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Genericity of zero Lyapunov exponents

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Abstract. We show that, for any compact surface, there is a residual (dense G_δ) set of C^1 area-preserving diffeomorphisms which either are Anosov or have zero Lyapunov exponents a.e. This result was announced by R. Mañé, but no proof was available. We also show that for any fixed ergodic dynamical system over a compact space, there is a residual set of continuous SL(2, $\mathbb{R}$)-cocycles which either are uniformly hyperbolic or have zero exponents a.e.

1. *Introduction*

Let M be a compact connected Riemannian two-dimensional C^∞ manifold without boundary and let μ be its normalized area. Denote by $\mathrm{Diff}^1_\mu(M)$ the set of all μ-preserving C^1 diffeomorphisms endowed with the C^1 topology.

For $f \in \mathrm{Diff}^1_\mu(M)$, the number

$$\lambda^+(f, x) = \lim_{n\to+\infty} \frac{1}{n} \log \|Df^n_x\|,$$

called the upper Lyapunov exponent, exists for almost every $x \in M$. Moreover, $\lambda^+(f, x) \geq 0$. Our main result is as follows.

THEOREM A. *There exists a residual subset $\mathcal{R} \subset \mathrm{Diff}^1_\mu(M)$ such that every $f \in \mathcal{R}$ either is Anosov or $\lambda^+(f, x) = 0$ for μ-almost every x.*

This theorem was announced by Ricardo Mañé (1948–1995) around 1983. He also announced its generalization to symplectic manifolds (with a somewhat more elaborate statement) in [**10**]. Although his proofs have never been published, a sketch of a proof of Theorem A appeared in 1995 [**11**]. We exploited ideas outlined there, together with new ingredients, to prove the theorem in the present paper.

We recall that the only surface that admits Anosov diffeomorphisms is the torus, see [**4**].

It is interesting to compare our results with another C^1-generic dichotomy for area-preserving diffeomorphisms that was obtained by Newhouse [**13**]: a generic diffeomorphism either is Anosov or the set of elliptic periodic points is dense in the surface.

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We will indicate by $\mathrm{LE}(f)$ the 'integrated Lyapunov exponent' of $f \in \mathrm{Diff}^1_\mu(M)$, that is,

$$\mathrm{LE}(f) = \int_M \lambda^+(f)\, d\mu.$$

Recall Ruelle's inequality $h_\mu(f) \le \mathrm{LE}(f)$ [**18**], where $h_\mu(f)$ is the metric entropy of f. As a corollary of the proof of Theorem A, we obtain that the functions

$$\mathrm{LE} \text{ and } h_\mu : \mathrm{Diff}^1_\mu(M) \to [0, \infty)$$

are not continuous.

A more general setting to study Lyapunov exponents consists of linear cocycles. In §5, we prove the analogue of Theorem A for this context. More precisely, let X be a compact space, let $T : X \to X$ be a homeomorphism and let μ be some ergodic measure for T. Given a continuous matrix function (called a linear cocycle) $A : X \to \mathrm{SL}(2, \mathbb{R})$, the following limit exists:

$$\mathrm{LE}(A) = \lim_{n\to+\infty} \frac{1}{n} \log \|A(T^{n-1}x) \cdots A(x)\| \ge 0$$

for μ-a.e. $x \in X$. We prove the following.

THEOREM C. *If T is ergodic then there is a residual set $\mathcal{R} \subset C^0(X, \mathrm{SL}(2, \mathbb{R}))$ such that, for every $A \in \mathcal{R}$, either A is uniformly hyperbolic or* $\mathrm{LE}(A) = 0$.

It is natural to ask whether Theorem A extends, for instance, to the C^2 topology or whether Theorem C extends to the C^1 topology. The answer to the first question is unknown, but the answer to the second is negative. For instance, Young [**19**] exhibits open subsets of $C^1(X, \mathrm{SL}(2, \mathbb{R}))$ consisting of non-uniformly hyperbolic cocycles with positive exponent, where the base transformations are linear automorphisms of the 2-torus.

Deciding whether a diffeomorphism (or a cocycle) has non-zero exponents is, in general, very hard. Our results above provide some explanation for that fact: having positive Lyapunov exponents is *not* a C^1-open condition.

A related very natural question in this setting is whether systems with non-zero Lyapunov exponents are typical, in a measure-theoretic sense. At this point, there is no general theorem in this direction.

2. *Preliminaries*

The manifold M and the measure μ will be fixed from here until the end of §4. Also, 'a.e.' will mean 'μ-almost every'.

2.1. *Oseledets' Theorem and Lyapunov exponents.*

Let us recall Oseledets' Theorem in the two-dimensional area-preserving case. A proof can be found in [**14**].

THEOREM 2.1. *Let $f \in \mathrm{Diff}^1_\mu(M)$. Then there exists a measurable function $x \mapsto \lambda^+(x) \ge 0$ such that*

$$\lambda^+(f, x) = \lim_{n\to+\infty} \frac{1}{n} \log \|Df^n(x)\|$$

for a.e. $x \in M$. Moreover, if $\mathcal{O}^+ = \{x : \lambda^+(f, x) > 0\}$ has positive measure then for a.e. $x \in \mathcal{O}^+$ there is a splitting $T_x M = E^u(x) \oplus E^s(x)$, depending measurably on x, such that for $v \in T_x M - \{0\}$

$$\lim_{n\to+\infty} \frac{1}{n} \log \|Df^n(x).v\| = \begin{cases} \lambda^+(f, x) & \text{if } v \notin E^s(x), \\ -\lambda^+(f, x) & \text{if } v \in E^s(x), \end{cases}$$

$$\lim_{n\to-\infty} \frac{1}{n} \log \|Df^n(x).v\| = \begin{cases} -\lambda^+(f, x) & \text{if } v \notin E^u(x), \\ \lambda^+(f, x) & \text{if } v \in E^u(x), \end{cases}$$

and

$$\lim_{n\to\pm\infty} \frac{1}{n} \log \sin \measuredangle(E^u(f^n x), E^s(f^n x)) = 0.$$

Remark. One can also define the lower Lyapunov exponent as

$$\lambda^-(f, x) = \lim_{n\to-\infty} \frac{1}{n} \log \|Df^n_x\|.$$

It satisfies $\lambda^-(f, x) = -\lambda^+(f, x)$.

The *integrated Lyapunov exponent* is defined by

$$\mathrm{LE}(f) = \int_M \lambda^+(f, x)\, d\mu(x).$$

PROPOSITION 2.1. *The function* LE : $\mathrm{Diff}^1_\mu(M) \to [0, \infty)$, *is upper semicontinuous. Moreover, it is given by*

$$\mathrm{LE}(f) = \inf_{n\ge1} \frac{1}{n} \int_M \log \|Df^n\|\, d\mu.$$

Proof. Given f, let $a_n(f) = \int_M \log \|Df^n\|\, d\mu$. Then the sequence $(a_n(f))$ is subadditive, that is, $a_{n+m} \le a_n + a_m$ for every m, n. Therefore,

$$\mathrm{LE}(f) = \lim_{n\to+\infty} \frac{a_n(f)}{n} = \inf_{n\ge1} \frac{a_n(f)}{n}.$$

Thus $\mathrm{LE}(\cdot)$ is the infimum of a sequence of continuous functions and therefore it is upper semicontinuous. □

2.2. *Avoiding periodic points and hyperbolic sets.* We call $\mathcal{R} \subset \mathrm{Diff}^1_\mu(M)$ a residual subset if it contains a countable intersection of open dense sets. The set $\mathrm{Diff}^1_\mu(M)$ is a Baire space, that is, every residual subset is dense. A property is said to be *generic* if it holds in a residual set.

We say that a measure-preserving transformation is *aperiodic* if the set of its periodic points has zero measure.

PROPOSITION 2.2. *For a dense set of $f \in \mathrm{Diff}^1_\mu(M)$, the following holds: every hyperbolic set for f has measure zero or one and f is aperiodic.*

Proof. Let $f_0 \in \mathrm{Diff}^1_\mu(M)$; we will find f close to f_0 with the required properties. Take a C^2 diffeomorphism $f_1 \in \mathrm{Diff}^1_\mu(M)$ C^1-close to f_0 (it exists, as proved by Zehnder [**20**]). Using Robinson's conservative version of the Kupka–Smale Theorem [**15**, Theorem 1.B.i], we find a C^2 diffeomorphism f, C^2-close to f_1, with countably many periodic points. If f is Anosov then we are done. Otherwise, we use the fact that the hyperbolic sets for a C^2 non-Anosov diffeomorphism have zero measure (this is a folklore theorem; for a proof for basic sets see [**3**]). $\square$

It follows from Proposition 2.1 that the set $\{f \in \mathrm{Diff}^1_\mu(M) : \mathrm{LE}(f) = 0\}$ is a countable intersection of open subsets $\{f : \mathrm{LE}(f) < 1/k\}$ of $\mathrm{Diff}^1_\mu(M)$. Hence to prove Theorem A we only need to show the following.

PROPOSITION 2.3. *Let $f \in \mathrm{Diff}^1_\mu(M)$ be aperiodic and such that every hyperbolic set for f has zero measure. Let $\mathcal{U}$ be a neighborhood of f in $\mathrm{Diff}^1_\mu(M)$ and let $\delta > 0$. Then there exists $g \in \mathcal{U}$ such that $\mathrm{LE}(g) < \delta$.*

Proposition 2.3, which is proved in §§3 and 4, also has the following consequence.

PROPOSITION 2.4. *The functions* LE *and* $h_\mu : \mathrm{Diff}^1_\mu(M) \to [0, \infty)$ *are not continuous.*

Proof. For any compact surface M, Katok [**8**] constructs area-preserving diffeomorphisms $f : M \to M$ with positive metric entropy. Moreover, one checks that the examples are homotopic to the identity (see the proof of Theorem B in [**8**]). In particular, they cannot be approximated by Anosov diffeomorphisms. Using Propositions 2.2, 2.3 and Ruelle's inequality $h_\mu \le \mathrm{LE}$, we conclude that Katok's examples are points of discontinuity of both functions h_μ and LE. $\square$

2.3. *Fixing coordinates, metrics and neighborhoods.* Now we establish some notation to be used until the end of §4.

Darboux's Theorem (see [**1**], for instance) gives that for each $x \in M$ there is an open set $V \ni x$ and a C^∞ diffeomorphism $\varphi : V \to \varphi(V) \subset \mathbb{R}^2$ such that the induced measure $\varphi_*(\mu)$ coincides with the usual Lebesgue measure in $\varphi(V) \subset \mathbb{R}^2$. Taking a finite cover of M by such domains, we obtain an atlas

$$\mathcal{A}^* = \{\varphi_i : V_i^* \to \mathbb{R}^2, i = 1, 2, \ldots, \alpha\}.$$

For technical reasons let us take open sets $V_i \subset M$ with $\overline{V_i} \subset V_i^*$ such that $\bigcup_i V_i = M$. Restricting the charts φ_i to these smaller domains, we obtain another atlas $\mathcal{A} = \bigcup_i \{\varphi_i : V_i \to \mathbb{R}^2\}$. We will also suppose that $\mu(\partial V_i) = 0 = \mu(\partial V_i^*)$ for each i.

For each point $x \in M$, let $i(x) = \min\{i : V_i \ni x\}$. We define a norm $\|\cdot\| = \|\cdot\|_x$ on each tangent space $T_x M$ by $\|v\| = \|D\varphi_{i(x)}(v)\|$. The Riemannian metric on M will not be used.

If $A : T_x M \to T_y M$ is a linear map, the norm $\|A\|$ is then defined in the usual way:

$$\|A\| = \sup_{0 \neq v \in T_x M} \frac{\|Av\|}{\|v\|}.$$

Using the charts $\varphi_{i(x)}$ and $\varphi_{i(y)}$, we may view A as a linear map $\mathbb{R}^2 \to \mathbb{R}^2$. This permits us, for example, to speak of the distance $\|A - B\|$ between two linear maps $A : T_{x_1}M \to T_{x_2}M$ and $B : T_{x_3}M \to T_{x_4}M$ whose base points are different. Precisely, we define

$$\|A - B\| = \|D_2 A D_1^{-1} - D_4 B D_3^{-1}\|, \quad \text{where } D_j = (D\varphi_{i(x_j)})_{x_j}.$$

If $x \in M$ and $r > 0$ is small, we define

$$B_r(x) = \varphi_{i(x)}^{-1}(B_r(\varphi_i(x))).$$

We will always assume that r is small enough so that $\overline{B_r(x)} \subset V_i^*$. The sets $B_r(x)$ will be called *disks*.

Given $\varepsilon_0 > 0$, we define the ε_0*-basic neighborhood* $\mathcal{U}(\mathrm{id}, \varepsilon_0)$ *of the identity* as the set of diffeomorphisms $h \in \mathrm{Diff}_\mu^1(M)$ such that we have $h(V_i) \subset V_i^*$ for every i, $h(x) \in B_{\varepsilon_0}(x)$ and $\|Dh_x - I\| < \varepsilon_0$ for every $x \in M$. Given $f \in \mathrm{Diff}_\mu^1(M)$, we define the ε_0*-basic neighborhood* $\mathcal{U}(f, \varepsilon_0)$ *of* f as the set of $g \in \mathrm{Diff}_\mu^1(M)$ such that $g \circ f^{-1} \in \mathcal{U}(\mathrm{id}, \varepsilon_0)$ or $f^{-1} \circ g \in \mathcal{U}(\mathrm{id}, \varepsilon_0)$. Notice that every neighborhood of f in $\mathrm{Diff}_\mu^1(M)$ contains a basic neighborhood.

3. *Construction of perturbations along an orbit segment*

The aim of this section is to prove the lemma below. In the next section we will deduce Theorem A from it.

MAIN LEMMA. *Let $f \in \mathrm{Diff}_\mu^1(M)$ be aperiodic and such that every hyperbolic set has zero measure. Let $\mathcal{U}$ be a neighborhood of f in $\mathrm{Diff}_\mu^1(M)$, $\delta > 0$ and $0 < k < 1$. Then there exists a measurable integer function $N : M \to \mathbb{N}$ with the following properties. For a.e. $x \in M$ and every integer $n \geq N(x)$ there exists $r = r(x, n)$ such that for every disk $U = B_{r'}(x)$ with $0 < r' < r$, there exist $g \in \mathcal{U}$ and a compact set $K \subset U$ such that:*

(i) *g equals f outside the set $\bigsqcup_{j=0}^{n-1} f^j(\overline{U})$ and the iterates $f^j(\overline{U})$, $0 \leq j \leq n$ are two-by-two disjoint;*
(ii) *$\mu(K)/\mu(U) > k$;*
(iii) *if $y \in K$ then $(1/n)\log\|Dg_y^n\| < \delta$.*

We will now outline some ideas in the proof of this lemma. We perturb the derivatives $Df_{f^j x}$ of f along the orbit segment. These perturbations are constructed in such a way that their product has small (i.e. not exponentially large) norm. To hinder the growth of these products, we send the expanding Oseledets direction to the contracting direction. This is possible in the absence of uniform hyperbolicity. Once the linear perturbations are constructed, we must find an area-preserving diffeomorphism g close to f having approximately the assigned derivatives. To guarantee that the diffeomorphism g exists and has the stated properties, some care is needed in the choice of the perturbations of $Df_{f^j x}$.

3.1. *Realizable sequences.*

As we mentioned, to prove the Main Lemma we will construct perturbations L_j of the linear maps $Df_{f^j x}$. These perturbations will be required to have the following property.

Definition. Given $f \in \mathrm{Diff}^1_\mu(M)$, a basic neighborhood $\mathcal{U} = \mathcal{U}(f, \varepsilon_0)$, $0 < k < 1$ and a non-periodic point $x \in M$, a sequence of (area-form-preserving) linear maps

$$T_x M \xrightarrow{L_0} T_{fx} M \xrightarrow{L_1} \cdots \xrightarrow{L_{n-1}} T_{f^n x} M$$

is called a *$(k, \mathcal{U})$-realizable sequence* of length n at x if the following holds. For every $\gamma > 0$ there is $r > 0$ such that if $U \subset B_r(x)$ is a non-empty open set then there are $g \in \mathcal{U}$ and a compact set $K \subset U$ such that:

(i) g equals f outside the set $\bigsqcup_{j=0}^{n-1} f^j(\overline{U})$ and the iterates $f^j(\overline{B_r(x)})$, $0 \le j \le n$ are two-by-two disjoint;

(ii) $\mu(K)/\mu(U) > k$;

(iii) if $y \in K$ then $\|Dg_{g^j y} - L_j\| < \gamma$ for every j.

In the following lemma we give some useful properties of realizable sequences.

LEMMA 3.1.

(1) *The sequence $\{Df_x, \dots, Df_{f^{n-1}(x)}\}$ is $(k, \mathcal{U})$-realizable for every k and $\mathcal{U}$.*

(2) *Let k, k_1, $k_2 \in (0, 1)$ be such that $1 - k = (1 - k_1) + (1 - k_2)$. If $\{L_0, \dots, L_{n-1}\}$ is a $(k_1, \mathcal{U})$-realizable sequence at x and $\{L_n, \dots, L_{n+m-1}\}$ is a $(k_2, \mathcal{U})$-realizable sequence at $f^n(x)$ then $\{L_0, \dots, L_{n+m-1}\}$ is $(k, \mathcal{U})$-realizable at x.*

(3) *Let $\{L_j : T_{f^j x} M \to T_{f^{j+1} x} M\}$ be a sequence of linear maps at a non-periodic point x. To prove that the sequence $\{L_j\}$ is $(k, \mathcal{U})$-realizable we only need to check the conditions for open sets U that are disks, $U = B_{r_0}(y) \subset B_r(x) \subset V$.*

Proof. For the first property, just take $g = f$.

For property (2), take $\gamma > 0$. Let $r_1 > 0$ be the radius associated to the first ($(k_1, \mathcal{U})$-realizable) sequence, and r_2 the radius associated to the second sequence. Let $0 < r < r_1$ be such that $f^n(B_r(x)) \subset B(f^n(x), r_2)$. Given an open set $U \subset B_r(x)$, the realizability of the first sequence gives us a diffeomorphism $g_1 \in \mathcal{U}$ and a set $K_1 \subset U$. Analogously, from the open set $f^n(U) \subset B(f^n(x), r_2)$ we can find $g_2 \in \mathcal{U}$ and $K_2 \subset f^n(U)$. Then define a diffeomorphism g as $g = g_1$ inside $U, \dots, f^{(n-1)}(U)$, $g = g_2$ inside $f^n(U), \dots, f^{(n+m-1)}(U)$, $g = f$ elsewhere. Also define a compact set $K = K_1 \cap g^{-n}(K_2) \subset U$. Then one can check that g and K satisfy the required properties.

Now let us prove (3). Let $\{L_j\}$ be a sequence at the point x and suppose that the conditions of realizability are satisfied for open sets U that are disks. That is, given $\gamma > 0$, there is an $r > 0$ such that for every disk $U \subset B(x, r)$ there are g and K verifying conditions (i)–(iii) of the definition of a realizable sequence. Fix the chart $(\varphi, V) = (\varphi_{i(x)}, V_{i(x)}) \in \mathcal{A}$ and take any open set $U \subset B_r(x)$. By Vitali's covering lemma (see, for instance, [**12**]), there is a countable family of disjoint closed disks in $\mathbb{R}^2$ covering $\varphi_i^{-1}(U)$ mod 0. Thus we can find a finite family of disks $U_i = B_{r_i}(y_i) \subset U$ with disjoint closures such that $\mu(U - \bigsqcup_i B_{r_i}(y_i))$ is as small as we please. For each disk U_i there are, by hypothesis, a perturbation $g_i \in \mathcal{U}(f, \varepsilon_0)$ and a set $K_i \subset U_i$ with the properties (i)–(iii) of realizability. Let $K = \bigcup_i K_i$ and define g as equal to g_i in each $f^j(U_i)$. Since the latter sets are disjoint, g is well-defined. Moreover, $g \in \mathcal{U}(f, \varepsilon_0)$ and the pair (g, K) satisfies the required properties (i)–(iii). □

We will denote by R_θ the rotation of angle θ,

$$R_\theta = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}.$$

The simple lemma below is the basic tool that will be used to construct all our realizable sequences.

LEMMA 3.2. *Let $\varepsilon_1 > 0$ and $0 < k < 1$. Then there exists $\alpha_0 > 0$ with the following properties. If $|\alpha| \le \alpha_0$ and $r > 0$, then there exists a C^1 area-preserving diffeomorphism $h : \mathbb{R}^2 \to \mathbb{R}^2$ such that:*
(i) $|z| \ge r$ *implies* $h(z) = z$*;*
(ii) $|z| \le \sqrt{k}r$ *implies* $h(z) = R_\alpha(z)$*;*
(iii) $|h(z)| = |z|$ *for all* z*;*
(iv) $|h(z) - z| \le \alpha r$ *for all* z*;*
(v) $\|Dh_z - I\| \le \varepsilon_1$ *for all* z*.*

Proof. First suppose $r = 1$. Let $F : \mathbb{R} \to \mathbb{R}$ be a C^∞ function such that $F(t) = 1$ for $t \le \sqrt{k}$, $F(t) = 0$ for $t \ge 1$ and $0 \le -F'(t) \le 2/(1 - \sqrt{k})$. Define h by $h(z) = R_{\alpha F(|z|)}(z)$. Then h is an area-preserving diffeomorphism satisfying properties (i)–(iv). If α_0 is small, (v) will hold for $|z| \le \sqrt{k}$. We still have to check (v) in the annulus $\sqrt{k} \le |z| \le 1$, where we can take polar coordinates (ρ, θ). The mapping h takes the form $(\rho, \theta) \mapsto (\rho, \theta + \alpha F(\rho))$. The Jacobian matrix of h is $\left(\begin{smallmatrix} 1 & 0 \\ \alpha F'(\rho) & 1 \end{smallmatrix}\right)$ and it is close to the identity if α_0 is small enough.

Now we claim that the same α_0 will work for any $r > 0$. Indeed, if h_1 is the diffeomorphism associated to $r = 1$ constructed above, let $h(z) = r h_1(r^{-1}z)$. Then (i)–(iv) obviously hold for h and, since $Dh_z = D(h_1)_{r^{-1}z}$, (v) also holds. □

The two lemmas below will be used to construct realizable sequences. Given $x \in M$ and $\theta \in \mathbb{R}$, consider the chart $\varphi = \varphi_{i(x)} : V_{i(x)} \to \mathbb{R}^2$ and the linear map $(D\varphi_x)^{-1} R_\theta D\varphi_x : T_x M \hookleftarrow$. We shall call this map *the rotation of angle θ at x* and also denote it by R_θ.

LEMMA 3.3. *Given $f \in \mathrm{Diff}^1_\mu(M)$, $\mathcal{U} = \mathcal{U}(f, \varepsilon_0)$ and $0 < k < 1$, there is an $\alpha_1 > 0$ with the following properties. Suppose that $x \in M$ is not periodic and $|\theta| \le \alpha_1$. Then $\{Df_x R_\theta\}$ and $\{R_\theta Df_x\}$ are $(k, \mathcal{U})$-realizable sequences of length 1 at x.*

Proof. We will prove that if $|\theta|$ is small then the sequence $\{Df_x R_\theta\}$ is realizable; for the other sequence the proof is similar. Let $r > 0$ be small. By Lemma 3.1(3), we only need to construct perturbations supported in disks $U = B_{r'}(y) \subset B_r(x)$. Now apply Lemma 3.2 to find, for each small angle θ, a diffeomorphism g and a disk $K \subset U$ with the required properties. Since r, and thus r', is small, g is near f. □

LEMMA 3.4. *Given $f \in \mathrm{Diff}^1_\mu(M)$, $\mathcal{U} = \mathcal{U}(f, \varepsilon_0)$ and $0 < k < 1$, there is an $\alpha_2 > 0$ with the following properties. Suppose that $x \in M$ is not periodic, $m \ge 2$, $R_{\theta_0} : T_x M \hookleftarrow$ and $R_{\theta_1} : T_{f^m(x)} M \hookleftarrow$ are rotations such that $|\theta_i| \le \alpha_2$. Then*

$$\{Df_x R_{\theta_0}, Df_{fx}, \ldots, Df_{f^{m-2}x}, R_{\theta_1} Df_{f^{m-1}x}\}$$

is a $(k, \mathcal{U})$-realizable sequence of length m at x.

Proof. Let k_0 be such that $1 - k_0 < \frac{1}{2}(1 - k)$. Apply Lemma 3.3 with k_0 in the place of k to obtain α_2. Now, if $|\theta_0|$, $|\theta_1| \leq \alpha_2$ then $\{Df_x R_{\theta_0}\}$ and $\{R_{\theta_1} Df_{f^{m-1}x}\}$ are $(k_0, \mathcal{U})$-realizable sequences of length 1. By items (1) and (2) of Lemma 3.1, $\{Df_x R_{\theta_0}, Df_{fx}, \ldots, Df_{f^{m-2}x}, R_{\theta_1} Df_{f^{m-1}x}\}$ is a $(k, \mathcal{U})$-realizable sequence. □

The lemma above is somewhat weak; we cannot use an arbitrary (say, m) number of rotations instead of just two. This difficulty will be overcome in the next section.

3.2. *Nested rotations.* Now we will deal with linear area-preserving transformations with an invariant ellipse. As will be shown in Lemma 3.7, suitable sequences of such elliptic rotations are realizable. The key point there is that those sequences can be arbitrarily long while k is kept controlled.

Our ellipses will always be filled ones. An ellipse $\mathcal{B} \subset \mathbb{R}^2$ has *eccentricity* E if it is the image of a disk under a transformation $L \in \mathrm{SL}(2, \mathbb{R})$ with $\|L\| = E$. (This is not quite the usual definition.) Thus,

$$E = \sqrt{\frac{\text{major axis}}{\text{minor axis}}}.$$

The following lemma is a slight generalization of Lemma 3.2.

LEMMA 3.5. *Let $\varepsilon_1 > 0$, $0 < k < 1$ and $E \geq 1$. Then there exists $\varepsilon > 0$ with the following properties. Let $\mathcal{B} \subset \mathbb{R}^2$ be an ellipse centered at the origin, of eccentricity $\leq E$ and diameter $\leq \varepsilon_1$, and let $L \in \mathrm{SL}(2, \mathbb{R})$ be a linear map with $\|L - I\| < \varepsilon$ and $A(\mathcal{B}) = \mathcal{B}$. Then there exists a C^1 area-preserving diffeomorphism $h : \mathbb{R}^2 \to \mathbb{R}^2$ such that if $\mathcal{B}'$ is the (smaller) ellipse $\mathcal{B}' = \sqrt{k}\mathcal{B}$ then:*

(i) *$z \notin \mathcal{B}$ implies $h(z) = z$;*
(ii) *$z \in \mathcal{B}'$ implies $h(z) = L(z)$;*
(iii) *h preserves all ellipses of the form $t\mathcal{B}$, $t > 0$;*
(iv) *$|h(z) - z| \leq \varepsilon_1$ for all z;*
(v) *$\|Dh_z - I\| \leq \varepsilon_1$ for all z.*

Proof. There is $M \in \mathrm{SL}(2, \mathbb{R})$ with $\|M\| = \|M^{-1}\| = E$ such that $M(\mathcal{B})$ is a disk. We apply Lemma 3.2 with $E^{-2}\varepsilon_1$ in the place of ε_1 to get $\alpha_0 > 0$. If $L \in \mathrm{SL}(2, \mathbb{R})$ preserves $\mathcal{B}$ and $\|L - I\| < \varepsilon$ then $\|MLM^{-1} - I\| < E^2\varepsilon$. Thus if $\varepsilon > 0$ is chosen small enough then MLM^{-1} is a rotation of angle α with $|\alpha| < \alpha_0$. Thus Lemma 3.2 gives a diffeomorphism h_0 and we define $h = M^{-1}h_0M$. The properties (i)–(iii) and (v) are easily seen to hold for h; (iv) is assured if we suppose that ε, and thus $|\alpha|$, is very small. □

The next lemma says that the image of a small ellipse by a C^1 diffeomorphism is approximately an ellipse.

LEMMA 3.6. *Let $h : \mathbb{R}^2 \to \mathbb{R}^2$ be a C^1 diffeomorphism with $h(0) = 0$. Let $M = Dh_0 : \mathbb{R}^2 \to \mathbb{R}^2$. Then, given $\eta > 0$ and $E > 1$ there exists $r > 0$ such that if $\mathcal{B}(y) \subset B_r(0)$ is an ellipse centered at y with eccentricity $\leq E$ and $\mathcal{B}(0) = \mathcal{B}(y) - y$ is the translated ellipse centered at zero, then we have*

$$(1 - \eta)M\mathcal{B}(0) + h(y) \subset h(\mathcal{B}(y)) \subset (1 + \eta)M\mathcal{B}(0) + h(y).$$

Proof. Let $g : \mathbb{R}^2 \to \mathbb{R}^2$ be such that $h = Mg$. Since g is C^1 smooth and $Dg_0 = I$, we have

$$g(z) - g(y) = z - y + \xi(z, y) \tag{$*$}$$

where

$$\lim_{(z,y)\to(0,0)} \frac{\xi(z, y)}{|z - y|} = 0.$$

($|\cdot|$ indicates the Euclidean norm in $\mathbb{R}^2$.) Choose $r > 0$ such that $|z|, |y| < r$ implies $|\xi(z, y)| \le E^{-2}\eta|z - y|$. Now let $\mathcal{B}(y) \subset B_r(0)$ be an ellipse with axes $2a$ and $2b$; $1 \le b/a \le E^2$. If $z \in \partial\mathcal{B}(y)$ then $|z - y| \le b$ and $|\xi(z, y)| \le E^{-2}\eta b \le \eta a$. Thus equation ($*$) gives that

$$g(\partial\mathcal{B}(y)) - g(y) \subset V_{\eta a}(\partial\mathcal{B}(0)),$$

where $V_\varepsilon(\cdot)$ denotes ε-neighborhood. To avoid confusion, we will temporarily denote the difference of sets by the symbol $\setminus$. We have the geometrical property

$$V_{\eta a}(\partial\mathcal{B}(0)) \subset (1 + \eta)\mathcal{B}(0) \setminus (1 - \eta)\mathcal{B}(0).$$

Therefore,

$$g(\partial\mathcal{B}(y)) - g(y) \subset (1 + \eta)\mathcal{B}(0) \setminus (1 - \eta)\mathcal{B}(0).$$

Applying the linear mapping M, we have

$$h(\partial\mathcal{B}(y)) - h(y) \subset (1 + \eta)M\mathcal{B}(0) \setminus (1 - \eta)M\mathcal{B}(0).$$

The lemma now follows from standard topological arguments. □

Now we use the above material to give another construction of realizable sequences. In the next lemma, $\mathbb{D} \subset T_xM$ will denote the unit disk $\{v : \|v\| < 1\}$.

LEMMA 3.7. *Given $f \in \mathrm{Diff}^1_\mu(M)$, $\mathcal{U} = \mathcal{U}(f, \varepsilon_0)$, $0 < k < 1$ and $E > 1$, there is $\varepsilon > 0$ with the following properties. Suppose that $x \in M$ is not periodic and there is $n \in \mathbb{N}$ such that $\|Df^j_x\| \le E$ for $j = 1, \dots, n$. If*

$$T_xM \xrightarrow{L_0} T_{fx}M \xrightarrow{L_1} \cdots \xrightarrow{L_{n-1}} T_{f^nx}M$$

are linear maps such that for every $j = 1, \dots, n$ we have
(i) $L_{j-1} \cdots L_0(\mathbb{D}) = Df^j_x(\mathbb{D})$ *and*
(ii) $\|L_j - Df_{f^jx}\| < \varepsilon$,
then $\{L_0, \dots, L_{n-1}\}$ is $(k, \mathcal{U})$-realizable at x.

Proof. Let f, ε_0, k and E be given. Let $\varepsilon > 0$ be given by Lemma 3.5, depending on $\varepsilon_1 = \varepsilon_0$, k and E.

Now let x, n and $\{L_0, \dots, L_{n-1}\}$ be as in the statement. We must prove that the sequence $\{L_0, \dots, L_{n-1}\}$ is $(k, \mathcal{U})$-realizable; so let $\gamma > 0$ be given.

We will consider the charts $\varphi_{i(f^jx)} : V_{i(f^jx)} \to \mathbb{R}^2$. To simplify notation, we write $\varphi_j : V_j \to \mathbb{R}^2$ instead.

Let $r_0 > 0$ be such that, for each $j = 0, 1 \dots, n$, we have:
(1) $f^j(\overline{B_{r_0}(x)}) \subset V^*_{i(f^j(x))}$;

(2) the sets $f^j(\overline{B_{r_0}(x)})$ are two-by-two disjoint;
(3) for every $z \in f^j(B_{r_0}(x))$ we have $\|Df_z - Df_{f^j x}\| < \gamma E^{-2}$;
(4) $\|Df^j_z\| < 2E$ for every $z \in B_{r_0}(x)$.

Using the charts, we can translate the problem to $\mathbb{R}^2$. Let f_j be the expression of f in charts in the neighborhood of $f^j(x)$, that is, $f_j = \varphi_{j+1} \circ f \circ \varphi_j^{-1}$. To simplify notation, we suppose that $\varphi_j(f^j(x)) = 0$.

Let $\widetilde{L}_j$ be the expression of L_j in charts, that is,

$$\widetilde{L}_j = (D\varphi_{j+1})_{f^{j+1}x} \cdot L_j \cdot [(D\varphi_j)_{f^j x}]^{-1}.$$

Let $M_j = (Df_j)_0 \in \mathrm{SL}(2, \mathbb{R})$. By hypothesis, we have $\|M_{j-1} \cdots M_0\| \le E$ for each j. Let $R_j \in \mathrm{SL}(2, \mathbb{R})$ be such that $\widetilde{L}_j = M_j R_j$. Since R_j preserves an ellipse of eccentricity $\le E$, we have $\|R_j\| \le E^2$.

Take k_0 such that $k < k_0 < 1$ and let $0 < \tau < 1$ be such that $\tau^{4n} k_0 > k$. Take $\eta > 0$ such that $\tau < 1 - \eta < 1 + \eta < \tau^{-1}$. Using Lemma 3.6 n times, we find $0 < r_1 < r_0$ with the following properties. If $\mathcal{B} \subset B_{r_1}(0)$ is an ellipse centered at a point y and with eccentricity $\le E$ then, for each $j = 0, \dots, n-1$,

$$(1-\eta) M_j(\mathcal{B} - y) \subset f_j(\mathcal{B}) - f_j(y) \subset (1+\eta) M_j(\mathcal{B} - y).$$

Finally, let $r = r_1/3E$. This is the r of the definition of realizable sequences.

By Lemma 3.1(3), in order to prove realizability we may restrict ourselves to sets U that are disks, so let $U = B_{r'}(x') \subset B_r(x)$, for some x' and r'.

Let, for $0 < t \le 1$, $\mathcal{B}^0_t = B_{tr'}(x')$. Let $\mathcal{B}^0_t - x'$ denote the translate of $\mathcal{B}^0_t$ centered at the origin. Define, for $t > 0$ and $j = 1, \dots, n$,

$$\mathcal{B}^j_t = M_{j-1} \cdots M_0(\mathcal{B}^0_t - x') + f_{j-1} \cdots f_0(x').$$

Then $\mathcal{B}^j_t$ is an ellipse centered at $f_{j-1} \cdots f_0(x')$ with eccentricity $\le E$. We have $M_{j-1} \cdots M_0(B_r(0)) \subset B_{Er}(0)$ and, by (4), $f_{j-1} \cdots f_0(B_r(0)) \subset B_{2Er}(0)$. Therefore,

$$\mathcal{B}^j_t \subset \mathcal{B}^j_1 \subset M_{j-1} \cdots M_0(B_r(0)) + f_{j-1} \cdots f_0(x') \subset B_{3Er}(0) \subset B_{r_1}(0).$$

This permits us to apply Lemma 3.6 to those ellipses and get the property

$$0 < t \le 1,\ 0 \le j \le n-1 \Longrightarrow \mathcal{B}^{j+1}_{\tau t} \subset f_j(\mathcal{B}^j_t) \subset \mathcal{B}^{j+1}_{\tau^{-1} t}.$$

By hypothesis (i), the linear map R_j preserves the ellipses $\mathcal{B}^j_t$. So we apply Lemma 3.5 with k_0, $\mathcal{B}^j_{\tau^n}$ and R_j in the place of k, $\mathcal{B}$ and L. The lemma gives us a C^1 area-preserving diffeomorphism $h_j : \mathbb{R}^2 \to \mathbb{R}^2$ such that:

(5) $z \notin \mathcal{B}^j_{\tau^n}$ implies $h_j(z) = z$;
(6) $z \in \overline{\mathcal{B}}^j_{\tau^n \sqrt{k_0}}$ implies $h_j(z) = R_j(z)$;
(7) $t \ge 0$ implies $h_j(\mathcal{B}^j_t) = \mathcal{B}^j_t$;
(8) $|h_j(z) - z| < \varepsilon_0$ for all z;
(9) $\|D(h_j)_z - I\| \le \varepsilon_0$ for all z.

For $0 \le j \le n-1$, let $S_j = \varphi_j^{-1}(\{z : h_j(z) \neq z\})$. Then we have

$$S_j \subset \varphi_j^{-1}(\mathcal{B}^j_{\tau^n}) \subset \varphi_j^{-1}[f_{j-1}(\mathcal{B}^{j-1}_{\tau^{n-1}})] \subset \cdots$$
$$\subset \varphi_j^{-1}[f_{j-1}\cdots f_0(\mathcal{B}^0_{\tau^{n-j}})] = f^j(B_{\tau^{n-j}r'}(x')) \subset f^j(U).$$

So, by (1) and $U \subset B_{r_0}(x)$, the sets $\overline{S_j}$ are disjoint. This permits us to define a diffeomorphism $g \in \mathrm{Diff}^1_\mu(M)$ as equal to f in $S_0 \sqcup \cdots \sqcup S_{n-1}$, and equal to $\varphi_{j+1}^{-1} \circ g_j \circ \varphi_j$ in S_j, where $g_j = f_j \circ h_j$.

Let us verify that g is the desired perturbation. First of all, by (8) and (9), we have $f^{-1} \circ g \in \mathcal{U}(\mathrm{id}, \varepsilon_0)$, that is, $g \in \mathcal{U}(f, \varepsilon_0)$.

We abbreviate $f^{(0)} = \mathrm{id}$, $f^{(j)} = f_{j-1}\cdots f_0$ and analogously for $g^{(j)}$. Condition (i) in the definition of realizable sequences is easy to check. We have seen that $f^j(U) \supset S_j$; this means that g_j equals f_j outside $f^j(U)$. Now define

$$K = \varphi_0^{-1}(\overline{\mathcal{B}}^0_{\tau^{2n}\sqrt{k_0}}) = \overline{B}_{\tau^{2n}\sqrt{k_0}}(x') \subset U.$$

Since φ_0 takes μ to the area in $\mathbb{R}^2$, we can calculate

$$\mu(U - K) = \pi r'^2 - \pi\tau^{4n}k_0 r'^2 = (1 - \tau^{4n}k_0)\mu(U) < (1-k)\mu(U),$$

which is condition (ii) in the definition of realizable sequences.

Since $g_j = f_j \circ h_j$ and h_j preserves $\mathcal{B}^j_t$ ellipses, we have (by induction in j)

$$g^{(j)}(\overline{\mathcal{B}}^0_t) \subset \overline{\mathcal{B}}^j_{\tau^{-j}\sqrt{k_0}t} \subset \overline{\mathcal{B}}^j_{\tau^{-n}\sqrt{k_0}t}$$

for every j and $0 < t < 1$. Setting $t = \tau^{2n}\sqrt{k_0}$, we get

$$g^j(K) \subset \varphi_j^{-1}(\overline{\mathcal{B}}^j_{\tau^n\sqrt{k_0}}).$$

To check condition (iii), take $y \in K$. Let $\tilde{y} = \varphi_0^{-1}(y)$. Then $g^{(j)}(\tilde{y}) \in \overline{\mathcal{B}}^j_{\tau^n\sqrt{k_0}}$ and so, by (6), $(Dh_j)_{g^{(j)}(\tilde{y})} = R_j$. Therefore,

$$\|Dg_{g^j y} - L_j\| = \|(Dg_j)_{g^{(j)}\tilde{y}} - \widetilde{L}_j\| = \|(Df_j)_{h_j(g^{(j)}\tilde{y})}R_j - (Df_j)_0 R_j\|.$$

Using (3) and $\|R_j\| \le E^2$ we get

$$\|Dg_{g^j y} - L_j\| \le \|(Df_j)_{h_j(g^{(j)}\tilde{y})} - (Df_j)_0\|\|R_j\| < \gamma,$$

proving the third condition and thus the lemma. □

3.3. *Sending E^u to E^s.* Here we use Lemmas 3.3, 3.4 and 3.7 to construct realizable sequences that send the expanding Oseledets direction to the contracting direction; this is the content of Lemma 3.8 below.

First we define some notation that will also be used in §3.4. Given a diffeomorphism $f \in \mathrm{Diff}^1_\mu(M)$, let $\mathcal{O}(f) \subset M$ be the full measure set given by Oseledets' Theorem. Define the following f-invariant sets:

$$\mathcal{O}^+(f) = \{x \in \mathcal{O}(f) : \lambda^+(f,x) > 0\},$$
$$\mathcal{O}^0(f) = \{x \in \mathcal{O}(f) : \lambda^+(f,x) = 0\}.$$

Now we define for $x \in \mathcal{O}^+(f)$ and $m \geq 1$,

$$\Delta(x, m) = \frac{\|Df_x^m|_{E^s(x)}\|}{\|Df_x^m|_{E^u(x)}\|}$$

and define the set

$$\Gamma_m(f) = \{x \in \mathcal{O}^+(f) : \Delta(x, m) \geq 1/2\}.$$

In informal words, the set $\Gamma_m(f)$, for large m, is the place where the lack of uniform hyperbolicity appears, and where the Oseledets directions can be mixed. More precisely, we have the following.

LEMMA 3.8. *Let $f \in \mathrm{Diff}^1_\mu(M)$, $\mathcal{U} = \mathcal{U}(f, \varepsilon_0)$ and $0 < k < 1$. Then there is a positive integer m such that, for every point $x \in \Gamma_m(f)$, there exists a $(k, \mathcal{U})$-realizable sequence $\{L_0, L_1, \ldots, L_{m-1}\}$ at x of length m such that*

$$L_{m-1} \cdots L_1 L_0(E^u(x)) = E^s(f^m(x)).$$

For the proof of Lemma 3.8, we will need two simple linear-algebraic lemmas.

LEMMA 3.9. *Given $\alpha_2 > 0$, there is a $c > 1$ with the following properties. Given a linear transformation $A : \mathbb{R}^2 \to \mathbb{R}^2$ and unit vectors s, $u \in \mathbb{R}^2$ such that $\|A(s)\|/\|A(u)\| > c$, there exists $\xi \in \mathbb{R}^2 - \{0\}$ such that*

$$\measuredangle(\xi, u) \leq \alpha_2 \quad \text{and} \quad \measuredangle(A(\xi), A(s)) \leq \alpha_2.$$

Proof. Let $a = \arcsin \alpha_2$ and $c = a^{-2}$. Let $\xi = u + as$. Since $\|u\| = \|s\| = 1$, we have $\sin \measuredangle(\xi, u) \leq a$. Let $\hat{u} = A(u)/\|A(u)\|$ and $\hat{s} = A(s)/\|A(s)\|$. The vector $A(\xi) = A(u) + aA(s)$ is parallel to

$$\left(a^{-1} \frac{\|A(u)\|}{\|A(s)\|}\right) \hat{u} + \hat{s}.$$

Therefore $\sin \measuredangle(A(\xi), \hat{s}) \leq a^{-1}\|A(u)\|/\|A(s)\| < a$. □

LEMMA 3.10. *Given $\alpha_1 > 0$ and $\hat{c} > 1$, there exists $E > 1$ with the following properties: Let $A \in \mathrm{SL}(2, \mathbb{R})$. If there exists a pair of unit vectors s, $u \in \mathbb{R}^2$ such that $\measuredangle(u, s) \geq \alpha_1$, $\measuredangle(A(u), A(s)) \geq \alpha_1$ and*

$$\frac{1}{\hat{c}} \leq \frac{\|A(s)\|}{\|A(u)\|} \leq \hat{c},$$

then $\|A\| \leq E$.

Proof. By substituting A by $B_2^{-1} A B_1$, where B_1 and B_2 are area-preserving changes of coordinates whose norms are bounded by some function of α_1, we may suppose that $s \perp u$ and $A(s) \perp A(u)$. Now we have $\|A(s)\| \cdot \|A(u)\| = \det A = 1$, so both $\|A(s)\|$ and $\|A(u)\|$ are bounded by $\hat{c}^{1/2}$. □

Proof of Lemma 3.8. We first define constants k_0, C, α_1, α_2, E, ε, C, β and m.

Let $k_0 \in (0, k)$. Let $C = \sup \|Df^{\pm 1}\|$. Let $\alpha_1 > 0$ and $\alpha_2 > 0$, depending on f, ε_0 and k_0 (in the place of k), be given by Lemmas 3.3 and 3.4, respectively. Let c, depending on α_2, be given by Lemma 3.9. We assume that $c > C^2$. Let $E > 1$, depending on α_1 and $\hat{c} = 2c^2$, be given by Lemma 3.10. Let $\varepsilon > 0$, depending on f, ε_0, k and E, be given by Lemma 3.7.

Choose $\beta > 0$ such that if $|\theta| \leq \beta$ then the rotation R_θ is close to the identity, $\|R_\theta - I\| < C^{-1}E^{-2}\varepsilon$. Let m be the least integer satisfying $m \geq 2\pi/\beta$.

Fix $x \in \Gamma_m(f)$. The rest of proof is divided into three cases.

First case. Suppose that the following condition holds:

$$\text{there exists } j_0 \in \{0, 1, \ldots, m\} \text{ such that } \measuredangle(E^u(f^{j_0}(x)), E^s(f^{j_0}(x))) < \alpha_1. \tag{I}$$

By Lemma 3.3, for every $y \in M$ and every $|\theta| < \alpha_1$ and every rotation R_θ at y (respectively at $f(y)$) the sequence $\{Df_y R_\theta\}$ (respectively $\{R_\theta Df_y\}$) of length 1 at y is $(k, \mathcal{U})$-realizable. If $j_0 < m$, we use this fact with $y = f^{j_0}(x)$ and $\theta = \pm\measuredangle(E^u(y), E^s(y))$ such that $R_\theta(E^u(y)) = E^s(y)$. So $\{Df_{f^{j_0}(x)} R_\theta\}$ is a $(k_0, \mathcal{U})$-realizable sequence of length 1 at y. By Lemma 3.2, items (i) and (ii), the sequence

$$\{L_0, \ldots, L_{m-1}\} = \{Df_x, \ldots, Df_{f^{j_0-1}(x)}, Df_{f^{j_0}(x)} R_\theta, Df_{f^{j_0+1}(x)}, \ldots, Df_{f^{m-1}(x)}\}$$

is a $(k, \mathcal{U})$-realizable sequence of length m at x. The product $L_{m-1} \cdots L_0$ sends $E^u(x)$ to $E^s(f^m(x))$, as required. If $j_0 = m$, we define

$$\{L_0, \ldots, L_{m-1}\} = \{Df_x, \ldots, Df_{f^{m-2}(x)}, R_\theta Df_{f^{m-1}(x)}\}$$

instead. Again, this is a $(k, \mathcal{U})$-realizable sequence whose product sends $E^u(x)$ to $E^s(f^m(x))$.

Second case. We assume the following condition:

$$\text{there exist } j_0, j_1 \in \{0, 1, \ldots, m\},\ j_0 < j_1, \text{ such that } \Delta(f^{j_0}(x), j_1 - j_0) > c. \tag{II}$$

Since $c > C^2$, we have $j_1 - j_0 > 1$. Let $A = Df^{j_1-j_0}_{f^{j_0}x}$, and take unit vectors $s \in E^s(f^{j_0}(x))$ and $u \in E^u(f^{j_0}(x))$. By (II), we have $\|As\|/\|Au\| > c$. Therefore, Lemma 3.9 gives a vector $\xi \in T_{f^{j_0}(x)}M$ such that

$$|\theta_0| = \measuredangle(\xi, E^s(f^{j_0}x)) \leq \alpha_2 \quad \text{and} \quad |\theta_1| = \measuredangle(Df^{j_1-j_0}_{f^{j_0}x}(\xi), E^u(f^{j_1}x)) \leq \alpha_2.$$

The signs of θ_0 and θ_1 are chosen so that $R_{\theta_0}(E^s(f^{j_0}x)) \ni \xi$ and $R_{\theta_1} Df^{j_1-j_0}_{f^{j_0}x}(\xi) \in E^u(f^{j_1}x)$. Applying Lemma 3.4, we conclude that the sequence

$$\{Df_{f^{j_0}x} R_{\theta_0}, Df_{f^{j_0+1}x}, \ldots, Df_{f^{j_1-2}x}, R_{\theta_1} Df_{f^{j_1-1}x}\},$$

of length $j_1 - j_0$ at $f^{j_0}(x)$, is $(k_0, \mathcal{U})$-realizable. Now define the sequence $\{L_0, \ldots, L_{m-1}\}$ of length m at x putting $L_{j_0} = Df_{f^{j_0}x} R_{\theta_0}$, $L_{j_1-1} = R_{\theta_1} Df_{f^{j_1}x}$ and all the others $L_j = Df_{f^j x}$. By Lemma 3.2, items (i) and (ii), this is a $(k, \mathcal{U})$-realizable sequence. Moreover, we have $L_{m-1} \cdots L_0(E^u(x)) = E^s(f^m(x))$.

Third case. We suppose that we are not in the previous cases, that is we assume:

$$\text{for every } j \in \{0, 1, \ldots, m\}, \quad \measuredangle(E^u(f^j(x)), E^s(f^j(x))) \geq \alpha_1 \tag{not I}$$

and

$$\text{for every } i, j \in \{0, 1, \ldots, m\} \text{ with } i < j, \quad \Delta(f^i(x), j - i) \leq c. \tag{not II}$$

We now use the assumption $x \in \Gamma_m(f)$, that is $\Delta(x, m) \geq 1/2$.

CLAIM. *For every $i, j \in \{0, 1, \ldots, m\}$ with $i < j$,*

$$1/(2c^2) \le \Delta(f^i(x), j - i) \le c.$$

Proof. The second inequality is just (not II). For the first,

$$\Delta(f^i(x), j - i) = \Delta(f^j(x), m - j)^{-1} \cdot \Delta(x, m) \cdot \Delta(x, i)^{-1} \ge 1/(2c^2).$$

It follows from the claim, condition (not I) and Lemma 3.10 that

$$\text{for every } j \in \{0, 1, \ldots, m\}, \quad \|Df_x^j\| \le E.$$

Choose numbers $\theta_0, \ldots, \theta_{m-1}$ such that $|\theta_j| \le \beta$ and $\sum_{j=0}^{m-1} \theta_j = \measuredangle(E^u(x), E^s(x))$. Define linear maps $L_j : T_{f^j x} M \to T_{f^{j+1} x} M$ by

$$L_j = Df_x^{j+1} R_{\theta_j} (Df_x^j)^{-1}.$$

Then we have:

(i) $L_{j-1} \cdots L_0 = Df_x^j R_{\theta_{j-1} + \cdots + \theta_0}$, therefore $L_{j-1} \cdots L_0(\mathbb{D}) = Df_x^j(\mathbb{D})$, where $\mathbb{D}$ is the unit disk in $T_x M$;

(ii) $\|L_j - Df_{f^j x}\| \le \|Df_{f^j x}\| \|Df_x^j R_{\theta_j} (Df_x^j)^{-1} - I\| \le C \|Df_x^j\|^2 \|R_{\theta_j} - I\| < \varepsilon$.

So we have constructed a sequence $\{L_0, \ldots, L_{m-1}\}$ of length m at x that, by Lemma 3.7, is $(k, \mathcal{U})$-realizable. Furthermore, we have

$$L_{m-1} \cdots L_0(E^u(x)) = Df_x^m R_{\sum_{j=0}^{m-1} \theta_j}(E^u(x)) = Df_x^m(E^s(x)) = E^s(f^m x),$$

as required. □

3.4. *Realizable sequences with small products.* In §3.3 we have defined sets $\Gamma_m(f)$, for $f \in \mathrm{Diff}^1_\mu(M)$ and $m \in \mathbb{N}$. We also define the following f-invariant sets:

$$\Omega_m(f) = \bigcup_{n \in \mathbb{Z}} f^n(\Gamma_m(f)).$$

LEMMA 3.11. *For $f \in \mathrm{Diff}^1_\mu(M)$ and $m \in \mathbb{N}$, let $H_m = \mathcal{O}^+(f) - \Omega_m(f)$. If H_m is not empty then its closure $\overline{H_m}$ is a hyperbolic set.*

Proof. The proof is quite standard. If $x \in H_m$ then $\Delta(x, m) \le 1/2$ and $\Delta(x, mi) \le 1/2^i$. Since $\|Df\|$ is bounded, there are constants $K > 1$ and $0 < \tau < 1$ such that

$$\Delta(x, n) \le K\tau^n \quad \text{for every } x \in H_m \text{ and } n \ge 1.$$

Fix $x \in H_m$ and let $v^u \in E^u(x)$, $v^s \in E^s(x)$ be unit vectors. Then

$$\begin{aligned} \|v^u - v^s\| &\ge \|Df_x^m\|^{-1} \|Df_x^m v^u - Df_x^m v^s\| \\ &\ge \|Df_x^m\|^{-1} [\|Df_x^m v^u\| - \|Df_x^m v^s\|] \ge \|Df_x^m\|^{-1} \|Df_x^m v^u\|/2. \end{aligned}$$

This shows that there exists a constant $a > 0$ such that $\theta(x) := \measuredangle(E^u(x), E^s(x)) \ge a$ for every $x \in H_m$.

Let $x \in H_m$ and $n \geq 1$. Since Df_x^n preserves the area form, we have

$$\sin\theta(x) = \|Df_x^m|_{E^s(x)}\| \cdot \|Df_x^m|_{E^u(x)}\| \sin\theta(f^n x).$$

Therefore,

$$\|Df_x^n|_{E^s(x)}\| = \sqrt{\frac{\sin\theta(x)}{\sin\theta(f^n x)}\Delta(x,n)} \leq \sqrt{\frac{K\tau^n}{\sin a}} = K_1\tau_1^n,$$

$$\|Df_x^{-n}|_{E^u(x)}\| = \sqrt{\frac{\sin\theta(x)}{\sin\theta(f^{-n}(x))}\Delta(f^{-n}(x),n)} \leq \sqrt{\frac{K\tau^n}{\sin a}} = K_1\tau_1^n,$$

for some constants $K_1 > 0$, $0 < \tau_1 < 1$. These inequalities imply that the bundles E^s, E^u are continuous on H_m, and have a unique continuous extension to the closure. □

We will need the following result, which may be thought of as a quantitative Poincaré's Recurrence Theorem.

LEMMA 3.12. *Let* $f \in \mathrm{Diff}^1_\mu(M)$. *Let* $\Gamma \subset M$ *be a measurable set with* $\mu(\Gamma) > 0$ *and let*

$$\Omega = \bigcup_{n\in\mathbb{Z}} f^n(\Gamma).$$

Take $\gamma > 0$. *Then there exists a measurable function* $N_0 : \Omega \to \mathbb{N}$ *such that for a.e.* $x \in \Omega$, *every* $n \geq N_0(x)$ *and every* $t \in [0,1]$ *there is some* $\ell \in \{0,1,\dots n\}$ *such that* $f^\ell(x) \in \Gamma$ *and* $|(\ell/n) - t| < \gamma$.

Proof. Let χ_Γ be the characteristic function of the set Γ. Consider the Birkhoff sums

$$s_n(x) = \sum_{j=0}^{n-1} \chi_\Gamma(f^j(x)).$$

CLAIM. *For a.e.* $x \in \Omega$, *the limit* $\lim_{n\to\infty}(1/n)s_n(x)$ *exists and is positive.*

Proof. Birkhoff's Theorem gives the existence; we are left to show the positivity. Let $Z \subset \Omega$ be the set where the limit is zero. Let $Z_0 = Z \cap \Gamma$. The a.e.-defined f-invariant function

$$\varphi = \lim_{n\to\infty} \frac{1}{n}\sum_{j=0}^{n-1} \chi_{Z_0} \circ f^j \leq \lim_{n\to\infty} \frac{s_n}{n}$$

vanishes in Z_0. This means that the set $P = \{\varphi > 0\}$ is disjoint from Z_0. On the other hand, $\varphi(x) > 0$ implies that some iterate of x is in Z_0. Since P is invariant, it follows that $P = \varnothing$. Thus $\varphi = 0$ a.e. and $\mu(Z_0) = \int \varphi\, d\mu = 0$. But $\Omega = \bigcup_{n\in\mathbb{Z}} f^n(\Gamma)$ means that $Z = \bigcup_{n\in\mathbb{Z}} f^n(Z_0)$ and therefore $\mu(Z) = 0$. The claim is proved.

Take $x \in \Omega$. Let $a = \lim_{n\to\infty}(1/n)s_n(x)$. Take $0 < \varepsilon < a$ such that $(a+\varepsilon)/(a-\varepsilon) < 1 + \gamma/2$. Choose (measurably) n_0 such that $n \geq n_0$ implies $|(s_n/n) - a| < \varepsilon$. Finally, take an integer

$$N_0(x) > \max\left\{\frac{2n_0}{\gamma(a-\varepsilon)}, \frac{4}{\gamma}\right\}.$$

Now, by contradiction, suppose that for some $n \geq N_0(x)$ there exists $t \in [0, 1]$ such that $f^\ell(x) \notin \Gamma$ for every $\ell \in (n(t-\gamma), n(t+\gamma))$. Let $[\ell_1, \ell_2]$ be the maximal closed subinterval of $(n(t-\gamma), n(t+\gamma)) \cap [0, n]$ with integer endpoints. Then $\ell_2 - \ell_1 > n\gamma - 2 > n\gamma/2$. If $\ell_1 \geq n_0$ then

$$a - \varepsilon < \frac{s_{\ell_2}}{\ell_2} = \frac{s_{\ell_1}}{\ell_2} \leq \frac{s_{\ell_1}}{\ell_1 + n\gamma/2} \leq \frac{s_{\ell_1}}{\ell_1(1+\gamma/2)} < \frac{a+\varepsilon}{1+\gamma/2} < a - \varepsilon,$$

a contradiction. Therefore, $\ell_1 < n_0$. We have $\ell_2 > n\gamma/2 > n_0/(a-\varepsilon) > n_0$, thus

$$a - \varepsilon < \frac{s_{\ell_2}}{\ell_2} < \frac{\ell_1}{\ell_1 + n\gamma/2} < \frac{n_0}{n\gamma/2} < a - \varepsilon,$$

again a contradiction. □

In Lemma 3.8 we have constructed realizable sequences that send E^u into E^s. Using this, we now construct realizable sequences whose products have 'small' norms.

LEMMA 3.13. *Let $f \in \mathrm{Diff}^1_\mu(M)$ be aperiodic and such that every hyperbolic set has zero measure, $\mathcal{U} = \mathcal{U}(f, \varepsilon_0)$, $\delta > 0$ and $0 < k < 1$. Then there exists a measurable integer function $N : M \to \mathbb{N}$ such that for a.e. $x \in M$ and every integer $n \geq N(x)$ there exists a realizable sequence*

$$\{L_0, \ldots, L_{n-1}\} = \{L_0^{(x,n)}, \ldots, L_{n-1}^{(x,n)}\}$$

of length n at x such that

$$\|L_{n-1} \cdots L_0\| < e^{\frac{4}{5}n\delta}.$$

Proof. Let $k_0 \in (0, k)$. Let $m \in \mathbb{N}$, depending on f, ε_0 and k_0 (in the place of k), be given by Lemma 3.8. By Lemma 3.11, the disjoint union $\mathcal{O}^0(f) \sqcup \Omega_m(f)$ has full measure. We will define the function $N : M \to \mathbb{N}$ separately on $\mathcal{O}^0(f)$ and $\Omega_m(f)$.

For each $x \in \mathcal{O}^0(f)$, take $N(x) \in \mathbb{N}$ such that $\|Df^n_x\| < e^{n\delta}$ for every $n \geq N(x)$. For some fixed $n \geq N(x)$, we define $L_j = L_j^{(x,n)} = Df_{f^j x}$ for $0 \leq j \leq n-1$. By Lemma 3.2, item (1), $\{L_0, \ldots, L_{n-1}\}$ is a $(k, \mathcal{U})$-realizable sequence of length n at x.

If $\mu(\mathcal{O}^0(f)) = 1$ then we are done. Suppose from now on that $\mu(\Omega_m(f)) > 0$. Let $C > \log \sup_{g \in \mathcal{U}, x \in M} \|Dg_x\|$. Apply Lemma 3.12 with $\Gamma = \Gamma_m(f)$, $\Omega = \Omega_m(f)$, and $\gamma = \delta/20C$ to find $N_0(x)$, depending measurably on $x \in \Omega_m(f)$, such that for every $n \geq N_0(x)$ and $t \in [0, 1]$ there is an $\ell \in \mathbb{N}$ with $f^\ell(x) \in \Gamma_m(f)$ and $|(\ell/n) - t| < \gamma$.

Fix $x \in \Omega_m(f)$. Consider the Lyapunov exponent $\lambda = \lambda^+(f, x) > 0$. If $\lambda < \delta$ then it suffices to take $N(x)$ large enough and define $\{L_j^{(x,n)}\}$ as the trivial realizable sequence (as we did when defining N on $\mathcal{O}^0(f)$). Thus, we can assume that $\lambda \geq \delta$. If x is contained in a certain full measure subset of $\Omega_m(f)$ (we will also assume this) then we can find $N_1(x) \geq N_0(x)$ such that if $j > \frac{1}{10}N_1(x)$ then

$$\left| \frac{1}{j} \log \|Df^j_x|_{E^u}\| - \lambda \right| < \frac{\delta}{20},$$

$$\left| \frac{1}{j} \log \|Df^j_x|_{E^s}\| + \lambda \right| < \frac{\delta}{20},$$

$$\left| \frac{1}{j} \log \sin \measuredangle(E^u(f^j x), E^s(f^j x)) \right| < \frac{\delta}{5}.$$

For each $j \geq 0$, take unit vectors $v_j^u \in E^u(f^j x)$ and $v_j^s \in E^s(f^j x)$. Let $\mathcal{B}_j = \{v_j^u, v_j^s\}$ be a basis of the space $T_{f^j x}M$. Also denote $\theta_j = \measuredangle(v_j^u, v_j^s)$. If $A : T_{f^{j_0}x}M \to T_{f^{j_1}x}M$ is any linear transformation, we will indicate by $\overline{A}$ its matrix relative to the bases $\mathcal{B}_{j_0}$ and $\mathcal{B}_{j_1}$. If

$$\overline{A} = \begin{pmatrix} a_{uu} & a_{su} \\ a_{us} & a_{ss} \end{pmatrix}$$

is such a matrix, we will write $\|A\|_{\max} = \max\{|a_{uu}|, |a_{su}|, |a_{us}|, |a_{ss}|\}$. We claim that there is a constant $K > 1$ such that

$$\begin{aligned} \|A\| &\leq K(\sin\theta_{j_0})^{-1}\|A\|_{\max}, \\ \|A\|_{\max} &\leq K(\sin\theta_{j_1})^{-1}\|A\| \end{aligned}$$

for every such A. To prove this fact, look at the matrix of the change of bases from $\mathcal{B}_j$ to the orthonormal basis $\{v_j^u, (v_j^u)^\perp\}$, that is,

$$\begin{pmatrix} 1 & \cos\theta_j \\ 0 & \sin\theta_j \end{pmatrix}.$$

The norm of its inverse is of the order of $(\sin\theta_j)^{-1}$.

Take $N(x) \geq N_1(x)$ such that

$$\frac{m}{N(x)} < \gamma \quad \text{and} \quad \frac{K^2 e^{Cm}}{\sin\theta_0} < e^{\frac{\delta}{5}N(x)}.$$

(Notice that θ_0 is a measurable function of x.) This defines the measurable function N on the set $\Omega_m(f)$. Now fix some $n \geq N(x)$. Let ℓ be as given by Lemma 3.12 with $t = \frac{1}{2}$, that is, such that $f^\ell(x) \in \Gamma_m(f)$ and $|(\ell/n) - \frac{1}{2}| < \gamma$. By Lemma 3.8, there is a $(k_0, \mathcal{U})$-realizable sequence $\{L_0', \ldots, L_{m-1}'\}$ at $f^\ell(x)$ of length m such that the product $\mathcal{L}' = L_{m-1}' \cdots L_0'$ satisfies $\mathcal{L}'(E^u(f^\ell(x))) = E^s(f^{\ell+m}(x))$. We now define the sequence $\{L_0, \ldots, L_{n-1}\}$ of length n at x as

$$\{Df_x, \ldots, Df_{f^{\ell-1}x}, L_0', \ldots, L_{m-1}', Df_{f^{\ell+m+1}x}, \ldots, Df_{f^n x}\}.$$

By Lemma 3.2, it is a $(k, \mathcal{U})$-realizable sequence. To complete the proof, we must estimate the norm of

$$\mathcal{L} = L_{n-1} \cdots L_0 = Df^{n-\ell-m}_{f^{\ell+m}(x)} \cdot \mathcal{L}' \cdot Df^\ell_x.$$

Write

$$\overline{Df^\ell_x} = \begin{pmatrix} e^{\mu_1 n} & 0 \\ 0 & e^{-\mu_2 n} \end{pmatrix}, \quad \overline{Df^{n-\ell-m}_{f^{\ell+m}(x)}} = \begin{pmatrix} e^{\mu_3 n} & 0 \\ 0 & e^{-\mu_4 n} \end{pmatrix}.$$

CLAIM. *For $i = 1, 2, 3, 4$ we have $|\mu_i - (\lambda/2)| < \delta/5$.*

Proof. We have $\mu_1 = (1/n)\log\|Df^\ell_x|_{E^u}\|$. But

$$\left|\frac{1}{\ell}\log\|Df^\ell_x|_{E^u}\| - \lambda\right| < \frac{\delta}{20} \quad \text{and} \quad \left|\frac{\ell}{n} - \frac{1}{2}\right| < \gamma.$$

Also, $\lambda \le C$. Through $|ab - a'b'| \le |a - a'||b| + |b - b'||a'|$, we obtain

$$\left|\mu_1 - \frac{\lambda}{2}\right| < \frac{\delta}{20} + C\gamma = \frac{2\delta}{20}\,.$$

Similarly for μ_2. Now,

$$\mu_3 = \frac{1}{n}\log\|Df^{n-\ell-m}_{f^{\ell+m}(x)}|_{E^u}\| = \frac{1}{n}\log\|Df^n_x|_{E^u}\| - \frac{1}{n}\log\|Df^{\ell+m}_x|_{E^u}\|.$$

Since

$$\left|\frac{1}{\ell+m}\log\|Df^{\ell+m}_x|_{E^u}\| - \lambda\right| < \frac{\delta}{20} \quad\text{and}\quad \left|\frac{\ell+m}{n} - \frac{1}{2}\right| < 2\gamma,$$

we have

$$\left|\frac{1}{n}\log\|Df^{\ell+m}_x|_{E^u}\| - \frac{\lambda}{2}\right| < \frac{\delta}{20} + 2C\gamma = \frac{3\delta}{20}.$$

Thus

$$\left|\mu_3 - \frac{\lambda}{2}\right| \le \left|\frac{1}{n}\log\|Df^n_x|_{E^u}\| - \lambda\right| + \left|\frac{1}{n}\log\|Df^{\ell+m}_x|_{E^u}\| - \frac{\lambda}{2}\right| < \frac{\delta}{20} + \frac{3\delta}{20} = \frac{\delta}{5}.$$

Similarly for μ_4. The claim is proved.

Since the sequence $\{L'_0, \dots, L'_{m-1}\}$ is $(k, \mathcal{U})$-realizable, we have $\|L'_i\| < e^C$ for each i. In particular, $\|\mathcal{L}'\| < e^{Cm}$ and

$$\|\mathcal{L}'\|_{\max} \le K(\sin\theta_{\ell+m})^{-1}\|\mathcal{L}'\| < Ke^{Cm}e^{\frac{1}{5}\delta n},$$

since $\sin\theta_{\ell+m} > e^{-\frac{1}{5}\delta(\ell+m)} > e^{-\frac{1}{5}\delta n}$. Now write

$$\overline{\mathcal{L}'} = \begin{pmatrix} b_{uu} & b_{su} \\ b_{us} & b_{ss} \end{pmatrix}.$$

Since $\mathcal{L}'(E^u_{f^\ell(x)}) = E^s_{f^{\ell+m}(x)}$, we have $b_{uu} = 0$. Computing the matrix product $\overline{\mathcal{L}} = \overline{Df^{n-\ell-m}_{f^{\ell+m}(x)}} \cdot \overline{\mathcal{L}'} \cdot \overline{Df^\ell_x}$, we obtain

$$\overline{\mathcal{L}} = \begin{pmatrix} 0 & e^{(-\mu_2+\mu_3)n}b_{su} \\ e^{(\mu_1-\mu_4)n}b_{us} & e^{(-\mu_2-\mu_4)n}b_{ss} \end{pmatrix}.$$

We have $\max\{\mu_1 - \mu_4, -\mu_2 + \mu_3, -\mu_2 - \mu_4\} < \frac{2}{5}\delta$. Therefore,

$$\|\mathcal{L}\|_{\max} \le e^{\frac{2}{5}\delta n}\|\mathcal{L}'\|_{\max} \le Ke^{Cm}e^{\frac{3}{5}\delta n}$$

and

$$\|\mathcal{L}\| \le K(\sin\theta_0)^{-1}\|\mathcal{L}\|_{\max} \le K^2(\sin\theta_0)^{-1}e^{Cm}e^{\frac{3}{5}\delta n} < e^{\frac{4}{5}\delta n}.$$

This concludes the proof. □

The Main Lemma follows easily from Lemma 3.13.

Proof of the Main Lemma. Of course, we can suppose that $\mathcal{U} = \mathcal{U}(f, \varepsilon_0)$ for some $\varepsilon_0 > 0$. Applying Lemma 3.13, we find the measurable function $N : M \to \mathbb{N}$. Fix x (in a full measure set) and $n \geq N(x)$. Then Lemma 3.13 also gives us a $(k, \mathcal{U})$-realizable sequence $\{L_0, \dots, L_{n-1}\}$ of length n at x such that

$$\|L_{n-1} \cdots L_0\| < e^{\frac{4}{5}n\delta}. \tag{$*$}$$

Take $\gamma > 0$ very small (depending on n). By the definition of a realizable sequence, there exists $r = r(x, n)$ with the following property. For every disk $U = B_{r'}(x)$, with $0 < r' < r$ there exist $g \in \mathcal{U}(f, \varepsilon_0)$ and a compact set $K \subset U$ such that:

(i) g equals f outside the set $\bigsqcup_{j=0}^{n-1} f^j(\overline{U})$ and the iterates $f^j(\overline{B_r(x)})$, $0 \leq j \leq n$ are two-by-two disjoint;

(ii) $\mu(K)/\mu(U) > k$;

(iii′) if $y \in K$ then $\|Dg_{g^j y} - L_j\| < \gamma$ for every $j = 0, 1, \dots, n-1$.

If γ is small enough then $(*)$ and (iii′) imply that:

(iii) if $y \in K$ then $\|Dg^n_y\| < e^{n\delta}$.

This completes the proof of the Main Lemma. □

4. *Proof of Theorem A*

4.1. *Preliminary definitions.* Fix some measure-preserving diffeomorphism f. If a measurable set $A \subset M$ and $n \in \mathbb{N}$ are such that the sets $A, f(A), \dots, f^{n-1}(A)$ are disjoint then we call the set

$$T = A \cup f(A) \cup \cdots \cup f^{n-1}(A)$$

a *tower* for f. The number n is called the *height* of the tower and the set A is called its *base*. A *castle* is a finite or countable union of two-by-two disjoint towers. The *base of the castle* is the union of the bases of its towers.

Given f and a positive measure set $A \subset M$, consider the (a.e. finite) return time $\tau : A \to \mathbb{N}$ defined by $\tau(x) = \inf\{n \geq 1 : f^n(x) \in A\}$. If we denote $A_n = \tau^{-1}(n)$ then $T_n = A_n \cup f(A_n) \cup \cdots \cup f^{n-1}(A_n)$ is a tower. Let $Q = \bigcup_{n \in \mathbb{Z}} f^n(A)$. Then Q is f-invariant and it is a castle with base A and towers T_n. We will call Q the *Kakutani castle* with base A.

We will need also the following.

LEMMA 4.1. *For every aperiodic invertible measure-preserving transformation f on a probability space X, every subset $U \subset X$ of positive measure and every $n \in \mathbb{N}$, there exists a positive measure set $V \subset U$ such that $V, f(V), \dots, f^n(V)$ are two-by-two disjoint. Moreover, V can be chosen maximal on 'the measure-theoretical sense'. (This means that no set that includes V and has larger measure than V has the stated properties.)*

Proof. We follow [**6**, p. 70]. Take $U_1 \subset U$ such that $\mu(U_1 \bigtriangleup f(U_1)) > 0$ (it exists because otherwise a.e. the point of U would be fixed). Then $V_1 = U_1 - f(U_1)$ has positive measure and $V_1 \cap f(V_1) = \varnothing$. Take $U_2 \subset V_1$ such that $\mu(U_2 \bigtriangleup f^2(U_2)) > 0$ and let $V_2 = U_2 - f^2(U_2)$. Continuing in this way we will find $V = V_n$ such that $V, f(V), \dots, f^n(V)$ are two-by-two disjoint. Suppose that the set $R_V = U - \bigcup_{j=-n}^{n} f^j(V)$ has positive measure; otherwise V is maximal. Take $V' \subset R_V$ such that $V', f(V'), \dots, f^n(V')$ are two-by-two disjoint. Continue in this way by transfinite induction. Since a disjoint class of positive measure sets is countable, the process will terminate at some countable ordinal. Hence we find a measurable set $V \subset U$ such that $\mu(R_V) = 0$. $\square$

Now we will prove Proposition 2.3 and thus Theorem A.

4.2. *First step. Construction of a castle Q.* Let $f \in \mathrm{Diff}^1_\mu(M)$ be aperiodic and such that every hyperbolic set for f has zero measure. Let $\mathcal{U}$ be a neighborhood of f in $\mathrm{Diff}^1_\mu(M)$ and let $\delta > 0$.

Take $0 < k < 1$ such that $1 - k < \delta^2$. Apply the Main Lemma to get a measurable function $N : M \to \mathbb{N}$ with the properties stated there. We define the sets $P_n = \{x \in M : N(x) \leq n\}$ for $n \in \mathbb{N}$. Obviously, $\mu(P_n) \to 1$. Fix $H \in \mathbb{N}$ such that $\mu(P_H^C) < \delta^2$, where P_H^C denotes the complementary set $M - P_H$.

Take $B \subset P_H$ such that $B, f(B), \ldots, f^{H-1}(B)$ are two-by-two disjoint and such that B is maximal in a measure-theoretical sense. Consider the following f-invariant set:

$$\widehat{Q} = \bigcup_{n \in \mathbb{Z}} f^n(B).$$

$\widehat{Q}$ is the Kakutani castle with base B. Notice that $\widehat{Q} \supset P_H$ mod 0 (by maximality of B) and hence $\mu(\widehat{Q}^C) < \delta^2$. Let $Q \subset \widehat{Q}$ be the (finite) castle consisting of all the towers of $\widehat{Q}$ with heights at most $3H$ floors. The following property will be important later.

LEMMA 4.2. $\mu(\widehat{Q} - Q) \leq 3\mu(P_H^C) < 3\delta^2$.

Proof. Write the castle as $\widehat{Q} = \bigsqcup_{i=H}^{\infty} T_i$ where $B = \bigsqcup_{i=H}^{\infty} B_i$ is the base and $T_i = \bigsqcup_{j=0}^{i-1} f^j(B_i)$ is the tower of height i floors. Take $i \geq 2H$ and $H \leq j \leq i - H$. The sets $f^j(B_i), \ldots, f^{j+H-1}(B_i)$ are disjoint and do not intersect $B \sqcup \cdots \sqcup f^{H-1}(B)$. Since B is maximal, we conclude that (see Figure 1)

$$i \geq 2H,\ H \leq j \leq i - H \Longrightarrow f^j(B_i) \subset P_H^C \bmod 0$$

(otherwise we could replace B by $B \sqcup (f^j(B_i) \cap P_H)$, contradicting the maximality of B). Thus,

$$i \geq 2H \text{ implies } \mu(T_i \cap P_H^C) \geq \sum_{j=H}^{i-H} \mu(f^j(B_i)) = \frac{i - 2H + 1}{i}\mu(T_i).$$

In particular,

$$i \geq 3H + 1 \text{ implies } \mu(T_i \cap P_H^C) > \tfrac{1}{3}\mu(T_i)$$

and so

$$\mu(\widehat{Q} - Q) = \sum_{i=3H+1}^{\infty} \mu(T_i) \leq \sum_{i=3H+1}^{\infty} 3\mu(T_i \cap P_H^C) = 3\mu\left(P_H^C \cap \bigsqcup_{i=3H+1}^{\infty} T_i\right) \leq 3\mu(P_H^C).$$

$\square$

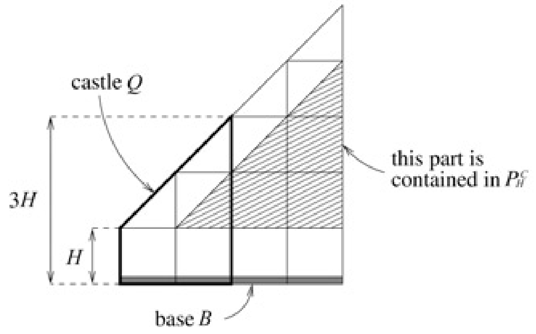

FIGURE 1. The castle $\widehat{Q}$.

4.3. *Second step. Construction of the perturbed diffeomorphism g.* Let $0 < \gamma < \delta^2 H^{-1}$. By the regularity of the measure μ, one can find a compact castle $J \subset Q$ such that $\mu(Q - J) < \gamma$ and it is of the *same type* as Q. Saying so we mean that the castles have the same number of towers and that the towers have the same heights. Since J is compact, we can find an *open* castle V containing J with $\mu(V - J) < \gamma$ and also being the same type as J. Hence ($\triangle$ denotes symmetric difference),

$$\mu(V \triangle Q) = \mu(V - Q) + \mu(Q - V) \leq \mu(V - J) + \mu(Q - J) < 2\gamma.$$

Denote by S the base of the castle $V \cap Q$. For each $x \in S$, let $n(x)$ be the height of the tower that contains x. We have $n(x) \geq H \geq N(x)$. Hence the Main Lemma gives, for a.e. $x \in S$, a radius $r(x) = r(x, n(x))$.

Reducing $r(x)$ if needed, we can suppose that the disk $\overline{B}_{r(x)}(x)$ is contained in the base of a tower in V, for a.e. $x \in S$. Using Vitali's Covering Lemma, we can find a finite collection of disjoint disks $U_i = B_{r_i}(x_i)$ with $x_i \in S$ and $0 < r_i < r(x_i)$ such that

$$\frac{\mu(S - \bigsqcup_i \overline{U_i})}{\mu(S)} < \gamma. \tag{$*$}$$

(Actually, Vitali's Lemma allows us only to a.e.-cover the set S restricted to each chart domain, so we have to cover S by chart domains first.)

Let $n_i = n(x_i)$. Notice that $n(x) = n_i$ for all $x \in U_i$. By the Main Lemma, for each i we can find a compact set $K_i \subset U_i$ and $g_i \in \mathcal{U}$ such that:

(i) $\mu(K_i)/\mu(U_i) > k$;
(ii) g_i equals f outside the set $\bigsqcup_{j=0}^{n_i-1} f^j(U_i)$;
(iii) $x \in K_i$ implies $\|D(g_i^{n_i})_x\| < e^{\delta n_i}$.

Let g be equal to g_i in the set $\bigsqcup_{j=0}^{n_i-1} f^j(U_i)$, for each i, and be equal to f outside. Since those sets are disjoint, $g \in \mathrm{Diff}^1_\mu(M)$ is a well-defined diffeomorphism. Moreover, $g \in \mathcal{U} = \mathcal{U}(f, \varepsilon_0)$.

Since each U_i is contained in the base of a tower in the castle V, V is also a castle for g. Moreover, we can define a g-castle U of the same type as V with base $\bigsqcup_i U_i$. Analogously, let K be the g-castle of the same type as V with base $\bigsqcup_i K_i$. Then, by definition,

$K \subset U \subset V$. We have $\mu(K)/\mu(U) > k$. Also, $(*)$ implies $\mu(U)/\mu(V \cap Q) > 1 - \gamma$. Thus

$$\mu(V - U) \leq \mu(V \cap Q - U) < \gamma \mu(V \cap Q) < \gamma$$

and

$$\mu(U \vartriangle Q) \leq \mu(U \vartriangle V) + \mu(V \vartriangle Q) < 3\gamma.$$

Summarizing, we have constructed a diffeomorphism $g \in \mathcal{U}$ and g-castles $K \subset U$ of the same type as the castle Q and such that:

(i) $\mu(U \vartriangle Q) < 3\gamma$ and $\mu(U - K) < 1 - k$;

(ii) g equals f outside U;

(iii) if x is in the base of K and $n(x)$ is the height of the tower of K that contains x, then we have $\|Dg_x^{n(x)}\| < e^{\delta n(x)}$.

4.4. *Third step. Estimation of* LE(g). We claim that LE(g) is small. To show it, we will use the following property from Proposition 2.1:

$$g \in \mathrm{Diff}^1_\mu(M) \text{ implies } \mathrm{LE}(g) = \inf_{N \geq 1} \frac{1}{N} \int_M \log \|Dg^N\| \, d\mu.$$

This allows one to conclude in finite time (i.e. without taking limits) that the integrated Lyapunov exponent is small.

Set $N = \delta^{-1} H$. (Of course, we can assume that $\delta^{-1} \in \mathbb{N}$.) We define the following 'good set':

$$G = \bigcap_{j=0}^{N-1} g^{-j}(K).$$

We will show that this set has almost full measure.

LEMMA 4.3. $\mu(G^C) < 16\delta$.

Proof. Consider the following sets.

$$S_1 = U - K, \quad S_2 = Q - U, \quad S_3 = \widehat{Q} - Q, \quad S_4 = M - \widehat{Q}.$$

Then $K^C \subset S_1 \cup S_2 \cup S_3 \cup S_4$ and

$$G^C \subset \bigcup_{i=1}^{4} \bigcup_{j=0}^{N-1} g^{-j}(S_i).$$

We will estimate the measure of each set $\bigcup_{j=0}^{N-1} g^{-j}(S_i)$ separately.

First, we have $\mu(S_2) \leq \mu(U \vartriangle Q) < 3\gamma$, and so

$$\mu\bigg(\bigcup_{j=0}^{N-1} g^{-j}(S_2)\bigg) \leq N\mu(S_2) < 3N\gamma < 3\delta.$$

Before continuing, we point out that if $X \subset M$ is any measurable set then

$$\mu\bigg(\bigcup_{j=0}^{N-1} g^{-j}(X)\bigg) \leq \mu\bigg(X \cup \bigcup_{j=1}^{N-1} (g^{-j}(X) - g^{-(j-1)}(X))\bigg)$$
$$= \mu(X) + (N-1)\mu(g^{-1}(X) - X).$$

Moreover, $\mu(g^{-1}(X) - X) = \mu(g(X) - X) = \mu(X - g(X))$, simply because μ is g-invariant.

We have

$$\mu\left(\bigcup_{j=0}^{N-1} g^{-j}(S_1)\right) \le \mu(S_1) + N\mu(S_1 - g(S_1)).$$

The set $S_1 = U - K$ is a g-castle whose towers have heights at least H. Hence its base, which contains the set $S_1 - g(S_1)$, measures at most $(1/H)\mu(S_1)$. Moreover, $\mu(S_1) < 1 - k = \delta^2$. Substituting these estimates, we get

$$\mu\left(\bigcup_{j=0}^{N-1} g^{-j}(S_1)\right) < \delta^2 + \frac{N\delta^2}{H} < \delta^2 + \delta < 2\delta.$$

We are going to treat the case $i = 3$ similarly. By Lemma 4.2, $\mu(S_3) < 3\delta^2$. The set $S_3 = \widehat{Q} - Q$ is an f-castle whose towers have heights at least $3H$. Hence $\mu(f(S_3) - S_3) \le (1/3H)\mu(S_3)$. Since g and f coincide in U^C, we have

$$g(S_3) = g(S_3 - U) \cup g(S_3 \cap U) \subset f(S_3) \cup g(S_3 \cap U).$$

So

$$\mu(g(S_3) - S_3) \le \mu(f(S_3) - S_3) + \mu(g(S_3 \cap U)) \le \frac{\mu(S_3)}{3H} + \mu(U - Q) < \frac{\delta^2}{H} + 3\gamma$$

and

$$\mu\left(\bigcup_{j=0}^{N-1} g^{-j}(S_3)\right) < 3\delta^2 + N\left(\frac{\delta^2}{H} + 3\gamma\right) < 3\delta^2 + \delta + 3N\gamma < 7\delta.$$

The set $S_4 = M - \widehat{Q}$ is f-invariant, thus

$$g(S_4) - S_4 = [g(S_4 - U) \cup g(S_4 \cap U)] - S_4 \subset g(S_4 \cap U) \subset g(U - Q).$$

Now, $\mu(S_4) < \delta^2$ (see the definition of $\widehat{Q}$) and $\mu(U - Q) \le \mu(U \triangle Q) < 3\gamma$. Thus,

$$\mu\left(\bigcup_{j=0}^{N-1} g^{-j}(S_4)\right) < \mu(S_4) + N\mu(U - Q) < \delta^2 + 3N\gamma < 4\delta.$$

Putting all estimates together, we get $\mu(G^C) < 2\delta + 3\delta + 7\delta + 4\delta = 16\delta$. □

Now we will show that $(1/N)\log\|Dg^N\|$ is small inside the set G. Let K_0 be the base of the castle K. For $y \in K_0$, let $n(y)$ be the height of the tower containing y. We know that

$$\|Dg_y^{n(y)}\| < e^{\delta n(y)}.$$

Now take $x \in G$. Since the heights of K-towers are at most $3H$, we can write

$$N = j_1 + n_1 + n_2 + \cdots + n_i + j_2$$

such that $0 \le j_1, j_2 \le 3H$, $1 \le n_1, \ldots, n_i \le 3H$, and the points

$$g^{j_1}(x), g^{j_1+n_1}(x), \ldots, g^{j_1+n_1+\cdots+n_i}(x)$$

are exactly the points of the orbit segment $x, g(x), \dots, g^{N-1}(x)$ which belong to K_0. Hence, if $C > \sup_{y\in M} \|Dg_y\|$ then

$$\|Dg_x^N\| < C^{j_1} e^{\delta n_1} \cdots e^{\delta n_i} C^{j_2} \leq C^{6H} e^{\delta N}$$

and so

$$\frac{1}{N} \log \|Dg_x^N\| < \frac{6H \log C}{N} + \delta = (6 \log C + 1)\delta.$$

Therefore,

$$\begin{aligned}
\mathrm{LE}(g) &\leq \int_M \frac{1}{N} \log \|Dg^N\|\, d\mu \\
&= \int_G \frac{1}{N} \log \|Dg^N\|\, d\mu + \int_{G^C} \frac{1}{N} \log \|Dg^N\|\, d\mu \\
&< (6 \log C + 1)\delta \cdot \mu(G) + (\log C)\mu(G^C) \\
&< (22 \log C + 1)\delta.
\end{aligned}$$

This proves Theorem A.

5. *Generic dichotomy for continuous cocycles*

5.1. *Cocycles.* Let (X, μ) be a non-atomic probability space and $T : X \hookleftarrow$ an automorphism of it. Denote by $\mathrm{SL}(2, \mathbb{R})$ the group of two-by-two real matrices with unit determinant. Let

$$L^\infty(X, \mathrm{SL}(2, \mathbb{R})) = \{A : X \to \mathrm{SL}(2, \mathbb{R}) \text{ measurable and essentially bounded}\}$$

and consider in this space the following metric:

$$\|A - B\|_\infty = \operatorname{ess\,sup} \|A(x) - B(x)\|.$$

Given $A \in L^\infty(X, \mathrm{SL}(2, \mathbb{R}))$, we denote for $x \in X$ and $n \in \mathbb{Z}$,

$$A^n(x) = \begin{cases} A(T^{n-1}x) \cdots A(x) & \text{if } n > 0, \\ I & \text{if } n = 0, \\ [A(T^n x)]^{-1} \cdots [A(T^{-1}x)]^{-1} & \text{if } n < 0. \end{cases}$$

Notice that the relation

$$A^{n+m}(x) = A^n(T^m x) \cdot A^m(x),$$

called the cocycle identity, is satisfied for $m, n \in \mathbb{Z}$. With some abuse, we call the function A a *cocycle*.

Oseledets' Theorem may be summarized in the present case as follows.

THEOREM 5.1. *Let T and A be as above. Then there exists a measurable function $x \mapsto \lambda^+(x) \geq 0$ such that*

$$\lambda^+(x) = \lim_{n\to+\infty} \frac{1}{n} \log \|A^n(x)\|$$

for μ-a.e. $x \in X$. Moreover, if $\mathcal{O}^+ = \{x : \lambda^+(x) > 0\}$ has positive measure then for a.e. $x \in \mathcal{O}^+$ there is a splitting $\mathbb{R}^2 = E^u(x) \oplus E^s(x)$, depending measurably on x, such that for $v \in \mathbb{R}^2 - \{0\}$

$$\lim_{n\to+\infty} \frac{1}{n} \log \|A^n(x) \cdot v\| = \begin{cases} \lambda^+(x) & \textit{if } v \notin E^s(x), \\ -\lambda^+(x) & \textit{if } v \in E^s(x), \end{cases}$$

$$\lim_{n\to-\infty} \frac{1}{n} \log \|A^n(x) \cdot v\| = \begin{cases} -\lambda^+(x) & \textit{if } v \notin E^u(x), \\ \lambda^+(x) & \textit{if } v \in E^u(x), \end{cases}$$

$$\lim_{n\to\pm\infty} \frac{1}{n} \log \sin \measuredangle(E^u(T^n x), E^s(T^n x)) = 0.$$

As before, we define the integrated Lyapunov exponent,

$$\mathrm{LE}(A) = \int_X \lambda^+ \, d\mu = \inf_{n\in\mathbb{N}} \frac{1}{n} \int_X \log \|A^n\| \, d\mu.$$

We now define the notion of uniform hyperbolicity for cocycles.

Definition. A cocycle $A \in L^\infty(X, \mathrm{SL}(2, \mathbb{R}))$ over $T : (X, \mu) \hookleftarrow$ is said to be *uniformly hyperbolic* if there exists, for a.e. $x \in M$, a splitting $\mathbb{R}^2 = E^u(x) \oplus E^s(x)$, which is measurable with respect to x, such that:

(i) $A(x).E^u(x) = E^u(T(x))$, $A(x).E^s(x) = E^s(T(x))$ for a.e. $x \in M$;

(ii) there exist constants $C > 0$ and $0 < \tau < 1$ such that

$$\|A^n(x) \cdot v^s\| \le C\tau^n \|v^s\|,$$
$$\|A^{-n}(x) \cdot v^u\| \le C\tau^n \|v^u\|,$$

for a.e. $x \in X$, $v^s \in E^s(x)$, $v^u \in E^u(x)$ and $n \ge 0$.

Remark. If A is uniformly hyperbolic then there is a constant $\alpha > 0$ such that $\measuredangle(E^u(x), E^s(x)) \ge \alpha$ for a.e. $x \in X$; see the proof of Lemma 3.11.

Remark. If A is uniformly hyperbolic then it has positive ($\ge -\log \tau$) Lyapunov exponent a.e. and the spaces E^u, E^s in the definition coincide a.e. with the spaces given by Oseledets Theorem.

Remark. The set $\mathcal{H} \subset L^\infty(X, \mathrm{SL}(2, \mathbb{R}))$ of uniformly hyperbolic cocycles is open. This can be shown by standard invariant cones techniques.

We have the following result.

THEOREM B. *If T is ergodic then there is a residual set $\mathcal{R} \subset L^\infty(X, \mathrm{SL}(2, \mathbb{R}))$ such that, for every $A \in \mathcal{R}$, either A is uniformly hyperbolic or* $\mathrm{LE}(A) = 0$.

From the above theorem we will deduce its continuous version. Now we suppose that X is a compact Hausdorff space, μ is a regular Borel probability measure on X and $T : (X, \mu) \hookleftarrow$ is an automorphism of X. (T is not assumed to be continuous.) In this setting, we denote

$$C(X, \mathrm{SL}(2, \mathbb{R})) = \{A : X \to \mathrm{SL}(2, \mathbb{R}) \text{ continuous}\},$$

endowed with the uniform convergence topology. Then we have the following.

THEOREM C. *Let X, μ and T be as above. If T is ergodic then there is a residual set $\mathcal{R} \subset C(X, \mathrm{SL}(2,\mathbb{R}))$ such that, for every $A \in \mathcal{R}$, either A is uniformly hyperbolic or* $\mathrm{LE}(A) = 0$.

5.2. Proof of Theorems B and C.

Proof of Theorem B. The proof is similar to Theorem A's, but it is easier in several aspects.

Let $A \in L^\infty(X, \mathrm{SL}(2,\mathbb{R}))$ be a non-uniformly hyperbolic cocycle with $\mathrm{LE}(A) > 0$, $\delta > 0$ and $\varepsilon > 0$. We have to show that there exists $\tilde{A} \in L^\infty(X, \mathrm{SL}(2,\mathbb{R}))$ with $\|\tilde{A} - A\|_\infty < \varepsilon$ and $\mathrm{LE}(\tilde{A}) < \delta$.

The first step is to prove an analogue of the Main Lemma.

LEMMA 5.1. *Given A, δ and ε as above, there exists a measurable function $N : M \to \mathbb{N}$ such that for a.e. $x \in X$ and every $n \geq N(x)$ there are $L_0, \ldots, L_{n-1} \in \mathrm{SL}(2,\mathbb{R})$ satisfying $\|L_j - A(T^j x)\| < \varepsilon$ and $\|L_{n-1} \cdots L_0\| < e^{n\delta}$. Moreover, the matrices L_j depend measurably on x and n.*

Proof. Since the proof of this lemma is essentially contained in the proof of the Main Lemma, we will be brief. Let $\mathcal{O} \subset X$ be the full measure set given by Oseledets' Theorem. For $m \in \mathbb{N}$, let

$$\Delta(x, m) = \frac{\|A^m(x)|_{E^s(x)}\|}{\|A^m(x)|_{E^u(x)}\|}, \quad \text{for } x \in \mathcal{O},$$

$$\Gamma_m(A) = \{x \in \mathcal{O} : \Delta(x, m) \geq 1/2\}, \quad \Omega_m(A) = \bigcup_{n \in \mathbb{Z}} T^n(\Gamma_m(A)).$$

Then, for every m, $\mu(\Omega_m(A)) = 1$ (same proof as Lemma 3.11).

Imitating the proof of Lemma 3.8, we can show that if m is large enough then for every $y \in \Gamma_m(A)$ there exist matrices $L'_0(y), \ldots, L'_{m-1}(y) \in \mathrm{SL}(2,\mathbb{R})$ such that:

(i) $\|L'_j(y) - A(T^j y)\| < \varepsilon$; and

(ii) $L'_{m-1}(y) \cdots L'_0(y)(E^u(y)) = E^s(T^m y)$.

(These perturbations may be taken in the form $L'_j(y) = A(T^j y) R_{\theta_j}$.)

Proceeding as in the proof of Lemma 3.13, we find a measurable function $N : X \to \mathbb{N}$ such that, for a.e. $x \in X$ and every $n \geq N(x)$, there is an integer $\ell \approx n/2$ such that $y = T^\ell x \in \Gamma_m(A)$ and

$$\|A^{n-\ell-m}(T^{\ell+m}(x)) \cdot L'_{m-1}(y) \cdots L'_0(y) \cdot A^\ell(x)\| < e^{n\delta}.$$

The matrices $L_0, \ldots, L_{n-1}$ are defined in the obvious way: $L_j = A(T^j(x))$ if $j < \ell$ or $j > \ell + m - 1$; and $L_j = L'_{j-\ell}(y)$ otherwise. □

We choose an integer H and a set B as in §4.2. Let $\widehat{Q}$ be the Kakutani castle with base B. Since T is ergodic and $\mu(B) > 0$, we have $\widehat{Q} = X$ mod 0. Let Q be the castle consisting of all the towers of $\widehat{Q}$ with heights of at most $3H$ floors. We have $\mu(Q^C) < 3\delta^2$.

Let Q_0 be the base of the castle Q. We apply Lemma 5.1 to a.e. point in Q_0 to find $\tilde{A} \in L^\infty(X, \mathrm{SL}(2,\mathbb{R}))$ such that:

(i) $\|\tilde{A} - A\|_\infty < \varepsilon$;

(ii) $\tilde{A} = A$ outside Q;

(iii) if $x \in Q_0$ and $n(x)$ is the height of the tower containing x then $\|\tilde{A}^n(x)\| < e^{n\delta}$.

Now let $N = \delta^{-1}H$ and $G = \bigcap_{j=0}^{N-1} T^{-j}(Q)$. Since Q^C is a castle with towers of height $\geq 3H$, we have

$$\mu(G^C) = \mu\bigg(\bigcup_{j=0}^{N-1} T^{-j}(Q^C)\bigg) \leq \mu(Q^C) + N\mu(T^{-1}(Q^C) - Q^C)$$
$$\leq \bigg(1 + \frac{N}{3H}\bigg)\mu(Q^C) < 3\delta^2 + \delta < 4\delta.$$

Let $C > \|A\|_\infty + \varepsilon > 1$. Then $\|\tilde{A}\|_\infty < C$ and if $x \in G$ then $\|\widetilde{A}^N(x)\| < C^{6H}e^{N\delta}$. Therefore

$$\mathrm{LE}(\tilde{A}) \leq \int_G \frac{1}{N}\log\|\widetilde{A}^N\|\,d\mu + \int_{G^C} \frac{1}{N}\log\|\tilde{A}^N\|\,d\mu$$
$$< (6\log C + 1)\delta + (\log C)\mu(G^C) < (10\log C + 1)\delta$$

and Theorem B is proved. □

Remark. An alternative proof of Theorem B, without using castles and the related estimates, can be given following Knill's [**9**] methods, using coboundary sets. This is done in [**2**].

Proof of Theorem C. Let $A \in C(X, \mathrm{SL}(2,\mathbb{R}))$ be a non-uniformly hyperbolic cocycle with $\mathrm{LE}(A) > 0$, $\delta > 0$ and $\varepsilon > 0$. We have to show that there exists $B \in C(X, \mathrm{SL}(2,\mathbb{R}))$ with $\|B - A\|_\infty < \varepsilon$ and $\mathrm{LE}(B) < \delta$.

By Theorem B, there exists $\tilde{A} \in L^\infty(X, \mathrm{SL}(2,\mathbb{R}))$ near A with $\mathrm{LE}(\tilde{A}) = 0$. Write $\tilde{A} = A \cdot (I + J)$ with $J \in L^\infty(X, M(2,\mathbb{R}))$ close to zero in order that if $J_{ij}(x)$ denote the entries of the matrix $J(x)$, then $\|J_{ij}\|_\infty = \sup_x |J_{ij}(x)| < \varepsilon$. Let $N \in \mathbb{N}$ be such that $(1/N)\int_X \log\|\widetilde{A}^N\|\,d\mu < \delta$ and let $\gamma = N^{-1}\delta$.

By Lusin's Theorem (see, for instance, [**16**]), there exists $J' \in C(X, M(2,\mathbb{R}))$ with $\mu\{x : J' \neq J\} < \gamma$ and $\|J'_{ij}\|_\infty \leq \|J_{ij}\|_\infty < \varepsilon$. Define $J''_{ij} = J'_{ij}$ if $(i,j) \neq (1,1)$ and take J''_{11} in order to have $\det(I + J'') = 1$. One can check that $\|J''_{11}\|_\infty < 2\varepsilon$ if we suppose $\varepsilon < 1/3$. Thus $\|J''\|_\infty \leq K\varepsilon$, where K is a constant. Let $B = A(I + J'')$. Then $B \in C(X, \mathrm{SL}(2,\mathbb{R}))$ and

$$\|B - A\|_\infty \leq \|A\|_\infty\|J''\|_\infty \leq K\|A\|_\infty\varepsilon.$$

That is, B is close to A.

Let $G_0 = \{x : B(x) = \tilde{A}(x)\}$ and $G = \bigcap_{j=0}^{N-1} T^{-j}(G_0)$. Then we have

$$\mu(G^C) = \mu\bigg(\bigcup_{j=0}^{N-1} T^{-j}(G_0^C)\bigg) \leq N\mu(G_0^C) \leq N\mu(\{x : J' \neq J\}) < \delta.$$

Moreover, if $C > \|B\|_\infty$ then

$$\mathrm{LE}(B) \leq \int_G \frac{1}{N}\log\|B^N\|\,d\mu + \int_{G^C} \frac{1}{N}\log\|B^N\|\,d\mu$$
$$\leq \int_G \frac{1}{N}\log\|\tilde{A}^N\|\,d\mu + (\log C)\mu(G^C)$$
$$< (1 + \log C)\delta$$

and we are done. □

5.3. The non-ergodic case. In the statements of Theorems B and C, T was assumed to be ergodic just for simplicity. We will state without proof the generalizations of these theorems to the non-ergodic case.

Again assume that X is a compact Hausdorff space, μ is a regular Borel probability measure on X and $T : (X, \mu) \hookleftarrow$ is an automorphism of X. One says that a T-invariant set $Y \subset X$ is uniformly hyperbolic if the restricted cocycle $(Y, T|_Y, A|_Y)$ is uniformly hyperbolic.

THEOREM C$'$. *Let X, μ and T be as above. Then there is a residual set $\mathcal{R} \subset C(X, \mathrm{SL}(2, \mathbb{R}))$ such that for every $A \in \mathcal{R}$ the following holds. If the invariant set $\mathcal{O}^+ = \{x \in X : \lambda^+(A, x) > 0\}$ has positive measure then there are invariant sets $H_1 \subset H_2 \subset \cdots \subset \mathcal{O}^+$ covering $\mathcal{O}^+$ mod 0 such that each set H_i is uniformly hyperbolic for (T, A).*

The generalization of Theorem B is entirely analogous.

5.4. Discontinuity of the Lyapunov exponents. Knill proves in [**9**] that LE : $L^\infty(X, \mathrm{SL}(2, \mathbb{R})) \to \mathbb{R}$ is discontinuous if T is aperiodic. We will analyze the continuous case.

Again, suppose that X is a compact Hausdorff space and μ is a regular Borel probability measure on X. Now let $T : (X, \mu) \hookleftarrow$ be an *ergodic homeomorphism* of X.

By Theorem C, $A \in C(X, \mathrm{SL}(2, \mathbb{R}))$ is a point of continuity of the function LE : $C(X, \mathrm{SL}(2, \mathbb{R})) \to \mathbb{R}$ if and only if either $\mathrm{LE}(A) = 0$ or A is uniformly hyperbolic. Moreover, it was proved in [**17**] that LE is even real-analytic when restricted to the open set of uniformly hyperbolic cocycles.

Thus, it is interesting to look for examples of continuous non-hyperbolic cocycles with positive exponent for any given aperiodic ergodic system (X, T, μ). These examples are easily constructed if the system is not uniquely ergodic, as is shown below.

PROPOSITION 5.1. *Let (X, T, μ) be as above. If T is not uniquely ergodic then the function* LE : $C(X, \mathrm{SL}(2, \mathbb{R})) \to \mathbb{R}$ *is discontinuous.*

Proof. Assume, without loss of generality, that the support of μ is X. Take another invariant measure ν. Take a continuous function $h : X \to \mathbb{R}$ such that $\int h\, d\mu \neq 0$ but $\int h\, d\nu = 0$. Consider the diagonal cocycle

$$A(x) = \begin{pmatrix} e^{h(x)} & 0 \\ 0 & e^{-h(x)} \end{pmatrix}.$$

We have $\mathrm{LE}(A) = |\int h\, d\mu|$. Moreover, for every $\varepsilon > 0$ and $n_0 > 0$ there is an $n > n_0$ such that the open set

$$\left\{ x \in X : \frac{1}{n} \left| \sum_{j=0}^{n-1} h(T^j x) \right| < \varepsilon \right\}$$

is not empty and thus its μ-measure is positive. This shows that A is not uniformly hyperbolic. □

For T an irrational rotation of the circle, some examples of continuous non-uniform hyperbolic cocycles with positive exponent were given by Herman; see [**7**, §4]. As a consequence, we obtain the following result.

PROPOSITION 5.2. *For every irrational rotation T of the n-torus $\mathbb{T}^n$, the function*

$$\mathrm{LE} : C(\mathbb{T}^n, \mathrm{SL}(2, \mathbb{R})) \to \mathbb{R}$$

is discontinuous.

Remark. Clearly, it suffices to consider the case $n = 1$.

This proposition generalizes a previous result of Furman [**5**], which says that there is some irrational rotation such that LE is discontinuous.

For a general aperiodic uniquely ergodic transformation T it is an open question whether such examples exist; see [**5**].

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