THE MULTIPLICATIVE ERGODIC THEOREM OF OSELEDETS

1. Statement

Let $(X, \mu, \mathcal{A})$ be a probability space, and let $T : X \to X$ be an ergodic measure-preserving transformation.

Given a measurable map $A : X \to \mathrm{GL}(d, \mathbb{R})$, we can define a skew-product $F : X \times \mathbb{R}^d \to X \times \mathbb{R}^d$ by $(x, v) \mapsto (T(x), A(x) \cdot v)$. Write $F^n(x) = (T^n(x), A^{(n)}(x) \cdot v)$. The map F is called a *linear cocycle*.

Fix some norm $\|\cdot\|$ on $\mathbb{R}^d$, and use the same symbol to indicate the induced operator norm.

Theorem 1 (Multiplicative Ergodic Theorem of Oseledets [O]). *Let T be an invertible bimeasurable ergodic transformation of the probability space (X, μ). Let $A : X \to \mathrm{GL}(d, \mathbb{R})$ be measurable such that $\log^+ \|A\|$ and $\log^+ \|A^{-1}\|$ are integrable. Then there exist numbers*

$$\lambda_1 > \cdots > \lambda_k \tag{1}$$

and for μ-almost every $x \in X$, there exist a splitting

$$\mathbb{R}^d = E_x^1 \oplus \cdots \oplus E_x^k \tag{2}$$

such that for every $1 \le i \le k$ we have $A(x) \cdot E_x^i = E_{T(x)}^i$ and

$$\lim_{n \to \pm\infty} \frac{1}{n} \log \|A^{(n)}(x) \cdot v_i\| = \lambda_i \quad \text{for all non-zero } v_i \in E_x^i. \tag{3}$$

The subspaces E_x^i are unique μ-almost everywhere, and they depend measurably on x.

The numbers (1) are called *Lyapunov exponents*, and the splitting (2) is called the *Lyapunov splitting*.

Invariance of the bundles E_x^i implies that their dimensions d_i are constant a.e. Measurability means the following: there exist measurable maps $u_1, \ldots, u_d : X \to \mathbb{R}^d$ such that for each $j = 1, \ldots, k$, the space E_x^i is spanned by the vectors $u_i(x)$ with $d_1 + \cdots + d_{i-1} < i \le d_1 + \cdots + d_i$.

Exercise i. If T is not assumed to be ergodic then a similar result still holds. The Lyapunov exponents (and their number) become measurable T-invariant a.e. defined functions. State and prove this assuming Theorem 1 and using ergodic decomposition.

Exercise ii. Use the last exercise, state and prove a version of Oseledets' Theorem for diffemorphisms of a compact manifold.

In the next section, we explain some additional conclusions that could be added to Oseledets' Theorem, and also state some results that will be need later. The proofs will take the rest of the note. Some exercises are given along the way. Those that are need for the proof of Theorem 1 are marked with the symbol ✓.

2. Additional information

2.1. **One-sided version.** There is a version of Theorem 1 for non-(necessarily)-invertible transformations:

Theorem 2. *Let T be an ergodic transformation of the probability space (X, μ). Let $A : X \to \mathrm{GL}(d, \mathbb{R})$ be measurable such that* $\log^+ \|A\|$ *and* $\log^+ \|A^{-1}\|$ *are integrable. Then there exist numbers*

$$\lambda_1 > \cdots > \lambda_k$$

and for μ-almost every $x \in X$, there exist a flag

$$\mathbb{R}^d = V_x^1 \supset V_x^2 \supset \cdots \supset V_x^k \supset V_x^{k+1} = \{0\} \tag{4}$$

depending measurably on the point, and invariant by the cocycle, such that

$$\lim_{n\to\infty} \frac{1}{n} \log \|A^{(n)}(x) \cdot v_i\| = \lambda_i \quad \textit{for all } v_i \in V_x^i \smallsetminus V_x^{i+1} . \tag{5}$$

We call (4) the *Lyapunov flag* of the cocycle.

Exercise iii. In the case T *is* invertible, the flag is given in terms of the Lyapunov splitting by $V_x^i = E_x^1 \oplus \cdots \oplus E_x^i$.

2.2. **Subadditivity.** We will use the following result (see e.g. [L] for a proof):

Theorem 3 (Kingman's Subadditive Ergodic Theorem)**.** *Assume T is ergodic. If $(\phi_n)_{n=1,2,\ldots}$ is a sequence of functions such that ϕ_1^+ is integrable and $\phi_{m+n} \le \phi_m + \phi_n \circ T^m$ for all $m, n \ge 1$. Then $\frac{1}{n}\phi_n$ converges a.e. to some $c \in \mathbb{R} \cup \{-\infty\}$. Moreover, $c = \inf_n \frac{1}{n} \int \phi_n$.*

An immediate corollary of Theorem 3 in the setting of cocycles is the existence of the limit $\lim_{n\to\infty} \frac{1}{n} \log \|A^{(n)}(x)\|$. (This is called the Furstenberg-Kesten Theorem.) It can be shown that this limit is the upper Lyapunov exponent λ_1 given by Oseledets' Theorem:

Addendum 4. *We have:*

$$\lambda_1 = \lim_{n\to+\infty} \frac{1}{n} \log \|A^{(n)}(x)\| = \lim_{n\to-\infty} \frac{1}{n} \log \mathbf{m}(A^{(n)}(x)) \,,$$
$$\lambda_k = \lim_{n\to+\infty} \frac{1}{n} \log \mathbf{m}(A^{(n)}(x)) = \lim_{n\to-\infty} \frac{1}{n} \log \|A^{(n)}(x)\| \,.$$

Recall that the *co-norm* of a linear map L is $\mathbf{m}(L) = \inf_{\|v\|=1} \|Lv\|$; it equals $\|L^{-1}\|^{-1}$ if L is invertible.

2.3. **Angles.** The angles between the spaces of the Lyapunov splitting are sub-exponential, in the sense made precise by the following:

Addendum 5. *Let $\mathbb{R}^d = E^1_x \oplus \cdots \oplus E^k_x$ be the Lyapunov splitting, and assume $k \geq 2$. Let $J_1 \sqcup J_2$ be a partition of the set of indices $\{1, \ldots, k\}$ into two disjoint non-empty sets. Let $F^i_x = \bigoplus_{j\in J_i} E^j_x$ for $i = 1, 2$. Then, for μ-a.e. x we have*

$$\lim_{n\to\pm\infty} \frac{1}{n} \log \sin \measuredangle\Big(F^1_{T^n(x)}, F^2_{T^n(x)}\Big) = 0.$$

We will deduce Addendum 5 directly from Oseledets Theorem. For that, we need the following result (that will also be usuful in the proof of the theorem itself):

Lemma 6. *Let $\phi : X \to \mathbb{R}$ be a measurable function such that $\phi \circ T - \phi$ is integrable in the extended sense*[1]. *Then*

$$\lim_{n\to\pm\infty} \frac{1}{|n|} \phi(T^n x) = 0 \quad \text{for a.e. } x \in X.$$

($n < 0$ is allowed if T is invertible.)

Proof. Let $\psi = \phi \circ T - \phi$ and let $c = \int \psi \in [-\infty, +\infty]$. By Birkhoff's Theorem,

$$\frac{\phi \circ T^n}{n} = \frac{\phi}{n} + \frac{1}{n} \sum_{j=0}^{n-1} \psi \circ T^j \to c \text{ a.e.}$$

If c were non-zero then we would have $|\phi(T^n x)| \to \infty$ for a.e. x. But, by the Poincaré's Recurrence Theorem, the set of points x which satisfy the latter condition has zero measure. Therefore $c = 0$, as we wanted to show. If T is invertible then $\phi \circ T^{-1} - \phi$ is quasi-integrable as well, so the previous reasoning applies. □

[1] ψ is said to be integrable in the extended sense if ψ^+ or ψ^- are integrable. Another exercise: Verify that the Birkhoff Theorem still applies.

Exercise iv. Let $L : \mathbb{R}^d \to \mathbb{R}^d$ be a invertible linear map and let v, w be non-zero vectors. Then

$$\frac{1}{\|L\|\,\|L^{-1}\|} \le \frac{\sin \measuredangle(Lv, Lw)}{\sin \measuredangle(v, w)} \le \|L\|\,\|L^{-1}\|.$$

Proof of Addendum 5. The function $\phi : X \to \mathbb{R}$ defined by $\phi(x) = \log \sin \measuredangle(F^1_x, F^2_x)$ is measurable. By Exercise iv, we have $|\phi(Tx) - \phi(x)| \le \log\|A(x)\| + \log\|A(x)^{-1}\|$, which is integrable, by the hypothesis of the Oseledets Theorem. So Lemma 6 gives $\phi(T^n(x))/n \to 0$ for a.e. x. □

2.4. **Conjugacy.** For a fixed $T : X \to X$, we say a measurable map $C : X \to \mathrm{GL}(d, \mathbb{R})$ is *tempered* if $\lim_{n\to\infty} \frac{1}{n} \log\|C^{\pm 1}(T^n x)\| = 0$.

We say that two cocycles A, $B : X \to \mathrm{GL}(d, \mathbb{R})$ are *conjugate* (or *equivalent*) if there is a tempered $C : X \to \mathrm{GL}(d, \mathbb{R})$ such that

$$B(x) = C(Tx)^{-1} A(x) C(x)$$

That is, $(T, B) = (C, id)^{-1} \circ (T, A) \circ (C, id)$.

Exercise v. If two cocycles are conjugate then they have the same Lyapunov exponents.

2.5. **Determinants.** Another addendum to Oseledets Theorem (that we will obtain along its proof) is this:

Addendum 7. *For a.e. x, the convergence in* (3) *is uniform over unit vectors in E^i_x.*

Using this, do the following:

Exercise vi. Let $\varnothing \ne J \subset \{1, \dots, k\}$ and $F_x = \bigoplus_{j\in J} E^j_x$. Prove that

$$\lim_{n\to\pm\infty} \frac{1}{n} \log|\det A^{(n)}(x)_{|F_x}| = \sum_{j\in J} \lambda_j \dim E^j\,.$$

3. Limsup Oseledets

The proof of Theorem 1 will take the rest of this note.

It is comparatively very easy to prove Theorem 2 if limits are replaced by lim sup's. Let

$$\bar\lambda(x, v) = \limsup_{n\to\infty} \frac{1}{n} \log\|A^{(n)}(x) \cdot v\|, \quad v \ne 0.$$

Lemma 8. *There exist numbers $\lambda_1 > \cdots > \lambda_k$ such that for a.e. x, $\bar\lambda(x, \cdot)$ takes only these values. For a.e. x there is a flag into linear spaces*

$$\mathbb{R}^d = V^1_x \supset V^2_x \supset \cdots \supset V^k_x \supset V^{k+1}_x = \{0\}$$

such that

$$v \in V_x^i \setminus V_x^{i+1} \;\Rightarrow\; \bar{\lambda}(x,v) = \lambda_i\,.$$

Moreover, these spaces depend measurably on the point, and are invariant by the cocycle.

Proof. First, the integrability of $\log^+ \|A^{\pm 1}\|$ implies that $\bar{\lambda}(x,v)$ is finite for a.e. x and all $v \neq 0$. It is easy to see that:

$$\bar{\lambda}(x,av) = \bar{\lambda}(x,v),$$

$$\bar{\lambda}(x,v+w) \leq \max\left(\bar{\lambda}(x,v), \bar{\lambda}(x,w)\right) \text{ with equality if } \bar{\lambda}(x,v) \neq \bar{\lambda}(x,w).$$

It follows that for any $t \in \mathbb{R}$,

$$V_x(t) = \left\{v \in \mathbb{R}^d;\ \bar{\lambda}(x,v) \leq t \text{ or } v = 0\right\}$$

is a linear subspace. Also, vectors with different $\bar{\lambda}$ are linearly independent, so for all x, $\bar{\lambda}(x,\cdot)$ takes at most d distinct values $\lambda_1(x) > \cdots > \lambda_{k(x)}(x)$. Let $V_x^i = V_x(\lambda_i(x))$.

Exercise vii (✓)**.** Show that everything is measurable and conclude the proof of the Lemma.

□

4. Interlude: Some ergodic theory of skew-products

Let P be a compact metric space. Let $S : X \times P \to X \times P$ be a skew-product map over T (this means that $\pi \circ S = T$ where $\pi : X \times P \to X$ is the projection) such that $S(x,\cdot) : P \to P$ is continuous for a.e. x.

Given $\phi : X \times P \to \mathbb{R}$, we write

$$\psi^{(n)} = \begin{cases} \psi + \psi \circ S + \cdots + \psi \circ S^{n-1} & \text{if } n > 0; \\ -\psi \circ S^{-1} - \psi \circ S^{-2} - \cdots - \psi \circ S^{n} & \text{if } n < 0 \text{ and } S \text{ is invertible.} \end{cases}$$

Let $\mathcal{F}$ be the set of measurable functions $\phi : X \times P \to \mathbb{R}$ such that $\phi(x,\cdot)$ is continuous for a.e. x and $x \in X \mapsto \sup_{u\in P} |\phi(x,u)|$ belongs to $L^1(\mu)$.

Exercise viii. Prove that the norm $\|\phi\| = \int \sup_u |\phi(x,u)|\, d\mu(x)$ makes $\mathcal{F}$ a separable Banach space.

Lemma 9. *Assume $\psi \in \mathcal{F}$ and $c \in \mathbb{R}$ are such that*

$$\text{for } \mu\text{-a.e. } x \in X, \quad \limsup_{n\to\infty} \frac{1}{n}\psi^{(n)}(x,u) \geq c \quad \text{for all } u \in P.$$

Then the assertion remains true with the $\limsup$ *replaced with a* $\liminf$ *that is uniform over P. More precisely, for μ-a.e.* $x \in X$ *and every* $\varepsilon > 0$ *there is* n_0 *such that*

$$\frac{1}{n}\psi^{(n)}(x,u) > c - \varepsilon \quad \textit{for all } u \in P \textit{ and } n > n_0.$$

Furthermore, if S is invertible then for a.e. x, $\liminf_{n\to-\infty} \frac{1}{n}\psi^{(n)}(x,u) \geq c$ *uniformly in u.*

Corollary 10. *Assume* $\psi \in \mathcal{F}$ *and* $c \in \mathbb{R}$ *are such that*

$$\textit{for } \mu\textit{-a.e. } x \in X, \quad \limsup_{n\to\infty} \frac{1}{n}\psi^{(n)}(x,u) = c \quad \textit{for all } u \in P.$$

Then the assertion remains true with $\limsup$ *replaced with uniform* $\lim$ *(or* $\lim_{n\to\pm\infty}$ *if S is invertible).*

The proof of Lemma 9 is a Krylov–Bogoliubov argument. For that, we need:

Exercise ix (✓)**.** Let $\mathcal{M}$ be the set of probability measures on $X \times P$ that project on μ (that is, $\pi_*\nu = \mu$). There is a topology on $\mathcal{M}$ which makes it compact, and such that a sequence ν_n converges to ν if and only if

$$\int \phi\, d\nu_n \to \int \phi\, d\nu \quad \text{for every } \phi \in \mathcal{F}.$$

Proof of Lemma 9. Let

$$I_n(x) = \inf_{u\in P} \psi^{(n)}(x,u).$$

Then I_n is measurable, I_1 is integrable, and $I_{n+m} \geq I_n \circ T^m + I_m$. So the Subadditive Ergodic Theorem 3 applies to $-I_n$, and thus $\frac{1}{n}I_n$ converges a.e. to a constant b.

For a.e. x, let $u_n(x)$ be such that $\psi^{(n)}(x, u_n(x)) = I_n(x)$.

Exercise x (✓)**.** It is possible to choose u_n measurable.

Take u_n measurable and define the following probability measures on $X \times P$:

$$\nu_n^0 = \int \delta_{(x,u_n(x))}\, d\mu(x) \quad \text{and} \quad \nu_n = \sum_{j=0}^{n-1} S_*^j \nu_n^0.$$

All this measures project to μ. By Exercise ix, there is a converging subsequence $\nu_{n_i} \to \nu$.

Exercise xi (✓)**.** ν is a S-invariant measure.[2]

[2]Hint: for any $\eta \in \mathcal{F}$, $|\int \eta d(F_*\nu_n - \nu_n)| \leq (2/n)\|\eta\|$, where $\|\cdot\|$ is the obvious norm on the Banach space $\mathcal{F}$.

Moreover, $\int \psi \, d\nu = \lim \int \psi \, d\nu_n = \lim \frac{1}{n} \int I_n \, d\mu = b$. By Birkhoff's Theorem, there is a set of full ν-measure formed by points (x, u) such that the sequence $\frac{1}{n}\psi^{(n)}(x, u)$ converges to b. By the assumption, the $\limsup$'s of those sequences are $\geq c$ ν-a.e. Therefore $b \geq c$. Since $b = \lim_{n\to\infty} n^{-1} \inf_u \psi^{(n)}(x, u)$, we obtain the desired uniform $\liminf$.

Notice that for any S-invariant probability measure ν', we have $\int \psi \, d\nu' \geq b$ (by Birkhoff again).

Now assume that S is invertible. Repeating the above reasoning for S^{-1}, we obtain a S-invariant probability ν' such that

$$\lim_{n\to-\infty} \inf_{u\in P} \frac{1}{n} \psi^{(n)}(x, u) = \int \psi \, d\nu'.$$

So $\liminf_{n\to-\infty} n^{-1}\psi^{(n)}(x, u) \geq \int \psi \, d\nu' \geq b \geq c$ uniformly with respect to u. □

5. The bundle of least stretch

Consider the exponents $\lambda_1 > \cdots > \lambda_k$ and the flag $V^1 \supset \cdots \supset V^k$ given by Lemma 8. We will focus the attention on λ_k and V^k. Write

$$\lambda_{\min}(A) = \lambda_k, \quad E_x = V^k_x$$

We call E the *bundle of least stretch* for A. It has positive dimension.

By definition, $\limsup_{n\to\infty} n^{-1} \log \|A^{(n)} v\| = \lambda_{min}(A)$ for a.e. x and all non-zero $v \in E_x$. Let us improve this information:

Lemma 11. *We have:*

$$\lim_{n\to\pm\infty} \frac{1}{n} \log \|A^{(n)}(x) \cdot v\| = \lambda_{\min}(A) \quad \textit{for a.e. } x \textit{ and all } v \in E_x \setminus \{0\}.$$

($n \to -\infty$ allowed if T is invertible.) Moreover the limit is uniform over unit vectors.

The conclusion of the lemma can be rephrased as:

$$\lim_{n\to\pm\infty} \frac{1}{n} \log \|A^{(n)}_{|E}(x)\| \lim_{n\to\pm\infty} \frac{1}{n} \log \mathbf{m}(A^{(n)}_{|E}(x)) = \lambda_{\min}(A)\,.$$

Proof of Lemma 11. Let $m = \dim E$. By conjugating with some measurable $C : X \to \mathrm{O}(d, \mathbb{R})$, we can assume that $E_x = \mathbb{R}^m \subset \mathbb{R}^d$.

Let P be the projective space of $\mathbb{R}^m$. Consider the induced skew-product $F : X \times P \to X \times P$ over T. Let $\psi(x, \bar{v}) = \log \frac{\|A(x)\cdot v\|}{\|v\|}$, where $\bar{v}$ denotes the element of P corresponding to v. Then

$$\frac{1}{n}\psi^{(n)}(x, \bar{v}) = \frac{1}{n} \sum_{j=0}^{n-1} \log \frac{\|A^{(j+1)}(x) \cdot v\|}{\|A^{(j)}(x) \cdot v\|} = \frac{1}{n} \log \frac{\|A^{(n)}(x) \cdot v\|}{\|v\|}$$

has a lim sup as $n \to \infty$ equal to $\lambda_{\min}$, for a.e. x and all $\bar{v} \in P$. Thus Corollary 10 implies that the lim sup can be replaced with uniform lim. ◻

By Lemma 8, we have

$$\limsup_{n\to\infty} \frac{1}{n} \log \|A^{(n)}(x) \cdot v\| \geq \lambda_{\min}(A) \quad \text{for a.e. } x \text{ and all } v \in \mathbb{R}^d \setminus \{0\}.$$

Using Lemma 9 and a similar argument as in the proof of Lemma 11, we conclude that this lim sup can by replaced by a lim inf that is uniform with respect to u. This can be rephrased as:

$$\liminf_{n\to+\infty} \frac{1}{n} \log \mathbf{m}(A^{(n)}(x)) \geq \lambda_{\min}(A)\,.$$

On the other hand,

$$\limsup_{n\to+\infty} \frac{1}{n} \log \mathbf{m}(A^{(n)}(x)) \leq \limsup_{n\to+\infty} \frac{1}{n} \log \mathbf{m}(A^{(n)}_{|E}(x)) \leq \lambda_{\min}(A)\,.$$

So we obtain

$$\lim_{n\to+\infty} \frac{1}{n} \log \mathbf{m}(A^{(n)}(x)) = \lambda_{\min}(A)\,.$$

This proves one of the assertions in Addendum 4. The others are proven similarly.

6. Interlude: How to make good use of asymptotic information

Let $\mathrm{M}(d)$ indicate the set of $d \times d$ real matrices.

Lemma 12. *Let $B : X \to \mathrm{M}(d)$ be measurable with $\log^+ \|B\|$ integrable. Assume $\gamma \in \mathbb{R}$ is such that*

$$\limsup_{n\to+\infty} \frac{1}{n} \|B^{(n)}(x)\| \leq \gamma \quad \text{a.e.}$$

Then for any $\varepsilon > 0$ there exists a measurable function $b_\varepsilon : X \to \mathbb{R}_+$ such that

$$\|B^{(n)}(T^i x)\| \leq b_\varepsilon(x) e^{(\gamma+\varepsilon)n+\varepsilon|i|} \quad \text{for a.e. } x \text{ and all } n \geq 0,\ i \in \mathbb{Z}.$$

($i < 0$ is allowed if T is invertible.)

Proof. The formula

$$c(x) = \sup_{n\geq 0} e^{-(\gamma+\varepsilon)n} \|B^{(n)}(x)\|$$

defines a measurable a.e. finite function. By definition, $\|B^{(n)}(x)\| \le c(x)e^{(\gamma+\varepsilon)n}$. Now we need to bound the expression $c(T^i x)$. We have

$$\begin{aligned} c(Tx) &= \sup_{n\ge 0} e^{-(\gamma+\varepsilon)n}\|B^{(n)}(Tx)\| \\ &\ge \|B(x)\|^{-1} \sup_{n\ge 0} e^{-(\gamma+\varepsilon)n}\|B^{(n+1)}(x)\| \\ &= e^{\gamma+\varepsilon}\|B(x)\|^{-1} \sup_{n\ge 1} e^{-(\gamma+\varepsilon)n}\|B^{(n)}(x)\| \end{aligned}$$

Either the rightmost sup equals $c(x)$ or else $c(x) = 1$ and then $c(Tx) \ge 1$. In any case, we have

$$c(Tx) \ge \min\left(e^{\gamma+\varepsilon}\|B(x)\|^{-1}, 1\right) c(x)\,.$$

and so

$$\log c \circ T - \log c \ge \min\left(\gamma + \varepsilon - \log^+ \|B\|, 0\right).$$

is quasi-integrable. So Lemma 6 applies, giving $\frac{1}{i}\log c \circ T^i \to 0$ a.e. as $i \to \pm\infty$. We apply the sup trick a second time: Let

$$b_\varepsilon(x) = \sup_{i\in\mathbb{Z}} e^{-\varepsilon|i|} c(T^i x)\,.$$

Then b_ε is the desired function. □

7. The quotient

As before, let E be the bundle of least stretch for A. Assume the norm $\|\cdot\|$ comes from an inner product. For each x, let $E_x^\perp$ be the orthogonal complement of E_x; this is a measurable subbundle. With respect to the splitting $\mathbb{R}^d = E^\perp \oplus E$, we write

$$A = \begin{pmatrix} B & 0 \\ C & A_{|E} \end{pmatrix}. \tag{6}$$

So $B(x) : E_x^\perp \to E_{Tx}^\perp$ gives a new cocycle.

Lemma 13. $\lambda_{\min}(B) \ge \lambda_{k-1}$. *In particular,* $\lambda_{\min}(B) > \lambda_{\min}(A)$.

Proof. Notice that $\mathbf{m}(B^{(n)}(x)) \ge \mathbf{m}(A^{(n)}(x))$.[3] So, by Addendum 4, $\lambda_{\min}(B) \ge \lambda_{\min}(A) = \lambda_k$.

Let $E' \subset E^\perp$ be the subbundle the bundle of least stretch for the cocicle B. The measurable bundle $F = E' \oplus E$ is A-invariant.

[3] Let w be the unit vector in $E_x^\perp$ the most contracted by $B^{(n)}(x)$, and let $v = [A^{(n)}(x)]^{-1} B^{(n)}(x) \cdot w$. By Pythagoras, $\|v\| \ge \|w\|$. Hence $\mathbf{m}(A^{(n)}(x)) \le \frac{\|A^{(n)}(x)v\|}{\|v\|} \le \frac{\|B^{(n)}(x)w\|}{\|w\|} = \mathbf{m}(B^{(n)}(x))\,.$

Fix $\varepsilon > 0$ small. We have

$$A^{(n)}_{|F} = \begin{pmatrix} B^{(n)}_{|E'} & 0 \\ C_n & A^{(n)}_{|E} \end{pmatrix}, \quad \text{where} \quad C_n(x) = \sum_{i=0}^{n-1} A^{(n-i+1)}_{|E}(T^{i+1}x)\, C(T^i x)\, B^{(i)}_{|E'}(x)\,.$$

Write $\sigma = \lambda_{\min}(B)$. Applying Lemma 12, we find measurable functions c, b, a such that for a.e. x and all $n, i \geq 0$,

$$\|C(T^i x)\| \leq c(x) e^{\varepsilon i}$$
$$\|B^{(n)}_{|E'}(x)\| \leq b(x) e^{(\varepsilon+\sigma)n}$$
$$\|A^{(n)}_{|E}(T^i x)\| \leq a(x) e^{(\varepsilon+\lambda_k)n+\varepsilon i}$$

Thus

$$\begin{aligned} \|C_n(x)\| &\leq n \max_{0\leq i\leq n-1} \|A^{(n-i+1)}_{|E}(T^{i+1}x)\|\, \|C(T^i x)\|\, \|B^{(i)}_{|E'}(x)\| \\ &\leq n d_1(x) \max_{0\leq i\leq n-1} e^{\varepsilon(i+1)+(\lambda_k+\varepsilon)(n-i-1)} e^{\varepsilon i} e^{(\sigma+\varepsilon)i} \\ &\leq n d_2(x) e^{4\varepsilon n} \max_{0\leq i\leq n-1} e^{(n-i-1)\lambda_k + \sigma i} \\ &\leq n d_3(x) e^{4\varepsilon n} e^{\sigma n}\,. \end{aligned}$$

(In the last step we used that $\sigma \geq \lambda_k$.) So, at a.e. point,

$$\begin{aligned} \lambda_{k-1} &\leq \limsup_{n\to\infty} \frac{1}{n} \log \|A^{(n)}_{|F}\| \leq \limsup_{n\to\infty} \frac{1}{n} \log \max\left(\|A^{(n)}_{|E}\|\, \|B^{(n)}_{|E'}\|\, \|C_n\|\right) \\ &= \max\left(\limsup_{n\to\infty} \frac{1}{n} \log \|A^{(n)}_{|E}\|, \limsup_{n\to\infty} \frac{1}{n} \log \|B^{(n)}_{|E'}\|, \limsup_{n\to\infty} \frac{1}{n} \log \|C_n\|\right) \\ &\leq \max(\lambda_k, \sigma, \sigma + 4\varepsilon). \end{aligned}$$

So $\lambda_{k-1} \leq \sigma + 4\varepsilon$. As $\varepsilon > 0$ is arbitrary, $\lambda_{k-1} \leq \sigma = \lambda_{\min}(B)$. □

Exercise xii. In the notation of the proof of Lemma 13, prove that $\lim n^{-1} \log \|A^{(n)}(x) \cdot v\|$ exists and equals $\lambda_{\min}(B)$ for all $v \in F \setminus E$. Also show that $\lambda_{\min}(B) = \lambda_{k-1}$ (that is, equality holds in Lemma 13) and $F = V^{k-1}$ is the penultimate space of the Lyapunov flag. Use induction to prove Theorem 2.

8. Construction of a complementary invariant bundle

Lemma 14. *There is a measurable invariant subbundle G of $\mathbb{R}^d$ such that $G \oplus E = \mathbb{R}^d$.*

Remark. Here is the first time we need T to be invertible.

Proof. Let G be any (non-necessarily invariant) measurable subbundle which is complementary to E, that is, such that $G \oplus E = \mathbb{R}^d$.

Let $L(x) : E_x^\perp \to E_x$ be linear maps depending measurably on x. The set of such maps is indicated by $\mathcal{L}$. The graph of L, given by $G_x = \{L(x)v + v;\ v \in E_x\}$, is a subbundle complementary to E.

The (forward) image of a subbundle G is the bundle whose fiber over x is $A(T^{-1}x) \cdot G_{T^{-1}x}$.

Let $\Gamma : \mathcal{L} \to \mathcal{L}$ be the *graph transform* so that the image of the graph of L is the graph of ΓL. Recalling (6), we see that

$$\Gamma L = D + \Phi L\,, \tag{7}$$

where $D \in \mathcal{L}$ and $\Phi : \mathcal{L} \to \mathcal{L}$ are given by:

$$D(x) = C(T^{-1}x)\,[B(T^{-1}x)]^{-1}\,, \quad (\Phi L)(x) = A_{|E}(T^{-1}(x))\,L(T^{-1}x)\,[B(T^{-1}x)]^{-1}\,.$$

To prove Lemma 14 we need to prove that Γ has a fixed point. We will prove that there is a measurable function $a : X \to \mathbb{R}$ and $\tau > 0$ such that

$$\|(\Phi^n D)(x)\| \le a(x)e^{-\tau n} \quad \text{for a.e. } x \text{ and all } n \ge 0. \tag{8}$$

Then the formula $L = \sum_{n=0}^\infty \Phi^n D$ will define a element of $\mathcal{L}$ which is fixed by the graph transform (7).

We have $(\Phi^n L)(x) = (A_{|E})^{(n)}(T^{-n}(x))\,L(T^{-n}x)\,B^{(-n)}(x)$, so

$$\|(\Phi^n D)(x)\| \le \|(A_{|E})^{(n)}(T^{-n}x)\|\,\|D(T^{-n}x)\|\,\|B^{(-n)}(x)\| \tag{9}$$

We will bound each of the three factors using Lemma 12. Fix a small $\varepsilon > 0$ be small (to be specified later). For the first factor in (9) we have

$$\|(A_{|E})^{(n)}(T^{-n}x)\| \le a_1(x)e^{n(\lambda_k+2\varepsilon)}\,, \quad \text{for some measurable } a_1.$$

Secondly,

$$\begin{aligned}\log^+\|D(x)\| &\le \log^+\|C(T^{-1}x)\| + \log\|B(T^{-1}x)^{-1}\| \\ &\le \log^+\|A(T^{-1}x)\| + \log\|A(T^{-1}x)^{-1}\|.\end{aligned}$$

So $\log^+\|D\|$ is integrable and Lemma 12 gives

$$\|D(T^{-n}x)\| \le a_2(x)e^{\varepsilon n}\,, \quad \text{for some measurable } a_2.$$

Now look to the third factor in (9). By Addendum 4 and Lemma 13,

$$\lim_{n\to+\infty} \frac{1}{n}\log\|B^{(-n)}(x)\| = -\lambda_{\min}(B) \le -\lambda_{k-1}\,.$$

The function $\log^+\|B^{-1}\| \le \log^+\|A^{-1}\|$ is integrable. Thus we can apply Lemma 12 to the cocycle B^{-1} over T^{-1} and conclude that

$$\|B^{(-n)}(x)\| \le a_3(x)e^{(-\lambda_{k-1}+\varepsilon)n}\,, \quad \text{for some measurable } a_3.$$

Putting the three bounds together in (9) we obtain

$$\|\Phi^n D(x)\| \le (a_1a_2a_3)(x)\,e^{(-\lambda_{k-1}+\lambda_k+4\varepsilon)}$$

Taking ε small we obtain the desired (8). This proves Lemma 14. □

9. Conclusion

Proof of Theorem 1 (and Addendum 7). Let $V^1 \supset \cdots \supset V^k$ be given by Lemma 8. Let $E^k = V^k$. By Lemma 11, (3) holds for $i = k$ (uniformly with respect to unit vectors). If $k = 1$ then we are done. Otherwise, apply Lemma 14 to find an invariant subbundle G complementary to E^k. Now consider the cocycle restricted to G; its Lyapunov flag is:

$$G = V^1 \cap G \supset \cdots \supset V^{k-1} \cap G \supset \{0\}$$

with corresponding exponents $\lambda_1 > \cdots > \lambda_{k-1}$. Then define $E^{k-1} = V^{k-1} \cap G$ and repeat the argument. □

10. Comments about the proof(s)

There are several proofs of the Oseledets' Theorem. The more conventional proofs use heavier Linear Algebra (exterior powers, singular values etc). The more conventional approach has the advantage of providing more useful information about the Lyapunov exponents and splittings: see the detailed treatment in [A], also [L].

The proof here is a combination of the proofs of [V] (which is on its turn based on [M]), and [W]. Lemmas 8 and 9 come from [W], and Lemma 14 comes from [V]. In [M, V], only the two-sided (i.e., invertible) case was considered, while [W] proves only the one-sided[4] version of the Oseledets' Theorem. Unlike [V, M], we used the Subadditive Ergodic Theorem (in the proof of Lemma 9)[5].

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Jairo Bochi May 27, 2008

[4] It is not clear to me how to deduce the two-sided theorem from its one-sided version without using exterior powers.

[5] Maybe this could be avoided, if desired.