + \Theta_1 + \cdots + \Theta_n$.
- Since every (non-stopped) random walk is transient, most orbits eventually reach $\pm\pi/2$.
- Thus we succeeded in sending the direction $\frac{\partial}{\partial p_1} \in E^u$ to the direction $\frac{\partial}{\partial q_1} \in E^c$ (for most points).

The end

THANK YOU!