Available online at www.sciencedirect.com
SCIENCE DIRECT®
C. R. Acad. Sci. Paris, Ser. I 342 (2006) 763–766
COMPTES RENDUS MATHEMATIQUE
http://france.elsevier.com/direct/CRASS1/

Dynamical Systems

A remark on conservative diffeomorphisms

Jairo Bochi [a,1], Bassam R. Fayad [b], Enrique Pujals [c]

[a] *Inst. Matemática, UFRGS, Av Bento Gonçalves 9500, 91509-900 Porto Alegre, Brazil*
[b] *LAGA, université Paris 13, 99, avenue J.-B. Clément, 93430 Villetaneuse, France*
[c] *IMPA, Estr. D. Castorina 110, 22460-320 Rio de Janeiro, Brazil*

Received 27 August 2004; accepted after revision 21 March 2006

Available online 18 April 2006

Presented by Étienne Ghys

Abstract

We show that a stably ergodic diffeomorphism can be C^1 approximated by a diffeomorphism having stably non-zero Lyapunov exponents. ***To cite this article: J. Bochi et al., C. R. Acad. Sci. Paris, Ser. I 342 (2006).***

Résumé

Une remarque sur les difféomorphismes conservatifs. On montre qu'un difféomorphisme stablement ergodique peut être C^1 approché par un difféomorphisme ayant des exposants de Lyapunov stablement non-nuls. ***Pour citer cet article : J. Bochi et al., C. R. Acad. Sci. Paris, Ser. I 342 (2006).***

Two central notions in Dynamical Systems are ergodicity and hyperbolicity. In many works showing that certain systems are ergodic, some kind of hyperbolicity (e.g. uniform, non-uniform or partial) is a main ingredient in the proof. In this note the converse direction is investigated.

Let M be a compact manifold of dimension $d \geqslant 2$, and let μ be a volume measure in M. Take $\alpha > 0$ and let $\mathrm{Diff}^{1+\alpha}_\mu(M)$ be the set of μ-preserving $C^{1+\alpha}$ diffeomorphisms, endowed with the C^1 topology. Let $\mathcal{SE} \subset \mathrm{Diff}^{1+\alpha}_\mu(M)$ be the set of stably ergodic diffeomorphisms (i.e., the set of diffeomorphisms such that every sufficiently C^1-close $C^{1+\alpha}$ conservative diffeomorphism is ergodic).

Our result answers positively a question of [8]:

Theorem 1. *There is an open and dense set $\mathcal{R} \subset \mathcal{SE}$ such that if $f \in \mathcal{R}$ then f is non-uniformly hyperbolic, that is, all Lyapunov exponents of f are non-zero. Moreover, every $f \in \mathcal{R}$ admits a dominated splitting $TM = E^+ \oplus E^-$, where E^+* (*resp.* E^-) *coincides a.e. with the sum of the Oseledets spaces corresponding to positive* (*resp. negative*) *Lyapunov exponents.*

E-mail addresses: jairo@mat.ufrgs.br (J. Bochi), fayadb@math.univ-paris13.fr (B.R. Fayad), enrique@impa.br (E. Pujals).
[1] J.B. was supported by CNPq-Profix during the preparation of this work. J.B. thanks the hospitality of the LAGA – université de Paris 13.

1631-073X/$ – see front matter
doi:10.1016/j.crma.2006.03.028

Remark 1. The set $\mathcal{SE}$ contains all Anosov diffeomorphisms, and many partially hyperbolic ones – see e.g. [10]. It is not true that every stably ergodic diffeomorphism can be approximated by a partially hyperbolic system, see [13,7].

Remark 2. Let $\mathcal{SE}'$ be the set of diffeomorphisms $f \in \mathcal{SE}$ such that every power f^k, $k \geqslant 2$, is ergodic. Then every f in $\mathcal{SE}' \cap \mathcal{R}$ is Bernoulli. This follows from Theorem 1 and Pesin theory (see Theorem 5.10 in [11]).

The proof of Theorem 1 has three steps:

1. A stably ergodic (or stably transitive) diffeomorphism f must have a dominated splitting. This is true because if it did not, [6] permits us to perturb f and create a periodic point whose derivative is the identity. Then, using the Pasting lemma from [1] (for which $C^{1+\alpha}$ regularity is an essential hypothesis), one breaks transitivity.
2. A result of [5] gives a perturbation of f such that the sum of the Lyapunov exponents 'inside' each of the bundles of the (finest) dominated splitting is non-zero.
3. Using a result of [4], we find another perturbation such that the Lyapunov exponents in each of the bundles become almost equal. (If we attempted to make the exponents exactly equal, we could not guarantee that the perturbation is $C^{1+\alpha}$.) Since the sum of the exponents in each bundle varies continuously, we conclude there are no zero exponents.

Remark 3. The perturbation techniques of [5] and [4] in fact don't assume ergodicity, but are only able to control the integrated Lyapunov exponents. That is why we have to assume stable ergodicity (in place of stable transitivity) in Theorem 1.

Remark 4. Theorem 1 is stated in C^1 topology because in higher topologies the technology from [6,5], and [4] is not available. The $C^{1+\alpha}$ diffeomorphisms come from [1]. To get our result in C^1 topology (which perhaps would be more natural) one has to solve the following problem: any diffeomorphism having a periodic point tangent to the identity may be C^1-approximated by a non-transitive diffeomorphism.

Remark 5. Some ideas of the present proof were already present in [9].

Let us recall briefly the definition and some properties of dominated splittings, see [6] for details. Let $f \in \mathrm{Diff}^1_\mu(M)$.

A Df-invariant splitting $TM = E^1 \oplus \cdots \oplus E^k$, with $k \geqslant 2$, is called a *dominated splitting* (over M) if there are constants $c, \tau > 0$ such that

$$\frac{\|Df^n(x)\cdot v_j\|}{\|Df^n(x)\cdot v_i\|} < c\,\mathrm{e}^{-\tau n} \tag{1}$$

for all $x \in M$, all $n \geqslant 1$, and all unit vectors $v_i \in E^i(x)$ and $v_j \in E^j(x)$, provided $i < j$. (One can also define in the same way a dominated splitting over an f-invariant set.)

A dominated splitting is always continuous, that is, the spaces $E_i(x)$ depend continuously on x. Also, a dominated splitting persists under C^1-perturbations of the map. More precisely, if g is sufficiently close to f, then g has a dominated splitting $E^1_g \oplus \cdots \oplus E^k_g$, called the *continuation*, with $\dim E^i_g = \dim E^i$ and which coincides with the given one when $g = f$. Moreover, $E^i_g(x)$ depends continuously on g (and x).

A dominated splitting $E^1 \oplus \cdots \oplus E^k$ is called the *finest dominated splitting* if there is no dominated splitting defined over all M with more than k bundles. If some dominated splitting exists, then the finest dominated splitting exists, is unique, and refines every dominated splitting.

The continuation of the finest dominated splitting is not necessarily the finest dominated splitting of the perturbed diffeomorphism. We call a dominated splitting for $f \in \mathrm{Diff}^{1+\alpha}_\mu(M)$ *stably finest* if it has a continuation which is the finest dominated splitting of every sufficiently C^1-close diffeomorphism of class $C^{1+\alpha}$. It is easy to see that diffeomorphisms with stably finest dominated splittings are (open and) dense among $C^{1+\alpha}$ diffeomorphisms with a dominated splitting.

Let $\lambda_1(f,x) \geqslant \cdots \geqslant \lambda_d(f,x)$ be the Lyapunov exponents of f (counted with multiplicity), defined for almost all x. (See e.g. [2] for definition and basic properties of Lyapunov exponents.) We write also

$$\lambda_i(f) = \int_M \lambda_i(f,x)\,\mathrm{d}\mu(x). \tag{2}$$

Assume f has a dominated splitting $E^1 \oplus \cdots \oplus E^k$. Then the Oseledets splitting is a measurable refinement of it. For simplicity of writing, we will say the exponent λ_p *belongs* to the bundle E^i if $d_1 + \cdots + d_{i-1} < p \leqslant d_1 + \cdots + d_i$, where $d_i = \dim E^i$. By (1), there is an uniform gap between Lyapunov exponents that belong to different bundles.

We now give the proof of Theorem 1 in detail. Let $\mathcal{R}$ be the set of $f \in \mathcal{SE}$ such that f has a dominated splitting $E^+ \oplus E^-$ with $\lambda_p(f) > 0 > \lambda_{p+1}(f)$, where $p = \dim E^+$. First we see that $\mathcal{R}$ is an open set. Indeed, given $f \in \mathcal{R}$, there is an open set $\mathcal{U} \ni f$ where the dominated splitting has a continuation, say $E_g^+ \oplus E_g^-$ for $g \in \mathcal{U}$. As λ_{p+1} is the top exponent in E^-, we can write

$$\lambda_{p+1}(g) = \inf_{n\in\mathbb{N}} \frac{1}{n} \int_M \log\bigl\| Dg^n(x)\big|_{E_g^-(x)}\bigr\|\,\mathrm{d}\mu(x). \tag{3}$$

Therefore $g \in \mathcal{U} \mapsto \lambda_{p+1}(g)$ is an upper semicontinuous function. Accordingly, $\lambda_{p+1}(g) < 0$ for all g sufficiently close to f. And analogously for λ_p, showing that $\mathcal{R}$ is open.

Next we show that $\mathcal{R}$ is dense in $\mathcal{SE}$. Take $f \in \mathrm{Diff}_\mu^{1+\alpha}(M)$ a stably ergodic diffeomorphism. As mentioned, this implies that f has a dominated splitting, see [1]. As remarked above, we can assume, after a perturbation of f if necessary, that f has a stably finest dominated splitting.

For all g sufficiently close to f, we denote by $E_g^1 \oplus \cdots \oplus E_g^k$ the finest dominated splitting of g. Let us indicate by $J_i(g)$ the sum of all Lyapunov exponents $\lambda_p(g)$ that belong to E_g^i. Then we can also write

$$J_i(g) = \int_M \log\bigl|\det Dg|_{E_g^i}\bigr|\,\mathrm{d}\mu. \tag{4}$$

In particular, $J_i(\cdot)$ is a continuous function in the neighborhood of f.

By the theorem from [5], up to C^1-perturbing f, we may assume $J_i(f) \neq 0$ for all i. (It is important to notice that the perturbed map can be taken of class $C^{1+\alpha}$ since so is the original f.)

In the last step we need the following proposition:

Proposition 1. *Let $f \in \mathcal{SE}$. Assume that f has a stably finest dominated splitting $E_f^1 \oplus \cdots \oplus E_f^k$. Then for all $\varepsilon > 0$ there exists a perturbation $g \in \mathrm{Diff}_\mu^{1+\alpha}(M)$ of f such that if the Lyapunov exponents $\lambda_p(g)$, $\lambda_q(g)$ belong to the same bundle E_g^i, then $|\lambda_p(g) - \lambda_q(g)| < \varepsilon$.*

Applying the proposition, we find g close to f such that all $\lambda_p(g)$ in E_g^i are close to $J_i(g)/\dim E^i$ and therefore are non-zero. This finishes the proof of Theorem 1, modulo giving the:

Proof of Proposition 1. For $f \in \mathrm{Diff}_\mu^{1+\alpha}(M)$ and $1 \leqslant p \leqslant d$, let us write $\Lambda_p(f) = \lambda_1(f) + \cdots + \lambda_p(f)$. Then $\Lambda_p(\cdot)$ is an upper semicontinuous function (see [2] or [4]). Since $\mathrm{Diff}_\mu^{1+\alpha}(M)$ is not a complete metric space, we cannot deduce that the set of continuity points of $\Lambda_p(\cdot)$ is dense. Nevertheless, for every $\varepsilon > 0$, the set

$$\mathcal{D}_{\varepsilon,p} = \bigl\{ f \in \mathrm{Diff}_\mu^{1+\alpha}(M);\ \exists \mathcal{U} \ni f \text{ open s.t. } \bigl|\Lambda_p(g_1) - \Lambda_p(g_2)\bigr| < \varepsilon\ \forall g_1, g_2 \in \mathcal{U} \bigr\}$$

is (open and) dense in $\mathrm{Diff}_\mu^{1+\alpha}(M)$. (This is an easy exercise using $\Lambda_p \geqslant 0$.) In particular, $\mathcal{D}_\varepsilon = \bigcap_{p=1}^d \mathcal{D}_{\varepsilon,p}$ is dense.

Now let $f \in \mathcal{SE}$ have a stably finest dominated spitting into k bundles. Fix $\varepsilon > 0$ and take $g \in \mathcal{D}_\varepsilon$ very C^1-close to f. We claim that g has the desired properties: for any $i = 1, \ldots, k$, if λ_p, λ_q belong to E_g^i then λ_p, λ_q are close. Clearly, it suffices to consider the case $q = p + 1$.

Consider the set $D_p(g)$ of points $x \in M$ such that there exists a dominated splitting $T_{\overline{o(g,x)}}M = F \oplus G$ over the closure of the g-orbit of x, with $\dim F = p$. Notice there is no dominated splitting $TM = F \oplus G$ (over M) with $\dim F = p$, because λ_p and λ_{p+1} belong to the same bundle of the finest dominated splitting of g. Thus no $x \in D_p(g)$ can have a dense orbit. In particular, $D_p(g)$ has zero measure. By Proposition 4.17 from [4], there exists a C^1-perturbation h of g such that

$$\begin{aligned}\Lambda_p(h) &< \Lambda_p(g) - \int\limits_{M\setminus D_p(g)} \frac{\lambda_p(g,x)-\lambda_{p+1}(g,x)}{2}\,\mathrm{d}\mu(x)+\varepsilon \\ &= \Lambda_p(g) - \frac{\lambda_p(g)-\lambda_{p+1}(g)}{2}+\varepsilon.\end{aligned}$$

(In the notation of [4], $\Gamma_p(g,\infty) = M \setminus D_p(g)$.) Because g is $C^{1+\alpha}$, the map h given by the proof of Proposition 4.17 in [4] is $C^{1+\alpha}$ as well. Since $g \in \mathcal{D}_{\varepsilon,p}$ and h is close to g, we have $|\Lambda_p(h) - \Lambda_p(g)| < \varepsilon$ and accordingly $\lambda_p(g) - \lambda_{p+1}(g) < 4\varepsilon$. $\square$

We close this note with some questions about what can be said in the absence of stable ergodicity. The following question (similar to one in [12]) is likely to have a positive answer:

Problem 1. Is it true that for the generic $f \in \mathrm{Diff}^1_\mu(M)$, either all Lyapunov exponents are zero at almost every point, or f is non-uniformly hyperbolic (i.e., all Lyapunov exponents are non-zero almost everywhere)?

Notice this is true if $\dim M = 2$, by [3] (later extended in [4]). Using the main result of the papers [4] and [5], it is not difficult to show that the dichotomy of Problem 1 holds true modulo an eventual positive answer to the following well known conjecture of A. Katok:

Problem 2. Is it true that the generic map $f \in \mathrm{Diff}^1_\mu(M)$ is ergodic?

References

[1] A. Arbieto, C. Matheus, A pasting lemma and some applications for conservative systems, Ergodic Theory Dynam. Systems, in press.
[2] L. Arnold, Random Dynamical Systems, Springer-Verlag, New York, 1998.
[3] J. Bochi, Genericity of zero Lyapunov exponents, Ergodic Theory Dynam. Systems 22 (2002) 1667–1696.
[4] J. Bochi, M. Viana, The Lyapunov exponents of generic volume preserving and symplectic maps, Ann. of Math. 161 (2005) 1423–1485.
[5] C. Bonatti, A.T. Baraviera, Removing zero Lyapunov exponents, Ergodic Theory Dynam. Systems 23 (2003) 1655–1670.
[6] C. Bonatti, L. Díaz, E. Pujals, A C^1-generic dichotomy for diffeomorphisms: weak forms of hyperbolicity or infinitely many sinks or sources, Ann. of Math. 158 (2003) 355–418.
[7] C. Bonatti, M. Viana, SRB measures for partially hyperbolic systems whose central direction is mostly contracting, Israel J. Math. 115 (2000) 157–193.
[8] K. Burns, D. Dolgopyat, Ya. Pesin, Partial hyperbolicity, Lyapunov exponents, and stable ergodicity, J. Statist. Phys. 108 (2002) 927–942.
[9] D. Dolgopyat, Ya. Pesin, Every compact manifold carries a completely hyperbolic diffeomorphism, Ergodic Theory Dynam. Systems 22 (2002) 409–435.
[10] M. Grayson, C. Pugh, M. Shub, Stably ergodic diffeomorphisms, Ann. of Math. 140 (1994) 295–329.
[11] F. Ledrappier, Propriétés ergodiques des mesures de Sinaï, Publ. Math. IHES 59 (1984) 163–188.
[12] M. Shub, A. Wilkinson, Pathological foliations and removable zero exponents, Invent. Math. 139 (2000) 495–508.
[13] A. Tahzibi, Stably ergodic diffeomorphisms which are not partially hyperbolic, Israel J. Math. 142 (2004) 315–344.