ON THE SUBADDITIVE ERGODIC THEOREM

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ABSTRACT. We present a simple proof of Kingman's Subadditive Ergodic Theorem that does not rely on Birkhoff's (Additive) Ergodic Theorem and therefore yields it as a corollary.

1. Statements

Throughout this note, let $(X, \mathcal{A}, \mu)$ be a fixed probability space and $T : X \to X$ be a fixed measurable map that preserves the measure μ.

Birkhoff's Ergodic Theorem ([B]). *Let $f_1 : X \to \mathbb{R}$ be an integrable function, and let*

$$f_n = \sum_{j=0}^{n-1} f_1 \circ T^j \quad \textit{for all } n \geq 1. \tag{1}$$

Then f_n/n converges a.e. to an integrable function f such that $\int f = \int f_1$.

Kingman's Subadditive Ergodic Theorem ([Ki]). *Let $f_n : X \to \overline{\mathbb{R}}$ be a sequence of measurable functions such that f_1^+ is integrable and*

$$f_{m+n} \leq f_m + f_n \circ T^m \quad \textit{for all } m,\, n \geq 1. \tag{2}$$

Then f_n/n converges a.e. to a function $f : X \to \overline{\mathbb{R}}$. Moreover, f^+ is integrable and

$$\int f = \lim_{n\to\infty} \tfrac{1}{n} \int f_n = \inf_n \tfrac{1}{n} \int f_n \in [-\infty, +\infty).$$

A sequence of functions f_n is called *subadditive* if it satisfies (2), and is called *additive* if equality holds in (2). Clearly every additive sequence takes the form (1).

In this note we will prove Kingman's Theorem and obtain Birkhoff's Theorem as a corollary.

2. Proof

Let $f_n : X \to \mathbb{R}$ be a subadditive sequence of functions with f_1^+ (and therefore f_n^+) in L^1. Using that $\int f_n$ is a subadditive sequence of extended-real numbers, it is an easy exercise to show that

$$\tfrac{1}{n} \int f_n \text{ converges as } n \to \infty \text{ to } L := \inf_n \tfrac{1}{n} \int f_n \text{ (which can be } -\infty).$$

Let $f_\flat$, $f_\sharp : X \to [-\infty, \infty)$ be the measurable functions defined by

$$f_\flat = \liminf_{n\to\infty} \frac{f_n}{n}\,, \qquad f_\sharp = \limsup_{n\to\infty} \frac{f_n}{n}\,.$$

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The plan of the proof is this: We will show that

$$\int f_\flat \geq L \geq \int f_\sharp . \tag{3}$$

In fact, the first inequality is the key one, and the second will be obtained as a consequence. Thus we obtain $f_\flat = f_\sharp$ a.e., at least in the case $L > -\infty$. The same is true in the case $L = -\infty$ by a simple truncation procedure, which allows us to conclude.

To begin the proof, notice that

$$f_\flat(x) \leq \liminf_{n\to\infty} \frac{f_1(x) + f_{n-1}(Tx)}{n} = f_\flat(Tx);$$

hence $T^{-1}(\{f_\flat \geq a\}) \subset \{f_\flat \geq a\}$ for each $a \in \overline{\mathbb{R}}$ and therefore $f_\flat \circ T = f_\flat$ a.e. Similarly for $f_\sharp$.

Now let us prove the first part of (3); it fact we show:

Lemma 1. *$\int f_\flat = L$.*

Proof. We will first consider the case where

$$\text{there exists } C \in \mathbb{R} \text{ such that } f_n \geq -Cn \text{ for all } n. \tag{4}$$

By Fatou's Lemma, $f_\flat$ is integrable, with $\int f_\flat \leq L$. Fix $\varepsilon > 0$ and consider the following increasing sequence of sets:

$$E_k = \Big\{x;\ \exists j \in \{1, \dots, k\} \text{ s.t. } \frac{f_j(x)}{j} < f_\flat(x) + \varepsilon\Big\}, \qquad k \in \mathbb{N}^+.$$

We have $\bigcup_k E_k = X$. Define an integrable function

$$\psi_k = \begin{cases} f_\flat + \varepsilon & \text{in } E_k, \\ f_1 & \text{in } E_k^c. \end{cases}$$

The heart of the proof is the following inequality:

$$f_n(x) \leq \sum_{i=0}^{n-k-1} \psi_k(T^i x) + \sum_{i=n-k}^{n-1} (\psi_k \vee f_1)(T^i x)\,, \quad \text{for a.e. } x \text{ and all } n \geq k. \tag{5}$$

To see this, fix a point x along whose orbit the function $f_\flat$ is constant. Define a sequence of integers

$$m_0 \leq n_1 < m_1 \leq n_2 < m_2 \leq \cdots$$

inductively as follows: Set $m_0 = 0$. Let n_j be the least integer greater or equal than m_{j-1} such that $T^{n_j} x$ belongs to the set E_k. By definition of this set, we can choose m_j such that $1 \leq m_j - n_j \leq k$ and

$$f_{m_j - n_j}(T^{n_j} x) \leq (m_j - n_j)(f_\flat(x) + \varepsilon)\,. \tag{6}$$

Now, given $n \geq k$, let ℓ be the biggest integer such that $m_\ell \leq n$. Using subadditivity, we write

$$f_n(x) \leq \sum f_1(T^i x) + \sum_{j=1}^{\ell} f_{m_j - n_j}(T^{n_j} x)\,, \tag{7}$$

where the first sum is over all i in the set $\bigcup_{j=0}^{\ell-1} [m_j, n_{j+1}) \cup [m_\ell, n)$. Each term $f_1(T^i x)$ with $i \in \bigcup_{j=0}^{\ell-1} [m_j, n_{j+1}) \cup [m_\ell, n_{\ell+1} \wedge n)$ equals $\psi_k(T^i x)$ (because $T^i x \in E_k^c$).

On the other hand, using (6), invariance of $f_\flat$ along the orbit, and the fact that $\psi_k \ge f_\flat + \varepsilon$, we get

$$f_{m_j - n_j}(T^{n_j} x) \le \sum_{i \in [n_j, m_j)} (f_\flat(T^i x) + \varepsilon) \le \sum_{i \in [n_j, m_j)} \psi_k(T^i x) .$$

Thus (7) becomes

$$f_n(x) \le \sum_{i=0}^{n_{\ell+1} \wedge n-1} \psi_k(T^i x) + \sum_{i=n_{\ell+1}}^{n-1} f_1(T^i x) .$$

Since $n_{\ell+1} > n - k$, (5) follows.

Integrating (5), we get $\int f_n \le (n-k) \int \psi_k + k \int (\psi_k \vee f_1)$. Dividing by n and making $n \to \infty$, we get $L \le \int \psi_k$. Then making $k \to \infty$, we get $L \le \int f_\flat + \varepsilon$. Since $\varepsilon > 0$ is arbitrary, we conclude that the lemma holds under the assumption (4).

Now let us consider the general case. For $C \in \mathbb{R}$, define functions

$$f_n^{(C)} = f_n \vee (-Cn) . \tag{8}$$

Then the sequence $f_n^{(C)}$ is subadditive and

$$f_\flat^{(C)} := \liminf_{n\to\infty} \frac{f_n^{(C)}}{n} = f_\flat \vee (-C), \quad f_\sharp^{(C)} := \limsup_{n\to\infty} \frac{f_n^{(C)}}{n} = f_\sharp \vee (-C). \tag{9}$$

Therefore, using the Monotone Convergence Theorem and the part of the lemma already obtained, we get

$$\int f_\flat = \inf_C \int f_\flat^{(C)} = \inf_C \inf_n \tfrac{1}{n} \int f_n^{(C)} = \inf_n \inf_C \tfrac{1}{n} \int f_n^{(C)} = \inf_n \tfrac{1}{n} \int f_n = L. \tag{10}$$ □

Lemma 2. *Let $g : X \to \mathbb{R}$ be an integrable function. Then $g \circ T^n / n \to 0$ a.e. as $n \to \infty$.*

This is usually presented as a consequence of Birkhoff's Theorem; but we provide a simple proof that does not rely on it:

Proof. It suffices to show that for every $\varepsilon > 0$, the set of $x \in X$ such that $|g(T^n x)| \ge \varepsilon n$ for infinitely many $n \in \mathbb{N}$ has zero measure. This follows from the Borel–Cantelli Lemma:

$$\begin{aligned}\sum_{n=1}^{\infty} \mu\{|g \circ T^n| \ge \varepsilon n\} &= \sum_{n=1}^{\infty} \mu\{|g| \ge \varepsilon n\} = \sum_{n=1}^{\infty} \sum_{k=n}^{\infty} \mu\{k \le \varepsilon^{-1}|g| < k+1\} \\ &= \sum_{k=1}^{\infty} k \, \mu\{k \le \varepsilon^{-1}|g| < k+1\} \le \int_{\{|g| > \varepsilon\}} \varepsilon^{-1}|g| < \infty. \end{aligned}$$ □

Lemma 3. *For any $k \in \mathbb{N}^+$,*

$$\limsup_{n\to\infty} \frac{f_{kn}}{n} = k \limsup_{n\to\infty} \frac{f_n}{n} \quad a.e.$$

Proof. The $\le$ inequality is obvious, so let us prove the reverse one. Fix k. For each $n \in \mathbb{N}^+$, write $n = km_n + r_n$ and $1 \le r_n \le k$. By subadditivity,

$$f_n \le f_{km_n} + g \circ T^{km_n} , \quad \text{where } g = f_1^+ \vee \cdots \vee f_k^+ .$$

As $n \to \infty$, we have $m_n \to \infty$; more precisely $m_n/n \to 1/k$. Since $g \in L^1$, Lemma 2 gives $g \circ T^{km_n}/n \to 0$ a.e. The result follows. □

Now let us prove the second part of (3); as mentioned, the idea is to deduce it from the first part. Again we first consider the case where (4) holds. Fix $k \in \mathbb{N}^+$. Let F_n be the n-th Birkhoff sum of $-f_k$ with respect to T^k, that is, $-\sum_{j=0}^{n-1} f_k \circ T^{jk}$. Then the sequence F_n is additive with respect to T^k. Moreover, $F_1 = -f_k \le Ck$, so $F_1^+ \in L^1$. Letting $F_\flat = \liminf F_n/n$, Lemma 1 gives $\int F_\flat \ge \lim \frac{1}{n} \int F_n$. By invariance, $\int \frac{F_n}{n} = -\int f_k$. On the other hand, using Lemma 3,

$$-F_\flat = \limsup_{n\to\infty} \frac{1}{n} \sum_{j=0}^{n-1} f_k \circ T^{jk} \ge \limsup_{n\to\infty} \frac{f_{kn}}{n} = k \limsup_{n\to\infty} \frac{f_n}{n} = k f_\sharp \,.$$

Thus $\int f_\sharp \le -\frac{1}{k} \int F_\flat \le \frac{1}{k} \int f_k$. This holds for every k; hence we proved that $\int f_\sharp \le L$ under assumption (4).

Now we deal with the general case. Again consider $f_n^{(C)}$ as in (8). By what we have already proved, the functions $f_\flat^{(C)}$ and $f_\sharp^{(C)}$ defined by (9) have the same integral, and thus they coincide almost everywhere. Since $f_\flat^{(C)} \to f_\flat$ and $f_\sharp^{(C)} \to f_\sharp$ as $C \to +\infty$, it follows that $f_\flat = f_\sharp$ a.e. This concludes the proof of Kingman's Theorem.

3. Comments

Lemma 1 by itself immediately implies Birkhoff's Theorem: applying it to $-f_1$ we get $\int f_\sharp \le L$ and thus $f_\flat = f_\sharp$ a.e. Also notice that the proof of the lemma wouldn't get any simpler under the assumption of additivity. Thus our proof of Kingman's Theorem is a modified proof of Birkhoff's, where the last inequality $\int f_\sharp \le L$ is deduced directly from $\int f_\flat \ge L$.

Except perhaps for that step, the other ingredients are not significantly new. Among the simplest proofs of Birkhoff's and Kingman's theorems that can be found in the literature we have those of [KeP] and [St], respectively. The former also establishes the equality $f_\flat = f_\sharp$ a.e. by showing that $\int f_\flat \ge L$. Our key inequality (5) is essentially contained in [St], and [KeP] is based on a similar estimate. Truncation, as in (8), appears in both papers. In fact, these approaches are descended from [KzW] – which in turn uses ideas of [Km].

Let us mention that [Sc] also obtains Kingman's Theorem (in fact, a generalization of it) without using Birkhoff's Theorem.

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References

[B] G. D. Birkhoff. Proof of the ergodic theorem. *Proc. Nat. Acad. Sci. USA*, 17:656–660, 1931.

[KzW] Y. Katznelson and B. Weiss. A simple proof of some ergodic theorems. *Israel J. Math.*, 42(4):291–296, 1982.

[KeP] M. Keane and K. Petersen. Easy and nearly simultaneous proofs of the ergodic theorem and maximal ergodic theorem. In *Dynamics & stochastics*, volume 48 of *IMS Lecture Notes Monogr. Ser.*, pages 248–251. Inst. Math. Statist., Beachwood, OH, 2006.

[Km] T. Kamae. A simple proof the ergodic theorem using non-standard analysis. *Israel J. Math.*, 42(4):284–290, 1982.

[Ki] J. F. C. Kingman. The ergodic theory of subadditive stochastic processes. *J. Roy. Statist. Soc. Ser. B*, 30:499–510, 1968.

[Sc] K. Schürger. Almost subadditive extensions of Kingman's ergodic theorem. *Ann. Probab.*, 19(4):1575–1586, 1991.

[St] J. M. Steele. Kingman's subadditive ergodic theorem. *Ann. Inst. H. Poincaré Probab. Statist.*, 25(1):93–98, 1989.

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