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The Lyapunov exponents of generic volume-preserving and symplectic maps

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To Jacob Palis, on his 60^{th} *birthday, with friendship and admiration.*

Abstract

We show that the integrated Lyapunov exponents of C^1 volume-preserving diffeomorphisms are simultaneously continuous at a given diffeomorphism only if the corresponding Oseledets splitting is trivial (all Lyapunov exponents are equal to zero) or else dominated (uniform hyperbolicity in the projective bundle) almost everywhere.

We deduce a sharp dichotomy for generic volume-preserving diffeomorphisms on any compact manifold: almost every orbit either is projectively hyperbolic or has all Lyapunov exponents equal to zero.

Similarly, for a residual subset of all C^1 symplectic diffeomorphisms on any compact manifold, either the diffeomorphism is Anosov or almost every point has zero as a Lyapunov exponent, with multiplicity at least 2.

Finally, given any set $S \subset \mathrm{GL}(d)$ satisfying an accessibility condition, for a residual subset of all continuous S-valued cocycles over any measure-preserving homeomorphism of a compact space, the Oseledets splitting is either dominated or trivial. The condition on S is satisfied for most common matrix groups and also for matrices that arise from discrete Schrödinger operators.

1. Introduction

Lyapunov exponents describe the asymptotic evolution of a linear cocycle over a transformation: positive or negative exponents correspond to exponential growth or decay of the norm, respectively, whereas vanishing exponents mean lack of exponential behavior.

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In this work we address two basic, *a priori* unrelated problems. One is to understand how frequently Lyapunov exponents vanish on typical orbits. The other, is to analyze the dependence of Lyapunov exponents as functions of the system. We are especially interested in dynamical cocycles, i.e. those given by the derivatives of conservative diffeomorphisms, but we discuss the general situation as well.

Several approaches have been proposed for proving existence of nonzero Lyapunov exponents. Let us mention Furstenberg [14], Herman [16], Kotani [17], among others. In contrast, we show here that vanishing Lyapunov exponents are actually very frequent: *for a residual* (*dense* G_δ) *subset of all volume-preserving* C^1 *diffeomorphisms*, *and for almost every orbit*, *all Lyapunov exponents are equal to zero or else the Oseledets splitting is dominated.* This extends to generic continuous S-valued cocycles over any transformation, where S is a set of matrices that satisfy an accessibility condition, for instance, a matrix group G that acts transitively on the projective space.

Domination, or uniform hyperbolicity in the projective bundle, means that each Oseledets subspace is more expanded/less contracted than the next, by a definite *uniform* factor. This is a very strong property. In particular, *domination implies that the angles between the Oseledets subspaces are bounded from zero*, *and the Oseledets splitting extends to a continuous splitting on the closure.* For this reason, it can often be excluded *a priori*:

Example 1. Let $f : S^1 \to S^1$ be a homeomorphism and μ be any invariant ergodic measure with $\operatorname{supp} \mu = S^1$. Let $\mathcal{N}$ be the set of all continuous $A : S^1 \to \mathrm{SL}(2, \mathbb{R})$ nonhomotopic to a constant. For a residual subset of $\mathcal{N}$, the Lyapunov exponents of the corresponding cocycle over (f, μ) are zero. That is because the cocycle has no invariant continuous subbundle if A is nonhomotopic to a constant.

These results generalize to arbitrary dimension the work of Bochi [4], where it was shown that generic area-preserving C^1 diffeomorphisms on any compact surface either are uniformly hyperbolic (Anosov) or have no hyperbolicity at all; both Lyapunov exponents equal zero almost everywhere. This fact was announced by Mañé [19], [20] in the early eighties.

Our strategy is to tackle the higher dimensional problem and to analyze the dependence of Lyapunov exponents on the dynamics. We obtain the following characterization of the continuity points of Lyapunov exponents in the space of volume-preserving C^1 diffeomorphisms on any compact manifold: *they must have all exponents equal to zero or else the Oseledets splitting must be dominated*, *over almost every orbit.* This is similar for continuous linear cocycles over any transformation, and in this setting the necessary condition is known to be sufficient.

The issue of continuous or differentiable dependence of Lyapunov exponents on the underlying system is subtle, and not well understood. See Ruelle [29] and also Bourgain, Jitomirskaya [9], [10] for a discussion and further references. We also mention the following simple application of the result just stated, in the context of quasi-periodic Schrödinger cocycles:

Example 2. Let $f : S^1 \to S^1$ be an irrational rotation. Given $E \in \mathbb{R}$ and a continuous function $V : S^1 \to \mathbb{R}$, let $A : S^1 \to \mathrm{SL}(2, \mathbb{R})$ be given by

$$A(\theta) = \begin{pmatrix} E - V(\theta) & -1 \\ 1 & 0 \end{pmatrix}.$$

Then the cocycle determined by (A, f) is a point of discontinuity for the Lyapunov exponents, as functions of $V \in C^0(S^1, \mathbb{R})$, if and only if the exponents are nonzero and E is in the spectrum of the associated Schrödinger operator. Compare [10]. This is because E is in the complement of the spectrum if and only if the cocycle is uniformly hyperbolic, which for $\mathrm{SL}(2, \mathbb{R})$-cocycles is equivalent to domination.

We extend the two-dimensional result of Mañé–Bochi also in a different direction, namely to symplectic diffeomorphisms on any compact symplectic manifold. Firstly, we prove that continuity points for the Lyapunov exponents either are uniformly hyperbolic or have at least two Lyapunov exponents equal to zero at almost every point. Consequently, *generic symplectic C^1 diffeomorphisms either are Anosov or have vanishing Lyapunov exponents with multiplicity at least* 2 *at almost every point.*

Topological results in the vein of our present theorems were obtained by Millionshchikov [22], in the early eighties, and by Bonatti, Díaz, Pujals, Ures [8], [12], in their recent characterization of robust transitivity for diffeomorphisms. A counterpart of the latter for symplectic maps was obtained by Newhouse [25] in the seventies, and was recently extended by Arnaud [1]. Also recently, Dolgopyat, Pesin [13, §8] extended the perturbation technique of [4] to one 4-dimensional case, as part of their construction of nonuniformly hyperbolic diffeomorphisms on any compact manifold.

1.1. *Dominated splittings.* Let M be a compact manifold of dimension $d \geq 2$. Let $f : M \to M$ be a diffeomorphism and $\Gamma \subset M$ be an f-invariant set. Suppose for each $x \in \Gamma$ one is given nonzero subspaces E^1_x and E^2_x such that $T_x M = E^1_x \oplus E^2_x$, the dimensions of E^1_x and E^2_x are constant, and the subspaces are Df-invariant: $Df_x(E^i_x) = E^i_{f(x)}$ for all $x \in \Gamma$ and $i = 1, 2$.

Definition 1.1. Given $m \in \mathbb{N}$, we say that $T_\Gamma\, M = E^1 \oplus E^2$ is an *m-dominated* splitting if for every $x \in \Gamma$,

$$\|Df^m_x|_{E^2_x}\| \cdot \|(Df^m_x|_{E^1_x})^{-1}\| \leq \tfrac{1}{2}\,. \tag{1.1}$$

We call $T_\Gamma M = E^1 \oplus E^2$ a *dominated splitting* if it is m-dominated for some $m \in \mathbb{N}$. Then we write $E^1 \succ E^2$.

Condition (1.1) means that, for typical tangent vectors, their forward iterates converge to E^1 and their backward iterates converge to E^2, at uniform exponential rates. Thus, E^1 acts as a global hyperbolic attractor, and E^2 acts as a global hyperbolic repeller, for the dynamics induced by Df on the projective bundle.

More generally, we say that a splitting $T_\Gamma M = E^1 \oplus \cdots \oplus E^k$, into any number of sub-bundles, is dominated if

$$E^1 \oplus \cdots \oplus E^j \succ E^{j+1} \oplus \cdots \oplus E^k \quad \text{for every } 1 \le j < k.$$

We say that a splitting $T_\Gamma M = E^1 \oplus \cdots \oplus E^k$, is *dominated at* x, for some point $x \in \Gamma$, if it is dominated when restricted to the orbit $\{f^n(x);\ n \in \mathbb{Z}\}$ of x.

1.2. *Dichotomy for volume-preserving diffeomorphisms.* Let μ be the measure induced by some volume form. We indicate by $\mathrm{Diff}^1_\mu(M)$ the set of all μ-preserving C^1 diffeomorphisms of M, endowed with the C^1 topology. Let $f \in \mathrm{Diff}^1_\mu(M)$. By the theorem of Oseledets [26], for μ-almost every point $x \in M$, there exist $k(x) \in \mathbb{N}$, real numbers $\hat\lambda_1(f,x) > \cdots > \hat\lambda_{k(x)}(f,x)$, and a splitting $T_x M = E^1_x \oplus \cdots \oplus E^{k(x)}_x$ of the tangent space at x, all depending measurably on the point x, such that

$$\lim_{n\to\pm\infty} \frac{1}{n} \log \|Df^n_x(v)\| = \hat\lambda_j(f,x) \quad \text{for all } v \in E^j_x \setminus \{0\}. \tag{1.2}$$

The *Lyapunov exponents* $\hat\lambda_j(f,x)$ also correspond to the limits of $1/(2n)\log\rho_n$ as $n \to \infty$, where ρ_n represents the eigenvalues of $Df^n(x)^* Df^n(x)$. Let $\lambda_1(f,x) \ge \lambda_2(f,x) \ge \cdots \ge \lambda_d(f,x)$ be the Lyapunov exponents in nonincreasing order and each repeated with multiplicity $\dim E^j_x$. Note that $\lambda_1(f,x) + \cdots + \lambda_d(f,x) = 0$, because f preserves volume. We say that the Oseledets splitting is *trivial* at x when $k(x) = 1$, that is, when all Lyapunov exponents vanish.

It should be stressed that these are purely asymptotic statements: the limits in (1.2) are far from being uniform, in general. However, our first main result states that for *generic* volume-preserving diffeomorphisms one does have a lot of uniformity, over every orbit in a full measure subset:

THEOREM 1. *There exists a residual set $\mathcal{R} \subset \mathrm{Diff}^1_\mu(M)$ such that, for each $f \in \mathcal{R}$ and μ-almost every $x \in M$, the Oseledets splitting of f is either trivial or dominated at x.*

For $f \in \mathcal{R}$ the ambient manifold M splits, up to zero measure, into disjoint invariant sets Z and D corresponding to trivial splitting and dominated splitting, respectively. Moreover, D may be written as an increasing union $D = \cup_{m\in\mathbb{N}} D_m$ of compact f-invariant sets, each admitting a dominated splitting of the tangent bundle.

If $f \in \mathcal{R}$ is ergodic then either $\mu(Z) = 1$ or there is $m \in \mathbb{N}$ such that $\mu(D_m) = 1$. The first case means that all the Lyapunov exponents vanish almost everywhere. In the second case, the Oseledets splitting extends continuously to a dominated splitting of the tangent bundle over the whole ambient manifold M.

Example 3. Let $f_t : N \to N$, $t \in S^1$, be a smooth family of volume-preserving diffeomorphisms on some compact manifold N, such that $f_t = \mathrm{id}$ for t in some interval $I \subset S^1$, and f_t is partially hyperbolic for t in another interval $J \subset S^1$. Such families may be obtained, for instance, using the construction of partially hyperbolic diffeomorphisms isotopic to the identity in [7]. Then $f : S^1 \times N \to S^1 \times N$, $f(t, x) = (t, f_t(x))$ is a volume-preserving diffeomorphism for which $D \supset S^1 \times J$ and $Z \supset S^1 \times I$.

Thus, in general we may have $0 < \mu(Z) < 1$. However, we ignore whether such examples can be made generic (see also Section 1.3).

Problem 1. Is there a residual subset of $\mathrm{Diff}^1_\mu(M)$ for which invariant sets with a dominated splitting have either zero or full measure?

Theorem 1 is a consequence of the following result about continuity of Lyapunov exponents as functions of the dynamics. For $j = 1, \ldots, d-1$, define

$$\mathrm{LE}_j(f) = \int_M [\lambda_1(f, x) + \cdots + \lambda_j(f, x)]\, d\mu(x).$$

It is well-known that the functions $f \in \mathrm{Diff}^1_\mu(M) \mapsto \mathrm{LE}_j(f)$ are upper semi-continuous (see Proposition 2.2 below). Our next main theorem shows that *lower* semi-continuity is much more delicate:

THEOREM 2. *Let $f_0 \in \mathrm{Diff}^1_\mu(M)$ be such that the map*

$$f \in \mathrm{Diff}^1_\mu(M) \mapsto \big(\mathrm{LE}_1(f), \ldots, \mathrm{LE}_{d-1}(f)\big) \in \mathbb{R}^{d-1}$$

is continuous at $f = f_0$. Then for μ-almost every $x \in M$, the Oseledets splitting of f_0 is either dominated or trivial at x.

The set of continuity points of a semi-continuous function on a Baire space is always a residual subset of the space (see e.g. [18, §31.X]); therefore Theorem 1 is an immediate corollary of Theorem 2.

Problem 2. Is the necessary condition in Theorem 2 also sufficient for continuity?

Diffeomorphisms with all Lyapunov exponents equal to zero almost everywhere, or else whose Oseledets splitting extends to a dominated splitting over the whole manifold, are always continuity points. Moreover, the answer is affirmative in the context of linear cocycles, as we shall see.

1.3. *Dichotomy for symplectic diffeomorphisms.* Now we turn ourselves to symplectic systems. Let (M^{2q}, ω) be a compact symplectic manifold without boundary. We denote by μ the volume measure associated to the volume form $\omega^q = \omega \wedge \cdots \wedge \omega$. The space $\mathrm{Sympl}^1_\omega(M)$ of all C^1 symplectic diffeomorphisms is a subspace of $\mathrm{Diff}^1_\mu(M)$. We also fix a smooth Riemannian metric on M, the particular choice being irrelevant for all purposes.

The Lyapunov exponents of symplectic diffeomorphisms have a symmetry property: $\lambda_j(f,x) = -\lambda_{2q-j+1}(f,x)$ for all $1 \le j \le q$. (That is because in this case the linear operator $Df^n(x)^* Df^n(x)$ is symplectic and so (see Arnold [3]) its spectrum is symmetric; the inverse of every eigenvalue is also an eigenvalue, with the same multiplicity.) In particular, $\lambda_q(x) \ge 0$ and $\mathrm{LE}_q(f)$ is the integral of the sum of all nonnegative exponents. Consider the splitting

$$T_x M = E^+_x \oplus E^0_x \oplus E^-_x,$$

where E^+_x, E^0_x, and E^-_x are the sums of all Oseledets spaces associated to positive, zero, and negative Lyapunov exponents, respectively. Then $\dim E^+_x = \dim E^-_x$ and $\dim E^0_x$ is even.

Theorem 3. *Let $f_0 \in \mathrm{Sympl}^1_\omega(M)$ be such that the map*

$$f \in \mathrm{Sympl}^1_\omega(M) \mapsto \mathrm{LE}_q(f) \in \mathbb{R}$$

is continuous at $f = f_0$. Then for μ-almost every $x \in M$, either $\dim E^0_x \ge 2$ or the splitting $T_x M = E^+_x \oplus E^-_x$ is hyperbolic along the orbit of x.

In the second alternative, what we actually prove is that the splitting is dominated at x. This is enough because, as we shall prove in Lemma 2.4, for symplectic diffeomorphisms dominated splittings into two subspaces of the same dimension are uniformly hyperbolic.

As in the volume-preserving case, the function $f \mapsto \mathrm{LE}_q(f)$ is continuous on a residual subset $\mathcal{R}_1$ of $\mathrm{Sympl}^1_\omega(M)$. Also, we show that there is a residual subset $\mathcal{R}_2 \subset \mathrm{Sympl}^1_\omega(M)$ such that for every $f \in \mathcal{R}_2$ either f is an Anosov diffeomorphism or all its hyperbolic sets have zero measure. Taking $\mathcal{R} = \mathcal{R}_1 \cap \mathcal{R}_2$, we obtain:

Theorem 4. *There exists a residual set $\mathcal{R} \subset \mathrm{Sympl}^1_\omega(M)$ such that every $f \in \mathcal{R}$ either is Anosov or has at least two zero Lyapunov exponents at almost every point.*

For $d = 2$ one recovers the two-dimensional result of Mañé–Bochi.

1.4. *Linear cocycles.* Now we comment on corresponding statements for linear cocycles. Let M be a compact Hausdorff space, μ a Borel regular probability measure, and $f : M \to M$ a homeomorphism that preserves μ. Given a continuous map $A : M \to \mathrm{GL}(d, \mathbb{R})$, one associates the linear cocycle

$$F_A : M \times \mathbb{R}^d \to M \times \mathbb{R}^d \ , \quad F_A(x, v) = (f(x), A(x)v). \tag{1.3}$$

Oseledets' theorem extends to this setting, and so does the concept of dominated splitting; see Sections 2.1 and 2.2.

One is often interested in classes of maps A whose values have some specific form, e.g., belong to some subgroup $G \subset \mathrm{GL}(d, \mathbb{R})$. To state our results in greater generality, we consider the space $C(M, S)$ of all continuous maps $M \to S$, where $S \subset \mathrm{GL}(d, \mathbb{R})$ is a fixed set. We endow the space $C(M, S)$ with the C^0-topology. We shall deal with sets S that satisfy an accessibility condition:

Definition 1.2. Let $S \subset \mathrm{GL}(d, \mathbb{R})$ be an embedded submanifold (with or without boundary). We call S *accessible* if for all $C_0 > 0$ and $\varepsilon > 0$, there are $\nu \in \mathbb{N}$ and $\alpha > 0$ with the following properties: Given ξ, η in the projective space $\mathbb{R}\mathrm{P}^{d-1}$ with $\sphericalangle(\xi, \eta) < \alpha$, and $A_0, \dots, A_{\nu-1} \in S$ with $\|A_i^{\pm 1}\| \le C_0$, there exist $\widetilde{A}_0, \dots, \widetilde{A}_{\nu-1} \in S$ such that $\|\widetilde{A}_i - A_i\| < \varepsilon$ and $\widetilde{A}_{\nu-1} \dots \widetilde{A}_0(\xi) = A_{\nu-1} \dots A_0(\eta)$.

Example 4. Let G be a closed subgroup $\mathrm{GL}(d, \mathbb{R})$ which acts transitively in the projective space $\mathbb{R}\mathrm{P}^{d-1}$. Then $S = G$ is accessible and, in fact, we may always take $\nu = 1$ in the definition. See Lemma 5.12. So the most common matrix groups are accessible, e.g., $\mathrm{GL}(d, \mathbb{R})$, $\mathrm{SL}(d, \mathbb{R})$, $\mathrm{Sp}(2q, \mathbb{R})$, as well as $\mathrm{SL}(d, \mathbb{C})$, $\mathrm{GL}(d, \mathbb{C})$ (which are isomorphic to subgroups of $\mathrm{GL}(2d, \mathbb{R})$). (Compact groups are not of interest in our context, because all Lyapunov exponents vanish identically.)

Example 5. The set of matrices of the type already mentioned in Example 2:

$$S = \left\{ \begin{pmatrix} t & -1 \\ 1 & 0 \end{pmatrix} ;\ t \in \mathbb{R} \right\} \subset \mathrm{GL}(2, \mathbb{R})$$

is accessible. To see this, let $\nu = 2$. If ξ and η are not too close to $\mathbb{R}(0, 1)$, then we may find a small perturbation $\widetilde{A}_0$ of A_0 such that $\widetilde{A}_0(\xi) = A_0(\eta)$, and let $\widetilde{A}_1 = A_1$. In the other case, $A_0(\xi)$ and $A_0(\eta)$ must be close to $\mathbb{R}(1, 0)$; then we take $\widetilde{A}_0 = A_0$ and find a suitable $\widetilde{A}_1$.

Theorem 5. *Let $S \subset \mathrm{GL}(d, \mathbb{R})$ be an accessible set. Then $A_0 \in C(M, S)$ is a point of continuity of*

$$C(M, S) \ni A \mapsto (\mathrm{LE}_1(A), \dots, \mathrm{LE}_{d-1}(A)) \in \mathbb{R}^{d-1}$$

if and only if the Oseledets splitting of the cocycle F_A at x is either dominated or trivial at μ-almost every $x \in M$.

Consequently, there exists a residual subset $\mathcal{R} \subset C(M,S)$ such that for every $A \in \mathcal{R}$ and at almost every $x \in X$, either all Lyapunov exponents of F_A are equal or the Oseledets splitting of F_A is dominated.

Corollary 1. *Assume (f, μ) is ergodic. For any accessible set $S \subset \mathrm{GL}(d, \mathbb{R})$, there exists a residual subset $\mathcal{R} \subset C(M,S)$ such that every $A \in \mathcal{R}$ either has all exponents equal at almost every point, or there exists a dominated splitting of $M \times \mathbb{R}^d$ which coincides with the Oseledets splitting almost everywhere.*

Theorem 5 and the corollary remain true if one replaces $C(M,S)$ by $L^\infty(M,S)$. We only need f to be an invertible measure-preserving transformation.

It is interesting that an accessibility condition of control-theoretic type was used by Nerurkar [24] to get nonzero exponents.

1.5. *Extensions, related problems, and outline of the proof.* Most of the results stated above were announced in [5]. Actually, our Theorems 3 and 4 do not give the full strength of Theorem 4 in [5]. The difficulty is that the symplectic analogue of our construction of realizable sequences is less satisfactory, unless the subspaces involved have the same dimension; see Remark 5.2. Thus, the following question remains open (see also Remark 2.5):

Problem 3. Is it true that the Oseledets splitting of generic symplectic C^1 diffeomorphisms is either trivial or partially hyperbolic at almost every point?

Problem 4. For generic smooth families $\mathbb{R}^p \to \mathrm{Diff}^1_\mu(M)$, $\mathrm{Sympl}^1_\omega(M)$, $C(M,S)$ (i.e. smooth in the parameters), what can be said of the Lebesgue measure of the subset of parameters corresponding to zero Lyapunov exponents?

Problem 5. What are the continuity points of Lyapunov exponents in $\mathrm{Diff}^{1+r}_\mu(M)$ or $C^r(M,S)$ for $r > 0$?

Problem 6. Is the generic volume-preserving C^1 diffeomorphism ergodic or, at least, does it have only a finite number of ergodic components?

The first question in Problem 6 was posed to us by A. Katok and the second one was suggested by the referee. The theorem of Oxtoby, Ulam [27] states that generic volume-preserving *homeomorphisms* are ergodic.

Let us close this introduction with a brief outline of the proof of Theorem 2. Theorems 3 and 5 follow from variations of these arguments, and the other main results are fairly direct consequences.

Suppose the Oseledets splitting is neither trivial nor dominated, over a positive Lebesgue measure set of orbits: for some i and for arbitrarily large m there exist iterates y for which

$$\|Df^m\mid_{E_y^{i-}}\|\,\|(Df^m\mid_{E_y^{i+}})^{-1}\| > \tfrac{1}{2} \tag{1.4}$$

where $E_y^{i+} = E_y^1 \oplus \cdots \oplus E_y^i$ and $E_y^{i-} = E_y^{i+1} \oplus \cdots \oplus E_y^{k(y)}$. The basic strategy is to take advantage of this fact to, by a small perturbation of the map, cause a vector originally in E_y^{i+} to move to E_z^{i-}, $z = f^m(y)$, thus "blending" different expansion rates.

More precisely, given a perturbation size $\varepsilon > 0$ we take m sufficiently large with respect to ε. Then, given $x \in M$, for n much bigger than m we choose an iterate $y = f^\ell(x)$, with $\ell \approx n/2$, as in (1.4). By composing Df with small rotations near the first m iterates of y, we cause the orbit of some $Df_x^\ell(v) \in E_y^{i+}$ to move to E_z^{i-}. In this way we find an ε-perturbation $g = f \circ h$ preserving the orbit segment $\{x, \ldots, f^n(x)\}$ and such that $Dg_x^s(v) \in E^{i+}$ during the first $\ell \approx n/2$ iterates and $Dg_x^s(v) \in E^{i-}$ during the last $n - \ell - m \approx n/2$ iterates. We want to conclude that Dg_x^n lost some expansion if compared to Df_x^n. To this end we compare the p^{th} exterior products of these linear maps, with $p = \dim E^{i+}$. While $\|\wedge^p(Df_x^n)\| \approx \exp(n(\lambda_1 + \cdots + \lambda_p))$ we see that

$$\|\wedge^p(Dg_x^n)\| \lesssim \exp\Big(n\big(\lambda_1 + \cdots + \lambda_{p-1} + \frac{\lambda_p + \lambda_{p+1}}{2}\big)\Big),$$

where the Lyapunov exponents are computed at (f, x). Notice that $\lambda_{p+1} = \hat{\lambda}_{i+1}$ is strictly smaller than $\lambda_p = \hat{\lambda}_i$. This local procedure is then repeated for a positive Lebesgue measure set of points $x \in M$. Using (see Proposition 2.2)

$$\mathrm{LE}_p(g) = \inf_n \frac{1}{n} \int \log \|\wedge^p(Dg^n)\|\, d\mu$$

and a Kakutani tower argument, we deduce that LE_p drops under such arbitrarily small perturbations, contradicting continuity.

Let us also comment on the way the C^1 topology comes into the proof. It is very important for our arguments that the various perturbations of the diffeomorphism close to each $f^s(y)$ do not interfere with each other, nor with the other iterates of x in the time interval $\{0, \ldots, n\}$. The way we achieve this is by rescaling the perturbation $g = f \circ h$ near each $f^s(y)$ if necessary, to ensure its support is contained in a sufficiently small neighborhood of the point. In local coordinates w for which $f^s(y)$ is the origin, rescaling corresponds to replacing $h(w)$ by $rh(w/r)$ for some small $r > 0$. Observe that this does not affect the value of the derivative at the origin nor the C^1 norm of the map, but it tends to increase C^r norms for $r > 1$.

This paper is organized as follows. In Section 2 we introduce several preparatory notions and results. In Section 3 we state and prove the main perturbation tool, the directions exchange Proposition 3.1. We use this proposition to prove Theorem 2 in Section 4, where we also deduce Theorem 1. Section 5 contains a symplectic version of Proposition 3.1. This is used in Section 6 to prove Theorem 3, from which we deduce Theorem 4. Similar ideas, in an easier form, are used in Section 7 to get Theorem 5.

2. Preliminaries

2.1. *Lyapunov exponents*, *Oseledets splittings.* Let M be a compact Hausdorff space and $\pi : \mathcal{E} \to M$ be a continuous finite-dimensional vector bundle endowed with a continuous Riemann structure. A *cocycle* over a homeomorphism $f : M \to M$ is a continuous transformation $F : \mathcal{E} \to \mathcal{E}$ such that $\pi \circ F = f \circ \pi$ and $F_x : \mathcal{E}_x \to \mathcal{E}_{f(x)}$ is a linear isomorphism on each fiber $\mathcal{E}_x = \pi^{-1}(x)$. Notice that (1.3) corresponds to the case when the vector bundle is trivial.

2.1.1. *Oseledets' theorem.* Let μ be any f-invariant Borel probability measure in M. The theorem of Oseledets [26] states that for μ-almost every point x there exists a splitting

$$\mathcal{E}_x = E_x^1 \oplus \cdots \oplus E_x^{k(x)}\,, \tag{2.1}$$

and real numbers $\hat\lambda_1(x) > \cdots > \hat\lambda_{k(x)}(x)$ such that $F_x(E_x^j) = E_{f(x)}^j$ and

$$\lim_{n\to\pm\infty} \frac{1}{n} \log \|F_x^n(v)\| = \hat\lambda_j(x)$$

for $v \in E_x^j \setminus \{0\}$ and $j = 1, \dots, k(x)$. Moreover, if J_1 and J_2 are any disjoint subsets of the set of indexes $\{1, \dots, k(x)\}$, then

$$\lim_{n\to\pm\infty} \frac{1}{n} \log \measuredangle\Big(\bigoplus\nolimits_{j\in J_1} E^j_{f^n(x)}, \bigoplus\nolimits_{j\in J_2} E^j_{f^n(x)}\Big) = 0. \tag{2.2}$$

Let $\lambda_1(x) \geq \lambda_2(x) \geq \cdots \geq \lambda_d(x)$ be the numbers $\hat\lambda_j(x)$, each repeated with multiplicity $\dim E_x^j$ and written in nonincreasing order. When the dependence on F matters, we write $\lambda_i(F,x) = \lambda_i(x)$. In the case when $F = Df$, we write $\lambda_i(f,x) = \lambda_i(F,x) = \lambda_i(x)$.

2.1.2. *Exterior products.* Given a vector space V and a positive integer p, let $\wedge^p(V)$ be the p^{th} exterior power of V. This is a vector space of dimension $\binom{d}{p}$, whose elements are called *p-vectors*. It is generated by the p-vectors of the form $v_1 \wedge \cdots \wedge v_p$ with $v_j \in V$, called the *decomposable p-vectors*. A linear map $L : V \to W$ induces a linear map $\wedge^p(L) : \wedge^p(V) \to \wedge^p(W)$ such that

$$\wedge^p(L)(v_1 \wedge \cdots \wedge v_p) = L(v_1) \wedge \cdots \wedge L(v_p).$$

If V has an inner product, then we always endow $\wedge^p(V)$ with the inner product such that $\|v_1 \wedge \cdots \wedge v_p\|$ equals the p-dimensional volume of the parallelepiped spanned by $v_1, \ldots, v_p$. See [2, §3.2.3].

More generally, there is a vector bundle $\wedge^p(\mathcal{E})$, with fibers $\wedge^p(\mathcal{E}_x)$, associated to $\mathcal{E}$, and there is a vector bundle automorphism $\wedge^p(F)$, associated to F. If the vector bundle $\mathcal{E}$ is endowed with a continuous inner product, then $\wedge^p(\mathcal{E})$ also is. The Oseledets data of $\wedge^p(F)$ can be obtained from that of F, as shown by the proposition below. For a proof, see [2, Th. 5.3.1].

Proposition 2.1. *The Lyapunov exponents* (*with multiplicity*) $\lambda_i^{\wedge p}(x)$, $1 \le i \le \binom{d}{p}$, *of the automorphism* $\wedge^p(F)$ *at a point* x *are the numbers*

$$\lambda_{i_1}(x) + \cdots + \lambda_{i_p}(x), \quad \textit{where } 1 \le i_1 < \cdots < i_p \le d.$$

Let $\{e_1(x), \ldots, e_d(x)\}$ *be a basis of* $\mathcal{E}_x$ *such that*

$$e_i(x) \in E_x^\ell \quad \textit{for } \dim E_x^1 + \cdots + \dim E_x^{\ell-1} < i \le \dim E_x^1 + \cdots + \dim E_x^\ell.$$

Then the Oseledets space $E_x^{j,\wedge p}$ *of* $\wedge^p(F)$ *corresponding to the Lyapunov exponent* $\hat\lambda_j(x)$ *is the sub-space of* $\wedge^p(\mathcal{E}_x)$ *generated by the* p*-vectors*

$$e_{i_1} \wedge \cdots \wedge e_{i_p}, \quad \textit{with } 1 \le i_1 < \cdots < i_p \le d \textit{ and } \lambda_{i_1}(x) + \cdots + \lambda_{i_p}(x) = \hat\lambda_j(x).$$

2.1.3. *Semi-continuity of integrated exponents.* Let us indicate $\Lambda_p(F,x) = \lambda_1(F,x) + \cdots + \lambda_p(F,x)$, for $p = 1, \ldots, d-1$. We define the *integrated Lyapunov exponent*

$$\mathrm{LE}_p(F) = \int_M \Lambda_p(F,x)\, d\mu(x).$$

More generally, if $\Gamma \subset M$ is a measurable f-invariant subset, we define

$$\mathrm{LE}_p(F,\Gamma) = \int_\Gamma \Lambda_p(F,x)\, d\mu(x).$$

By Proposition 2.1, $\Lambda_p(F,x) = \lambda_1(\wedge^p F, x)$ and so $\mathrm{LE}_p(F,\Gamma) = \mathrm{LE}_1(\wedge^p(F),\Gamma)$. When $F = Df$, we write $\Lambda_p(f,x) = \Lambda_i(F,x)$ and $\mathrm{LE}_p(f,\Gamma) = \mathrm{LE}_p(F,\Gamma)$.

Proposition 2.2. *If* $\Gamma \subset M$ *is a measurable* f*-invariant subset then*

$$\mathrm{LE}_p(F,\Gamma) = \inf_{n \ge 1} \frac{1}{n} \int_\Gamma \log \|\wedge^p(F_x^n)\|\, d\mu(x).$$

Proof. The sequence $a_n = \int_\Gamma \log\|\wedge^p(F_x^n)\|\, d\mu$ is sub-additive ($a_{n+m} \le a_n + a_m$); therefore $\lim \frac{a_n}{n} = \inf \frac{a_n}{n}$. □

As a consequence of Proposition 2.2, the map $f \in \mathrm{Diff}^1_\mu(M) \mapsto \mathrm{LE}_p(f)$ is upper semi-continuous, as mentioned in the introduction.

2.2. *Dominated splittings.* Let $\Gamma \subset M$ be an f-invariant set. A splitting $\mathcal{E}_\Gamma = E^1 \oplus E^2$ is *dominated* for F if it is F-invariant, the dimensions of E^i_x are constant on Γ, and there exists $m \in \mathbb{N}$ such that, for every $x \in \Gamma$,

$$\frac{\|F^m_x|_{E^2_x}\|}{\mathbf{m}(F^m_x|_{E^1_x})} \le \frac{1}{2}\,. \tag{2.3}$$

We denote $\mathbf{m}(L) = \|L^{-1}\|^{-1}$ the *co-norm* of a linear isomorphism L. The dimension of the space E^1 is called the *index* of the splitting.

A few elementary properties of dominated decompositions follow. The proofs are left to the reader.

Transversality. If $\mathcal{E}_\Gamma = E^1 \oplus E^2$ is a dominated splitting then the angle $\sphericalangle(E^1_x, E^2_x)$ is bounded away from zero, over all $x \in \Gamma$.

Uniqueness. If $\mathcal{E}_\Gamma = E^1 \oplus E^2$ and $\mathcal{E}_\Gamma = \hat{E}^1 \oplus \hat{E}^2$ are dominated decompositions with $\dim E^i = \dim \hat{E}^i$ then $E^i = \hat{E}^i$ for $i = 1, 2$.

Continuity. A dominated splitting $\mathcal{E}_\Gamma = E^1 \oplus E^2$ is continuous, and extends continuously to a dominated splitting over the closure of Γ.

2.3. *Dominance and hyperbolicity for symplectic maps.* We just recall a few basic notions that are needed in this context, referring the reader to Arnold [3] for definitions and fundamental properties of symplectic forms, manifolds, and maps.

Let (V, ω) be a symplectic vector space of dimension $2q$. Given a subspace $W \subset V$, its *symplectic orthogonal* is the space (of dimension $2q - \dim W$)

$$W^\omega = \{w \in W;\ \omega(v, w) = 0 \text{ for all } v \in V\}.$$

The subspace W is called *symplectic* if $W^\omega \cap W = \{0\}$; that is, $\omega|_{W\times W}$ is a nondegenerate form. W is called *isotropic* if $W \subset W^\omega$, that is, $\omega|_{W\times W} \equiv 0$. The subspace W is called *Lagrangian* if $W = W^\omega$; that is, it is isotropic and $\dim W = q$.

Now let (M, ω) be a symplectic manifold of dimension $d = 2q$. We also fix in M a Riemannian structure. For each $x \in M$, let $J_x : T_xM \to T_xM$ be the anti-symmetric isomorphism defined by $\omega(v, w) = \langle J_x v, w\rangle$ for all $v, w \in T_xM$. Denote

$$C_\omega = \sup_{x\in M} \|J_x^{\pm 1}\|. \tag{2.4}$$

In particular,

$$|\omega(v, w)| \le C_\omega \|v\|\,\|w\| \quad \text{for all } v, w \in T_xM. \tag{2.5}$$

LEMMA 2.3. *If E, $F \subset T_xM$ are two Lagrangian subspaces with $E \cap F = \{0\}$ and $\alpha = \sphericalangle(E, F)$ then*:

(1) *For every $v \in E \setminus \{0\}$ there exists $w \in F \setminus \{0\}$ such that*

$$|\omega(v,w)| \geq C_\omega^{-1} \sin\alpha \, \|v\| \, \|w\|.$$

(2) *If $S : T_xM \to T_yM$ is any symplectic linear map and $\beta = \sphericalangle(S(E), S(F))$ then*

$$C_\omega^{-2} \sin\alpha \leq \mathbf{m}(S|_E) \, \|S|_F\| \leq C_\omega^2 (\sin\beta)^{-1}.$$

Proof. To prove part 1, let $p : T_xM \to F$ be the projection parallel to E. Given a nonzero $v \in E$, take $w = p(J_x v)$. Since E is isotropic, $\omega(v,w) = \omega(v, J_x v) = \|J_x v\|^2 \geq C_\omega^{-1} \|v\| \, \|J_x v\|$. Also $\|w\| \leq \|p\| \, \|J_x v\|$ and $\|p\| = 1/\sin\alpha$, so that the claim follows.

To prove part 2, take a nonzero $v \in E$ such that $\|Sv\|/\|v\| = \mathbf{m}(S|_E)$ and let w be as in part 1. Then

$$C_\omega^{-1} \sin\alpha \, \|v\| \, \|w\| \leq |\omega(v,w)| = |\omega(Sv, Sw)| \leq C_\omega \|Sv\| \, \|Sw\|.$$

Thus $\mathbf{m}(S|_E) \, \|Sw\|/\|w\| \geq C_\omega^{-2} \sin\alpha$, proving the lower inequality in part 2. The upper inequality follows from the lower one applied to $S(F)$, $S(E)$ and S^{-1} in the place of E, F, and S, respectively. ☐

LEMMA 2.4. *Let $f \in \mathrm{Sympl}^1_\omega(M)$, and let x be a regular point. Assume that $\lambda_q(f,x) > 0$, that is, there are no zero exponents. Let E^+_x and E^-_x be the sum of all Oseledets subspaces associated to positive and to negative Lyapunov exponents, respectively. Then*

(1) *The subspaces E^+_x and E^-_x are Lagrangian.*

(2) *If the splitting $E^+ \oplus E^-$ is dominated at x then E^+ is uniformly expanding and E^- is uniformly contracting along the orbit of x.*

Proof. To prove part 1, we only have to show that the spaces E^+_x and E^-_x are isotropic. Take vectors $v_1, v_2 \in E^-_x$. Take $\varepsilon > 0$ with $\varepsilon < \lambda_q(f,x)$. For every large n and $i = 1, 2$, we have $\|Df^n_x v_i\| \leq e^{-n\varepsilon} \|v_i\|$. Hence, by (2.5),

$$|\omega(v_1, v_2)| = |\omega(Df^n_x v_1, Df^n_x v_2)| \leq C_\omega e^{-2n\varepsilon} \|v_1\| \, \|v_2\|,$$

that is, $\omega(v_1, v_2) = 0$. A similar argument, iterating backward, gives that E^+_x is isotropic.

Now assume that $E^+ \succ E^-$ at x. Let $\alpha > 0$ be a lower bound for $\sphericalangle(E^+, E^-)$ along the orbit of x, and let $C = C_\omega^2 (\sin\alpha)^{-1}$. By domination, there exists $m \in \mathbb{N}$ such that

$$\frac{\|Df^m_{f^n(x)}|_{E^-}\|}{\mathbf{m}(Df^m_{f^n(x)}|_{E^+})} < \frac{1}{4C}, \quad \text{for all } n \in \mathbb{Z}.$$

By part 2 of Lemma 2.4, we have $C^{-1} \leq \mathbf{m}(Df^m_{f^n(x)}|_{E^+})\,\|Df^m_{f^n(x)}|_{E^-}\| \leq C$. Therefore

$$\mathbf{m}(Df^m_{f^n(x)}|_{E^+}) > 2 \quad \text{and} \quad \|Df^m_{f^n(x)}|_{E^-}\| < \tfrac{1}{2} \quad \text{for all } n \in \mathbb{Z}.$$

This proves part 2. □

Remark 2.5. More generally, existence of a dominated splitting implies partial hyperbolicity: If $E \oplus \widehat{F}$ is a *dominated splitting*, with $\dim E \leq \dim \widehat{F}$, then $\widehat{F}$ *splits invariantly* as $\widehat{F} = C \oplus F$, with $\dim F = \dim E$. Moreover, the *splitting* $E \oplus C \oplus F$ is *dominated*, E is *uniformly expanding*, and F is *uniformly contracting*. This fact was pointed out by Mañé in [20]. Proofs appeared recently in Arnaud [1], for dimension 4, and in [6], for arbitrary dimension.

2.4. *Angle estimation tools.* Here we collect a few useful facts from elementary linear algebra. We begin by noting that, given any one-dimensional subspaces A, B, and C of $\mathbb{R}^d$, then

$$\begin{aligned}\sin\sphericalangle(A,B)\,\sin\sphericalangle(A+B,C) &= \sin\sphericalangle(C,A)\,\sin\sphericalangle(C+A,B)\\ &= \sin\sphericalangle(B,C)\,\sin\sphericalangle(B+C,A).\end{aligned}$$

Indeed, this quantity is the 3-dimensional volume of the parallelepiped with unit edges in the directions A, B and C. As a corollary, we get:

LEMMA 2.6. *Let A, B and C be subspaces (of any dimension) of $\mathbb{R}^d$. Then*

$$\sin\sphericalangle(A,B+C) \geq \sin\sphericalangle(A,B)\,\sin\sphericalangle(A+B,C).$$

Let v, w be nonzero vectors. For any $\alpha \in \mathbb{R}$, $\|v+\alpha w\| \geq \|v\|\sin\sphericalangle(v,w)$, with equality when $\alpha = \langle v,w\rangle/\|w\|^2$. Given $L \in \mathrm{GL}(d,\mathbb{R})$, let $\beta = \langle Lv,Lw\rangle/\|Lw\|^2$ and $z = v+\beta w$. By the previous remark, $\|z\| \geq \|v\|\sin\sphericalangle(v,w)$ and $\|Lz\| = \|Lv\|\sin\sphericalangle(Lv,Lw)$. Therefore

$$\sin\sphericalangle(Lv,Lw) = \frac{\|Lz\|}{\|Lv\|} \geq \frac{\mathbf{m}(L)\|v\|}{\|Lv\|}\,\sin\sphericalangle(v,w). \tag{2.6}$$

As a consequence of (2.6), we have:

LEMMA 2.7. *Let $L : \mathbb{R}^d \to \mathbb{R}^d$ be a linear map and let v, w be nonzero vectors. Then*

$$\frac{\mathbf{m}(L)}{\|L\|} \leq \frac{\sin\sphericalangle(Lv,Lw)}{\sin\sphericalangle(v,w)} \leq \frac{\|L\|}{\mathbf{m}(L)}.$$

Thus $\|L\|/\mathbf{m}(L)$ measures how much angles can be distorted by L. At last, we give a bound for this quantity when $d = 2$.

LEMMA 2.8. *Let $L : \mathbb{R}^2 \to \mathbb{R}^2$ be an invertible linear map and let $v, w \in \mathbb{R}^2$ be linearly independent unit vectors. Then*

$$\frac{\|L\|}{\mathbf{m}(L)} \le 4 \max\left\{\frac{\|Lv\|}{\|Lw\|}, \frac{\|Lw\|}{\|Lv\|}\right\} \frac{1}{\sin\sphericalangle(v,w)} \frac{1}{\sin\sphericalangle(Lv,Lw)}.$$

Proof. We may assume that L is not conformal, for in the conformal case the left-hand side is 1 and the inequality is obvious. Let $\mathbb{R}s$ be the direction most contracted by L, and let θ, $\phi \in [0, \pi]$ be the angles that the directions $\mathbb{R}v$ and $\mathbb{R}w$, respectively, make with $\mathbb{R}s$. Suppose that $\|Lv\| \ge \|Lw\|$. Then $\phi \le \theta$ and so $\sphericalangle(v, w) \le 2\theta$. Hence

$$\|Lv\| \ge \|L\| \sin\theta \ge \tfrac{1}{2}\|L\| \sin 2\theta \ge \tfrac{1}{2}\|L\| \sin\sphericalangle(v,w).$$

Moreover, $|\det L| = \mathbf{m}(L)\|L\|$ and

$$\|Lv\|\,\|Lw\| \sin\sphericalangle(Lv,Lw) = |\det L| \sin\sphericalangle(v,w).$$

The claim is an easy consequence of these relations. □

2.5. *Coordinates*, *metrics*, *neighborhoods.* Let (M, ω) be a symplectic manifold of dimension $d = 2q \ge 2$. According to Darboux's theorem, there exists an atlas $\mathcal{A}^* = \{\varphi_i : V_i^* \to \mathbb{R}^d\}$ of *canonical* local coordinates, that is, such that

$$(\varphi_i)_*\omega = dx_1 \wedge dx_2 + \cdots + dx_{2q-1} \wedge dx_{2q}$$

for all i. Similarly, cf. [23, Lemma 2], given any volume structure β on a d-dimensional manifold M, one can find an atlas $\mathcal{A}^* = \{\varphi_i : V_i^* \to \mathbb{R}^d\}$ consisting of charts φ_i such that

$$(\varphi_i)_*\beta = dx_1 \wedge \cdots \wedge dx_d\,.$$

In either case, assuming M is compact one may choose $\mathcal{A}^*$ finite. Moreover, we may always choose $\mathcal{A}^*$ so that every V_i^* contains the closure of an open set V_i, such that the restrictions $\varphi_i : V_i \to \mathbb{R}^d$ still form an atlas of M. The latter will be denoted $\mathcal{A}$. Let $\mathcal{A}^*$ and $\mathcal{A}$ be fixed once and for all.

By compactness, there exists $r_0 > 0$ such that for each $x \in M$, there exists $i(x)$ such that the Riemannian ball of radius r_0 around x is contained in $V_{i(x)}$. For definiteness, we choose $i(x)$ smallest with this property. For technical convenience, when dealing with the point x we express our estimates in terms of the Riemannian metric $\|\cdot\| = \|\cdot\|_x$ defined on that ball of radius r_0 by $\|v\| = \|D\varphi_{i(x)}v\|$. Observe that these Riemannian metrics are (uniformly) equivalent to the original one on M, and so there is no inconvenience in replacing one by the other.

We may also view any linear map $A : T_{x_1}M \to T_{x_2}M$ as acting on $\mathbb{R}^d$, using local charts $\varphi_{i(x_1)}$ and $\varphi_{i(x_2)}$. This permits us to speak of the distance

$\|A - B\|$ between A and another linear map $B : T_{x_3}M \to T_{x_4}M$ whose base points are different:

$$\|A - B\| = \|D_2 A D_1^{-1} - D_4 B D_3^{-1}\|, \quad \text{where } D_j = (D\varphi_{i(x_j)})_{x_j}.$$

For $x \in M$ and $r > 0$ small (relative to r_0), $B_r(x)$ will denote the ball of radius r around x relative to the new metric. In other words, $B_r(x) = \varphi_{i(x)}^{-1}\big(B(\varphi_{i(x)}(x), r)\big)$. We assume that r is small enough so that the closure of $B_r(x)$ is contained in $V_{i(x)}^*$.

Definition 2.9. Let $\varepsilon_0 > 0$. The ε_0-*basic neighborhood* $\mathcal{U}(\mathrm{id}, \varepsilon_0)$ of the identity in $\mathrm{Diff}_\mu^1(M)$, or in $\mathrm{Sympl}_\omega^1(M)$, is the set $\mathcal{U}(\mathrm{id}, \varepsilon_0)$ of all $h \in \mathrm{Diff}_\mu^1(M)$, or $h \in \mathrm{Sympl}_\omega^1(M)$, such that $h^{\pm 1}(\overline{V}_i) \subset V_i^*$ for each i and

$$h(x) \in B_{\varepsilon_0}(x) \quad \text{and} \quad \|Dh_x - I\| < \varepsilon_0 \quad \text{for every } x \in M.$$

For a general $f \in \mathrm{Diff}_\mu^1(M)$, or $f \in \mathrm{Sympl}_\omega^1(M)$, the ε_0-basic neighborhood $\mathcal{U}(f, \varepsilon_0)$ is defined by: $g \in \mathcal{U}(f, \varepsilon_0)$ if and only if $f^{-1} \circ g \in \mathcal{U}(\mathrm{id}, \varepsilon_0)$ or $g \circ f^{-1} \in \mathcal{U}(\mathrm{id}, \varepsilon_0)$.

2.6. *Realizable sequences.* The following notion, introduced in [4], is crucial to the proofs of Theorems 1 through 4. It captures the idea of sequence of linear transformations that can be (almost) realized *on subsets with large relative measure* as tangent maps of diffeomorphisms close to the original one.

Definition 2.10. Given $f \in \mathrm{Diff}_\mu^1(M)$ or $f \in \mathrm{Sympl}_\omega^1(M)$, constants $\varepsilon_0 > 0$, and $0 < \kappa < 1$, and a nonperiodic point $x \in M$, we call a sequence of linear maps (volume-preserving or symplectic)

$$T_xM \xrightarrow{L_0} T_{fx}M \xrightarrow{L_1} \dots \xrightarrow{L_{n-1}} T_{f^n x}M$$

an (ε_0, κ)-*realizable sequence of length* n *at* x if the following holds:

For every $\gamma > 0$ there is $r > 0$ such that the iterates $f^j(\overline{B}_r(x))$ are two-by-two disjoint for $0 \le j \le n$, and given any nonempty open set $U \subset B_r(x)$, there are $g \in \mathcal{U}(f, \varepsilon_0)$ and a measurable set $K \subset U$ such that

(i) g equals f outside the disjoint union $\bigsqcup_{j=0}^{n-1} f^j(\overline{U})$;

(ii) $\mu(K) > (1 - \kappa)\mu(U)$;

(iii) if $y \in K$ then $\big\|Dg_{g^j y} - L_j\big\| < \gamma$ for every $0 \le j \le n-1$.

Some basic properties of realizable sequences are collected in the following:

LEMMA 2.11. *Let $f \in \mathrm{Diff}_\mu^1(M)$ or $f \in \mathrm{Sympl}_\omega^1(M)$, $x \in M$ not periodic and $n \in \mathbb{N}$.*

(1) *The sequence $\{Df_x, \dots, Df_{f^{n-1}(x)}\}$ is (ε_0, κ)-realizable for every ε_0 and κ (called a* trivial *realizable sequence).*

(2) *Let* $\kappa_1,\ \kappa_2 \in (0,1)$ *be such that* $\kappa = \kappa_1 + \kappa_2 < 1$. *If* $\{L_0, \dots, L_{n-1}\}$ *is* $(\varepsilon_0, \kappa_1)$*-realizable at* x, *and* $\{L_n, \dots, L_{n+m-1}\}$ *is* $(\varepsilon_0, \kappa_2)$*-realizable at* $f^n(x)$, *then* $\{L_0, \dots, L_{n+m-1}\}$ *is* (ε_0, κ)*-realizable at* x.

(3) *If* $\{L_0, \dots, L_{n-1}\}$ *is* (ε_0, κ)*-realizable at* x, *then* $\{L_{n-1}^{-1}, \dots, L_0^{-1}\}$ *is an* (ε_0, κ)*-realizable sequence at* $f^n(x)$ *for the diffeomorphism* f^{-1}.

Proof. The first claim is obvious. For the second one, fix $\gamma > 0$. Let r_1 be the radius associated to the $(\varepsilon_0, \kappa_1)$-realizable sequence, and r_2 be the radius associated to the $(\varepsilon_0, \kappa_2)$-realizable sequence. Fix $0 < r < r_1$ such that $f^n(B_r(x)) \subset B(f^n(x), r_2)$. Then the $f^j(\overline{B}_r(x))$ are two-by-two disjoint for $0 \le j \le n+m$. Given an open set $U \subset B_r(x)$, the realizability of the first sequence gives us a diffeomorphism $g_1 \in \mathcal{U}(f, \varepsilon_0)$ and a measurable set $K_1 \subset U$. Analogously, for the open set $f^n(U) \subset B(f^n(x), r_2)$ we find $g_2 \in \mathcal{U}(f, \varepsilon_0)$ and a measurable set $K_2 \subset f^n(U)$. Then define a diffeomorphism g as $g = g_1$ inside $U \cup \cdots \cup f^{n-1}(U)$ and $g = g_2$ inside $f^n(U) \cup \cdots \cup f^{n+m-1}(U)$, with $g = f$ elsewhere. Consider also $K = K_1 \cap g^{-n}(K_2)$. Using the fact that g preserves volume, one checks that g and K satisfy the conditions in Definition 2.10. For claim 3, notice that $\mathcal{U}(f, \varepsilon_0) = \mathcal{U}(f^{-1}, \varepsilon_0)$. $\square$

The next lemma makes it simpler to verify that a sequence is realizable: we only have to check the conditions for certain open sets $U \subset B_r(x)$.

Definition 2.12. A family of open sets $\{W_\alpha\}$ in $\mathbb{R}^d$ is a *Vitali covering* of $W = \cup_\alpha W_\alpha$ if there is $C > 1$ and for every $y \in W$, there are sequences of sets $W_{\alpha_n} \ni y$ and positive numbers $s_n \to 0$ such that

$$B_{s_n}(y) \subset W_{\alpha_n} \subset B_{Cs_n}(y) \quad \text{for all } n \in \mathbb{N}.$$

A family of subsets $\{U_\alpha\}$ of M is a *Vitali covering* of $U = \cup_\alpha U_\alpha$ if each U_α is contained in the domain of some chart $\varphi_{i(\alpha)}$ in the atlas $\mathcal{A}$, and the images $\{\varphi_{i(\alpha)}(U_\alpha)\}$ form a Vitali covering of $W = \varphi(U)$, in the previous sense.

LEMMA 2.13. *Let* $f \in \mathrm{Diff}^1_\mu(M)$ *or* $f \in \mathrm{Sympl}^1_\omega(M)$, *and set* $\varepsilon_0 > 0$ *and* $\kappa > 0$. *Consider any sequence* $L_j : T_{f^j(x)}M \to T_{f^{j+1}(x)}M$, $0 \le j \le n-1$ *of linear maps at a nonperiodic point* x, *and let* $\varphi : V \to \mathbb{R}^d$ *be a chart in the atlas* $\mathcal{A}$, *with* $V \ni x$. *Assume the conditions in Definition* 2.10 *are valid for every element of some Vitali covering* $\{U_\alpha\}$ *of* $B_r(x)$. *Then the sequence* L_j *is* (ε_0, κ)*-realizable.*

Proof. Let U be an arbitrary open subset of $B_r(x)$. By Vitali's covering lemma (see [21]), there is a countable family of two-by-two disjoint sets U_α covering U up to a zero Lebesgue measure subset. Thus we can find a finite family of U_α with disjoint closures such that $\mu(U - \bigsqcup_\alpha U_\alpha)$ is as small as we please. For each U_α there are, by hypothesis, a perturbation $g_\alpha \in \mathcal{U}(f, \varepsilon_0)$ and a measurable set $K_\alpha \subset U_\alpha$ with the properties (i)–(iii) of Definition 2.10. Let $K = \bigcup K_\alpha$ and define g as being equal to g_α on each $f^j(U_\alpha)$ with $0 \le j \le n-1$. Then $g \in \mathcal{U}(f, \varepsilon_0)$ and the pair (g, K) have the properties required by Definition 2.10. ☐

3. Geometric consequences of nondominance

The aim of this section is to prove the following key result, from which we shall deduce Theorem 2 in Section 4:

Proposition 3.1. *When $f \in \mathrm{Diff}^1_\mu(M)$, $\varepsilon_0 > 0$ and $0 < \kappa < 1$, if $m \in \mathbb{N}$ is sufficiently large then the following holds: Let $y \in M$ be a nonperiodic point and assume that there is a nontrivial splitting $T_yM = E \oplus F$ such that*

$$\frac{\|Df^m_y|_F\|}{\mathbf{m}(Df^m_y|_E)} \ge \frac{1}{2}\,.$$

Then there exists an (ε_0, κ)-realizable sequence $\{L_0, \dots, L_{m-1}\}$ at y of length m and there are nonzero vectors $v \in E$ and $w \in Df^m_y(F)$ such that

$$L_{m-1} \dots L_0(v) = w.$$

3.1. *Nested rotations.* Here we present some tools for the construction of realizable sequences. The first one yields sequences of length 1:

Lemma 3.2. *Given $f \in \mathrm{Diff}^1_\mu(M)$, $\varepsilon_0 > 0$, $\kappa > 0$, there exists $\varepsilon > 0$ with the following properties:*

Suppose there are a nonperiodic point $x \in M$, a splitting $T_xM = X \oplus Y$ with $X \perp Y$ and $\dim Y = 2$, and an elliptic linear map $\widehat{R} : Y \to Y$ with $\|\widehat{R} - I\| < \varepsilon$. Consider the linear map $R : T_xM \to T_xM$ given by $R(u+v) = u + \widehat{R}(v)$, for $u \in X$, $v \in Y$. Then $\{Df_x R\}$ is an (ε_0, κ)-realizable sequence of length 1 at x and $\{R\, Df_{f^{-1}(x)}\}$ is an (ε_0, κ)-realizable sequence of length 1 at the point $f^{-1}(x)$.

We call a linear isomorphism of a 2-dimensional space *elliptic* if its eigenvalues are not real; this means the map is a rotation, relative to some basis of the space.

We also need to construct long realizable sequences. Part 2 of Lemma 2.11 provides a way to do this, by concatenation of shorter sequences. However, simple concatenation is far too crude for our purposes because it worsens κ; the relative measure of the set where the sequence can be (almost) realized decreases when the sequence increases. This problem is overcome by Lemma 3.3 below, which allows us to obtain certain nontrivial realizable sequences with arbitrary length while keeping κ controlled.

In short terms, we do concatenate several length 1 sequences, of the type given by Lemma 3.2, but we also require that the supports of successive perturbations be mapped one to the other. More precisely, there is a domain $\mathcal{C}_0 \subset T_xM$ invariant under the sequence, in the sense that $L_{j-1} \dots L_0(\mathcal{C}_0) = Df_x^j(\mathcal{C}_0)$ for all j. Following [4], where a similar notion was introduced for the 2-dimensional setting, we call such L_j *nested rotations*. When $d > 2$ the domain $\mathcal{C}_0$ is not compact; indeed it is the product $\mathcal{C}_0 = X_0 \oplus \mathcal{B}_0$ of a codimension 2 subspace X_0 by an ellipse $\mathcal{B}_0 \subset X_0^\perp$.

Let us fix some terminology to be used in the sequel. If E is a vector space with an inner product and F is a subspace of E, we endow the quotient space E/F with the inner product that makes $v \in F^\perp \mapsto (v+F) \in E/F$ an isometry. If E' is another vector space, any linear map $L : E \to E'$ induces a linear map $L/F : E/F \to E'/F'$, where $F' = L(F)$. If E' has an inner product, then we indicate by $\|L/F\|$ the usual operator norm.

Lemma 3.3. *When $f \in \mathrm{Diff}^1_\mu(M)$, $\varepsilon_0 > 0$, $\kappa > 0$, there exists $\varepsilon > 0$ with the following properties: Suppose there are a nonperiodic point $x \in M$, an integer $n \geq 1$, and, for $j = 0, 1, \dots, n-1$,*

- *codimension 2 spaces $X_j \subset T_{f^j(x)}M$ such that $X_j = Df_x^j(X_0)$;*
- *ellipses $\mathcal{B}_j \subset (T_{f^j(x)}M)/X_j$ centered at zero with $\mathcal{B}_j = (Df_x^j/X_0)(\mathcal{B}_0)$;*
- *linear maps $\widehat{R}_j : (T_{f^j(x)}M)/X_j \to (T_{f^j(x)}M)/X_j$ such that $\widehat{R}_j(\mathcal{B}_j) \subset \mathcal{B}_j$ and $\|\widehat{R}_j - I\| < \varepsilon$.*

Consider the linear maps $R_j : T_{f^j(x)}M \to T_{f^j(x)}M$ such that R_j restricted to X_j is the identity, $R_j(X_j^\perp) = X_j^\perp$ and $R_j/X_j = \widehat{R}_j$. Define

$$L_j = Df_{f^j(x)}R_j : T_{f^j(x)}M \to T_{f^{j+1}(x)}M \quad \text{for } 0 \leq j \leq n-1.$$

Then $\{L_0, \dots, L_{n-1}\}$ is an (ε_0, κ)-realizable sequence of length n at x.

We shall prove Lemma 3.3 in Section 3.1.2. Notice that Lemma 3.2 is contained in Lemma 3.3: take $n = 1$ and use also part 3 of Lemma 2.11. Actually, Lemma 3.2 also follows from the forthcoming Lemma 3.4.

3.1.1. *Cylinders and rotations.* We call a *cylinder* any affine image $\mathcal{C}$ in $\mathbb{R}^d$ of a product $B^{d-i} \times B^i$, where B^j denotes a ball in $\mathbb{R}^j$. If ψ is the affine map, the *axis* $\mathcal{A} = \psi(B^{d-i} \times \{0\})$ and the *base* $\mathcal{B} = \psi(\{0\} \times B^i)$ are ellipsoids. We also write $\mathcal{C} = \mathcal{A} \oplus \mathcal{B}$. The cylinder is called *right* if $\mathcal{A}$ and $\mathcal{B}$ are perpendicular. The case we are most interested in is when $i = 2$.

The present section contains three preliminary lemmas that we use in the proof of Lemma 3.3. The first one explains how to rotate a right cylinder, while keeping the complement fixed. The assumption $a > \tau b$ means that the cylinder $\mathcal{C}$ is thin enough, and it is necessary for the C^1 estimate in part (ii) of the conclusion.

LEMMA 3.4. *Given* $\varepsilon_0 > 0$ *and* $0 < \sigma < 1$, *there is* $\varepsilon > 0$ *with the following properties*: *Suppose there are a splitting* $\mathbb{R}^d = X \oplus Y$ *with* $X \perp Y$ *and* $\dim Y = 2$, *a right cylinder* $\mathcal{A} \oplus \mathcal{B}$ *centered at the origin with* $\mathcal{A} \subset X$ *and* $\mathcal{B} \subset Y$, *and a linear map* $\widehat{R} : Y \to Y$ *such that* $\widehat{R}(\mathcal{B}) = \mathcal{B}$ *and* $\|\widehat{R} - I\| < \varepsilon$. *Then there exists* $\tau > 1$ *such that the following holds*:

Let $R : \mathbb{R}^d \to \mathbb{R}^d$ *be the linear map defined by* $R(u + v) = u + \widehat{R}v$, *for* $u \in X$, $v \in Y$. *For* $a, b > 0$ *consider the cylinder* $\mathcal{C} = a\mathcal{A} \oplus b\mathcal{B}$. *If* $a > \tau b$ *and* $\operatorname{diam} \mathcal{C} < \varepsilon_0$ *then there is a* C^1 *volume-preserving diffeomorphism* $h : \mathbb{R}^d \to \mathbb{R}^d$ *satisfying*

(i) $h(z) = z$ *for every* $z \notin \mathcal{C}$ *and* $h(z) = R(z)$ *for every* $z \in \sigma\mathcal{C}$;

(ii) $\|h(z) - z\| < \varepsilon_0$ *and* $\|Dh_z - I\| < \varepsilon_0$ *for all* $z \in \mathbb{R}^d$.

Proof. We choose $\varepsilon > 0$ small enough so that

$$\frac{18\varepsilon}{1 - \sigma} < \varepsilon_0. \tag{3.1}$$

Let $\mathcal{A}$, $\mathcal{B}$, X, Y, $\widehat{R}$, R be as in the statement of the lemma. Let $\{e_1, \ldots, e_d\}$ be an orthonormal basis of $\mathbb{R}^d$ such that $e_1, e_2 \in Y$ are in the directions of the axes of the ellipse $\mathcal{B}$ and $e_j \in X$ for $j = 3, \ldots, d$. We shall identify vectors $v = xe_1 + ye_2 \in Y$ with the coordinates (x, y). Then there are constants $\lambda \geq 1$ and $\rho > 0$ such that $\mathcal{B} = \{(x, y);\ \lambda^{-2}x^2 + \lambda^2 y^2 \leq \rho^2\}$. Relative to the basis $\{e_1, e_2\}$, let

$$H_\lambda = \begin{pmatrix} \lambda & 0 \\ 0 & \lambda^{-1} \end{pmatrix} \quad \text{and} \quad R_\alpha = \begin{pmatrix} \cos\alpha & -\sin\alpha \\ \sin\alpha & \cos\alpha \end{pmatrix}.$$

The assumption $\widehat{R}(\mathcal{B}) = \mathcal{B}$ implies that $\widehat{R} = H_\lambda R_\alpha H_\lambda^{-1}$ for some α. Besides, the condition $\|\widehat{R} - I\| < \varepsilon$ implies

$$\lambda^2 |\sin\alpha| \leq \|(\widehat{R} - I)(0, 1)\| < \varepsilon. \tag{3.2}$$

Let $\varphi : \mathbb{R} \to \mathbb{R}$ be a C^∞ function such that $\varphi(t) = 1$ for $t \leq \sigma$, $\varphi(t) = 0$ for $t \geq 1$, and $0 \leq -\varphi'(t) \leq 2/(1 - \sigma)$ for all t. Define smooth maps $\psi : Y \to \mathbb{R}$ and $\tilde{g}_t : Y \to Y$ by

$$\psi(x, y) = \alpha\varphi(\sqrt{x^2 + y^2}) \quad \text{and} \quad \tilde{g}_t(x, y) = R_{\varphi(t)\psi(x,y)}(x, y).$$

On the one hand, $\tilde{g}_t(x, y) = (x, y)$ if either $t \geq 1$ or $x^2 + y^2 \geq 1$. On the other hand, $\tilde{g}_t(x, y) = R_\alpha(x, y)$ if $t \leq \sigma$ and $x^2 + y^2 \geq \sigma^2$. We are going to check that the derivative of $\tilde{g}_t$ is close to the identity if ε is close to zero; note that

$|\sin\alpha|$ is also close to zero, by (3.2). We have

$$D(\tilde{g}_t)_{(x,y)} = \begin{pmatrix}\cos(t\psi) & -\sin(t\psi)\\ \sin(t\psi) & \cos(t\psi)\end{pmatrix} + \begin{pmatrix}-x\sin(t\psi) - y\cos(t\psi)\\ x\cos(t\psi) - y\sin(t\psi)\end{pmatrix} \cdot \begin{pmatrix}t\partial_x\psi & t\partial_y\psi\end{pmatrix}$$
$$= R_{t\psi(x,y)} + t\left[R_{\pi/2+t\psi(x,y)}(x,y)\right] \cdot D\psi_{(x,y)}.$$

Consider $0 \le t \le 1$ and $x^2 + y^2 \le 1$. Then

$$\begin{aligned}\|D(\tilde{g}_t)_{(x,y)} - I\| &= \|R_{t\psi(x,y)} - I\| + \|R_{\pi/2+t\psi(x,y)}(x,y)\| \cdot \|D\psi_{(x,y)}\|\\ &\le \left|\sin\left(t\psi(x,y)\right)\right| + \left\|\left(2\alpha x\varphi'(x^2+y^2)\ ,\ 2\alpha y\varphi'(x^2+y^2)\right)\right\|.\end{aligned}$$

Taking ε small enough, we may suppose that $\alpha \le 2|\sin\alpha|$. In view of the choice of φ and ψ, this implies

$$\|D(\tilde{g}_t)_{(x,y)} - I\| \le |\sin\alpha| + 4|\alpha|/(1-\sigma) \le 9|\sin\alpha|/(1-\sigma). \tag{3.3}$$

We also need to estimate the derivative with respect to t:

$$\|\partial_t\tilde{g}(x,y)\| \le \left\|\varphi'(t)\psi(x,y)R_{\pi/2+t\psi(x,y)}(x,y)\right\| \le 4|\sin\alpha|/(1-\sigma). \tag{3.4}$$

Now define $g_t : Y \to Y$ by $g_t = H_\lambda \circ \tilde{g}_t \circ H_\lambda^{-1}$. Each g_t is an area-preserving diffeomorphism equal to the identity outside $\mathcal{B}$. Thus

$$\|g_t(x,y) - (x,y)\| < \operatorname{diam}\mathcal{B}, \tag{3.5}$$

for every $(x,y) \in \mathcal{B}$. Moreover, $g_t = \widehat{R} = H_\lambda R_\alpha H_\lambda^{-1}$ on $\sigma\mathcal{B}$ for all $t \le \sigma$. By (3.3),

$$\|D(g_t)_{(x,y)} - I\| = \left\|H_\lambda\big(D(\tilde{g}_t)_{(\lambda^{-1}x,\lambda y)} - I\big)H_\lambda^{-1}\right\| \le \lambda^2\left(\frac{9|\sin\alpha|}{1-\sigma}\right),$$

and, applying (3.2) and (3.1), we deduce that

$$\|D(g_t)_{(x,y)} - I\| < \frac{9\varepsilon}{1-\sigma} < \frac{\varepsilon_0}{2} \tag{3.6}$$

for all $(x,y) \in \mathcal{B}$. Similarly, by (3.4),

$$\|\partial_t g_t(x,y)\| \le \lambda^2\|\partial_t\tilde{g}_t(\lambda^{-1}x,\lambda y)\| \le \lambda^2\left(\frac{4|\sin\alpha|}{1-\sigma}\right) < \frac{\varepsilon_0}{2}\,. \tag{3.7}$$

Now let $Q : X \to \mathbb{R}$ be a quadratic form such that $\mathcal{A} = \{u \in X; Q(u) \le 1\}$, and let $q : \mathbb{R}^d \to X$ and $p : \mathbb{R}^d \to Y$ be the orthogonal projections. Given $a, b > 0$, define $h : \mathbb{R}^d \to \mathbb{R}^d$ by

$$h(z) = z' + b g_{a^{-2}Q(z')}(b^{-1}z''), \quad \text{where } z' = q(z) \text{ and } z'' = p(z).$$

It is clear that h is a volume-preserving diffeomorphism. The subscript $t = a^{-2}Q(z')$ is designed so that $t \le 1$ if and only if $z' \in a\mathcal{A}$. Then $h(z) = z$ if either $z' \notin a\mathcal{A}$ or $z'' \notin b\mathcal{B}$. Moreover, $h(z) = z' + \widehat{R}(z'') = R(z)$ if $z' \in \sigma a\mathcal{A}$

and $z'' \in \sigma b\mathcal{B}$. This proves property (i) in the statement. The hypothesis $\operatorname{diam}\mathcal{C} < \varepsilon_0$ and (3.5) give

$$\begin{aligned}\|h(z) - z\| &= b\|g_{a^{-2}Q(z')}(b^{-1}z'') - b^{-1}z''\| \\ &< b \operatorname{diam}\mathcal{B} \le \operatorname{diam}(a\mathcal{A} \oplus b\mathcal{B}) < \varepsilon_0\end{aligned}$$

which is the first half of (ii). Finally, fix $\tau > 1$ such that $\|DQ_u\| \le \tau\|u\|$ for all $u \in \mathbb{R}^d$, and assume that $a > \tau b$. Clearly,

$$Dh = q + \frac{b}{a^2}(\partial_t g)(DQ)q + (Dg)p.$$

By (3.6), (3.7), and the fact that $\|q\| = \|p\| = 1$ (these are orthogonal projections),

$$\begin{aligned}\|Dh - I\| &\le \|\frac{b}{a^2}(\partial_t g)(DQ)q\| + \|(Dg - I)p\| \\ &\le \frac{b}{a^2}\,\|\partial_t g\|\,\tau a\,\|q\| + \|Dg - I\|\,\|p\| < \varepsilon_0\,.\end{aligned}$$

This completes the proof of property (ii) and the lemma. □

The second of our auxiliary lemmas says that the image of a small cylinder by a C^1 diffeomorphism h contains the image by Dh of a slightly shrunk cylinder. Denote $\mathcal{C}(y, \rho) = \rho\mathcal{C} + y$, for each $y \in \mathbb{R}^d$ and $\rho > 0$.

LEMMA 3.5. *Let $h : \mathbb{R}^d \to \mathbb{R}^d$ be a C^1 diffeomorphism with $h(0) = 0$, $\mathcal{C} \subset \mathbb{R}^d$ be a cylinder centered at* 0, *and $0 < \lambda < 1$. Then there exists $r > 0$ such that for any $\mathcal{C}(y, \rho) \subset B_r(0)$,*

$$h(\mathcal{C}(y, \rho)) \supset Dh_0(\mathcal{C}(0, \lambda\rho)) + h(y).$$

Proof. Fix a norm $\|\cdot\|_0$ in $\mathbb{R}^d$ for which $\mathcal{C} = \{z \in \mathbb{R}^d;\ \|z\|_0 < 1\}$. Such a norm exists because $\mathcal{C}$ is convex and $\mathcal{C} = -\mathcal{C}$. Let $H = Dh_0$ and $g : \mathbb{R}^d \to \mathbb{R}^d$ be such that $h = H \circ g$. Since g is C^1 and $Dg_0 = I$, we have

$$g(z) - g(y) = z - y + \xi(z, y) \quad \text{with} \quad \lim_{(z,y)\to(0,0)} \frac{\xi(z, y)}{\|z - y\|_0} = 0.$$

Choose $r > 0$ such that $\|z\|, \|y\| \le r \Rightarrow \|\xi(z, y)\|_0 < (1 - \lambda)\|z - y\|_0$ (where $\|\cdot\|$ denotes the Euclidean norm in $\mathbb{R}^d$). Now suppose $\mathcal{C}(y, \rho) \subset B_r(0)$, and let $z \in \partial\mathcal{C}(y, \rho)$. Then $\|z - y\|_0 = \rho$ and

$$\|g(z) - g(y)\|_0 \ge \|z - y\|_0 - \|\xi(z, y)\|_0 > \lambda\rho.$$

This proves that the sets $g(\partial\mathcal{C}(y, \rho)) - g(y)$ and $\lambda\mathcal{C}$ are disjoint. Applying the linear map H, we find that $h(\partial\mathcal{C}(y, \rho)) - h(y)$ and $\lambda H\mathcal{C}$ are disjoint. From topological arguments, $h(\mathcal{C}(y, \rho)) - h(y) \supset \lambda H\mathcal{C}$. □

The third lemma says that a linear image of a sufficiently thin cylinder contains a right cylinder with almost the same volume. The idea is shown in Figure 1. The proof of the lemma is left to the reader.

LEMMA 3.6. *Let* $\mathcal{A}\oplus\mathcal{B}$ *be a cylinder centered at the origin,* $L:\mathbb{R}^d\to\mathbb{R}^d$ *be a linear isomorphism,* $\mathcal{A}_1 = L(\mathcal{A})$ *and* $\mathcal{B}_1 = p(L(\mathcal{B}))$, *where* p *is the orthogonal projection onto the orthogonal complement of* $\mathcal{A}_1$. *Then, given any* $0<\lambda<1$, *there exists* $\tau>1$ *such that if* $a>\tau b$,

$$L\big(a\mathcal{A}\oplus b\mathcal{B}\big)\supset \lambda a\mathcal{A}_1\oplus b\mathcal{B}_1.$$

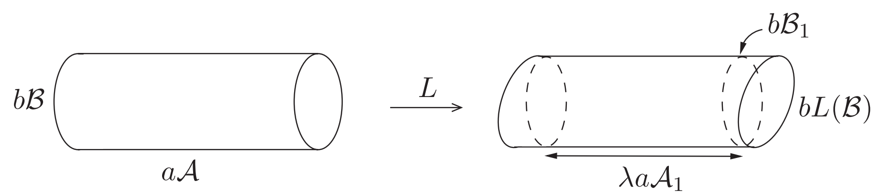

Figure 1: Truncating a thin cylinder to make it right

3.1.2. *Proof of the nested rotations Lemma* 3.3. Let f, ε_0, and κ be given. Define $\sigma=(1-\kappa)^{1/2d}$ and then take $\varepsilon>0$ as given by Lemma 3.4. Now let x, n, X_j, $\mathcal{B}_j$, $\widehat{R}_j$, R_j, L_j be as in the statement. We want to prove that $\{L_0,\dots,L_n\}$ is an (ε_0,κ)-realizable sequence of length n at x; cf. Definition 2.10.

In short terms, we use Lemma 3.4 to construct the realization g at each iterate. The subset $U\smallsetminus K$, where we have no control over the approximation, has two sources: Lemma 3.4 gives $h=R$ only on a slightly smaller cylinder $\sigma\mathcal{C}$; and we need to straighten out (Lemma 3.5) and to "rightify" (Lemma 3.6) our cylinders at each stage. These effects are made small by consideration of cylinders that are small and very thin. That is how we get $U\smallsetminus K$ with relative volume less than κ, independently of n.

For clearness we split the proof into three main steps:

Step 1. Fix any $\gamma>0$. We explain how to find $r>0$ as in Definition 2.10. We consider local charts $\varphi_j: V_j\to\mathbb{R}^d$ with $\varphi_j=\varphi_{i(f^jx)}$ and $V_j=V_{i(f^jx)}$, as introduced in Section 2.5. Let $r'>0$ be small enough so that

- $f^j(\overline{B}_{r'}(x))\subset V_j^*$ for every $j=0,1\dots,n$;
- the sets $f^j(\overline{B}_{r'}(x))$ are two-by-two disjoint;
- $\|Df_z-Df_{f^j(x)}\|\,\|R_j\|<\gamma$ for every $z\in f^j(B_{r'}(x))$ and $j=0,1\dots,n$.

We use local charts to translate the situation to $\mathbb{R}^d$. Let $f_j = \varphi_{j+1} \circ f \circ \varphi_j^{-1}$ be the expression of f in local coordinates near $f^j(x)$ and $f^{j+1}(x)$. To simplify the notation, we suppose that each φ_j has been composed with a translation to ensure $\varphi_j(f^j(x)) = 0$ for all j. Up to identification of tangent spaces via the charts φ_j and φ_{j+1}, we have $L_j = (Df_j)_0 R_j$.

Let $\mathcal{A}_0 \subset X_0$ be any ellipsoid centered at the origin (a ball, for example), and let $\mathcal{A}_j = Df^j_x(\mathcal{A}_0)$ for $j \geq 1$. We identify $(T_{f^j(x)}M)/X_j$ with $X_j^\perp$, so that we may consider $\mathcal{B}_j \subset X_j^\perp$. In these terms, the assumption $\mathcal{B}_j = (Df^j_x/X_0)(\mathcal{B}_0)$ means that $\mathcal{B}_j$ is the orthogonal projection of $Df^j_x(\mathcal{B}_0)$ onto $X_j^\perp$.

Fix $0 < \lambda < 1$ close enough to 1 so that $\lambda^{4n(d-1)} > 1 - \kappa$. Let $\tau_j > 1$ be associated to the data $(\mathcal{A}_j \oplus \mathcal{B}_j, (Df_j)_0, \lambda)$ by Lemma 3.6; if $a > \tau_j b$ then

$$(Df_j)_0(a\mathcal{A}_j \oplus b\mathcal{B}_j) \supset \lambda a \mathcal{A}_{j+1} \oplus b\mathcal{B}_{j+1}. \tag{3.8}$$

Let $\tau_j' > 1$ be associated to the data $(\varepsilon_0, \sigma, X_j \oplus X_j^\perp, \mathcal{A}_j \oplus \mathcal{B}_j, \widehat{R}_j)$ by Lemma 3.4. Fix $a_0 > 0$ and $b_0 > 0$ such that

$$a_0 > b_0 \lambda^{-n} \max\{\tau_j, \tau_j';\ 0 \leq j \leq n-1\}. \tag{3.9}$$

For $0 \leq j \leq n$, define $\mathcal{C}_j = \lambda^{2j} a_0 \mathcal{A}_j \oplus \lambda^j b_0 \mathcal{B}_j$. For $z \in \mathbb{R}^d$ and $\rho > 0$, denote $\mathcal{C}_j(z,\rho) = \rho \mathcal{C}_j + z$. Applying Lemma 3.5 to the data $(f_j, \mathcal{C}_j, \lambda)$ we get $r_j > 0$ such that

$$\mathcal{C}(z,\rho) \subset B_{r_j}(0) \quad \Rightarrow \quad f_j(\mathcal{C}_j(z,\rho)) \supset (Df_j)_0(\mathcal{C}_j(0,\lambda\rho)) + f_j(z). \tag{3.10}$$

Now take $r > 0$ such that $r < r'$ and, for each $j = 1, \ldots, m-1$,

$$f_{j-1} \ldots f_0(B_r(0)) \subset B_{r_j}(0). \tag{3.11}$$

Step 2. Let U be fixed. We find $g \in \mathcal{U}(f, \varepsilon_0)$ and $K \subset U$ as in Definition 2.10. We take advantage of Lemma 2.13: it suffices to consider open sets of the form $U = \varphi_0^{-1}(\mathcal{C}_0(y_0,\rho))$, because the cylinders $\mathcal{C}_0(y_0,\rho)$ contained in $B_r(0)$ constitute a Vitali covering.

We claim that, for each $j = 0, 1, \ldots, m-1$, and every $t \in [0,\rho]$,

$$\mathcal{C}_j(y_j, t) \subset f_{j-1} \ldots f_0(B_r(0)) \tag{3.12}$$

and

$$f_j(\mathcal{C}_j(y_j,t)) \supset \mathcal{C}_{j+1}(y_{j+1}, t). \tag{3.13}$$

For $j = 0$, relation (3.12) means $\mathcal{C}_0(y_0, t) \subset B_r(0)$, which is true by assumption. We proceed by induction. Assume (3.12) holds for some $j \geq 0$. Then, by (3.11) and (3.10),

$$\begin{aligned} f_j(\mathcal{C}_j(y_j,t)) &\supset (Df_j)_0(\mathcal{C}_j(0,\lambda t)) + y_{j+1} \\ &= (Df_0)_0 \left[(\lambda^{2j+1} t a_0 \mathcal{A}_j) \oplus (\lambda^{j+1} t b_0 \mathcal{B}_j) \right] + y_{j+1}. \end{aligned}$$

Relation (3.9) implies that $\lambda^{2j+1}ta_0 > \tau_j(\lambda^{j+1}tb_0)$. So, we may use (3.8) to conclude that

$$f_j(\mathcal{C}_j(y_j,t)) \supset (\lambda^{2j+2}ta_0\mathcal{A}_j) \oplus (\lambda^{j+1}tb_0\mathcal{B}_0) + y_{j+1} = \mathcal{C}_{j+1}(y_{j+1},t).$$

This proves that (3.13) holds for the same value of j. Moreover, it is clear that if (3.13) holds for all $0 \le i \le j$ then (3.12) is true with $j+1$ in the place of j. This completes the proof of (3.12) and (3.13).

Condition (3.9) also implies $\lambda^{2j}a_0 > \tau_j'(\lambda^j b_0)$. So, we may use Lemma 3.4 (centered at y_j) to find a volume-preserving diffeomorphism $h_j : \mathbb{R}^d \to \mathbb{R}^d$ such that

(1) $h_j(z) = z$ for all $z \notin \mathcal{C}_j(y_j,\rho)$ and $h_j(z) = y_j + R_j(z-y_j)$ for all $z \in \mathcal{C}_j(y_j,\sigma\rho)$ and, consequently,

$$h_j(\mathcal{C}_j(y_j,\sigma\rho)) = \mathcal{C}_j(y_j,\sigma\rho) \quad \text{and} \quad h_j(\mathcal{C}_j(y_j,\rho)) = \mathcal{C}_j(y_j,\rho). \tag{3.14}$$

(1) $\|h_j(z)-z\| < \varepsilon_0$ and $\|(Dh_j)_z - I\| < \varepsilon_0$ for all $z \in \mathbb{R}^d$.

R_j is the linear map $T_{f^j(x)} \to T_{f^{j+1}(x)}$ in the statement of the theorem or, more precisely, its expression in local coordinates φ_j. Let

$$S_j = \varphi_j^{-1}(\{z;\ h(z) \ne z\}) \subset M.$$

By Property 1 above and the inclusion (3.12),

$$S_j \subset \varphi_j^{-1}(f_{j-1}\dots f_0(B_r(0))) = f^j(B_r(x)).$$

In particular, the sets S_j have pairwise disjoint closures. This permits us to define a diffeomorphism $g \in \mathrm{Diff}^1_\mu(M)$ by

$$g = \begin{cases} \varphi_{j+1}^{-1} \circ (f_j \circ h_j) \circ \varphi_j & \text{on } S_j \text{ for each } 0 \le j \le n-1 \\ f & \text{outside } S_0 \sqcup \cdots \sqcup S_{n-1}. \end{cases}$$

Property 2 above gives that $f^{-1} \circ g \in \mathcal{U}(\mathrm{id},\varepsilon_0)$, and so $g \in \mathcal{U}(f,\varepsilon_0)$.

Step 3. Now we define $K \subset U$ and check the conditions (i)–(iii) in Definition 2.10. By construction, $h_j = \mathrm{id}$ outside $C_j(y_j,\rho)$, and so

$$\varphi_{j+1}^{-1} \circ (f_j \circ h_j) \circ \varphi_j = f \quad \text{outside } \varphi^{-j}(C_j(y_j,\rho)).$$

Using (3.13) and (3.14), we have $\varphi^{-j}(C_j(y_j,\rho)) \subset f^j(U)$ for all $0 \le j \le n-1$. Recall that $U = \varphi_0^{-1}(\mathcal{C}_0(y_0,\rho))$. Hence, $g = j$ outside the disjoint union $\sqcup_{j=0}^{n-1} f^j(U)$. This proves condition (i).

Define $K = g^{-n}(\varphi_n^{-1}(\mathcal{C}_n(y_n,\sigma\rho)))$. Using (3.13) and (3.14) in the same way as before, we see that $K \subset U$. Also, since all the maps f, g, h_j, φ_j are volume-preserving, and all the cylinders $\mathcal{C}_j(y_j,\rho)$, $\mathcal{C}_j(y_j,\sigma\rho)$, are right,

$$\frac{\mathrm{vol}\,K}{\mathrm{vol}\,U} = \frac{\mathrm{vol}(\sigma\rho\,\lambda^{2n}a\mathcal{A}_n \oplus \sigma\rho\,\lambda^n b\mathcal{B}_n)}{\mathrm{vol}(\rho a\mathcal{A}_0 \oplus \rho b\mathcal{B}_0)} = \frac{(\lambda^{2n}\sigma)^{d-2}\,\mathrm{vol}\,\mathcal{A}_n\,(\lambda^n\sigma)^2\ \mathrm{vol}\,\mathcal{B}_n}{\mathrm{vol}\,\mathcal{A}_0\ \mathrm{vol}\,\mathcal{B}_0}.$$

Notice also that $\operatorname{vol}\mathcal{A}_n \operatorname{vol}\mathcal{B}_n = \operatorname{vol}\mathcal{A}_0 \operatorname{vol}\mathcal{B}_0$, since the cylinders

$$Df_x^n(\mathcal{A}_0 \oplus \mathcal{B}_0)$$

and $\mathcal{A}_n \oplus \mathcal{B}_n$ differ by a sheer. So, the right-hand side is equal to $\lambda^{2n(d-1)}\sigma^d$. Now, this expression is larger than $1-\kappa$, because we have chosen $\sigma = (1-\kappa)^{1/2}$ and $\lambda > (1-\kappa)^{1/4n(d-1)}$. This gives condition (ii).

Finally, let $z \in K$. Recall that $L_j = Df_{f^j(x)}R_j$. Moreover, $(Dh_j)_{\varphi_j g^j(z)} = R_j$ (we continue to identify R_j with its expression in the local chart φ_j), because

$$g^j(z) \in g^{-n+j}(\varphi_n^{-1}(\mathcal{C}_n(y_n, \sigma\rho))) \subset \varphi_j^{-1}(\mathcal{C}_j(y_j, \sigma\rho)).$$

Therefore, writing $z_j = h_j(\varphi_j(g^j(z)))$ for simplicity,

$$\big\|Dg_{g^j(z)} - L_j\big\| = \big\|D(f_j)_{z_j}R_j - D(f_j)_0 R_j\big\| \le \big\|D(f_j)_{z_j} - D(f_j)_0\big\|\big\|R_j\big\| < \gamma.$$

The last inequality follows from our choice of r'. This gives condition (iii) in Definition 2.10. The proof of Lemma 3.3 is complete. □

Remark 3.7. This last step explains why it is technically more convenient to require $\|Dg_{g^j(z)} - L_j\| < \gamma$, rather than $Dg_{g^j(z)} = L_j$, when defining realizable sequence.

3.2. *Proof of the directions interchange Proposition* 3.1. First, we define some auxiliary constants. Fix $0 < \kappa' < \frac{1}{2}\kappa$. Let $\varepsilon_1 > 0$, depending on f, ε_0 and κ', be given by Lemma 3.2. Let $\varepsilon_2 > 0$, depending on f, ε_0 and κ, be given by Lemma 3.3. Take $\varepsilon = \min\{\varepsilon_1, \varepsilon_2\}$. Fix $\alpha > 0$ such that $\sqrt{2}\sin\alpha < \varepsilon$. Take

$$K \ge (\sin\alpha)^{-2} \quad \text{and} \quad K \ge \max\big\{\|Df_x\|/\mathbf{m}(Df_x);\ x \in M\big\}. \tag{3.15}$$

Let $\beta > 0$ be such that

$$8\sqrt{2}\, K \sin\beta < \varepsilon \sin^6\alpha. \tag{3.16}$$

Finally, assume $m \in \mathbb{N}$ satisfies $m \ge 2\pi/\beta$.

Let $y \in M$ be a nonperiodic point and $T_yM = E \oplus F$ be a splitting as in the hypothesis:

$$\frac{\|Df_y^m|_F\|}{\mathbf{m}(Df_y^m|_E)} \ge \frac{1}{2}\,. \tag{3.17}$$

We write $E_j = Df_y^j(E)$ and $F_j = Df_y^j(F)$ for $j = 0, 1, \ldots, m$. The proof is divided in three cases. Lemma 3.2 suffices for the first two; in the third step we use the full strength of Lemma 3.3.

First case. Suppose there exists $\ell \in \{0, 1, \ldots, m\}$ such that

$$\sphericalangle(E_\ell, F_\ell) < \alpha. \tag{3.18}$$

Fix ℓ as above. Take unit vectors $\xi \in E_\ell$ and $\eta \in F_\ell$ such that $\sphericalangle(\xi,\eta) < \alpha$. Let $Y = \mathbb{R}\xi \oplus \mathbb{R}\eta$ and $X = Y^\perp$. Let $\widehat{R} : Y \to Y$ be a rotation such that $\widehat{R}(\xi) = \eta$. Then $\|\widehat{R} - I\| = \sqrt{2}\ \sin\sphericalangle(\xi,\eta) < \varepsilon$. Let $R : T_{f^\ell(y)}M \to T_{f^\ell(y)}M$ be such that R preserves both X and Y, $R|_X = I$ and $R|_Y = \widehat{R}$.

Consider first $\ell < m$. By Lemma 3.2, the length 1 sequence $\{Df_{f^\ell(y)}R\}$ is (κ', ε_0)-realizable at $f^\ell(y)$. Using part 2 of Lemma 2.11 we conclude that

$$\{L_0, \dots L_{m-1}\} = \{Df_y, \dots, Df_{f^{\ell-1}(y)}, Df_{f^\ell(y)}R, Df_{f^{\ell+1}(y)}, \dots, Df_{f^{m-1}(y)}\}$$

is a (κ, ε_0)-realizable sequence of length m at y. The case $\ell = m$ is similar. By Lemma 3.2, the length 1 sequence $\{R\,Df_{f^{m-1}(y)}\}$ is (κ', ε_0)-realizable at $f^{m-1}(y)$. Then, by part 2 of Lemma 2.11,

$$\{L_0, \dots L_{m-1}\} = \{Df_y, \dots, Df_{f^{m-2}(y)}, R\,Df_{f^{m-1}(y)}\}$$

is a (κ, ε_0)-realizable sequence of length m at y. In either case, $L_{m-1} \dots L_0$ sends the vector $v = Df^{-\ell}(\xi) \in E_0$ to a vector w collinear to $Df^{m-\ell}(\eta) \in F_m$.

Second case. Assume there exist $k, \ell \in \{0, \dots, m\}$, with $k < \ell$, such that

$$\frac{\|Df^{\ell-k}_{f^k(y)}|_{F_k}\|}{\mathbf{m}(Df^{\ell-k}_{f^k(y)}|_{E_k})} > K. \tag{3.19}$$

Fix k and ℓ as above. Let $\xi \in E_k$, $\eta \in F_k$ be unit vectors such that

$$\|Df^{\ell-k}(\xi)\| = \mathbf{m}(Df^{\ell-k}|_{E_k}) \quad \text{and} \quad \|Df^{\ell-k}(\eta)\| = \|Df^{\ell-k}|_{F_k}\|$$

($Df^{\ell-k}$ is always meant at the point $f^k(y)$). Define also unit vectors

$$\xi' = \frac{Df^{\ell-k}(\xi)}{\|Df^{\ell-k}(\xi)\|} \in E_\ell \quad \text{and} \quad \eta' = \frac{Df^{\ell-k}(\eta)}{\|Df^{\ell-k}(\eta)\|} \in F_\ell\,.$$

Let $\xi_1 = \xi + (\sin\alpha)\eta$. Then $\theta = \sphericalangle(\xi, \xi_1) \le \alpha$. In particular, if $\widehat{R} : \mathbb{R}\xi \oplus \mathbb{R}\eta \to \mathbb{R}\xi \oplus \mathbb{R}\eta$ is a rotation of angle $\pm\theta$, sending $\mathbb{R}\xi$ to $\mathbb{R}\xi_1$ then

$$\|\widehat{R} - I\| = \sqrt{2}\ \sin\theta < \varepsilon.$$

Let $Y = \mathbb{R}\xi \oplus \mathbb{R}\eta$ and $X = Y^\perp$. Let $R : T_{f^k(y)}M \to T_{f^k(y)}M$ be such that R preserves both X and Y, with $R|_X = I$ and $R|_Y = \widehat{R}$. By Lemma 3.2, the length 1 sequence $\{Df_{f^k(y)}R\}$ is (κ', ε_0)-realizable at $f^k(y)$. Let $\eta_1' = s\xi' + \eta'$, where

$$s = \frac{1}{\sin\alpha}\,\frac{\|Df^{\ell-k}(\xi)\|}{\|Df^{\ell-k}(\eta)\|} = \frac{1}{\sin\alpha}\,\frac{\mathbf{m}(Df^{\ell-k}|_{E_k})}{\|Df^{\ell-k}|_{F_k}\|}.$$

Then the vectors $Df^{\ell-k}\xi_1$ and η_1 are collinear. Besides, $s < 1/(K\sin\alpha) < \sin\alpha$, because of (3.15) and (3.19). Hence, $\theta' = \sphericalangle(\eta_1', \eta) < \alpha$. Then, as before, there exists $R' : T_{f^\ell(y)}M \to T_{f^\ell(y)}M$ such that $R'(\mathbb{R}\eta_1') = \mathbb{R}\eta$ and $\{R'\,Df_{f^{\ell-1}(y)}\}$ is a (κ', ε_0)-realizable sequence of length 1 at $f^{\ell-1}(y)$.

Notice that (3.15) and (3.19) imply $\ell - 1 > k$. Then we may define a sequence $\{L_0, \dots, L_{m-1}\}$ of linear maps as follows:

$$L_j = \begin{cases} Df_{f^k(y)}\, R & \text{for } j = k \\ R'\, Df_{f^{\ell-1}(y)} & \text{for } j = \ell - 1 \\ Df_{f^j(y)} & \text{for all other } j. \end{cases}$$

By parts 1 and 2 of Lemma 2.11, this is a (κ, ε_0)-realizable sequence of length m at y. By construction, $L_{m-1} \dots L_0$ sends $v = Df^{-k}(\xi) \in E_0$ to a vector w collinear to $Df^{m-\ell}(\eta') \in F_m$.

Third case. We suppose that we are not in the previous cases, that is, we assume

$$\text{(3.20)} \qquad \text{for every } j \in \{0, 1, \dots, m\}, \quad \measuredangle(E_j, F_j) \geq \alpha,$$

and

$$\text{(3.21)} \qquad \text{for every } i, j \in \{0, \dots, m\} \text{ with } i < j, \quad \frac{\|Df^{j-i}_{f^i(y)}|_{F_i}\|}{\mathbf{m}(Df^{j-i}_{f^i(y)}|_{E_i})} \leq K.$$

We now use the assumption (3.17), and the choice of m in (3.16). Take unit vectors $\xi \in E_0$ and $\eta \in F_0$ such that $\|Df^m \xi\| = \mathbf{m}(Df^m|_{E_0})$ and $\|Df^m \eta\| = \|Df^m|_{F_0}\|$ (Df^m is always computed at y). Let also $\eta' = Df^m(\eta)/\|Df^m(\eta)\| \in F_m$.

Define $G_0 = E_0 \cap \xi^\perp$ and $G_j = Df^j_y(G_0) \subset E_j$ for $0 < j \leq m$. Dually, define $H_m = F_m \cap \eta'^\perp$ and $H_j = Df^{j-m}(H_m) \subset F_j$ for $0 \leq j < m$. In addition, consider unit vectors $v_j \in E_j \cap G_j^\perp$ and $w_j \in F_j \cap H_j^\perp$ for $0 \leq j \leq m$. These vectors are uniquely defined up to a choice of sign, and $v_0 = \pm\xi$ and $w_m = \pm\eta'$. See Figure 2.

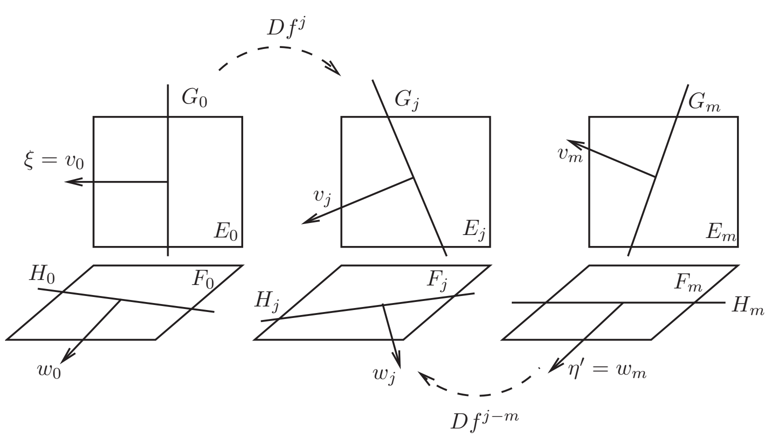

Figure 2: Setup for application of the nested rotations lemma

For $j = 0, \dots, m$, define

$$X_j = G_j \oplus H_j \quad \text{and} \quad Y_j = \mathbb{R}v_j \oplus \mathbb{R}w_j\,.$$

The spaces X_j are invariant: $Df_{f^j(y)}(X_j) = X_{j+1}$ (the Y_j are not). We shall prove, using (3.20) and (3.21), that the maps $Df^j_y/X_0 : T_yM/X_0 \to T_{f^j(y)}M/X_j$ do not distort angles too much:

LEMMA 3.8. *For every* $j = 0, 1, \dots, m$,

$$\frac{\|Df^j_y/X_0\|}{\mathbf{m}(Df^j_y/X_0)} \le \frac{8K}{\sin^6 \alpha}.$$

Let us postpone the proof of this fact for a while, and proceed preparing the application Lemma 3.3. Let $\mathcal{B}_0 \subset (T_yM)/X_0$ be a ball and $\mathcal{B}_j = (Df^j_y/X_0)(\mathcal{B}_0)$ for $0 < j \le m$. Since $m\beta \ge 2\pi$, it is possible to choose numbers $\theta_0, \dots, \theta_{m-1}$ such that $|\theta_j| \le \beta$ for all j and

$$\sum_{j=0}^{m-1} \theta_j = \sphericalangle(v_0 + X_0, w_0 + X_0). \tag{3.22}$$

Let $P_j : (T_yM)/X_0 \to (T_yM)/X_0$ be the rotation of angle θ_j. Define linear maps $\widehat{R}_j : (T_{f^j(y)}M)/X_j \to (T_{f^j(y)}M)/X_j$ by

$$\widehat{R}_j = \left(Df^j_y/X_0\right) P_j \left(Df^j_y/X_0\right)^{-1}.$$

Since P_j preserves the ball $\mathcal{B}_0$, we have $\widehat{R}_j(\mathcal{B}_j) = \mathcal{B}_j$ for all j. Moreover,

$$\|\widehat{R}_j - I\| \le \frac{\|Df^j_y/X_0\|}{\mathbf{m}(Df^j_y/X_0)}\, \|P_j - I\| \le \frac{8K}{\sin^6 \alpha}\, \sqrt{2} \sin\beta < \varepsilon,$$

by Lemma 3.8, the relation $\|P_j - I\| \le \sqrt{2}\, \sin\beta$, and our choice (3.16) of β.

Applying Lemma 3.3 to these data $(\varepsilon_0, \kappa, x = y, n = m, X_j, \mathcal{B}_j, \widehat{R}_j)$ we obtain an (ε_0, κ)-realizable sequence $\{L_0, \dots, L_{m-1}\}$ at the point y, with $L_j|_{X_j} = Df_{f^j(y)}|_{X_j}$ and

$$L_j/X_j = (Df_{f^j(y)}/X_j)\, \widehat{R}_j = (Df^{j+1}_y/X_j)\, P_j\, (Df^j_y/X_0)^{-1}.$$

Let $\mathcal{L} = L_{m-1} \dots L_0$. Then $\mathcal{L}/X_0 = (Df^m_y/X_0)P_{m-1} \dots P_0$. In particular, by (3.22),

$$\mathcal{L}(v_0 + X_0) = (Df^m_y/X_0)(w_0 + X_0) = Df^m_y(w_0) + X_m\,.$$

Recall that $X_m = G_m \oplus H_m$ by definition. Then we may write

$$\mathcal{L}(v_0) = Df^m_y(w_0) + u_m + u'_m$$

with $u_m \in G_m$ and $u'_m \in H_m$. Let $u_0 = (Df^m_y)^{-1}(u_m) \in G_0 \subset X_0 \cap E_0$. Since $\mathcal{L}$ equals Df^m_y on X_0, we have $\mathcal{L}(u_0) = u_m$. This means that the vector

$v = v_0 - u_0 \in E_0$ is sent by $\mathcal{L}$ to the vector $Df_y^m(w_0) + u'_m \in F_m$. This finishes the third and last case of Proposition 3.1.

Now we are left to give the:

Proof of Lemma 3.8. Recall that $X_j = G_j \oplus H_j$, $G_j \subset E_j$ and $H_j \subset F_j$, $v_j \in E_j$, $w_j \in F_j$, and $v_j \perp G_j$, $w_j \perp H_j$. Hence, by (3.20),

$$\begin{aligned} \sphericalangle(X_j, v_j) = \sphericalangle(H_j, v_j) \geq \sphericalangle(F_j, E_j) \geq \alpha \quad \text{and} \\ \sphericalangle(X_j \oplus \mathbb{R}v_j, w_j) = \sphericalangle(\mathbb{R}v_j \oplus G_j, w_j) \geq \sphericalangle(E_j, F_j) \geq \alpha. \end{aligned}$$

Using Lemma 2.6 with $A = X_j$, $B = \mathbb{R}v_j$, $C = \mathbb{R}w_j$, we deduce the following lower bound for the angle between the spaces X_j and $Y_j = \mathbb{R}v_j \oplus \mathbb{R}w_j$:

$$\sin \sphericalangle(X_j, Y_j) \geq \sin \sphericalangle(X_j, v_j) \sin \sphericalangle(\mathbb{R}v_j \oplus X_j, w_j) \geq \sin^2 \alpha.$$

Let $\pi_j : Y_j \to (T_{f^j(y)}M)/X_j$ be the canonical map $\pi_j(w) = w + X_j$. Then π_j is an isomorphism, $\|\pi_j\| = 1$ and

$$\|\pi_j^{-1}\| = 1/\sin \sphericalangle(Y_j, X_j) \leq 1/\sin^2 \alpha \tag{3.23}$$

(the quotient space has the norm that makes $X_j^\perp \ni w \mapsto w + X_j$ an isometry).

Now let $p_j : T_{f_j(y)}M \to Y_j$ be the projection onto Y_j associated to the splitting $T_{f_j(y)}M = X_j \oplus Y_j$. Let $\mathcal{D}_j : Y_j \to Y_{j+1}$ be given by $\mathcal{D}_j = p_{j+1} \circ (Df_{f^j(y)}|_{Y_j})$. Define

$$\mathcal{D}^{(j)} : Y_0 \to Y_j \quad \text{by} \quad \mathcal{D}^{(j)} = \mathcal{D}_{j-1} \circ \cdots \circ \mathcal{D}_0 = p_j \circ (Df_y^j|_{Y_0}).$$

We claim that the following inequalities hold:

$$\frac{1}{2K} < \frac{\|\mathcal{D}^{(j)}(w_0)\|}{\|\mathcal{D}^{(j)}(v_0)\|} \leq K \quad \text{for every } j \text{ with } 0 \leq j \leq m. \tag{3.24}$$

To prove this, consider the matrix of $\mathcal{D}_j$ relative to bases $\{v_j, w_j\}$ and $\{v_{j+1}, w_{j+1}\}$:

$$\mathcal{D}_j = \begin{pmatrix} a_j & 0 \\ 0 & b_j \end{pmatrix}.$$

Then $\|\mathcal{D}^{(j)}(v_0)\| = |a_{j-1} \ldots a_0|$ and $\|\mathcal{D}^{(j)}(w_0)\| = |b_{j-1} \ldots b_0|$, since v_j and w_j are unit vectors. Moreover, for $0 \leq i < j \leq m$ we have

$$\begin{aligned} |a_{j-1} \ldots a_i| &= \|p_j \circ Df_{f^i(y)}^{j-i}(v_i)\| = \|p_i \circ Df_{f^j(y)}^{-(j-i)}(v_j)\|^{-1}, \\ |b_{j-1} \ldots b_i| &= \|p_j \circ Df_{f^i(y)}^{j-i}(w_i)\| = \|p_i \circ Df_{f^j(y)}^{-(j-i)}(w_j)\|^{-1}. \end{aligned}$$

Recall that $v_s \in E_s$ and $w_s \in F_s$ for all s. When restricted to E_s (or F_s), the map p_s is the orthogonal projection to the direction of v_s (or w_s). In particular, $\|p_i|_{E_i}\| = \|p_j|_{F_j}\| = 1$ and so

$$|a_{j-1} \ldots a_i| \geq \|Df_{f^j(y)}^{-(j-i)}|_{E_j}\|^{-1} = \mathbf{m}(Df_{f^i(y)}^{j-i}|_{E_i})$$

and

$$|b_{j-1}\dots b_i| \le \|Df^{j-i}_{f^iy}|_{F_i}\|\,.$$

Using (3.21), we obtain

$$\frac{|b_{j-1}\dots b_i|}{|a_{j-1}\dots a_i|} \le K \quad \text{for all } 0 \le i \le j \le m. \tag{3.25}$$

Taking $i = 0$ gives the upper inequality in (3.24). For the same reasons, and the definitions of $v_0 = \xi$ and $w_m = \eta' = Df^m_y(\eta)/\|Df^m_y(\eta)\|$, we also have

$$|a_{m-1}\dots a_0| \le \|Df^m_y(v_0)\| = \|Df^m_y(\xi)\| = \mathbf{m}(Df^m|_{E_0}),$$
$$|b_{m-1}\dots b_0| \ge \|Df^{-m}_{f^m(y)}(w_m)\|^{-1} = \|Df^m_y(\eta)\| = \|Df^m|_{F_0}\|\,.$$

Now (3.17) translates into

$$\frac{|b_{m-1}\dots b_0|}{|a_{m-1}\dots a_0|} > \frac{1}{2}.$$

Combining this inequality and (3.25), with i, j replaced with j, m, we find

$$\frac{|b_{j-1}\dots b_0|}{|a_{j-1}\dots a_0|} = \frac{|b_{m-1}\dots b_0|}{|a_{m-1}\dots a_0|} \Big/ \frac{|b_{m-1}\dots b_j|}{|a_{m-1}\dots a_j|} > \frac{1/2}{K},$$

which is the remaining inequality in (3.24).

Now, combining Lemma 2.8 with (3.24) and $\sphericalangle(v_s, w_s) \ge \alpha$, we get

$$\frac{\|\mathcal{D}^{(j)}\|}{\mathbf{m}(\mathcal{D}^{(j)})} \le \frac{8K}{\sin^2\alpha}\,.$$

Moreover, $Df^j_y/X_0 = \pi_j \circ \mathcal{D}^{(j)} \circ \pi_0^{-1}$. So, using the relation (3.23),

$$\frac{\|Df^j_y/X_0\|}{\mathbf{m}(Df^j_y/X_0)} \le \frac{\|\pi_j\|}{\mathbf{m}(\pi_j)} \cdot \frac{\|\mathcal{D}^{(j)}\|}{\mathbf{m}(\mathcal{D}^{(j)})} \cdot \frac{\|\pi_0\|}{\mathbf{m}(\pi_0)} \le \frac{8K}{\sin^6\alpha}.$$

This finishes the proof of Lemma 3.8. $\square$

The proof of Proposition 3.1 is now complete. $\square$

4. Proofs of Theorems 1 and 2

Let us define some useful invariant sets. Given $f \in \mathrm{Diff}^1_\mu(M)$, let $\mathcal{O}(f)$ be the set of the regular points, in the sense of the theorem of Oseledets. Given $p \in \{1, \dots, d-1\}$ and $m \in \mathbb{N}$, let $\mathcal{D}_p(f, m)$ be the set of points x such that there is an m-dominated splitting of index p along the orbit of x. That is, $x \in \mathcal{D}_p(f, m)$ if and only if there exists a splitting $T_{f^nx}M = E_n \oplus F_n$ $(n \in \mathbb{Z})$ such that for all $n \in \mathbb{Z}$, $\dim E_n = p$, $Df_{f^nx}(E_n) = E_{n+1}$, $Df_{f^nx}(F_n) = F_{n+1}$ and

$$\frac{\|Df^m_{f^n(x)}|_{F_n}\|}{\mathbf{m}(Df^m_{f^n(x)}|_{E_n})} \le \frac{1}{2}.$$

By Section 2.2, $\mathcal{D}_p(f,m)$ is a closed set. Let

$$\begin{aligned}
\Gamma_p(f,m) &= M \setminus \mathcal{D}_p(f,m),\\
\Gamma_p^\sharp(f,m) &= \{x \in \Gamma_p(f,m) \cap \mathcal{O}(f);\ \lambda_p(f,x) > \lambda_{p+1}(f,x)\},\\
\Gamma_p^*(f,m) &= \{x \in \Gamma_p^\sharp(f,m);\ x \text{ is not periodic}\}.
\end{aligned}$$

Define also

$$\Gamma_p(f,\infty) = \bigcap_{m\in\mathbb{N}} \Gamma_p(f,m) \quad \text{and} \quad \Gamma_p^\sharp(f,\infty) = \bigcap_{m\in\mathbb{N}} \Gamma_p^\sharp(f,m).$$

It is clear that all these sets are invariant under f.

LEMMA 4.1. *For every f and p, the set $\Gamma_p^\sharp(f,\infty)$ contains no periodic points of f. In other words, $\bigcap_{m\in\mathbb{N}} \left(\Gamma_p^\sharp(f,m) \setminus \Gamma_p^*(f,m)\right) = \varnothing$.*

Proof. Suppose that $x \in \mathcal{O}(f)$ is periodic, say, of period n, and $\lambda_p(f,x) > \lambda_{p+1}(f,x)$. The eigenvalues of Df_x^n are $\nu_1,\dots,\nu_d$, with $|\nu_i| = e^{n\lambda_i(f,x)}$. Let E (resp. F) be the sum of the eigenspaces of Df_x^n associated to the eigenvalues $\nu_1,\dots,\nu_p$ (resp. $\nu_{p+1},\dots,\nu_d$). Then the splitting $T_xM = E \oplus F$ is Df_x^n-invariant. Spreading it along the orbit of x, we obtain a dominated splitting. That is, $x \in \mathcal{D}_p(f,m)$ for some $m \in \mathbb{N}$, and so $x \notin \Gamma_p^\sharp(f,\infty)$. □

4.1. *Lowering the norm along an orbit segment.* Recall that $\Lambda_p(f,x) = \lambda_1(x) + \cdots + \lambda_p(x)$ for each $p = 1,\dots,d$.

PROPOSITION 4.2. *Let $f \in \mathrm{Diff}_\mu^1(M)$, $\varepsilon_0 > 0$, $\kappa > 0$, $\delta > 0$ and $p \in \{1,\dots,d-1\}$. Then, for every sufficiently large $m \in \mathbb{N}$, there exists a measurable function $N : \Gamma_p^*(f,m) \to \mathbb{N}$ with the following properties: For almost every $x \in \Gamma_p^*(f,m)$ and every $n \geq N(x)$ there exists an (ε_0,κ)-realizable sequence $\{\widehat{L}_0^{(x,n)},\dots,\widehat{L}_{n-1}^{(x,n)}\}$ at x of length n such that*

$$\frac{1}{n}\log\|\wedge^p(\widehat{L}_{n-1}^{(x,n)}\dots\widehat{L}_0^{(x,n)})\| \leq \frac{\Lambda_{p-1}(f,x)+\Lambda_{p+1}(f,x)}{2} + \delta.$$

Proof. Fix f, ε_0, κ, δ and p. For clearness, we divide the proof into two parts:

Part 1. Definition of $N(\cdot)$ and the sequence $\widehat{L}_j^{(x,n)}$. Fix f, ε_0, κ, δ and p. Assume $m \in \mathbb{N}$ is sufficiently large so that the conclusion of Proposition 3.1 holds for f, ε_0 and $\frac{1}{2}\kappa$ (in the place of κ). To simplify the notation, let $\Gamma = \Gamma_p^*(f,m)$. We may suppose that $\mu(\Gamma) > 0$; otherwise there is nothing to prove. Consider the splitting $T_\Gamma M = E \oplus F$, where E is the sum of the Oseledets subspaces corresponding to the first p Lyapunov exponents $\lambda_1 \geq \cdots \geq \lambda_p$ and F is the sum of the subspaces corresponding to the other exponents $\lambda_{p+1} \geq \cdots \geq \lambda_d$. This makes sense since $\lambda_p > \lambda_{p+1}$ on Γ. Let $A \subset \Gamma$ be the set of points y such that the nondomination condition (3.17) holds. By definition of $\Gamma = \Gamma_p^*(f, m)$,

$$\Gamma = \bigcup_{n \in \mathbb{Z}} f^n(A). \tag{4.1}$$

Let $\lambda_i^{\wedge p}(x)$, $1 \le i \le \binom{d}{p}$, denote the Lyapunov exponents of the cocycle $\wedge^p(Df)$ over f, in nonincreasing order. Let V_x denote the Oseledets subspace associated to the upper exponent $\lambda_1^{\wedge p}(x)$ and let H_x be the sum of all other Oseledets subspaces. This gives us a splitting $\wedge^p(TM) = V \oplus H$. By Proposition 2.1,

$$\lambda_1^{\wedge p}(x) = \lambda_1(x) + \cdots + \lambda_{p-1}(x) + \lambda_p(x),$$
$$\lambda_2^{\wedge p}(x) = \lambda_1(x) + \cdots + \lambda_{p-1}(x) + \lambda_{p+1}(x).$$

If $x \in \Gamma$ then $\lambda_p(x) > \lambda_{p+1}(x)$ and so $\lambda_1^{\wedge p}(x) > \lambda_2^{\wedge p}(x)$. That is, the subspace V_x is one-dimensional.

For almost every $x \in \Gamma$, Oseledets' theorem gives $Q(x) \in \mathbb{N}$ such that for all $n \ge Q(x)$, we have:

- $\frac{1}{n} \log \frac{\|\wedge^p(Df_x^n)v\|}{\|v\|} < \lambda_1^{\wedge p}(x) + \delta$ for every $v \in V_x \setminus \{0\}$;
- $\frac{1}{n} \log \frac{\|\wedge^p(Df_x^n)w\|}{\|w\|} < \lambda_2^{\wedge p}(x) + \delta$ for every $w \in H_x \setminus \{0\}$;
- $\frac{1}{n} \log \sin \sphericalangle(V_{f^n x}, H_{f^n x}) > -\delta$.

For $q \in \mathbb{N}$, let $B_q = \{x \in \Gamma;\ Q(x) \le q\}$. Then $B_q \uparrow \Gamma$; that is, the B_q form a nondecreasing sequence and their union is a full measure subset of Γ. Define $C_0 = \varnothing$ and

$$C_q = \bigcup_{n \in \mathbb{Z}} f^n(A \cap f^{-m}(B_q)). \tag{4.2}$$

Since $f^{-m}(B_q) \uparrow \Gamma$ and the definition (4.1), we have $C_q \uparrow \Gamma$. To prove Proposition 4.2 we must define the function N on Γ. We are going to define it on each of the sets $C_q \setminus C_{q-1}$ separately. From now on, let $q \in \mathbb{N}$ be fixed.

We need the following recurrence result, proved in [4, Lemma 3.12].

Lemma 4.3. *Let $f \in \mathrm{Diff}_\mu^1(M)$. Let $A \subset M$ be a measurable set with $\mu(A) > 0$, and let $\Gamma = \cup_{n \in \mathbb{Z}} f^n(A)$. Fix any $\gamma > 0$. Then there exists a measurable function $N_0 : \Gamma \to \mathbb{N}$ such that for almost every $x \in \Gamma$, and for all $n \ge N_0(x)$ and $t \in (0, 1)$, there exists $\ell \in \{0, 1, \ldots, n\}$ such that $t - \gamma \le \ell/n \le t + \gamma$ and $f^\ell(x) \in A$.*

Let c be a strict upper bound for $\log \|\wedge^p(Df)\|$ and $\gamma = \min\{c^{-1}\delta, 1/10\}$. Using (4.2) and Lemma 4.3, we find a measurable function $N_0^{(q)} : C_q \to \mathbb{N}$ such that for almost every $x \in C_q$, every $n \geq N_0^{(q)}(x)$ and every $t \in (0,1)$ there is $\ell \in \{0,1,\ldots,n\}$ with $|\ell/n - t| < \gamma$ and $f^\ell x \in A \cap f^{-m}(B_q)$. We define $N(x)$ for $x \in C_q \setminus C_{q-1}$ as the least integer such that

$$N(x) \geq \max\{N_0^{(q)}(x),\ 10Q(x),\ m\gamma^{-1},\ \delta^{-1}\log[4/\sin\sphericalangle(V_x, H_x)]\}.$$

Now fix a point $x \in C_q \setminus C_{q-1}$ and $n \geq N(x)$. We will now construct the sequence $\{\widehat{L}_j^{(x,n)}\}$. Since $n \geq N_0^{(q)}(x)$, there exists $\ell \in \mathbb{N}$ such that

$$\left|\frac{\ell}{n} - \frac{1}{2}\right| < \gamma \quad \text{and} \quad y = f^\ell(x) \in A \cap f^{-m}(B_q).$$

Since $y \in A$, where the nondomination condition (3.17) holds, Proposition 3.1 gives a sequence $\{L_0, \ldots, L_{m-1}\}$ which is $(\varepsilon_0, \frac{1}{2}\kappa)$-realizable , such that there are nonzero vectors $v_0 \in E_y$, $w_0 \in F_{f^m(y)}$ for which

$$L_{m-1} \ldots L_0(v_0) = w_0\,.$$

We form the sequence $\{\widehat{L}_0^{(x,n)}, \ldots, \widehat{L}_{n-1}^{(x,n)}\}$ of length n by concatenating

$$\{Df_{f^i(x)};\ 0 \leq i < \ell\}, \quad \{L_0, \ldots, L_{m-1}\}, \quad \{Df_{f^i(x)};\ \ell + m \leq i < m\}.$$

According to parts 1 and 2 of Lemma 2.11, the concatenation is an (ε_0, κ)-realizable sequence at x.

Part 2. Estimation of $\|\wedge^p(\widehat{L}_{n-1}^{(x,n)} \ldots \widehat{L}_0^{(x,n)})\|$. Write $\wedge^p(\widehat{L}_{n-1}^{(x,n)} \ldots \widehat{L}_0^{(x,n)}) = D_1\mathcal{L}D_0$, with $D_0 = \wedge^p(Df_x^\ell)$, $D_1 = \wedge^p(Df^{n-\ell-m}_{f^{\ell+m}(x)})$, and $\mathcal{L} = \wedge^p(L_{m-1} \ldots L_0)$. The key observation is:

Lemma 4.4. *The map* $\mathcal{L} : \wedge^p(T_y M) \to \wedge^p(T_{f^m(y)}M)$ *satisfies* $\mathcal{L}(V_y) \subset H_{f^m(y)}$.

Proof. Proposition 2.1 describes the spaces V and H. Let $z \in \Gamma$ and consider a basis $\{e_1(z), \ldots e_d(z)\}$ of $T_z M$ such that

$$e_i(x) \in E_x^j \quad \text{for } \dim E_x^1 + \cdots + \dim E_x^{j-1} < i \leq \dim E_x^1 + \cdots + \dim E_x^j.$$

Then V_z is the space generated by $e_1(z) \wedge \cdots \wedge e_p(z)$ and H_z is generated by the vectors $e_{i_1}(z) \wedge \cdots \wedge e_{i_p}(z)$ with $1 \leq i_1 < \cdots < i_p \leq d$, $i_p > p$. Also notice that $\{e_1(z), \ldots, e_p(z)\}$ and $\{e_{p+1}(z), \ldots, e_d(z)\}$ are bases for the spaces E_z and F_z, respectively. Consider the vectors $v_0 \in E_y$ and $w_0 = L(v_0) \in F_{f^m y}$, where $L = L_{m-1} \ldots L_0$. There is $\nu \in \{1, \ldots, p\}$ such that

$$\{v_0, e_1(y), \ldots, e_{\nu-1}(y), e_{\nu+1}(y), \ldots, e_p(y)\}$$

is a basis for E_y. Therefore V_y is generated by the vector

$$v_0 \wedge e_1(y) \wedge \cdots \wedge e_{\nu-1}(y) \wedge e_{\nu+1}(y) \wedge \cdots \wedge e_p(y),$$

which is mapped by $\mathcal{L} = \wedge^p(L)$ to

$$w_0 \wedge Le_1(y) \wedge \cdots \wedge Le_{\nu-1}(y) \wedge Le_{\nu+1}(y) \wedge \cdots \wedge Le_p(y), \tag{4.3}$$

Write w_0 as a linear combination of vectors $e_{p+1}(f^m(y)), \ldots, e_d(f^m(y))$ and write each $Le_i(y)$ as a linear combination of vectors $e_1(f^m(y)), \ldots, e_d(f^m(y))$. Substituting in (4.3), we get a linear combination of $e_{i_1}(f^m(y)) \wedge \cdots \wedge e_{i_p}(f^m(y))$ where $e_1(f^m(y)) \wedge \cdots \wedge e_p(f^m(y))$ does *not* appear. This proves that the vector in (4.3) belongs to $H_{f^m(y)}$. $\square$

To carry on the estimates, we introduce a more convenient norm: For x_0, $x_1 \in \Gamma$ we represent a linear map $T : \wedge^p(T_{x_0}M) \to \wedge^p(T_{x_1}M)$ by its matrix

$$T = \begin{pmatrix} T^{++} & T^{+-} \\ T^{-+} & T^{--} \end{pmatrix}$$

with respect to the splittings $T_{x_0}M = V_{x_0} \oplus H_{x_0}$ and $T_{x_1}M = V_{x_1} \oplus H_{x_1}$. Then we define

$$\|T\|_{\max} = \max\left\{\|T^{++}\|, \|T^{+-}\|, \|T^{-+}\|, \|T^{--}\|\right\}.$$

The following elementary lemma relates this norm to the original one $\|T\|$ (that comes from the metric in $\wedge^p(T_\Gamma M)$).

Lemma 4.5. *Let* $\theta_{x_0} = \sphericalangle(V_{x_0}, H_{x_0})$ *and* $\theta_{x_1} = \sphericalangle(V_{x_1}, H_{x_1})$. *Then*:

(1) $\|T\| \le 4\,(\sin\theta_{x_0})^{-1}\,\|T\|_{\max}$;

(2) $\|T\|_{\max} \le (\sin\theta_{x_1})^{-1}\,\|T\|$.

Proof. Let $v = v_+ + v_- \in V_{x_0} \oplus H_{x_0}$. We have $\|v_*\| \le \|v\|/\sin\theta_{x_0}$ for $* = +$ and $* = -$. Thus,

$$\|Tv\| \le \|T^{++}v_+\| + \|T^{++}v_-\| + \|T^{--}v_+\| + \|T^{--}v_-\| \le 4\|T\|_{\max}\|v\|/\sin\theta_{x_0}.$$

This proves part 1. The proof of part 2 is similar. Let $v_+ \in V_{x_0}$. Its image splits as $Tv_+ = T^{++}v_+ + T^{-+}v_+ \in V_{x_1} \oplus H_{x_1}$. Hence,

$$\|T^{*+}v_+\| \le \|Tv_+\|(\sin\theta_{x_1})^{-1} \le \|T\|\,\|v_+\|(\sin\theta_{x_1})^{-1}$$

for $* = +$ and $* = -$. Together with a corresponding estimate for $T^{*-}v_-$, $v_- \in H_{x_0}$, this gives part 2. $\square$

For the linear maps we were considering, the matrices have the form:

$$D_i = \begin{pmatrix} D_i^{++} & 0 \\ 0 & D_i^{--} \end{pmatrix},\ i = 0, 1, \quad \text{and} \quad \mathcal{L} = \begin{pmatrix} 0 & \mathcal{L}^{+-} \\ \mathcal{L}^{-+} & \mathcal{L}^{--} \end{pmatrix}:$$

$D_i^{+-} = 0 = D_i^{-+}$ because V and H are $\wedge^p(Df)$-invariant, and $\mathcal{L}^{++} = 0$ because of Lemma 4.4. Then

$$\wedge^p(\widehat{L}_{n-1} \ldots \widehat{L}_0) = \begin{pmatrix} 0 & D_1^{++}\mathcal{L}^{+-}D_0^{--} \\ D_1^{--}\mathcal{L}^{-+}D_0^{++} & D_1^{--}\mathcal{L}^{--}D_0^{--} \end{pmatrix}. \tag{4.4}$$

LEMMA 4.6. *For $i=0,1$, $x \in C_q \setminus C_{q-1}$ and $n \geq N(x)$,*

$$\log \|D_i^{++}\| < \tfrac{1}{2}n(\lambda_1^{\wedge p}(x)+5\delta) \quad \textit{and} \quad \log \|D_i^{--}\| < \tfrac{1}{2}n(\lambda_2^{\wedge p}(x)+5\delta).$$

Proof. Since $\ell > (\frac{1}{2}-\gamma)n > \frac{1}{10}n \geq \frac{1}{10}N(x) \geq Q(x)$, we have

$$\log \|D_0^{++}\| = \log \|\wedge^p(Df_x^\ell)|_{V_x}\| < \ell(\lambda_1^{\wedge p}(x)+\delta),$$
$$\log \|D_0^{--}\| = \log \|\wedge^p(Df_x^\ell)|_{H_x}\| < \ell(\lambda_2^{\wedge p}(x)+\delta).$$

Let λ be either $\lambda_1^{\wedge p}(x)$ or $\lambda_2^{\wedge p}(x)$. Using $\gamma\lambda < \gamma c \leq \delta$ and $\gamma < 1$, we find

$$\ell(\lambda+\delta) < n(\tfrac{1}{2}+\gamma)(\lambda+\delta) < n(\tfrac{1}{2}\lambda+\tfrac{1}{2}\delta+\delta+\delta) = \tfrac{1}{2}n(\lambda+5\delta).$$

This proves the case $i=0$. We have $n-\ell-m > n(\frac{1}{2}-\gamma) - n\gamma > \frac{1}{10}n \geq Q(x) \geq q$. Also $f^\ell(x) \in f^{-m}(B_q)$, and so $Q(f^{\ell+m}(x)) \leq q$. Therefore

$$\log \|D_1^{++}\| = \log \|\wedge^p(Df_{f^{\ell+m}x}^{n-\ell-m})|_{V_{f^{\ell+m}x}}\| < (n-\ell-m)(\lambda_1^{\wedge p}(x)+\delta),$$
$$\log \|D_1^{--}\| = \log \|\wedge^p(Df_{f^{\ell+m}x}^{n-\ell-m})|_{H_{f^{\ell+m}x}}\| < (n-\ell-m)(\lambda_2^{\wedge p}(x)+\delta).$$

As before, $(n-\ell-m)(\lambda+\delta) < n(\frac{1}{2}+\gamma)(\lambda+\delta)\ \frac{1}{2}n(\lambda+5\delta)$. This proves the case $i=1$. ∎

LEMMA 4.7. $\log \|\mathcal{L}\|_{\max} < 2n\delta$.

Proof. Since the sequence $\{L_0, \dots, L_{m-1}\}$ is realizable, each L_j is close to the value of Df at some point. Therefore we may assume that $\log \|\wedge^p(L_j)\| < c$. In particular, $\log \|\mathcal{L}\| < mc \leq nc\gamma \leq n\delta$. We have $\ell + m \geq n(\frac{1}{2}-\gamma) \geq \frac{1}{10}n \geq Q(x)$. So $\log[1/\sin\measuredangle(V_{f^{\ell+m}x}, H_{f^{\ell+m}x})] < \delta$ and, by part 2 of Lemma 4.5, $\log \|\mathcal{L}\|_{\max} < 2n\delta$. ∎

Using Lemmas 4.6 and 4.7, we bound each of the entries in (4.4):

$$\log \|D_1^{++}\mathcal{L}^{+-}D_0^{--}\| < \tfrac{1}{2}n(\lambda_1^{\wedge p}(x)+\lambda_2^{\wedge p}(x)+14\delta),$$
$$\log \|D_1^{--}\mathcal{L}^{-+}D_0^{++}\| < \tfrac{1}{2}n(\lambda_1^{\wedge p}(x)+\lambda_2^{\wedge p}(x)+14\delta),$$
$$\log \|D_1^{--}\mathcal{L}^{--}D_0^{--}\| < \tfrac{1}{2}n(2\lambda_2^{\wedge p}(x)+14\delta).$$

The third expression is smaller than either of the first two, and so we get

$$\log \|\wedge^p(\widehat{L}_{n-1}\dots\widehat{L}_0)\|_{\max} < n\Big(\frac{\lambda_1^{\wedge p}(x)+\lambda_2^{\wedge p}(x)}{2}+7\delta\Big).$$

Therefore, by part 1 of Lemma 4.5 and $\log[4/\sin\measuredangle(V_x,H_x)] < n\delta$,

$$\log \|\wedge^p(\widehat{L}_{n-1}\dots\widehat{L}_0)\| < n\Big(\frac{\lambda_1^{\wedge p}(x)+\lambda_2^{\wedge p}(x)}{2}+8\delta\Big).$$

We also have $\lambda_1^{\wedge p}(x)+\lambda_2^{\wedge p}(x) = \Lambda_{p-1}(f,x)+\Lambda_{p+1}(f,x)$. This proves Proposition 4.2 (replace δ with $\delta/8$ in the proof). ∎

4.2. *Globalization.* The following proposition renders global the construction of Proposition 4.2:

Proposition 4.8. *Let* $f \in \mathrm{Diff}^1_\mu(M)$, $\varepsilon_0 > 0$, $p \in \{1, \dots, d-1\}$ *and* $\delta > 0$. *Then there exist* $m \in \mathbb{N}$ *and a diffeomorphism* $g \in \mathcal{U}(f, \varepsilon_0)$ *that equals* f *outside the open set* $\Gamma_p(f,m)$ *such that*

$$\int_{\Gamma_p(f,m)} \Lambda_p(g,x)\,d\mu(x) < \delta + \int_{\Gamma_p(f,m)} \frac{\Lambda_{p-1}(f,x) + \Lambda_{p+1}(f,x)}{2}\,d\mu(x).$$

We need some preparatory terminology:

Definition 4.9. Let $f \in \mathrm{Diff}^1_\mu(M)$. An *$f$-tower* (or simply *tower*) is a pair of measurable sets (T, T_b) such that there is a positive integer n, called the *height* of the tower, such that the sets $T_\mathrm{b}, f(T_\mathrm{b}), \dots, f^{n-1}(T_\mathrm{b})$ are pairwise disjoint and their union is T. Now, T_b is called the *base* of the tower.

An *f-castle* (or simply *castle*) is a pair of measurable sets (Q, Q_b) such that there exists a finite or countable family of pairwise disjoint towers $(T_i, T_{i\mathrm{b}})$ such that $Q = \bigcup T_i$ and $Q_\mathrm{b} = \bigcup T_{i\mathrm{b}}$ and Q_b is called the *base* of the castle.

A castle (Q, Q_b) is a *sub-castle* of a castle (Q', Q'_b) if $Q_\mathrm{b} \subset Q'_\mathrm{b}$ and for every point $x \in Q_\mathrm{b}$, the towers of (Q, Q_b) and (Q', Q'_b) that contain x have equal heights. In particular, $Q \subset Q'$.

We shall frequently omit reference to the base of a castle Q in our notation.

Definition 4.10. Given $f \in \mathrm{Diff}^1_\mu(M)$ and a positive measure set $A \subset M$, consider the return time $\tau : A \to \mathbb{N}$ defined by $\tau(x) = \inf\{n \geq 1;\ f^n(x) \in A\}$. If we denote $A_n = \tau^{-1}(n)$ then $T_n = A_n \cup f(A_n) \cup \cdots \cup f^{n-1}(A_n)$ is a tower. Consider the castle Q, with base A, given by the union of the towers T_n. This is called the *Kakutani castle* with base A.

Note that $Q = \bigcup_{n \in \mathbb{Z}} f^n(A)$ mod 0; in particular the set Q is invariant.

Proof of Proposition 4.8. Let f, ε_0, p and δ be given. For simplicity, we write

$$\phi(x) = \frac{\Lambda_{p-1}(f,x) + \Lambda_{p+1}(f,x)}{2}\,.$$

Step 1. Construction of families of castles $\widehat{Q}_i \supset Q_i$. Let $\kappa = \delta^2$. Take $m \in \mathbb{N}$ large enough so that the conclusion of Proposition 4.2 holds: There exists a measurable function $N : \Gamma^*_p(f,m) \to \mathbb{N}$ such that for a.e. $x \in \Gamma^*_p(f,m)$ and every $n \geq N(x)$ there exists an (ε_0, κ)-realizable sequence $\{\widehat{L}^{(x,n)}_0, \dots, \widehat{L}^{(x,n)}_{n-1}\}$ at x of length n such that

$$\frac{1}{n} \log \|\wedge^p(\widehat{L}^{(x,n)}_{n-1} \dots \widehat{L}^{(x,n)}_0)\| \leq \phi(x) + \delta. \tag{4.5}$$

We shall also (see Lemma 4.1) assume that m is large enough so that

$$\mu\big(\Gamma^\sharp_p(f,m) \smallsetminus \Gamma^*_p(f,m)\big) < \delta. \tag{4.6}$$

Let $C > \sup_{g\in\mathcal{U}(f,\varepsilon_0)} \sup_{y\in M} \log \|Dg_y\|^p$ and $\ell = \lceil C/\delta \rceil$. For $i = 1, 2, \ldots, \ell$, let

$$Z^i = \{x \in \Gamma_p^*(f,m);\ (i-1)\delta \le \phi(x) < i\delta\}.$$

Each Z^i is an f-invariant set. Since $\phi < C$, we have $\Gamma_p^*(f,m) = \bigsqcup_{i=1}^{\ell} Z^i$. Define the sets $Z_n^i = \{x \in Z^i;\ N(x) \le n\}$ for $n \in \mathbb{N}$ and $1 \le i \le \ell$. Obviously, $Z_n^i \uparrow Z^i$ when $n \to \infty$. Fix $H \in \mathbb{N}$ such that, for all $i = 1, 2, \ldots, \ell$,

$$\mu(Z^i \smallsetminus Z_H^i) < \delta^2 \mu(Z^i). \tag{4.7}$$

Using the fact that $\Lambda_p(f)$ equals ϕ in the f-invariant set $\Gamma_p(f,m) \smallsetminus \Gamma_p^\sharp(f,m)$, and Proposition 2.2, we may also assume that H is large enough so that

$$\int_{\Gamma_p(f,m)\smallsetminus\Gamma_p^\sharp(f,m)} \frac{1}{n} \log \|\wedge^p(Df^n)\| < \delta + \int_{\Gamma_p(f,m)\smallsetminus\Gamma_p^\sharp(f,m)} \phi \tag{4.8}$$

for all $n \ge H$.

A measure-preserving transformation is *aperiodic* if the set of periodic points has zero measure. The following result was proved in [4, Lemma 4.1]:

Lemma 4.11. *For every aperiodic invertible measure-preserving transformation f on a probability space X, every subset $U \subset X$ of positive measure, and every $n \in \mathbb{N}$, there exists a positive measure set $V \subset U$ such that the sets V, $f(V), \ldots, f^n(V)$ are two-by-two disjoint. Besides, V can be chosen to be maximal* in the measure-theoretical sense: *no set that includes V and has larger measure than V has the stated properties.*

By definition of the set $\Gamma_p^*(f,m)$, the map $f : \Gamma_p^*(f,m) \to \Gamma_p^*(f,m)$ is aperiodic. So, by Lemma 4.11, for each i there is $B^i \subset Z_H^i$ such that $B^i, f(B^i), \ldots, f^{H-1}(B^i)$ are two-by-two disjoint and such that B^i is maximal for these properties (in the measure-theoretical sense). Consider the following f-invariant set:

$$\widehat{Q}^i = \bigcup_{n\in\mathbb{Z}} f^n(B^i).$$

$\widehat{Q}^i$ is the Kakutani castle with base B^i. It is contained in Z^i and, by the maximality of B^i, it contains Z_H^i up to a zero-measure subset. Thus, by (4.7),

$$\mu(Z^i \smallsetminus \widehat{Q}^i) < \delta^2 \mu(Z^i). \tag{4.9}$$

Let $Q^i \subset \widehat{Q}^i$ be the sub-castle consisting of all the towers of $\widehat{Q}^i$ with heights at most $3H$ floors. The following is a key property of the construction:

Lemma 4.12. *For each $i = 1, 2, \ldots, \ell$, there exists the relation* $\mu(\widehat{Q}^i \smallsetminus Q^i) \le 3\mu(Z^i \smallsetminus Z_H^i)$.

Proof. We follow [4, Lemma 4.2] and split the castle $\widehat{Q}^i$ into towers as $\widehat{Q}^i = \bigsqcup_{k=H}^{\infty} T_k^i$ where $B^i = \bigsqcup_{k=H}^{\infty} B_k^i$ is the base $\widehat{Q}_{\mathrm{b}}^i$ and $T_k^i = \bigsqcup_{j=0}^{k-1} f^j(B_k^i)$

is the tower with base B^i_k and of height k floors. Take $k \geq 2H$ and $H \leq j \leq k - H$. The sets $f^j(B^i_k), \ldots f^{j+H-1}(B^i_k)$ are disjoint and do not intersect $B^i \sqcup \cdots \sqcup f^{H-1}(B^i)$. Since B^i is maximal, we conclude that

$$k \geq 2H \text{ and } \; H \leq j \leq k - H \quad \Rightarrow \quad \mu(f^j(B^i_k) \cap Z^i_H) = 0$$

(otherwise we could replace B^i with $B^i \sqcup (f^j(B^i_k) \cap Z^i_H)$, contradicting the maximality of B^i). Thus

$$k \geq 2H \quad \Rightarrow \quad \mu(T^i_k \smallsetminus Z^i_H) \geq \sum_{j=H}^{k-H} \mu(f^j(B^i_k)) = \frac{k-2H+1}{k} \mu(T^i_k).$$

In particular,

$$k \geq 3H+1 \Rightarrow \mu(T^i_k \smallsetminus Z^i_H) > \frac{1}{3} \mu(T^i_k)$$

and so

$$\begin{aligned} \mu(\widehat{Q}^i \smallsetminus Q^i) = \sum_{k=3H+1}^{\infty} \mu(T^i_k) &\leq \sum_{k=3H+1}^{\infty} 3\mu(T_k \smallsetminus Z^i_H) \\ &= 3\mu\left(\bigsqcup_{k=3H+1}^{\infty} T^i_k \smallsetminus Z^i_H \right) \leq 3\mu(Z^i \smallsetminus Z^i_H), \end{aligned}$$

as claimed. □

Step 2. Construction of the diffeomorphism g.

LEMMA 4.13. *For almost every $x \in \Gamma^*_p(f, m)$ and every $n \geq N(x)$, there exists $r > 0$ such that for every ball $U = B_{r'}(x)$ with $0 < r' < r$ there exist $h \in \mathcal{U}(f, \varepsilon_0)$ and a measurable set $K \subset B_{r'}(x)$ such that*

(i) *h equals f outside $\sqcup_{j=0}^{n-1} f^j(B_{r'}(x))$;*

(ii) $\mu(K) > (1-\kappa)\mu(B_{r'}(x))$;

(iii) *if $y \in K$ then $\frac{1}{n} \log \|\wedge^p(Dh^n_y)\| < \phi(x) + 2\delta$.*

Proof. Fix x and $n \geq N(x)$. Recall that the point x is not periodic. Let $\gamma > 0$ be very small. Since the sequence $\{\widehat{L}^{(x,n)}_j\}$ given by Proposition 4.2 is (κ, ε_0)-realizable, there exists $r > 0$ such that for every ball $U = B_{r'}(x)$ with $0 < r' < r$ there exists $h \in \mathcal{U}(f, \varepsilon_0)$ satisfying condition (i) above and there exists $K \subset B_{r'}(x)$ satisfying condition (ii) and

$$y \in K \text{ and } \; 0 \leq j \leq n-1 \quad \Rightarrow \quad \|Dh_{h^j y} - \widehat{L}^{(x,n)}_j\| < \gamma.$$

Taking γ small enough, this inequality and (4.5) imply

$$y \in K \;\Rightarrow\; \frac{1}{n} \log \|\wedge^p(Dh^n_y)\| < \frac{1}{n} \log \|\wedge^p(\widehat{L}^{(x,n)}_{n-1} \ldots \widehat{L}^{(x,n)}_0)\| + \delta \leq \phi(x) + 2\delta,$$

as claimed in the lemma. □

LEMMA 4.14. *Fix $\gamma > 0$. There exists $g \in \mathcal{U}(f, \varepsilon_0)$ and for each $i = 1, 2, \ldots, \ell$ there exist a g-castle U^i and a g-sub-castle K^i such that*:

(i) *the U^i are open, pairwise disjoint, and contained in $\Gamma_p(f, m)$;*

(ii) *$\mu(U^i \smallsetminus Q^i) < 2\gamma\mu(Z^i)$ and $\mu(Q^i \smallsetminus U^i) < 2\gamma\mu(Z^i)$;*

(iii) *$\mu(U^i \smallsetminus K^i) < 2\kappa\mu(Z^i)$;*

(iv) *$g(U^i) = f(U^i)$ and g equals f outside $\bigsqcup_{i=1}^{\ell} U^i$;*

(v) *if y is in the base of K^i and $n(y)$ is the height of the tower of K^i that contains x then*

$$\frac{1}{n(y)} \log \|\wedge^p(Dg_y^{n(y)})\| < i\delta + 2\delta.$$

Proof. By the regularity of the measure μ, one can find a compact sub-castle $J^i \subset Q^i$ such that

$$\mu(Q^i \smallsetminus J^i) < \gamma\mu(\widehat{Q}^i). \tag{4.10}$$

Since the J^i are compact and disjoint we can find *open* pairwise disjoint castles V^i such that each V^i contains J^i as a sub-castle, is contained in the open and invariant set $\Gamma_p(f, m)$, and

$$\mu(V^i \smallsetminus J^i) < \gamma\mu(\widehat{Q}^i). \tag{4.11}$$

For each $x \in J^i_{\mathrm{b}}$, let $n(x)$ be the height of the tower that contains x. J^i_{b} is contained in Z^i_H, so that $N(x) \le H \le n(x)$. Let $r(x) > 0$ be the radius given by Lemma 4.13, with $n = n(x)$. This is defined for almost every $x \in J^i_{\mathrm{b}}$. Reducing $r(x)$ if needed, we suppose that the ball $\overline{B}_{r(x)}(x)$ is contained in the base of a tower in V^i (with the same height).

Using Vitali's covering lemma[1], we can find a finite collection of disjoint balls $U^i_k = B_{r_{k,i}}(x_{k,i})$, with $x_{k,i} \in J^i_{\mathrm{b}}$ and $0 < r_{k,i} < r(x_{k,i})$, such that

$$\mu\Big(J^i_{\mathrm{b}} \smallsetminus \bigsqcup_k \overline{U^i_k}\Big) < \gamma\mu(J^i_{\mathrm{b}}). \tag{4.12}$$

Let $n_{k,i} = n(x_{k,i})$. Notice that $n(x) = n_{k,i}$ for all $x^i \in U^i_k$.

Now we apply Lemma 4.13 to each ball U^i_k. We get, for each k, a measurable set $K^i_k \subset U^i_k$ and a diffeomorphism $h_{k,i} \in \mathcal{U}(f, \varepsilon_0)$ such that (in 3 we use the fact that $x_{k,i} \in Z^i$)

(1) $h_{k,i}$ equals f outside the set $\bigsqcup_{j=0}^{n^i_k - 1} f^j(U^i_k)$;
(2) $\mu(K^i_k) > (1 - \kappa)\, \mu(U^i_k)$;
(3) if $y \in K^i_k$ then $\frac{1}{n_{k,i}} \log \|\wedge^p(Dh_{k,i}^{n_{k,i}})_y\| < \phi(x_{k,i}) + 2\delta < i\delta + 2\delta$.

[1] First, cover the basis J^i_{b} of the castle by chart domains.

Let g be equal to $h_{k,i}$ in the set $\bigsqcup_{j=0}^{n_k^i-1} f^j(U_i^k)$, for each i and k, and be equal to f outside. Since those sets are disjoint, $g \in \mathrm{Diff}_\mu^1(M)$ is a well-defined diffeomorphism. Each $h_{k,i}$ belongs to $\mathcal{U}(f, \varepsilon_0)$ and so g also does.

Since each U_k^i is contained in the base of a tower in the castle V^i, V^i is also a castle for g. Let U^i be the g-sub-castle of V^i with base $\sqcup_k U_k^i$. Analogously, let K^i be the g-sub-castle of U^i with base $\sqcup_k K_k^i$.

It remains to prove claims (ii) and (iii) in the lemma. Making use of the castle structures, relation (4.12) and item 2 above imply, respectively,

$$\mu(J^i \smallsetminus U^i) < \gamma\mu(J^i) \quad \text{and} \quad \mu(U^i \smallsetminus K^i) < \kappa\mu(U^i). \tag{4.13}$$

By (4.11) and $\widehat{Q}^i \subset Z^i$,

$$\mu(U^i \smallsetminus Q^i) < \mu(V^i \smallsetminus J^i) < \gamma\mu(\widehat{Q}^i) \le \gamma\mu(Z^i). \tag{4.14}$$

This implies the first part of item (ii). Combining the first part of (4.13) with (4.10),

$$\mu(Q^i \smallsetminus U^i) < \mu(Q^i \smallsetminus J^i) + \mu(J^i \smallsetminus U^i) < 2\gamma\mu(\widehat{Q}^i) \le 2\gamma\mu(Z^i).$$

This proves the second part of item (ii). Finally, the second inequality in (4.13) and

$$\mu(U^i) < \mu(Q^i) + \mu(U^i \smallsetminus Q^i) < (1+\gamma)\mu(\widehat{Q}^i) < 2\mu(Z^i).$$

imply item (iii). The lemma is proved. $\square$

Step 3. Conclusion of the proof of Proposition 4.8. Let $U = \bigsqcup_{i=1}^{\ell} U^i$ and $Q = \bigsqcup_{i=1}^{\ell} Q^i$ and $\widehat{Q} = \bigsqcup_{i=1}^{\ell} \widehat{Q}^i$. Set $N = H\delta^{-1}$. (Of course, we can assume that $\delta^{-1} \in \mathbb{N}$.) Let

$$G = \bigsqcup_{i=1}^{\ell} G^i \quad \text{where} \quad G^i = Z^i \cap \bigcap_{j=0}^{N-1} g^{-j}(K^i)$$

for each $i = 1, 2, \ldots, \ell$. The next lemma means that on G we managed to reduce some time N exponent:

LEMMA 4.15. *If $x \in G^i$ then*

$$\frac{1}{N} \log \|\wedge^p(Dg_x^N)\| < i\delta + (6C+2)\delta.$$

Proof. For $y \in K_{\mathrm{b}}^i$, let $n(y)$ be the height of the g-tower containing y. Take $x \in G$; say $x \in G^i$. Since the heights of towers of K^i are less than $3H$, we can write

$$N = k_1 + n_1 + n_2 + \cdots + n_j + k_2$$

so that $0 \le k_1, k_2 < 3H$, $1 \le n_1, \ldots, n_j < 3H$, and the points

$$x_1 = g^{k_1}(x), \ x_2 = g^{k_1+n_1}(x), \ \ldots, \ x_{j+1} = g^{k_1+n_1+\cdots+n_j}(x)$$

are exactly the points (of the orbit segment $x, g(x), \dots, g^{N-1}(x)$) which belong to K^i_{b}. Write

$$\|\wedge^p(Dg^N_x)\| \le \|\wedge^p(Dg^{k_1}_x)\| \; \|\wedge^p(Dg^{n_1}_{x_1})\| \; \dots \; \|\wedge^p(Dg^{n_j}_{x_j})\| \; \|\wedge^p(Dg^{k_2}_{x_{j+1}})\|.$$

Using item (v) of Lemma 4.14, and our choice of $N = H\delta^{-1}$, we get

$$\begin{aligned}\log \|\wedge^p(Dg^N_x)\| &< k_1 C + (n_1 + \dots + n_j)(i\delta + 2\delta) + k_2 C \\ &< 6HC + N(i\delta + 2\delta) < N(6C\delta + i\delta + 2\delta),\end{aligned}$$

as claimed. □

We also use the fact that G covers most of $U \cup \Gamma^*_p(f,m)$, as asserted by the next lemma whose proof we postpone for a while.

Lemma 4.16. *Let $\gamma = \delta^2/(\ell H)$ as in Lemma 4.14. Then*

$$\mu\big(U \cup \Gamma^*_p(f,m) \setminus G\big) < 12\delta.$$

Continuing with the proof of Proposition 4.8, write $\psi(x) = \frac{1}{N}\log\|\wedge^p(Dg^N_x)\|$. Since g leaves invariant the set $\Gamma_p(f,m)$, Proposition 2.2 gives

$$\int_{\Gamma_p(f,m)} \Lambda_p(g) \le \int_{\Gamma_p(f,m)} \psi.$$

We split the integral on the right-hand side as

$$\begin{aligned}\int_{\Gamma_p(f,m)} \psi &= \int_{\Gamma_p(f,m)\setminus(U\cup\Gamma^\sharp_p(f,m))} \psi + \int_{(U\cup\Gamma^\sharp_p(f,m))\setminus G} \psi + \int_G \psi \\ &= \text{(I)} + \text{(II)} + \text{(III)}.\end{aligned}$$

Outside U, g equals f and so ψ equals $\frac{1}{N}\log\|\wedge^p(Df^N)\|$. Thus

$$\text{(I)} \le \int_{\Gamma_p(f,m)\setminus\Gamma^\sharp_p(f,m)} \frac{1}{N}\log\|\wedge^p(Df^N)\| < \delta + \int_{\Gamma_p(f,m)\setminus\Gamma^\sharp_p(f,m)} \phi,$$

by (4.8). By Lemma 4.16 and (4.6), $\mu\big((U\cup\Gamma^\sharp_p(f,m))\setminus G\big) < 13\delta$. Since $\psi < C$, we have (II) $\le 13C\delta$. By Lemma 4.15,

$$\text{(III)} = \sum_{i=1}^{\ell}\int_{G^i}\psi \le \sum_{i=1}^{\ell}(i\delta + (6C+2)\delta)\mu(G^i) < (6C+3)\delta + \sum_{i=1}^{\ell}(i-1)\delta\mu(G^i).$$

Since $\phi \ge (i-1)\delta$ inside $Z^i \supset G^i$, we have

$$\text{(III)} < (6C+3)\delta + \int_{\Gamma^*_p(f,m)} \phi.$$

Summing the three terms, we get the conclusion of Proposition 4.8 (replace δ

with $\delta/(18C+4)$ throughout the arguments):

$$\int_{\Gamma_p(f,m)} \Lambda_p(g) < (18C+4)\delta + \int_{\Gamma_p(f,m)} \phi.$$

This completes the proof of the proposition, modulo proving Lemma 4.16. $\square$

Step 4. Proof of Lemma 4.16. The following observations will be useful in the proof: If $X \subset M$ is a measurable set and $N \in \mathbb{N}$, then

$$\mu\Big(\bigcup_{j=0}^{N-1} g^{-j}(X)\Big) \le \mu(X) + (N-1)\mu\big(g^{-1}(X) \setminus X\big). \tag{4.15}$$

Moreover, $\mu\big(g^{-1}(X) \setminus X\big) = \mu\big(X \setminus g^{-1}(X)\big)$.

Proof of Lemma 4.16. We shall prove first that

$$\mu(\widehat{Q}^i \setminus G^i) < 10\delta\mu(Z^i). \tag{4.16}$$

Since $\widehat{Q}^i \subset Z^i$, we have $\widehat{Q}^i \setminus G^i \subset \widehat{Q}^i \cap \bigcup_{j=0}^{N-1} g^{-j}(M \setminus K^i)$. Substituting

$$M \setminus K^i \subset (U^i \setminus K^i) \cup (Q^i \setminus U^i) \cup (\widehat{Q}^i \setminus Q^i) \cup (M \setminus \widehat{Q}^i),$$

we obtain

$$\begin{aligned}\widehat{Q}^i \setminus G^i \subset \bigcup_{j=0}^{N-1} g^{-j}(U^i \setminus K^i) \ \cup \ \bigcup_{j=0}^{N-1} g^{-j}(Q^i \setminus U^i) \ \cup \ \bigcup_{j=0}^{N-1} g^{-j}(\widehat{Q}^i \setminus Q^i) \\ \cup \ \left[\widehat{Q}^i \cap \bigcup_{j=1}^{N-1} g^{-j}(M \setminus \widehat{Q}^i)\right] = (\mathrm{I}) \cup (\mathrm{II}) \cup (\mathrm{III}) \cup (\mathrm{IV}).\end{aligned}$$

Let us bound the measure of each of these sets. The second one is easy: by Lemma 4.14(ii) and our choices $\gamma = \delta^2/\ell H$ and $N = H/\delta$,

$$\mu(\mathrm{II}) \le N\mu(Q^i \setminus U^i) < 2N\gamma\mu(Z^i) < \delta\mu(Z^i).$$

The other terms are more delicate.

The set $X_1 = U^i \setminus K^i$ is a g-castle whose towers have heights at least H. Hence its base, which contains $X_1 \setminus g(X_1)$, measures at most $\frac{1}{H}\mu(X_1)$. By (4.15), we get

$$\mu(\mathrm{I}) < \Big(1 + \frac{N}{H}\Big)\mu(X_1) < 2\delta^{-1}\mu(X_1).$$

By Lemma 4.14(iii), we have $\mu(X_1) < 2\kappa\mu(Z^i) = 2\delta^2\mu(Z^i)$. So, $\mu(\mathrm{I}) < 4\delta\mu(Z^i)$.

Let $X_3 = \widehat{Q}^i \setminus Q^i$. By Lemma 4.12 and (4.7), we have $\mu(X_3) < \delta^2\mu(Z_i)$. Since f and g differ only in U, we have

$$g(X_3) \setminus X_3 \subset [f(X_3) \setminus X_3] \cup g(X_3 \cap U) = (\mathrm{V}) \cup (\mathrm{VI}).$$

Since X_3 is an f-castle whose towers have heights of at least $3H$,

$$\mu(\mathrm{V}) = \mu(X_3 \setminus f(X_3)) \le \frac{1}{3H}\mu(X_3).$$

Since $X_3 \cap U \subset \bigsqcup_k (U^k \setminus Q^k)$, Lemma 4.14(ii) gives $\mu(\mathrm{VI}) \le 2\ell\gamma\mu(Z^i)$. Combining the estimates of $\mu(\mathrm{V})$, $\mu(\mathrm{VI})$, $\mu(X_3)$ with (4.15) and the definitions of N and γ, we have

$$\begin{aligned}\mu(\mathrm{III}) &< \mu(X_3) + N\Big(\frac{1}{3H}\mu(X_3) + 2\ell\gamma\mu(Z^i)\Big)\\ &< \Big(1+\frac{1}{3\delta}\Big)\mu(X_3) + 2\delta\mu(Z^i) < 3\delta\mu(Z^i).\end{aligned}$$

We also have

$$(\mathrm{IV}) = \widehat{Q}^i \setminus \bigcap_{j=1}^{N-1} g^{-j}(\widehat{Q}^i) \subset \bigcup_{j=1}^{N-1} \big(g^{j-1}(\widehat{Q}^i) \setminus g^{-j}(\widehat{Q}^i)\big).$$

In particular, $\mu(\mathrm{IV}) \le (N-1)\mu(\widehat{Q}^i \setminus g^{-1}(\widehat{Q}^i))$. Notice that $\widehat{Q}^i \setminus g^{-1}(\widehat{Q}^i) \subset \sqcup_k U^k$ (since $\widehat{Q}^i$ is f-invariant). Therefore

$$\widehat{Q}^i \setminus g^{-1}(\widehat{Q}^i) \subset [\widehat{Q}^i \cap \sqcup_{k\ne i} U^k] \cup [U^i \setminus g^{-1}(\widehat{Q}^i)] = (\mathrm{VII}) \cup (\mathrm{VIII}).$$

Combining

$$(\mathrm{VII}) \subset \bigsqcup_{k\ne i}(U^k \setminus \widehat{Q}^k) \subset \bigsqcup_{k\ne i}(U^k \setminus Q^k)$$

with Lemma 4.14(ii) we obtain $\mu(\mathrm{VII}) \le 2(\ell-1)\gamma\mu(Z^i)$. Using the fact that $g(U^i) = f(U^i)$ and $\widehat{Q}^i = f(\widehat{Q}^i)$, we also get

$$\mu(\mathrm{VIII}) = \mu(g(U^i) \setminus \widehat{Q}^i) = \mu(U^i \setminus \widehat{Q}^i) \le \mu(U^i \setminus Q^i) < 2\gamma\mu(Z^i).$$

Altogether, $\mu(\widehat{Q}^i \setminus g^{-1}(\widehat{Q}^i)) < 2\ell\gamma\mu(Z^i)$ and $\mu(\mathrm{IV}) \le 2N\ell\gamma\mu(\widehat{Q}_i) < 2\delta\mu(Z_i)$.

Summing the four parts, we obtain (4.16). Now

$$\begin{aligned}\mu\big(U \cup \Gamma_p^*(f,m) \setminus G\big) &\le \mu(\Gamma_p^*(f,m) \setminus \widehat{Q}) + \mu(U \setminus \widehat{Q}) + \mu(\widehat{Q} \setminus G)\\ &= \mu(\mathrm{IX}) + \mu(\mathrm{X}) + \mu(\mathrm{XI}).\end{aligned}$$

Using (4.9), Lemma 4.14, and (4.16), respectively, we get

$$\begin{aligned}\mu(\mathrm{IX}) &\le \sum_i \mu(Z^i \setminus \widehat{Q}^i) < \delta^2 < \delta,\\ \mu(\mathrm{X}) &\le \mu(U \setminus Q) \le \sum_i \mu(U^i \setminus \widehat{Q}^i) < 2\gamma < \delta,\\ \mu(\mathrm{XI}) &\le \sum_i \mu(\widehat{Q}^i \setminus G) < 10\delta.\end{aligned}$$

Summing the three parts, we conclude the proof of Lemma 4.16. $\square$

4.3. *End of the proofs of Theorems* 1 *and* 2. We give an explicit lower bound for the discontinuity "jump" of the semi-continuous function $\mathrm{LE}_p(\cdot)$. Denote, for each $p = 1, \dots, d$,

$$J_p(f) = \int_{\Gamma_p(f,\infty)} \frac{\lambda_p(f,x) - \lambda_{p+1}(f,x)}{2} \, d\mu(x).$$

PROPOSITION 4.17. *Given $f \in \mathrm{Diff}^1_\mu(M)$ and $p \in \{1, \dots, d-1\}$, and given any $\varepsilon_0 > 0$ and $\delta > 0$, there exists a diffeomorphism $g \in \mathcal{U}(f, \varepsilon_0)$ such that*

$$\int_M \Lambda_p(g,x) \, d\mu(x) < \int_M \Lambda_p(f,x) \, d\mu(x) - J_p(f) + \delta.$$

Proof. Let f, p, ε_0 and δ be as in the statement. Using Proposition 4.8, we find $m \in \mathbb{N}$ and $g \in \mathcal{U}(f, \varepsilon_0)$ such that $g = f$ outside $\Gamma_p(f,m)$ and

$$\int_{\Gamma_p(f,m)} \Lambda_p(g) < \delta + \int_{\Gamma_p(f,m)} \frac{\Lambda_{p-1}(f) + \Lambda_{p+1}(f)}{2}.$$

Then

$$\begin{aligned}\int_M \Lambda_p(g) &= \int_{\Gamma_p(f,m)} \Lambda_p(g) + \int_{M \setminus \Gamma_p(f,m)} \Lambda_p(g) \\ &< \delta + \int_{\Gamma_p(f,m)} \frac{\Lambda_{p-1}(f) + \Lambda_{p+1}(f)}{2} + \int_{M \setminus \Gamma_p(f,m)} \Lambda_p(f).\end{aligned}$$

Since $\Gamma_p(f,\infty) \subset \Gamma_p(f,m)$, and the integrand is nonnegative,

$$\begin{aligned}&\int_{\Gamma_p(f,m)} \left(\Lambda_p(f) - \frac{\Lambda_{p-1}(f) + \Lambda_{p+1}(f)}{2} \right) \\ &\qquad\qquad \geq \int_{\Gamma_p(f,\infty)} \left(\Lambda_p(f) - \frac{\Lambda_{p-1}(f) + \Lambda_{p+1}(f)}{2} \right) = J_p(f).\end{aligned}$$

Therefore, the previous inequality implies

$$\int_M \Lambda_p(g) < \delta - J_p(f) + \int_M \Lambda_p(f),$$

as we wanted to prove. □

Theorem 2 follows easily from Proposition 4.17:

Proof of Theorem 2. Let $f \in \mathrm{Diff}^1_\mu(M)$ be a point of continuity of $\mathrm{LE}_p(\cdot)$ for all $p = 1, \dots, d-1$. Then $J_p(f) = 0$ for every p. This means that $\lambda_p(f,x) = \lambda_{p+1}(f,x)$ for almost every $x \in \Gamma_p(f,\infty)$. Let $x \in M$ be an Oseledets regular point. If all Lyapunov exponents of f at x vanish, there is nothing to prove. Otherwise, for any p such that $\lambda_p(f,x) > \lambda_{p+1}(f,x)$, the point $x \notin \Gamma_p(f,\infty)$ (except for a zero-measure set of x) . This means that $x \in \mathcal{D}_p(f,m)$ for some m: there is a dominated splitting of index p, $T_{f^n x}M = E_n \oplus F_n$, $n \in \mathbb{Z}$ along the orbit of x. Clearly, domination implies that E_n is the sum of the Oseledets subspaces of f, at the point $f^n x$, associated to the Lyapunov exponents $\lambda_1(f,x), \ldots, \lambda_p(f,x)$, and F_n is the sum of the spaces associated to the other exponents. Since this holds whenever $\lambda_p(f,x)$ is bigger than $\lambda_{p+1}(f,x)$, it proves that the Oseledets splitting is dominated at x. □

Theorem 1 is an immediate consequence:

Proof of Theorem 1. The function $f \mapsto \mathrm{LE}_j(f)$ is semi-continuous for every $j = 1, \ldots, d-1$; see Section 2.1.3. Hence, there exists a residual subset $\mathcal{R}$ of $\mathrm{Diff}^1_\mu(M)$ such that any $f \in \mathcal{R}$ is a point of continuity for $f \mapsto (\mathrm{LE}_1(f), \ldots, \mathrm{LE}_{d-1}(f))$. By Theorem 2, every point of continuity satisfies the conclusion of Theorem 1. □

5. Consequences of nondominance for symplectic maps

Here we prove a symplectic analogue of Proposition 3.1:

Proposition 5.1. *Given $f \in \mathrm{Sympl}^1_\omega(M)$, $\varepsilon_0 > 0$ and $0 < \kappa < 1$, if $m \in \mathbb{N}$ is large enough then the following holds*:

Let $y \in M$ be a nonperiodic point and suppose there exists a nontrivial splitting $T_y M = E \oplus F$ into two Lagrangian *spaces such that*

$$\frac{\|Df^m_y|_F\|}{\mathbf{m}(Df^m_y|_E)} \geq \frac{1}{2}. \tag{5.1}$$

Then there exists an (ε_0, κ)-realizable sequence $\{L_0, \ldots, L_{m-1}\}$ at y of length m and there are nonzero vectors $v \in E$, $w \in Df^m_y(F)$ such that $L_{m-1} \ldots L_0(v) = w$.

Remark 5.2. The hypothesis that E and F are Lagrangian subspaces in Proposition 5.1 is the sole reason why Theorem 4 is weaker than what is stated in [5].

In subsections 5.1 and 5.2 we prove two results, namely, Lemmas 5.3 and 5.8, that are used in subsection 5.3 to prove Proposition 5.1. In Section 6 we prove Theorems 3 and 4 using Proposition 5.1.

5.1. *Symplectic realizable sequences of length* 1. First, we recall some elementary facts: Let $(\cdot,\cdot)$ denote the usual hermitian inner product in $\mathbb{C}^q$. Up to identification of $\mathbb{C}^q$ with $\mathbb{R}^{2q}$, the standard inner product in $\mathbb{R}^{2q}$ is $\mathrm{Re}(\cdot,\cdot)$ and the standard symplectic form in $\mathbb{R}^{2q}$ is $\mathrm{Im}(\cdot,\cdot)$. The unitary group $\mathrm{U}(q)$ is subgroup of $\mathrm{GL}(q,\mathbb{C})$ formed by the linear maps that preserve the hermitian product. When $R \in \mathrm{U}(q)$ is a map $R : \mathbb{R}^{2q} \to \mathbb{R}^{2q}$, then R is both symplectic and orthogonal.

If $R : T_xM \to T_xM$ is an ω-preserving linear map, we shall call R *unitary* if it preserves the inner product in T_xM induced from the Euclidean inner product in $\mathbb{R}^{2q}$ by the chart $\varphi_{i(x)}$ (recall subsection 2.5).

The next lemma constructs realizable sequences of length 1:

LEMMA 5.3. *Given $f \in \mathrm{Sympl}^1_\omega(M)$, $\varepsilon_0 > 0$, $\kappa > 0$, there exists $\varepsilon > 0$ with the following properties: Suppose given a nonperiodic point $x \in M$ and a unitary map $R : T_xM \to T_xM$ with $\|R - I\| < \varepsilon$. Then $\{Df_xR\}$ is an (ε_0, κ)-realizable sequence of length 1 at the point x and $\{RDf_{f^{-1}(x)}\}$ is an (ε_0, κ)-realizable sequence of length 1 at the point $f^{-1}(x)$.*

Before starting the proof, let us make some remarks.

Remark 5.4. Let $H : \mathbb{R}^{2q} \to \mathbb{R}$ be a smooth function such that the corresponding Hamiltonian flow $\varphi^t : \mathbb{R}^{2q} \to \mathbb{R}^{2q}$ is globally defined for every $t \in \mathbb{R}$. Let $\psi : \mathbb{R} \to \mathbb{R}$ be a smooth function, and define $\widetilde{H} = \psi \circ H$. Then the Hamiltonian flow $(\tilde{\varphi}^t)$ of $\widetilde{H}$ is globally defined and is given by $\tilde{\varphi}^t(x) = \varphi^{\psi'(H(x))t}(x)$.

If $R \in \mathrm{U}(q)$ then all its eigenvalues belong to the unit circle in $\mathbb{C}$. Moreover, there exists an orthonormal basis of $\mathbb{C}^q$ formed by eigenvectors of R. If $J \subset \mathbb{R}$ is an interval, we define S_J as the set of matrices $R \in \mathrm{U}(q)$ whose eigenvalues can be written as $e^{i\theta_1}, \dots, e^{i\theta_q}$, with all $\theta_k \in J$. There is $C_0 > 0$, depending only on q, such that if $\varepsilon > 0$ and $R \in S_{[-\varepsilon,\varepsilon]}$ then $\|R - I\| \le C_0\varepsilon$. It is convenient to consider first the case where the arguments of the eigenvalues of R all have the same sign and are comparable:

LEMMA 5.5. *Given $\varepsilon_0 > 0$ and $0 < \sigma < 1$, there exists $\varepsilon > 0$ with the following properties: Given $R \in S_{[-3\varepsilon,-\varepsilon]} \cup S_{[\varepsilon,3\varepsilon]}$, there exists a bounded open set $U \subset \mathbb{R}^{2q}$ such that $0 \in \sigma U \subset U$, and there exists a C^1 symplectomorphism $h : \mathbb{R}^{2q} \to \mathbb{R}^{2q}$ such that*

(i) *$h(z) = z$ for every $z \notin U$ and $h(z) = R(z)$ for every $z \in \sigma U$;*

(ii) *$\|Dh_z - I\| < \varepsilon_0$ for all $z \in \mathbb{R}^{2q}$.*

Proof. Let ε_0 and σ be given. Let $\varepsilon > 0$ be a small number, to be specified later. Take $R \in S_{[\varepsilon,3\varepsilon]}$; the other possibility is tackled in a similar way. Let $\{v_1, \dots, v_q\}$ be an orthonormal basis of eigenvectors of R, with associated eigenvalues $e^{i\theta_1}, \dots, e^{i\theta_q}$, and all $\varepsilon \le \theta_k \le 3\varepsilon$. Up to replacing R with SRS^{-1}, for some $S \in \mathrm{U}(q)$, we may assume that the basis $\{v_1, \dots, v_q\}$ coincides with the standard basis of $\mathbb{C}^q$. Therefore R assumes the form

$$R(z_1, \dots, z_q) = (e^{i\theta_1} z_1, \dots, e^{i\theta_q} z_q).$$

Let $H : \mathbb{C}^q \to \mathbb{R}$ be given by $H(z) = \frac{1}{2}\sum_k \theta_k |z_k|^2$. Then R is the time 1 map of the Hamiltonian flow of H. Besides, since $\max\theta_k \le 3\min\theta_k$, there is C_1

depending only on q such that

$$\|z\|\,\|DH_z\| \le C_1 H(z) \quad \text{for all } z \in \mathbb{C}^q. \tag{5.2}$$

Let $\tau : \mathbb{R} \to \mathbb{R}$ be a smooth function such that $\tau(s) = 1$ for $s \le \sigma^2$, $\tau(s) = 0$ for $s \ge 1$, and $0 \le -\tau'(s) \le 2/(1-\sigma^2)$ for all s. Let $\psi(s) = \int_0^s \tau(u)\,du$ and let $\widetilde{H} = \psi \circ H$. By Remark 5.4, the time 1 map h of the Hamiltonian flow of $\widetilde{H}$ is

$$h(z) = (e^{i\theta_1 \tau(H(z))} z_1, \dots, e^{i\theta_k \tau(H(z))} z_k).$$

Then $h(z) = R(z)$ if $H(z) \le \sigma^2$ and $h(z) = z$ if $H(z) \ge 1$. Moreover, a direct calculation gives

$$\|Dh_z - I\| \le \frac{C_2 \varepsilon \|z\|\,\|DH_z\|}{1-\sigma^2} + 3\varepsilon$$

where $C_2 = C_2(q)$. Due to (5.2), we can take $\varepsilon = \varepsilon(\varepsilon_0, \sigma)$ such that the right-hand side is less than ε_0 whenever $H(z) < 1$. Since H is positive definite, the set $U = \{z \in \mathbb{C}^q;\ H(z) < 1\}$ is bounded. □

Remark 5.6. We may assume that the set U in Lemma 5.5 has arbitrarily small diameter. Indeed, if $a > 0$ then we may replace U with $\widetilde{U} = aU$ and h with $\widetilde{h}(z) = ah(a^{-1}z)$. Notice that $D\widetilde{h}_z = Dh_{a^{-1}z}$, so that $\widetilde{h}$ is a symplectomorphism and satisfies property (ii) of the lemma.

LEMMA 5.7. *Given $\varepsilon_0 > 0$ and $0 < \sigma < 1$, there exists $\varepsilon > 0$ with the following properties: Given $R \in S_{[-\varepsilon,\varepsilon]}$, there exists a bounded open set $U \subset \mathbb{R}^{2q}$ such that $0 \in \sigma U \subset U$, a measurable set $K \subset U$ with $\operatorname{vol}(U \setminus K) < 3(1-\sigma^d)\operatorname{vol}(U)$, and a C^1 symplectomorphism $h : \mathbb{R}^{2q} \to \mathbb{R}^{2q}$ such that*

(i) *$h(z) = z$ for every $z \notin U$ and $Dh_z = R$ for every $z \in K$;*

(ii) *$\|Dh_z - I\| < \varepsilon_0$ for all $z \in \mathbb{R}^{2q}$.*

Proof. Let ε be as given by Lemma 5.5. Write $R \in S_{[-\varepsilon,\varepsilon]}$ as a product $R = R_+ R_-$, with $R_+ \in S_{[\varepsilon,3\varepsilon]}$ and $R_- = e^{-2\varepsilon i} I$. Applying Lemma 5.5 to $R_\pm$, with ε_0 replaced with $\varepsilon_0/2$, we obtain sets $U_\pm$ and symplectomorphisms $h_\pm$. Let $U = U_+$. Consider the family $\mathcal{F}$ of all sets of the form $aU_- + b$, with $a > 0$ and $b \in \mathbb{R}^{2q}$, that are contained in U. This is a Vitali covering of U, and so we may find a finite number of disjoint sets $U_-^i = a_i U_- + b_i \in \mathcal{F}$ that cover U except for a set of volume $(1-\sigma^d)\operatorname{vol}(U)$. Using Lemma 5.5 and Remark 5.6, for each i we find a symplectomorphism h_-^i such that $h_-^i = \mathrm{id}$ outside U_-^i and $D(h_-^i)_z = R_-$ for $z \in K_i = a_i \sigma U_- + b_i$, and $D(h_-^i)_z$ is uniformly close to I. Let $K = (\sigma U) \cap \sqcup_i K^i$. Define $h = h_+ \circ h_-^i$ inside each U_-^i, and $h = h_+$ outside. Then K and h have the desired properties. □

Proof of Lemma 5.3. Given ε_0 and κ, choose σ close to 1 so that $3(1-\sigma^d) < \kappa$. Remark 5.6 also applies to Lemma 5.7: the set U may be taken with arbitrarily small diameter. Using Lemma 2.13, we conclude that the sequences $\{Df_x R\}$ and $\{R\, Df_{f^{-1}(x)}\}$ are (ε_0, κ)-realizable as stated. $\square$

5.2. *Symplectic nested rotations.* In this subsection we prove an analogue of Lemma 3.3 for symplectic maps:

LEMMA 5.8. *Given $f \in \mathrm{Sympl}^1_\omega(M)$, $\varepsilon_0 > 0$, $\kappa > 0$, $E > 1$, and $0 < \gamma \leq \pi/2$, there exists $\beta > 0$ with the following properties: Suppose there is a nonperiodic point $x \in M$, a number $n \in \mathbb{N}$, and a two-dimensional symplectic subspace $Y_0 \subset T_xM$ such that:*

- $\|Df^j|_{Y_0}\| / \mathbf{m}(Df^j|_{Y_0}) \leq E^2$ *for every $j = 1, \ldots, n$;*
- $\sphericalangle(X_j, Y_j) \geq \gamma$ *for each $j = 0, \ldots, n-1$ where $X_0 = Y_0^\omega$, $X_j = Df_x^j(X_0)$, and $Y_j = Df_x^j(Y_0)$.*

Let $\theta_0, \ldots, \theta_{n-1} \in [-\beta, \beta]$ and let $S_0, \ldots, S_{n-1} : Y_0 \to Y_0$ be the rotations of the plane Y_0 by angles $\theta_0, \ldots, \theta_{n-1}$. Let linear maps

$$T_xM \xrightarrow{L_0} T_{fx}M \xrightarrow{L_1} \ldots \xrightarrow{L_{n-1}} T_{f^n(x)}M$$

be defined by $L_j(v) = Df_{f^j(x)}(v)$ for $v \in X_j$ and $L_j(w) = (Df_y^{j+1}) \cdot S_j \cdot (Df_y^j)^{-1}(w)$ for $w \in Y_j$. Then $\{L_0, \ldots, L_{n-1}\}$ is an (ε_0, κ)-realizable sequence of length n at the point x.

We begin by proving a perturbation lemma that corresponds to Lemma 3.4:

LEMMA 5.9. *Given $\varepsilon_0 > 0$ and $0 < \sigma < 1$, there is $\varepsilon > 0$ with the following properties: Suppose there exist a splitting $\mathbb{R}^{2q} = X \oplus Y$ with $\dim Y = 2$, $X^\omega = Y$ and $X \perp Y$, an ellipsoid $\mathcal{A} \subset X$ centered at the origin, and a unitary map $R \in \mathrm{U}(q)$ with $R|_X = I$ and $\|R - I\| < \varepsilon$.*

Then there exists $\tau > 1$ such that the following holds. Let $\mathcal{B}$ be the unit ball in Y. For a, $b > 0$ consider the cylinder $\mathcal{C} = \mathcal{C}_{a,b} = a\mathcal{A} \oplus b\mathcal{B}$. If $a > \tau b$ and $\operatorname{diam} \mathcal{C} < \varepsilon_0$ *then there is a C^1 symplectomorphism $h : \mathbb{R}^{2q} \to \mathbb{R}^{2q}$ satisfying:*

(i) *$h(z) = z$ for every $z \notin \mathcal{C}$ and $h(z) = R(z)$ for every $z \in \sigma\mathcal{C}$;*

(ii) *$\|h(z) - z\| < \varepsilon_0$ and $\|Dh_z - I\| < \varepsilon_0$ for all $z \in \mathbb{R}^{2q}$.*

Remark 5.10. If $H : \mathbb{R}^{2q} \to \mathbb{R}$ is a smooth function with bounded $\|DH\|$ and $\|D^2H\|$, then the associated Hamiltonian flow $\varphi^t : \mathbb{R}^{2q} \to \mathbb{R}^{2q}$ is defined for every time $t \in \mathbb{R}$, and

$$\|\varphi^t(z) - z\| \leq |t| \sup \|DH\|, \qquad \|(D\varphi^t)_z - I\| \leq \exp(|t| \sup \|D^2H\|) - 1$$

for every $z \in \mathbb{R}^{2q}$ and $t \in \mathbb{R}$.

Proof of Lemma 5.9. Given ε_0 and σ, let $\bar{t} > 0$ be small, to be specified later. Let $\varepsilon > 0$ be such that $\varepsilon < \sqrt{2}\sin\bar{t}$. Let X, Y, $\mathcal{A}$, $\mathcal{B}$, and R be as in the statement. Let $A : X \to X$ be a linear map such that $A(\mathcal{A})$ is the unit ball in X. We define $\tau = \|A\|$.

Let $H : \mathbb{R}^{2q} \to \mathbb{R}$ be defined by $H(x,y) = H(y) = \frac{1}{2}\|y\|^2$, where (x,y) are coordinates with respect to the splitting $X \oplus Y$. The Hamiltonian flow of H is a linear flow $(R_t)_t$, where R_t is a rotation of angle t in the plane Y, with axis X. In particular, $\|R_t - I\| = \sqrt{2}\,|\sin t|$ and there exists t_0 with $|t_0| < \bar{t}$ such that $R_{t_0} = R$.

Take numbers a, $b > 0$ such that $a/b > \tau$ and the cylinder $\mathcal{C} = a\mathcal{A} \oplus b\mathcal{B}$ has diameter less than ε_0. We are going to construct another Hamiltonian $\widetilde{H}$ which is equal to H inside $\sigma\mathcal{C}$ and constant outside $\mathcal{C}$. The symplectomorphism h will be defined as the time t_0 of the Hamiltonian flow associated to $\widetilde{H}$.

For this we need a few auxiliary functions. Let $\zeta : \mathbb{R} \to [0,1]$ be a smooth function such that:

- $\zeta(t) = 1$ for $t \le \sigma$ and $\zeta(t) = 0$ for $t \ge 1$;
- $|\zeta'(t)| \le 10/(1-\sigma)$ and $|\zeta''(t)| \le 10/(1-\sigma)^2$.

Define $\psi : X \to [0,1]$ by $\psi(x) = \zeta(a^{-1}\|Ax\|)$. Then

$$\psi(x) = 1 \text{ for } x \in \sigma a\mathcal{A} \quad \text{and} \quad \psi(x) = 0 \text{ for } x \notin a\mathcal{A}. \tag{5.3}$$

Let K_1 be an upper bound for the norms of the first and second derivatives of the function $x \in X \mapsto \zeta(\|x\|)$. (Notice that K_1 depends only on σ.) Then we have

$$\|D\psi\| \le K_1 a^{-1}\|A\| \quad \text{and} \quad \|D^2\psi\| \le K_1 a^{-2}\|A\|^2.$$

Now define $\rho : \mathbb{R} \to \mathbb{R}$ by $\rho(s) = \int_0^s \zeta$ and then let $\phi : Y \to \mathbb{R}$ be given by $\phi(y) = \frac{1}{2}b^2\rho(b^{-1}\|y\|)^2$. Then

$$\phi(y) = H(y) \text{ for } y \in \sigma b\mathcal{B} \quad \text{and} \quad \phi(y) = c \text{ for } y \notin b\mathcal{B}, \tag{5.4}$$

where $0 < c < \frac{1}{2}b^2$ is a constant. Besides, we can find $K_2 > 0$, depending only on σ, such that

$$\|D\phi\| \le K_2 b \quad \text{and} \quad \|D^2\phi\| \le K_2.$$

Define $\widetilde{H} : \mathbb{R}^{2q} \to \mathbb{R}$ by $\widetilde{H}(x,y) = c - \psi(x)(c - \phi(y))$. Then, by (5.3) and (5.4),

$$\begin{gathered} x \in \sigma a\mathcal{A} \quad \text{and} \quad y \in \sigma b\mathcal{B} \Rightarrow \widetilde{H}(x,y) = H(y), \\ x \notin a\mathcal{A} \quad \text{or} \quad y \notin b\mathcal{B} \Rightarrow \widetilde{H}(x,y) = c. \end{gathered} \tag{5.5}$$

The derivatives of $\widetilde{H}$ are (write $v = v_x + v_y \in X \oplus Y$ and analogously for w)

$$\begin{aligned} D\widetilde{H}_{(x,y)}(v) &= -\,(c - \phi(y))D\psi_x(v_x) + \psi(x)D\phi_y(v_y), \\ D^2\widetilde{H}_{(x,y)}(v,w) &= -\,(c - \phi(y))D^2\psi_x(v_x,w_x) + D\psi_x(v_x)D\phi_y(w_y) \\ &\quad + D\psi_x(w_x)D\phi_y(v_y) + \psi(x)D^2\phi_y(v_y,w_y). \end{aligned}$$

Using the previous bounds we obtain

$$\|D^2\widetilde{H}\| \leq \tfrac{1}{2}K_1 b^2 a^{-2}\|A\|^2 + 2K_1K_2 b a^{-1}\|A\| + K_2.$$

Since $a/b > \|A\|$, we conclude that $\|D^2\widetilde{H}\|$ is bounded by some K that depends only on σ.

Take $h : \mathbb{R}^{2q} \to \mathbb{R}^{2q}$ to be the time t_0 map of the Hamiltonian flow associated to $\widetilde{H}$. Property (i) in Lemma 5.9 follows from (5.5). Since $\operatorname{diam}\mathcal{C} < \varepsilon_0$, we have $\|h(z) - z\| < \varepsilon_0$ for all z. By Remark 5.10, $\|Dh_z - I\| \leq e^{t_0 K} - 1 \leq e^{\bar{t}K} - 1 < \varepsilon_0$, if $\bar{t} = \bar{t}(\varepsilon_0, \sigma)$ is small enough. This proves (ii) and the lemma. □

An ellipse $\mathcal{B}$ contained in a 2-dimensional symplectic subspace $Y \subset \mathbb{R}^{2q}$ and centered at the origin has *eccentricity* E if it is the image of the unit ball under a linear transformation $B : Y \to Y$ with $\|B\|/\mathbf{m}(B) = E^2$. If a map $\widehat{R} : Y \to Y$ preserves the ellipse $\mathcal{B}$, then $B^{-1}\widehat{R}B$ is a rotation of the plane Y of some angle θ. In this case we say that $\widehat{R}$ *rotates the ellipse* $\mathcal{B}$ *through angle* θ.

The following statement is a more flexible version of Lemma 5.9. In fact, it follows from Lemma 5.9 just by a change of the inner product.

LEMMA 5.11. *Given* $\varepsilon_0 > 0$, $0 < \sigma < 1$, $\gamma > 0$ *and* $E > 1$, *there is* $\beta > 0$ *with the following properties*: *Suppose there are given*:

- *a splitting* $\mathbb{R}^{2q} = X \oplus Y$ *with* $\dim Y = 2$, $X^\omega = Y$ *and* $\sphericalangle(X, Y) \geq \gamma$;
- *an ellipsoid* $\mathcal{A} \subset X$ *centered at the origin*;
- *an ellipse* $\mathcal{B} \subset Y$ *centered at the origin and with eccentricity at most* E;
- *a map* $\widehat{R} : Y \to Y$ *that rotates* $\mathcal{B}$ *through angle* θ, *with* $|\theta| < \beta$.

Then there exists $\tau > 1$ *such that the following holds. Let* $R : \mathbb{R}^{2q} \to \mathbb{R}^{2q}$ *be the linear map defined by* $R(v) = v$ *if* $v \in X$ *and* $R(w) = \widehat{R}(w)$ *if* $w \in Y$. *For* $a, b > 0$ *consider the cylinder* $\mathcal{C} = \mathcal{C}_{a,b} = a\mathcal{A} \oplus b\mathcal{B}$. *If* $a > \tau b$ *and* $\operatorname{diam}\mathcal{C} < \varepsilon_0$ *then there is a* C^1 *symplectomorphism* $h : \mathbb{R}^{2q} \to \mathbb{R}^{2q}$ *satisfying*:

(i) $h(z) = z$ *for every* $z \notin \mathcal{C}$ *and* $h(z) = R(z)$ *for every* $z \in \sigma\mathcal{C}$;

(ii) $\|h(z) - z\| < \varepsilon_0$ *and* $\|Dh_z - I\| < \varepsilon_0$ *for all* $z \in \mathbb{R}^{2q}$.

Now Lemma 5.8 is proved in the same way as we proved Lemma 3.3, using Lemmas 5.11 and 3.5. The argument is even a bit simpler since no truncation (as in Lemma 3.6) is necessary, since we assume that the angles $\sphericalangle(X_j, Y_j)$ are bounded from zero. The details are left to the reader.

5.3. *Proof of Proposition* 5.1. We use the following lemma, which was also needed for example 4 in the introduction:

LEMMA 5.12. *Let $G \subset \mathrm{GL}(d,\mathbb{R})$ be a closed group which acts transitively in $\mathbb{R}\mathrm{P}^{d-1}$. Then for every $\varepsilon_1 > 0$ there exists $\alpha > 0$ such that if v_1, $v_2 \in \mathbb{R}\mathrm{P}^{d-1}$ satisfy $\sphericalangle(v_1, v_2) < \alpha$ then there exists $R \in G$ such that $\|R - I\| < \varepsilon_1$ and $R(v_1) = v_2$.*

Proof. For $\delta > 0$, let $U_\delta = \{R;\ R \in G,\ \|R - I\| < \delta\}$. Given $\varepsilon > 0$, fix $\delta > 0$ such that if R_1, $R_2 \in U_\delta$ then $R_2 R_1^{-1} \in U_{\varepsilon_1}$. The hypothesis on the group implies that for any $w \in \mathbb{R}\mathrm{P}^{d-1}$, the map $G \to \mathbb{R}\mathrm{P}^{d-1}$ given by $A \mapsto A(w)$ is open (this follows from [15, Th. II.3.2]). Therefore, for any $\delta > 0$, the set $U_\delta(w) = \{Rw;\ R \in U_\delta\}$ is an open neighborhood of w. Cover $\mathbb{R}\mathrm{P}^{d-1}$ by some finite union $U_\delta(w_1) \cup \cdots \cup U_\delta(w_k)$. Now take two directions v_1, $v_2 \in \mathbb{R}\mathrm{P}^{d-1}$ sufficiently close. Then both belong to some $U_\delta(w_i)$, and so there are R_1, $R_2 \in U_\delta$ such that $v_1 = R_1 w_i$ and $v_2 = R_2 w_i$. Therefore $R = R_2 R_1^{-1}$ belongs to U_{ε_1} and $Rv_1 = v_2$. $\square$

Proof of Proposition 5.1. Let f, ε_0, κ be given. Fix $0 < \kappa' < \frac{1}{2}\kappa$. Let $\varepsilon > 0$, depending on f, ε_0, κ', be given by Lemma 5.3. Let $\alpha > 0$, depending on $\varepsilon_1 = \varepsilon$ and $G = \mathrm{U}(q)$, be given by Lemma 5.12. Take K satisfying $K \geq (\sin\alpha)^{-2}$ and $K \geq \max_x \|Df_x\|/\mathbf{m}(Df_x)$. Let $E > 1$ and $\gamma > 0$ be given by

$$E^2 = 8C_\omega^4 K(\sin\alpha)^{-4} \quad \text{and} \quad \sin\gamma = \tfrac{1}{2}C_\omega^{-14}K^{-2}\sin^9\alpha,$$

where C_ω is as in (2.4). Let $\beta > 0$ be given by Lemma 5.8. Finally, let $m \geq 2\pi/\beta$. The proof is divided into three cases.

First case. Suppose that there exists $\ell \in \{0, 1, \ldots, m\}$ such that

$$\sphericalangle(E_\ell, F_\ell) < \alpha. \tag{5.6}$$

Fix ℓ as above and take unit vectors $\xi \in E_\ell$, $\eta \in F_\ell$ such that $\sphericalangle(\xi, \eta) < \alpha$. By Lemma 5.12, there exists a unitary transformation $R : T_{f^\ell(y)}M \to T_{f^\ell(y)}M$ such that $\|R - I\| < \varepsilon$ and $R(\xi) = \eta$. By Lemma 5.3, the sequences $\{Df_{f^\ell(x)}\, R\}$ and $\{R\, Df_{f^{\ell-1}(x)}\}$ are (κ', ε_0)-realizable. Define $\{L_0, \ldots L_{m-1}\}$ as

$$\{Df_y, \ldots, Df_{f^{\ell-1}(y)}, Df_{f^\ell(y)}\, R, Df_{f^{\ell+1}(y)}, \ldots, Df_{f^{m-1}(y)}\}$$

if $\ell < m$ and as $\{Df_y, \ldots, Df_{f^{m-2}(y)}, R\, Df_{f^{m-1}(y)}\}$ if $\ell = m$. In either case, this is a (κ, ε_0)-realizable sequence of length m at y, whose product $L_{m-1} \ldots L_0$ sends the direction $\mathbb{R}Df^{-\ell}(\xi) \subset E_0$ to the direction $\mathbb{R}Df^{m-\ell}(\eta) \subset F_m$.

Second case. Assume that there exist $k, \ell \in \{0, \ldots, m\}$ with $k < \ell$ and

$$\frac{\|Df^{\ell-k}_{f^k(y)}|_{F_k}\|}{\mathbf{m}(Df^{\ell-k}_{f^k(y)}|_{E_k})} > K. \tag{5.7}$$

The proof of this case is easily adapted from the second case in the proof of Proposition 3.1. We leave the details to the reader.

Third case. We suppose that we are not in the previous cases, that is,

$$\text{for every } j \in \{0, 1, \ldots, m\}, \quad \sphericalangle(E_j, F_j) \geq \alpha, \tag{5.8}$$

and

$$\text{for every } i, j \in \{0, \ldots, m\} \text{ with } i < j, \quad \frac{\|Df^{j-i}_{f^i(y)}|_{F_i}\|}{\mathbf{m}(Df^{j-i}_{f^i(y)}|_{E_i})} \leq K. \tag{5.9}$$

By (5.8) and Lemma 2.3, we have, for all $i, j \in \{0, \ldots, m\}$ with $i < j$,

$$C_\omega^{-2} \sin\alpha \leq \mathbf{m}(Df^{j-i}|_{E_i})\, \|Df^{j-i}|_{F_i}\| \leq C_\omega^2 (\sin\alpha)^{-1}. \tag{5.10}$$

This, together with (5.9), gives

$$\mathbf{m}(Df^{j-i}|_{E_i}) \geq C_\omega^{-1} K^{-1/2} (\sin\alpha)^{1/2}, \tag{5.11}$$

$$\|Df^{j-i}|_{F_i}\| \leq C_\omega K^{1/2} (\sin\alpha)^{-1/2}. \tag{5.12}$$

Also, by (5.10) and the main assumption (5.1),

$$\mathbf{m}(Df^m|_{E_0}) \leq 2^{1/2} C_\omega (\sin\alpha)^{-1/2}, \tag{5.13}$$

$$\|Df^m|_{F_0}\| \geq 2^{-1/2} C_\omega^{-1} (\sin\alpha)^{1/2}. \tag{5.14}$$

Let $v_0 \in E_0$ be such that $\|v_0\| = 1$ and $\|Df^m_y(v_0)\| = \mathbf{m}(Df^m_y|_{E_0})$. Using Lemma 2.3.1, take $w_0 \in F_0$ with $\|w_0\| = 1$ such that $|\omega(v_0, w_0)| \geq C_\omega^{-1} \sin\alpha$. Let $G_0 = E_0 \cap w_0^\omega$ and $H_0 = F_0 \cap v_0^\omega$. (By v^ω we mean $(\mathbb{R}v)^\omega$.) Notice that $E_0 = \mathbb{R}v_0 \oplus G_0$ and $F_0 = \mathbb{R}w_0 \oplus H_0$. Let $X_0 = G_0 \oplus H_0$ and $Y_0 = \mathbb{R}v_0 \oplus \mathbb{R}w_0$. Then $X_0 = Y_0^\omega$. Let, for $j = 1, \ldots, m$,

$$\begin{aligned} v_j &= Df^j(v_0)/\|Df^j(v_0)\|, & G_j &= Df^j(G_0), & X_j &= Df^j(X_0),\\ w_j &= Df^j(w_0)/\|Df^j(w_0)\|, & H_j &= Df^j(H_0), & Y_j &= Df^j(Y_0) \end{aligned}$$

(all the derivatives are at y). By (2.5),

$$C_\omega^{-1} \sin\alpha \leq |\omega(v_0, w_0)| = |\omega(Df^m v_0, Df^m w_0)| \leq C_\omega \|Df^m v_0\|\, \|Df^m w_0\|.$$

Thus $\|Df^m w_0\| \geq C_\omega^{-2} \sin\alpha \cdot \mathbf{m}(Df^m|_{E_0})^{-1}$ and, by (5.10),

$$\|Df^m w_0\| \geq C_\omega^{-4} \sin^2\alpha \cdot \|Df^m|_{F_0}\|; \tag{5.15}$$

that is, w_0 is "almost" the most expanded vector by Df^m in F_0. Hence, by (5.1),

$$\frac{\|Df^m w_0\|}{\|Df^m v_0\|} \geq C_\omega^{-4} \sin^2\alpha \frac{\|Df^m|_{F_0}\|}{\mathbf{m}(Df^m|_{E_0})} \geq \tfrac{1}{2} C_\omega^{-4} \sin^2\alpha.$$

This and (5.9) imply that, for each $j = 1, \ldots, m$,

$$K \geq \frac{\|Df^j w_0\|}{\|Df^j v_0\|} \geq \frac{\|Df^m w_0\| / \|Df^{m-j}|_{F_j}\|}{\|Df^m v_0\| / \mathbf{m}(Df^{m-j}|_{E_j})} \geq \tfrac{1}{2} C_\omega^{-4} K^{-1} \sin^2\alpha.$$

Therefore, by (5.8) and Lemma 2.8,

$$\frac{\|Df^j|_{Y_0}\|}{\mathbf{m}(Df^j|_{Y_0})} \le 8C_\omega^4 K(\sin\alpha)^{-4} = E^2. \tag{5.16}$$

We now deduce some angle estimates. First, we claim that

$$\sin\sphericalangle(v_0, G_0) \ge C_\omega^{-2}\sin\alpha \quad \text{and} \quad \sin\sphericalangle(w_0, H_0) \ge C_\omega^{-2}\sin\alpha. \tag{5.17}$$

Indeed, write $v_0 = u+u'$ with $u' \in G_0$ and $u \perp G_0$. Since G_0 is skew-orthogonal to w_0,

$$C_\omega^{-1}\sin\alpha \le |\omega(v_0, w_0)| = |\omega(u, w_0)| \le C_\omega\|u\|.$$

That is, $\sin\sphericalangle(v_0, G_0) = \|u\| \ge C_\omega^{-2}\sin\alpha$. Analogously we prove the other inequality in (5.17). Next, we estimate $\sin\sphericalangle(v_j, G_j)$ and $\sin\sphericalangle(w_j, H_j)$ for $j = 1, \dots, m$. For this we use relation (2.6) from subsection 2.4, which gives:

$$\sin\sphericalangle(v_j, G_j) \ge \frac{\mathbf{m}(Df^j|_{E_0})}{\|Df^j v_0\|}\sin\sphericalangle(v_0, G_0), \tag{5.18}$$

$$\sin\sphericalangle(w_j, H_j) \ge \frac{\|Df^j w_0\|}{\|Df^j|_{F_0}\|}\sin\sphericalangle(w_0, H_0). \tag{5.19}$$

By (5.11) and (5.13),

$$\|Df^j v_0\| = \frac{\|Df^m v_0\|}{\|Df^{m-j} v_j\|} \le \frac{\mathbf{m}(Df^m|_{E_0})}{\mathbf{m}(Df^{m-j}|_{E_j})} \le 2^{1/2}C_\omega^2 K^{1/2}(\sin\alpha)^{-1}$$

for each $j = 1, \dots, m$. So, by (5.11) again,

$$\frac{\|Df^j v_0\|}{\mathbf{m}(Df^j|_{E_0})} \le 2^{1/2}C_\omega^3 K(\sin\alpha)^{-3/2}.$$

This, together with (5.17) and (5.18), gives

$$\sin\sphericalangle(v_j, G_j) \ge 2^{-1/2}C_\omega^{-5}K^{-1}(\sin\alpha)^{5/2}. \tag{5.20}$$

Similarly, by (5.15), (5.12), and (5.14),

$$\|Df^j w_0\| = \frac{\|Df^m w_0\|}{\|Df^{m-j} w_j\|} \ge C_\omega^{-4}\sin^2\alpha\,\frac{\|Df^m|_{F_0}\|}{\|Df^{m-j}|_{F_0}\|} \ge 2^{-1/2}C_\omega^{-6}K^{-1/2}\sin^3\alpha.$$

Then, using (5.12) again, we get

$$\frac{\|Df^j w_0\|}{\|Df^j|_{F_0}\|} \ge 2^{-1/2}C_\omega^{-7}K^{-1}(\sin\alpha)^{7/2}.$$

By (5.17) and (5.19),

$$\sin\sphericalangle(w_j, H_j) \ge 2^{-1/2}C_\omega^{-9}K^{-1}(\sin\alpha)^{9/2}. \tag{5.21}$$

Now we use Lemma 2.6 three times:

$$\begin{aligned}\sin\sphericalangle(Y_j,X_j) &\geq \sin\sphericalangle(v_j,X_j)\,\sin\sphericalangle(w_j,\mathbb{R}v_j\oplus X_j)\\ &\geq \sin\sphericalangle(v_j,G_j)\,\sin\sphericalangle(E_j,H_j)\,\sin\sphericalangle(w_j,\mathbb{R}v_j\oplus X_j)\\ &\geq \sin\sphericalangle(v_j,G_j)\,\sin\sphericalangle(w_j,H_j)\,\sin^2\sphericalangle(E_j,H_j)\end{aligned}$$

So, using (5.20), (5.21), and $\sphericalangle(E_j,H_j)\geq\alpha$, we obtain

$$\sin\sphericalangle(X_j,Y_j)\geq \tfrac{1}{2}C_\omega^{-14}K^{-2}\sin^9\alpha=\sin\gamma. \tag{5.22}$$

Relations (5.16) and (5.22) permit us to apply Lemma 5.8. Since $m\beta\geq 2\pi$, it is possible to choose numbers $\theta_0,\ldots,\theta_{m-1}$ such that $0\leq\theta_j\leq\beta$ and $\sum\theta_j=\sphericalangle(v_0,w_0)$. Let S_j and L_j be as in Lemma 5.8. We have $L_{m-1}\ldots L_0|_{Y_0}=(Df^m|_{Y_0})S_{m-1}\ldots S_0$, so that $L_{m-1}\ldots L_0(\mathbb{R}v_0)=\mathbb{R}w_m$. This completes the proof of Proposition 5.1. □

6. Proofs of Theorems 3 and 4

Given $f\in\mathrm{Diff}^1_\mu(M)$ and $m\in\mathbb{N}$, let $\mathcal{D}(f,m)$ be the (closed) set of points x such that there is an m-dominated splitting of index $q=d/2$ along the orbit of x. Let $\Gamma(f,m)=M\setminus\mathcal{D}(f,m)$ and let $\Gamma^*(f,m)$ be the set of points $x\in\Gamma(f,m)$ which are regular, not periodic, and satisfy $\lambda_q(f,x)>0$. Let also $\Gamma(f,\infty)=\bigcap_{m\in\mathbb{N}}\Gamma(f,m)$.

We have the following symplectic analogues of Propositions 4.2, 4.8 and 4.17:

Proposition 6.1. *Let $f\in\mathrm{Sympl}^1_\omega(M)$, $\varepsilon_0>0$, $\delta>0$, and $0<\kappa<1$. If $m\in\mathbb{N}$ is sufficiently large, then there exists a measurable function $N:\Gamma^*(f,m)\to\mathbb{N}$ such that for a.e. $x\in\Gamma^*(f,m)$ and every $n\geq N(x)$ there exists an (ε_0,κ)-realizable sequence $\{\widehat{L}_0,\ldots,\widehat{L}_{n-1}\}$ at x of length n such that*

$$\frac{1}{n}\log\|\wedge^q(\widehat{L}_{n-1}\ldots\widehat{L}_0)\|\;\leq\;\Lambda_{q-1}(f,x)+\delta.$$

Proposition 6.2. *Let $f\in\mathrm{Sympl}^1_\omega(M)$, $\varepsilon_0>0$ and $\delta>0$. Then there exist $m\in\mathbb{N}$ and a diffeomorphism $g\in\mathcal{U}(f,\varepsilon_0)$ that equals f outside the open set $\Gamma(f,m)$ and such that*

$$\int_{\Gamma(f,m)}\Lambda_q(g,x)\,d\mu(x)<\delta+\int_{\Gamma(f,m)}\Lambda_{q-1}(f,x)\,d\mu(x).$$

Proposition 6.3. *Given $f\in\mathrm{Sympl}^1_\omega(M)$, let*

$$J(f)=\int_{\Gamma(f,\infty)}\lambda_q(f,x)\,d\mu(x).$$

Then for every $\varepsilon_0 > 0$ and $\delta > 0$, there exists a diffeomorphism $g \in \mathcal{U}(f, \varepsilon_0)$ such that

$$\int_M \Lambda_q(g,x)\, d\mu(x) < \int_M \Lambda_q(f,x)\, d\mu(x) - J(f) + \delta.$$

The proofs of these propositions are exactly the same as those of the corresponding results in Section 4, in the following logical order:

Proposition 5.1 $\Rightarrow$ Proposition 6.1 $\Rightarrow$ Proposition 6.2 $\Rightarrow$ Proposition 6.3.

Concerning the first implication, notice that if $x \in \Gamma^*(f,m)$ then, by Lemma 2.4, the spaces E_x^+ and E_x^- (that correspond to positive and negative Lyapunov exponents) are Lagrangian, so that Proposition 5.1 applies.

6.1. *Conclusion of the proofs of Theorems* 3 *and* 4.

Proof of Theorem 3. Let $f \in \mathrm{Sympl}_\omega^1(M)$ be a point of continuity of the map $\mathrm{LE}_q(\cdot)$. By Proposition 6.3, $J(f) = 0$, that is, $\lambda_q(f,x) = 0$ for a.e. $x \in \Gamma(f,\infty)$. Let $x \in M$ be a regular point. If $\lambda_q(f,x) > 0$, we have (if we exclude a zero measure set of x) $x \notin \Gamma(f,\infty)$. This means that there is a dominated splitting, $T_{f^n(x)}M = E_n \oplus F_n$, $n \in \mathbb{Z}$ of index q, along the orbit of x. Then E_n is the sum of the Oseledets spaces of f, at the point $f^n x$, associated to the Lyapunov exponents $\lambda_1(f,x), \ldots, \lambda_q(f,x)$, and F_n is the sum of the spaces associated to the other exponents. By part 2 of Lemma 2.4, the splitting $T_{f^n(x)}M = E_n \oplus F_n$, $n \in \mathbb{Z}$, is hyperbolic. $\square$

The next proposition is used to deduce Theorem 4 from Theorem 3.

PROPOSITION 6.4. *There is a residual subset $\mathcal{R}_2 \subset \mathrm{Sympl}_\omega^1(M)$ such that if $f \in \mathcal{R}_2$ then either f is Anosov or every hyperbolic set of f has measure* 0.

Proof. This is a modification of an argument from [20]. We use the fact, proved in [30], that C^2 diffeomorphisms are dense in the space $\mathrm{Sympl}_\omega^1(M)$. Another key ingredient is that the hyperbolic sets of any C^2 non-Anosov diffeomorphism have zero measure. We comment on the latter near the end.

For each open set $U \subset M$ with $\overline{U} \neq M$ and each $f \in \mathrm{Sympl}_\omega^1(M)$, consider the maximal f-invariant set inside $\overline{U}$,

$$\Lambda_f(U) = \bigcap_{n\in\mathbb{Z}} f^n(\overline{U}).$$

For $\varepsilon > 0$, let $D(\varepsilon, U)$ be the set of diffeomorphisms $f \in \mathrm{Sympl}_\omega^1(M)$ such that at least one of the following properties is satisfied:

(i) There is a neighborhood $\mathcal{U}$ of f such that $\Lambda_g(U)$ is not hyperbolic for all $g \in \mathcal{U}$;

(ii) $\mu(\Lambda_f(U)) < \varepsilon$.

Clearly, the set $D(\varepsilon, U)$ is open. Moreover, it is dense. Indeed, if f does not satisfy (i) then there is g close to f such that $\Lambda_g(U)$ is hyperbolic. Take $f_1 \in C^2$ close to g in $\mathrm{Sympl}^1_\omega(M)$. Then $\Lambda_{f_1}(U)$ is hyperbolic with measure zero, and so $f_1 \in D(\varepsilon, U)$. This proves denseness. Hence the set

$$\begin{aligned} D(U) &= \cap_{\varepsilon>0} D(\varepsilon, U) \\ &\supset \{f \in \mathrm{Sympl}^1_\omega(M);\ \Lambda_f(U) \text{ is hyperbolic} \Rightarrow \mu(\Lambda_f(U)) = 0\} \end{aligned}$$

is residual. Now take $\mathcal{B}$ a countable basis of open sets of M and let $\widehat{\mathcal{B}}$ be the set of all finite unions of sets in $\mathcal{B}$. The set

$$\mathcal{R}_2 = \bigcap_{U \in \widehat{\mathcal{B}}, \overline{U} \neq M} D(U)$$

is residual in $\mathrm{Sympl}^1_\omega(M)$ and the hyperbolic sets for every non-Anosov $f \in \mathcal{R}$ have zero measure.

Finally, we explain why all hyperbolic sets of a C^2 non-Anosov diffeomorphism have zero measure. This is well-known for hyperbolic *basic* sets; see [11]. We just outline the arguments in the general case. Suppose f has a hyperbolic set Λ with $\mu(\Lambda) > 0$. Using absolute continuity of the unstable lamination, we get that $\mu_u(W^u_\varepsilon(x) \cap \Lambda) > 0$ for some $x \in \Lambda$, where μ_u denotes Lebesgue measure along unstable manifolds. By bounded distortion and a density point argument, we find points $x_k \in \Lambda$ such that $\mu_u(W^u_\varepsilon(x_k) \setminus \Lambda)$ converges to zero. Taking an accumulation point x_0 we get that $W^u_\varepsilon(x_0) \subset \Lambda$. We may suppose that every point of Λ is in the support of $\mu|\Lambda$. In particular, there are recurrent points of Λ close to x_0. Applying the shadowing lemma, we find a hyperbolic periodic point p_0 close to x_0. In particular, $W^s_\varepsilon(p_0)$ intersects $W^u_\varepsilon(x_0)$ transversely. Using the λ-lemma we conclude that the whole $W^u(p_0)$ is contained in Λ. Define Λ_0 as the closure of the unstable manifold of the orbit of p_0. This is a hyperbolic set contained in Λ, and it consists of entire unstable manifolds. Hence, $W^s(\Lambda_0)$ is an open neighborhood of Λ_0. Using the fact that f preserves volume, we check that $f(W^s_\varepsilon(\Lambda_0)) = W^s_\varepsilon(\Lambda_0)$. This implies that $W^s(\Lambda_0) = \Lambda_0$ and so, by connectedness, Λ_0 must be the whole M. Consequently, f is Anosov. □

Proof of Theorem 4. It suffices to take $\mathcal{R} = \mathcal{R}_1 \cap \mathcal{R}_2$ with $\mathcal{R}_1$ a residual set of continuity points of $f \mapsto \mathrm{LE}_q(f)$, and $\mathcal{R}_2$ as in Proposition 6.4. □

7. Proof of Theorem 5

Let M be a compact Hausdorff space, μ a Borel regular measure and $f : M \to M$ a homeomorphism preserving the measure μ. Let S be an accessible subset of $\mathrm{GL}(d, \mathbb{R})$, according to Definition 1.2.

The following result provides an analogue of Proposition 3.1:

Proposition 7.1. *Given $A \in C(M,S)$ and $\varepsilon > 0$, if $\widetilde{m} \in \mathbb{N}$ is large enough then the following holds*:

Let $y \in M$ be a nonperiodic point and suppose it is given a nontrivial splitting $\mathbb{R}^d = E \oplus F$ such that

$$\frac{\|A^{\widetilde{m}}(y)|_F\|}{\mathbf{m}(A^{\widetilde{m}}(y)|_E)} \geq \frac{1}{2}. \tag{7.1}$$

Then there exist matrices $L_0, \dots, L_{\widetilde{m}-1} \in S$, with $\|L_j - A(f^j y)\| < \varepsilon$, such that $L_{\widetilde{m}-1} \dots L_0(v) = w$ for some nonzero vectors $v \in E$ and $w \in A^{\widetilde{m}}(y)(F)$.

Proof. Let C_0 be such that $\|A(x)^{\pm 1}\| \leq C_0$ for all $x \in M$. Let $\nu \in \mathbb{N}$ and $\alpha > 0$, depending on C_0 and ε, be as in Definition 1.2. Let

$$K = \max\{(\sin\alpha)^{-1}, C_0^2\}, \quad \text{and} \quad C = 8C_0^2 K (\sin\alpha)^{-2}.$$

Given $\widetilde{m}$ large, let $m \in \mathbb{N}$ be such that $1 \leq \widetilde{m} - \nu m \leq \nu$. We assume $\widetilde{m}$ is large enough so that $m > 2C/\alpha$.

Now take y, E and F as in the statement. For $j = 0, 1, \dots, m-1$, let $A_j = A(f^{\nu j} y)$, $E_j = A^{\nu j}(x)(E)$, $F_j = A^{\nu j}(x)(F)$. (We disregard times which are not multiples of ν.) As before, we divide the rest of the proof into three cases:

First case. There exists $\ell \in \{0, \dots, m\}$ such that

$$\sphericalangle(E_\ell, F_\ell) < \alpha. \tag{7.2}$$

With ℓ as above, we take $\xi \in E_\ell$, $\eta \in F_\ell$ such that $\sphericalangle(\xi, \eta) < \alpha$. By definition of accessibility, there are $\widetilde{A}_0, \dots, \widetilde{A}_{\nu-1}$ such that $\|\widetilde{A}_i - A(f^{\nu\ell+i}x)\| < \varepsilon$ and $\widetilde{A}_{\nu-1} \dots \widetilde{A}_0(\mathbb{R}\xi) = A^\nu(f^{\nu\ell}y)(\mathbb{R}\eta)$. We define $L_j = \widetilde{A}_{j-\nu\ell}$ for $\nu\ell \leq j < \nu(\ell+1)$ and $L_j = A(f^j y)$ for the remaining j's. Then $\{L_0, \dots, L_{\widetilde{m}-1}\}$ has the required properties.

Second case. Assume that there exist k, $\ell \in \{0, \dots, m\}$ such that $k < \ell$ and

$$\frac{\|A_{\ell-1} \dots A_k|_{F_k}\|}{\mathbf{m}(A_{\ell-1} \dots A_k|_{E_k})} > K. \tag{7.3}$$

Once more, this is similar to the second case in Propositions 3.1 and 5.1. We leave it to the reader to spell-out the details.

Third case. We suppose that we are not in the previous cases, that is, we assume

$$\text{for every } j \in \{0, 1, \dots, m\}, \quad \sphericalangle(E_j, F_j) \geq \alpha, \tag{7.4}$$

and

$$\text{for every } i, j \in \{0, \dots, m\} \text{ with } i < j, \quad \frac{\|A_{j-1} \dots A_i|_{F_i}\|}{\mathbf{m}(A_{j-1} \dots A_i|_{E_i})} \leq K. \tag{7.5}$$

Take unit vectors $\xi \in E_0$ and $\eta \in F_0$ such that

$$\|A_{m-1} \dots A_0(\xi)\| = \mathbf{m}(A_{m-1} \dots A_0|_{E_0})$$

and

$$\|A_{m-1} \dots A_0(\eta)\| = \|A_{m-1} \dots A_0|_{F_0}\|.$$

Let $\xi_j = A_{j-1} \dots A_0(\xi)$, $\eta_j = A_{j-1} \dots A_0(\eta)$ and $Y_j = \mathbb{R}\xi_j \oplus \mathbb{R}\eta_j$.

The assumption (7.1) gives

$$\frac{\|A_{m-1} \dots A_0(\eta)\|}{\|A_{m-1} \dots A_0(\xi)\|} \geq \frac{\|A^{\widetilde{m}}(y)|_F\| \,/\, \|A^{\widetilde{m}-\nu m}(y)|_F\|}{\mathbf{m}(A^{\widetilde{m}}(y)|_E) \,/\, \mathbf{m}(A^{\widetilde{m}-\nu m}(y)|_E)} \geq \frac{1}{2C_0^{2\nu}}.$$

By (7.5), for any $j = 1, \dots, m$,

$$K \geq \frac{\|A_{j-1} \dots A_0(\eta)\|}{\|A_{j-1} \dots A_0(\xi)\|} \geq \frac{\|A_{m-1} \dots A_0(\eta)\| \,/\, \|A_{m-1} \dots A_j\|}{\|A_{m-1} \dots A_0(\xi)\| \,/\, \mathbf{m}(A_{m-1} \dots A_j)} \geq \frac{1}{2C_0^{2\nu} K}.$$

This, together with (7.4) and Lemma 2.8 implies that, for all $j = 1, \dots, m$,

$$\frac{\|A_{j-1} \dots A_0|_{Y_0}\|}{\mathbf{m}(A_{j-1} \dots A_0|_{Y_0})} \leq C. \tag{7.6}$$

Assign orientations to the planes Y_j such that each $A_j|_{Y_j} : Y_j \to Y_{j+1}$ preserves orientation. Let P_j be the projective space of Y_j, with the induced orientation. Let $v_j = \mathbb{R}\xi_j$ and $w_j = \mathbb{R}\eta_j \in P_j$. For each $z \in P_j$, let $[z] \in [0, \pi)$ be the oriented angle between z and v_j. Now, $z \mapsto [z]$ is a bijection and $[z] \mapsto [A_j z]$ is monotonic. If $L : Y_0 \to Y_j$ is any linear map then, by Lemma 2.7,

$$0 < [z_2] - [z_1] \leq \frac{\pi}{2} \implies \frac{[Lz_2] - [Lz_1]}{[z_2] - [z_1]} \leq \frac{2}{\pi} \cdot \frac{\|L\|}{\mathbf{m}(L)}. \tag{7.7}$$

We define directions $u_0 \in P_0, \dots, u_m \in P_m$ by recurrence as follows: Let $[u_0] = 0$ and

$$[u_{j+1}] = \min\{[w_{j+1}], [A_j(u_j + \alpha)]\}. \tag{7.8}$$

We claim that $[u_m] = [w_m]$. Indeed, $[A_j u_j] \leq [u_{j+1}] \leq [w_{j+1}]$ for each $j < m$. Therefore, defining $[z_j] = [(A_{j-1} \dots A_0)^{-1} u_j]$, we have

$$0 = [z_0] \leq [z_1] \leq \dots \leq [z_m] \leq [w_0] < \pi.$$

In particular, $[z_{i+1}] - [z_i] < \pi/m$ for some $i = 0, \dots, m-1$. Hence, by (7.6) and (7.7),

$$[A_i^{-1} u_{i+1}] - [u_i] = [A_{i-1} \dots A_0 z_{i+1}] - [A_{i-1} \dots A_0 z_i] < 2C/m \leq \alpha.$$

Due to (7.8), this is only possible if $[u_{i+1}] = [w_{i+1}]$. This implies $[u_m] = [w_m]$.

Now, $\sphericalangle(A_j^{-1} u_{j+1}, u_j) < \alpha$ for each j, so that we can find $\widetilde{A}_{j,0}, \dots, \widetilde{A}_{j,\nu-1} \in S$ such that $\|\widetilde{A}_{j,k} - A(f^{\nu j+k} x)\| < \varepsilon$ and $\widetilde{A}_{j,\nu-1} \dots \widetilde{A}_{j,0}(\mathbb{R}u_j) = \mathbb{R}u_{j+1}$. We define the sequence $\{L_0, \dots, L_{\widetilde{m}-1}\}$ as

$$\{\widetilde{A}_{0,0}, \dots, \widetilde{A}_{j,\nu-1}, \dots, \widetilde{A}_{m-1,0}, \dots, \widetilde{A}_{m-1,\nu-1}, A(f^{\nu m} x), \dots, A(f^{\widetilde{m}-1} x)\}.$$

Then $L_{\widetilde{m}-1} \dots L_0(v_0) \in A^{\widetilde{m}}(y)(F)$. $\square$

Next we define sets $\Gamma_p(A,m)$, $\Gamma_p^*(A,m)$, $\Gamma_p^\sharp(A,m)$ for $p \in \{1,\dots,d-1\}$ and $m \in \mathbb{N}$, in the same way as in Section 4, with the obvious adaptations. Lemma 4.1 also applies in the present context.

Proposition 7.2. *Given $A \in C(M,S)$, $\varepsilon > 0$, $\delta > 0$, and $p \in \{1,\dots, d-1\}$, if $m \in \mathbb{N}$ is sufficiently large then there exists a measurable function $N : \Gamma_p^*(A,m) \to \mathbb{N}$ such that for a.e. $x \in \Gamma_p^*(A,m)$ and every $n \geq N(x)$ there exist matrices $\widehat{L}_0, \dots, \widehat{L}_{n-1} \in S$ such that $\|\widehat{L}_j - A(f^j x)\| < \varepsilon$ and*

$$\frac{1}{n} \log \|\wedge^p(\widehat{L}_{n-1} \dots \widehat{L}_0)\| \leq \frac{\Lambda_{p-1}(A,x) + \Lambda_{p+1}(A,x)}{2} + \delta.$$

The proof is the same as that for Proposition 4.2.

Proposition 7.3. *Let $A \in C(M,S)$, $\varepsilon_0 > 0$, $p \in \{1,\dots,d-1\}$ and $\delta > 0$. Then there exist $m \in \mathbb{N}$ and a cocycle $B \in C(M,S)$, with $\|B - A\|_\infty < \varepsilon_0$, that equals A outside the open set $\Gamma_p(A,m)$ and is such that*

$$\int_{\Gamma_p(A,m)} \Lambda_p(B,x)\, d\mu(x) < \delta + \int_{\Gamma_p(A,m)} \frac{\Lambda_{p-1}(A,x) + \Lambda_{p+1}(A,x)}{2}\, d\mu(x).$$

The proof of Proposition 7.3 is not just an adaptation of that of Proposition 4.8, because Vitali's lemma may not apply to M. We begin by proving a weaker statement, in Lemma 7.4. Let $L^\infty(M,S)$ denote the set of bounded measurable functions from M to S. Oseledets' theorem also applies for cocycles in $L^\infty(M,S)$.

Lemma 7.4. *Let $A \in C(M,S)$, with $\varepsilon_0 > 0$, $p \in \{1,\dots,d-1\}$ and $\delta > 0$. Then there exist $m \in \mathbb{N}$ and a cocycle $\widetilde{B} \in L^\infty(M,S)$, with $\|\widetilde{B} - A\|_\infty < \varepsilon_0/2$, that equals A outside the open set $\Gamma_p(A,m)$ and such that*

$$\int_{\Gamma_p(A,m)} \Lambda_p(\widetilde{B},x)\, d\mu(x) < \delta + \int_{\Gamma_p(A,m)} \frac{\Lambda_{p-1}(A,x) + \Lambda_{p+1}(A,x)}{2}\, d\mu(x).$$

Sketch of proof. We shall explain the necessary modifications of the proof of Proposition 4.8. The sets Z^i, $\widehat{Q}^i$ and Q^i are defined as before. In Lemma 4.14, the castles U^i and K^i become equal to Q^i (as κ and γ were 0). We decompose each base Q^i_{b} into finitely many disjoint measurable sets U^i_k with small diameter. In each tower with base U^i_k we construct the perturbation $\widetilde{B}$ using Proposition 7.2, taking $\widetilde{B}$ constant in each floor. The definitions of N and G^i are the same. In Lemma 4.16 several bounds (those involving κ or γ) become trivial. Then one concludes the proof in the same way as before. □

Proof of Proposition 7.3. Let A, ε_0, p and δ be as in the statement. Let m and $\widetilde{B}$ be given by Lemma 7.4. Let $N \in \mathbb{N}$ be such that

$$\int_{\Gamma_p(A,m)} \frac{1}{N} \log \|\wedge^p(\widetilde{B}^N(x))\| \, d\mu < 2\delta + \int_{\Gamma_p(A,m)} \frac{\Lambda_{p-1}(A,x) + \Lambda_{p+1}(A,x)}{2} \, d\mu.$$

Let $\gamma = N^{-1}\delta$. Using Lusin's theorem (see [28]) and the fact that S is a manifold, one finds a continuous $B : M \to S$ such that $B = \widetilde{B} = A$ outside the open set $\Gamma_p(A,m)$, the norm $\|B - \widetilde{B}\|_\infty < \varepsilon_0/2$, and the set $E = \{x \in M;\ B(x) \neq \widetilde{B}(x)\}$ has measure $\mu(E) < \gamma$. Let

$$G = \bigcap_{j=0}^{N-1} f^{-j}\big(\Gamma_p(A,m) \setminus E\big) \subset \Gamma_p(A,m).$$

Then $\mu\big(\Gamma_p(A,m) \setminus G\big) \leq N\mu(E) < \delta$. Now, letting C be an upper bound for $\log \|\wedge^p(\widetilde{B}(x))\|$, we have

$$\begin{aligned} \int_{\Gamma_p(A,m)} \Lambda_p(B,x) \, d\mu &\leq \int_{\Gamma_p(A,m)} \frac{1}{N} \log \|\wedge^p(B^N(x))\| \, d\mu \\ &< C\delta + 2\delta + \int_{\Gamma_p(A,m)} \frac{\Lambda_{p-1}(A,x) + \Lambda_{p+1}(A,x)}{2} \, d\mu. \end{aligned}$$

Up to replacement of δ with $\delta/(C+2)$, this completes the proof. $\square$

Using Proposition 7.3, one concludes the proof of Theorem 5 exactly as in subsection 4.3. The fact that either vanishing of the exponents or dominance of the splitting is also a sufficient condition for continuity is an easy consequence of semi-continuity of Lyapunov exponents and robustness of dominated splittings under small perturbations of the cocycle.

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