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A generic C^1 map has no absolutely continuous invariant probability measure

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Abstract

Let M be a smooth compact manifold of any dimension. We consider the set of C^1 maps $f : M \to M$ which have no absolutely continuous (with respect to Lebesgue) invariant probability measure. We show that this is a residual set in the C^1 topology.

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1. Statement

Let M be a smooth compact manifold (maybe with boundary, maybe disconnected) of any dimension $d \geqslant 1$. Let m be some (smooth) volume probability measure in M. Let $C^1(M, M)$ be the set of C^1 maps $M \to M$, endowed with the C^1 topology. Given $f \in C^1(M, M)$, we say that μ is an *acim for* f if μ is an f-invariant probability measure which is absolutely continuous with respect to m.

Theorem 1. *The set $\mathcal{R}$ of C^1 maps $f : M \to M$ which have no acim is a residual (dense G_δ) subset of $C^1(M, M)$.*

Since the set of all expanding maps and the set of all diffeomorphisms are open subsets of $C^1(M, M)$, we have the following immediate consequences.

(i) The C^1-generic expanding map has no acim.
(ii) The C^1-generic diffeomorphism has no acim.

Result (i) was previously obtained in the case where M is a circle by Quas [Q]. Of course, (i) does not hold in the $C^{1+\text{Hölder}}$ topology.

It seems possible that result (ii) holds in higher topologies. An old result by Livsic and Sinai implies that the C^∞-generic Anosov map has no acim, see [LS] and also [C]. (In fact, the existence of a single periodic point of the Anosov map over which the Jacobian is different from 1 prohibits the existence of an acim.) On the other hand, the existence of acim is certainly not rare (in the probabilistic sense) among smooth enough diffeomorphisms of tori close to translations (by KAM).

In the course of the proof, we will need a generalization of the usual Rokhlin tower lemma to non-invariant measures. That result, theorem 2, may be of independent interest.

2. Proof

In sections 2.1–2.4 we give some results that are combined to prove theorem 1 in section 2.5.

2.1. Criterium for existence of acim

The following shows that the set of maps that do not have an acim is a G_δ subset of $C^1(M, M)$.

Lemma 1. *A map $f \in C^1(M, M)$ has no acim iff for every $\varepsilon > 0$ there exists a compact set $K \subset M$ and $N \in \mathbb{N}$ such that*

$$m(K) > 1 - \varepsilon \qquad \text{and} \qquad m(f^N(K)) < \varepsilon.$$

Proof. Assume that f has an acim μ. Let $\varepsilon > 0$ be such that $m(Z) \leqslant \varepsilon$ implies $\mu(Z) < 1/2$. Now assume that $K \subset M$ is a compact set such that $m(f^N K) < \varepsilon$ for some $N \in \mathbb{N}$. Then $\mu(K) \leqslant \mu(f^N K) < 1/2$, so $m(M \setminus K) > \varepsilon$.

Next assume that f has no acim. Let μ be a limit point of the sequence of measures $\frac{1}{n}(m + f_* m + \cdots + f_*^{n-1} m)$; then μ is f-invariant. Let $\mu = \mu_{\mathrm{ac}} + \mu_{\mathrm{sing}}$ be the Lebesgue decomposition of μ relative to m. Since f is C^1, $f_*\mu_{\mathrm{sing}}$ is singular, and it follows that μ_{ac} and μ_{sing} are f-invariant. But f is assumed to have no acim, so $\mu = \mu_{\mathrm{sing}}$. Thus there exists $Z \subset M$ such that $m(Z) = 1$ and $\mu(Z) = 0$. Given any $\varepsilon > 0$, take a compact set L and an open set V such that $L \subset Z \subset V$, $m(L) > 1 - \varepsilon$ and $\mu(V) < \varepsilon/2$. Let ϕ be a continuous function such that $\chi_L \leqslant \phi \leqslant \chi_V$. For some sequence $n_j \to \infty$ we have

$$\frac{1}{n_j} \sum_{i=0}^{n_j - 1} \int f^i \circ \phi \, \mathrm{d}m \to \int \phi \, \mathrm{d}\mu < \varepsilon/2.$$

In particular, there exists N such that $m(f^{-N} L) \leqslant \int f^N \circ \phi \, \mathrm{d}m < \varepsilon/2$. Take a compact $K \subset M \setminus f^{-N} L$ such that $m(M \setminus K) < \varepsilon$. Then $m(f^N K) \leqslant m(M \setminus L) < \varepsilon$. $\square$

2.2. A non-invariant Rokhlin lemma

Theorem 2. *Let $f : M \to M$ be a C^1-endomorphism of a compact manifold, and let m be normalized Lebesgue measure. Assume that $m(C_f \cup P_f) = 0$, where C_f is the set of critical points and P_f is the set of periodic points. Given any $\varepsilon_0 > 0$ and n_0, $\ell \in \mathbb{N}$ with $\ell \leqslant n_0$, there exists a measurable set $U \subset M$ such that $f^{-i}(U) \cap U = \varnothing$ for $1 \leqslant i < n_0$,*

$$\sum_{i=0}^{n_0 - 1} m(f^{-i}(U)) > 1 - \varepsilon_0, \qquad \text{and} \qquad \sum_{i=0}^{\ell - 1} m(f^{-i}(U)) < \frac{\ell}{n_0} + \varepsilon_0. \tag{1}$$

Notice that if the map f were assumed to preserve the measure m, the theorem would be an immediate consequence of the well-known Rokhlin lemma (for non-invertible maps, see [HS]).

The proof of theorem 2 will occupy the rest of this subsection. Let f be fixed from now on. Let $\mathcal{M}$ be the σ-algebra of measurable sets. Since f is C^1, $Y \in \mathcal{M}$ implies $f(Y) \in \mathcal{M}$. Given $Z \in \mathcal{M}$, we denote

$$\hat{Z} = \bigcup_{i=0}^{\infty} f^{-i}(Z).$$

We say that $Z \in \mathcal{M}$ is N-good (where $N \in \mathbb{N}$) if $Z \cap f^{-i}(Z) = \varnothing$ for $0 \leqslant i < N$.

Claim 1. If A and B are N-good sets then the set

$$C = (A \setminus \hat{B}) \cup (B \setminus (A \setminus \hat{B})^\wedge)$$

is N-good and satisfies $\hat{C} \supset A \cup B$.

Proof. Let $A' = A \setminus \hat{B}$; then $(A' \cup B)^\wedge = (A \cup B)^\wedge$. Let $B' = B \setminus \hat{A}'$; then $(A' \cup B')^\wedge = (A' \cup B)^\wedge$. That is, the set $C = A' \cup B'$ satisfies $\hat{C} = (A \cup B)^\wedge$. Using that A' and B' are N-good, $A' \cap \hat{B}' = \varnothing$ and $\hat{A}' \cap B' = \varnothing$, we see that C is N-good. $\square$

We say that $Z \in \mathcal{M}$ is N-saturated if $f^{-N}(f^N(Z)) = Z$. The N-saturated sets form the σ-algebra $f^{-N}\mathcal{M}$.

Claim 2. For each $N \in \mathbb{N}$ there exists a countable cover (modulo sets of zero m-measure) $M = \bigcup B_k$ such that each B_k is N-good and N-saturated.

Proof. Since $m(P_f) = 0$, there is a countable cover $M = \bigcup A_k$, where the sets A_k are N-good. Take $B_k = f^{-N}(A_k)$. $\square$

Claim 3. For every $\varepsilon > 0$ and $N \in \mathbb{N}$ there exists a set W which is N-good, N-saturated and $m(\hat{W}) > 1 - \varepsilon$.

Proof. Let B_k be the sets given by claim 2. Define inductively sets C_k: take $C_1 = B_1$, and for $k > 0$, let C_{k+1} be the N-good set given by claim 1 such that $\widehat{C_{k+1}} \supset C_k \cup B_{k+1}$. Then for all k we have that C_k is N-saturated and $\hat{C}_k \supset \bigcup_{j=1}^{k} B_j$. Finally, take $W = C_k$ for some large k. $\square$

Claim 4. For every $\varepsilon > 0$ and $N \in \mathbb{N}$, there exists a N-good set V such that $m(V) < \varepsilon$ and $m(\hat{V}) > 1 - \varepsilon$.

Proof. Increasing N if necessary, we assume $N > 1/\varepsilon$. Take W as in claim 3. Notice that the sets $W, f(W), \ldots, f^{N-1}(W)$ are disjoint. Take $0 \leqslant i \leqslant N-1$ such that $m(f^i(W)) \leqslant 1/N$. Let $V = f^i(W)$; then $\hat{V} \supset \hat{W}$. Since W is N-good and N-saturated, V is N-good. $\square$

Claim 5. For any $i \geqslant 0$, $f_*^i m$ is absolutely continuous with respect to m.

Proof. Clearly it suffices to consider $i = 1$. Let $Z \in \mathcal{M}$ be such that $m(f^{-1}(Z)) > 0$. Since $m(C_f) = 0$, we can find an open set $U \subset M \setminus C_f$ such that $f|U$ is a C^1-diffeomorphism and $m(f^{-1}(Z) \cap U) > 0$. Then $f(f^{-1}(Z) \cap U)$ and hence Z, has positive measure. $\square$

Proof of theorem 2. Let ℓ, n_0 and ε_0 be given. By claim 5, there exists $\varepsilon > 0$ such that

$$Z \in \mathcal{M},\ m(Z) < \varepsilon \ \Rightarrow\ m\left(\bigcup_{i=0}^{2n_0-1} f^{-i} Z\right) < \frac{\varepsilon_0}{2}.$$

Let V be given by claim 4 with $N = n_0$. For $i \geqslant 0$, let $V_i = f^{-i}(V)$ and

$$V_i^* = V_i \setminus \bigcup_{j=0}^{i-1} V_j.$$

For each $0 \leqslant j < n_0$, let

$$S_j = \sum_{k=j}^{j+\ell-1} \sum_{\substack{i \geqslant 0 \\ i = k \bmod n_0}} m(V_i^*).$$

We have $\sum_{j=0}^{n_0-1} S_j = \ell \cdot m(\hat{V})$, so there exists some j_0 for which $S_{j_0} \leqslant \ell/n_0$. Define

$$U = \bigsqcup_{\substack{i \geqslant n_0 \\ i = j_0 \bmod n_0}} V_i^*.$$

Noting that $f^{-j}(V_i^*) \subset V_{i+j}^* \cup V_0 \cup V_1 \cup \cdots \cup V_j$ for $0 \leqslant j < n_0$, we see that U is n_0-good. Also,

$$m\left(\bigsqcup_{j=0}^{\ell-1} f^{-j}(U)\right) \leqslant m(V_0 \cup \cdots \cup V_{\ell-1}) + S_{j_0} < \frac{\ell}{n_0} + \frac{\varepsilon_0}{2}.$$

Finally, since $f^{-j}(V_i^*) \supset V_{i+j}^*$, we have

$$m\left(\bigsqcup_{j=0}^{n_0-1} f^{-j}(U)\right) \geqslant m\left(\bigsqcup_{i=2n_0}^{\infty} V_i^*\right) > 1 - \varepsilon - \frac{\varepsilon_0}{2} \geqslant 1 - \varepsilon_0. \qquad \square$$

Remark 1. We used only the following assumptions about f and m:

- $(M, \mathcal{M}, m)$ is a Lebesgue space and $f : M \to M$ is measurable;
- f is aperiodic: $m(P_f) = 0$;
- f is non-singular with respect to m: for $Y \in \mathcal{M}$, we have $m(Y) = 0$ if and only if $m(f^{-1}(Y)) = 0$;
- f is forward-measurable: $Y \in \mathcal{M}$ implies $f(Y) \in \mathcal{M}$. (In fact, we can always replace f by a isomorphic copy which is forward-measurable: see [R].)

Remark 2 (addendum to theorem 2). The set U can be taken open, and with $f^{-i}(\bar{U}) \cap \bar{U} = \varnothing$, $0 \leqslant i < n_0$.

Indeed, take a compact set $K \subset U$ with $m(U \setminus K)$ very small. Then take an open set $U_0 \supset K$ with $m(U_0 \setminus K)$ very small and such that $\overline{U_0}, \ldots, f^{-n_0+1}(\overline{U_0})$ are disjoint. Bearing in mind claim 5, we see that (1) holds with U_0 in the place of U.

2.3. Linearization

Fix an atlas of M formed by charts that take the restricted volume on M to Lebesgue measure on $\mathbb{R}^d$. Fix also a family of pairs (A_i, ϕ_i) such that the $A_i \subset M \setminus \partial M$ are disjoint open sets compactly contained in the domain of the chart ϕ_i, and $\sum m(A_i) = 1$. We call the A_i *basic blocks*.

We shall say that a map $f : M \to M$ is *locally linear on an open set* $V \subset M \setminus \partial M$ if, for each connected component W of V, there exists both W and $f(W)$ contained in basic blocks and if under the corresponding change in coordinates the map $f : W \to f(W)$ becomes the restriction of an affine map $\mathbb{R}^d \to \mathbb{R}^d$.

Lemma 2. *If $f : M \to M$ is a C^1 map and $U \subset M \setminus \partial M$ is open then for every $\gamma > 0$ there exists a C^1-map $\tilde{f} : M \to M$ which is C^1-close to f and equals f outside U, and there exists an open set $V \subset U$ such that $m(V)/m(U) > 1-\gamma$ and $\tilde{f}$ is locally linear on V. Furthermore, if the set C_f of critical points of f has zero Lebesgue measure then $\tilde{f}$ can be taken to be a local diffeomorphism on V.*

Proof. Up to reducing U a little, we can assume each connected component of U, as well as its image by f, is contained in a basic block.

To simplify writing, from now on we assume $U \subset \mathbb{R}^d$, and m is Lebesgue measure on $\mathbb{R}^d$. Given $\gamma > 0$, let $\delta > 0$ be such that $(1-\delta)^d < \gamma/2$. Fix a C^1 bump function $\rho : \mathbb{R}^d \to [0, 1]$ such that $\rho(x) = 1$ if $\|x\| \leqslant 1-\delta$, $\rho(x) = 0$ if $\|x\| \geqslant 1$ (where $\|\cdot\|$ is the Euclidian norm on $\mathbb{R}^d$).

Let $r_0 > 0$ be small. By Vitali's lemma, we can find finitely many disjoint balls $B(p_i, r_i) \Subset U$ of radii $r_i < r_0$, such that the Lebesgue measure of their union is greater than $(1-\gamma/2)m(U)$. Define $\tilde{f}$ on each $B(p_i, r_i)$ by

$$\tilde{f}(x) = f(x) + \rho(r_i^{-1}(x-p_i)) \cdot [-f(x) + f(p_i) + Df(p_i)\cdot(x-p_i)],$$

and $\tilde{f} = f$ on $U \setminus \bigsqcup_i B(p_i, r_i)$.

Then $\tilde{f}$ is locally linear on $V = \bigsqcup_i B(p_i, (1-\delta)r_i)$. If r_0 is sufficiently small, then $\tilde{f}$ is C^1-close to f.

If $m(C_f) = 0$ we take each p_i such that $Df(p_i)$ is an isomorphism. □

2.4. Perturbation of a sequence of linear maps

Lemma 3. *Given $\varepsilon > 0$ and $0 < \delta < 1$, there exists $k \in \mathbb{N}$ such that given any of sequence linear isomorphisms*

$$\mathbb{R}^d \xrightarrow{L_n} \mathbb{R}^d \xrightarrow{L_{n-1}} \cdots \xrightarrow{L_1} \mathbb{R}^d,$$

with $n \geqslant k$, there exists $\tau_0 > 0$ such that for any $0 < \tau < \tau_0$ the following holds true.

Define boxes

$$\begin{aligned} U_0 &= [-1, 1]^{d-1} && \times[-\tau, \tau], \\ V_0 &= [-(1-\delta), 1-\delta]^{d-1} && \times[-(1-\delta)\tau, (1-\delta)\tau], \\ W_0 &= [-1, 1]^{d-1} && \times[-\delta\tau, \delta\tau]. \end{aligned}$$

Define also $U_i = L_i^{-1}U_{i-1}$, $V_i = L_i^{-1}V_{i-1}$, $W_i = L_i^{-1}W_{i-1}$, for $1 \leqslant i \leqslant n$. Then there exist C^1-diffeomorphisms $H_i : \mathbb{R}^d \to \mathbb{R}^d$ with derivative ε-close to id *and with $H_i =$* id *outside U_i, such that for all i with $k \leqslant i \leqslant n$ we have*

$$L_{i-k+1} \circ H_{i-k+1} \circ \cdots \circ L_{i-1} \circ H_{i-1} \circ L_i \circ H_i(V_i) \subset W_{i-k}\,. \tag{2}$$

Moreover, for $1 \leqslant i \leqslant n$, H_i only depends on ε, δ, τ and $L_1, \ldots, L_i$ (but not on $L_{i+1}, \ldots, L_n$).

In the following proof of lemma 3, we will assume $d \geqslant 2$, leaving for the reader the easy adaptation to the case $d = 1$ (where any τ_0 works).

In the proof we will need lemmas 4 and 5. We write $\mathbb{R}^{d-1} = \mathbb{R}^{d-1} \times \{0\} \subset \mathbb{R}^d$. Also, we call a subset B of a finite-dimensional vector space V a *ball* if there exists a norm on V such that B is the closed unit ball on V with respect to that norm.

Lemma 4. *For every $\varepsilon > 0$ and $0 < \delta < 1$, there exists $0 < \kappa < 1$ with the following properties: given any ball $\mathcal{C} \subset \mathbb{R}^{d-1}$, there exists $\tau^* > 0$ such that if $0 < \tau < \tau^*$ then there* *exists a diffeomorphism* $H : \mathbb{R}^d \to \mathbb{R}^d$ *satisfying the following:*

- H *has derivative* ε*-close to the identity;*
- H *equals the identity outside* $\mathcal{C} \times [-\tau, \tau]$*;*
- *if* $(z, t) \in (1-\delta)(\mathcal{C} \times [-\tau, \tau])$ *then* $H(z, t) = (z, \kappa t)$*.*

Proof. Given ε and δ, let κ be such that

$$(1-\kappa)(1+2\delta^{-1}) < \varepsilon.$$

Now let $\mathcal{C} = \{z \in \mathbb{R}^{d-1};\ \|z\|_* \leqslant 1\}$, where $\|\cdot\|_*$ is a norm in $\mathbb{R}^{d-1}$. Let $C > 0$ be such that $\|v\|_* \leqslant C\|v\|$, where $\|\cdot\|$ is Euclidian norm. Let $\tau^* = C^{-1}$.

Then, given $0 < \tau < \tau^*$, take a bump function $\rho : \mathbb{R} \to [0, 1]$ such that $\rho(x) = 1$ for $|x| \leqslant 1-\delta$, $\rho(x) = 0$ for $|x| \geqslant 1$ and $|\rho'| < 2\delta^{-1}$. Define

$$H(z, t) = (z, [1 - (1-\kappa)\cdot\rho(\tau^{-1}t)\cdot\rho(\|z\|_*)]t), \qquad z \in \mathbb{R}^{d-1}, \quad t \in \mathbb{R}.$$

Then

$$\left|\frac{\partial H}{\partial t} - 1\right| \leqslant (1-\kappa)\cdot\rho(\|z\|_*)\cdot[\rho(\tau^{-1}t) + |\rho'(\tau^{-1}t)\cdot\tau^{-1}t|] < \varepsilon.$$

And if $v \in \mathbb{R}^{d-1}$ then

$$\begin{aligned}\|DH(z,t)\cdot(v,0)\| &\leqslant |t|\cdot(1-\kappa)\cdot\rho(\tau^{-1}t)\cdot|\rho'(\|z\|_*)|\cdot\|v\|_* \\ &\leqslant \tau^*(1-\kappa)\cdot 2\delta^{-1}\cdot C\|v\| \leqslant \varepsilon\|v\|.\end{aligned}$$ □

Let $e_d = (0, \ldots, 0, 1) \in \mathbb{R}^d$ and $\langle\cdot,\cdot\rangle$ be the standard inner product on $\mathbb{R}^d$. The easy proof of the following lemma is left to the reader.

Lemma 5. *Let* $L : \mathbb{R}^d \to \mathbb{R}^d$ *be a linear isomorphism such that* $L(\mathbb{R}^{d-1}) = \mathbb{R}^{d-1}$*. Let* $\beta = |\langle L(e_d), e_d\rangle|$*. Let* $\mathcal{C} \subset \mathbb{R}^{d-1}$ *be a ball, and let* $\lambda > 1$*. Then there exists* $\tau' > 0$ *such that for any* $0 < \tau < \tau'$ *we have*

$$(\lambda^{-1}L(\mathcal{C})) \times [-\beta\tau, \beta\tau] \subset L(\mathcal{C} \times [-\tau, \tau]) \subset (\lambda L(\mathcal{C})) \times [-\beta\tau, \beta\tau].$$

Proof of lemma 3. Let $\kappa = \kappa(\varepsilon, \delta/2) > 0$ be given by lemma 4. We take $k \in \mathbb{N}$ such that $\kappa^k < \delta/(1-\delta)$.

Now take $n \geqslant k$ and $L_1, \ldots, L_n$ as in the statement of the lemma. Rotating coordinates if necessary, we can assume $L_i \cdot \mathbb{R}^{d-1} = \mathbb{R}^{d-1}$ for all i. Let $\mathcal{C}_0 = [-1, 1]^{d-1}$ and $\mathcal{C}_i = L_i^{-1}\cdots L_1^{-1}\cdot\mathcal{C}_0$ for $i \geqslant 1$. Let $\alpha_0 = 1$ and $\alpha_i = |\langle L_i^{-1}\cdots L_1^{-1}e_d, e_d\rangle|$. Write, for $a > 0$ and $b > 0$,

$$\mathcal{B}_i[a, b] = (a\mathcal{C}_i) \times [-\alpha_i b, \alpha_i b].$$

Fix $\lambda > 1$ so that

$$\lambda^{3n} < (1-\delta)^{-1}(1-\delta/2). \tag{3}$$

Let $\tau_i^* > 0$ be associated with the ball $\mathcal{C} = \mathcal{C}_i$ by lemma 4. Let $\tau_i' > 0$ be associated with the linear map L_i^{-1}, the ball $\mathcal{C}_i$ and λ by lemma 5. Let

$$\tau_0 = \lambda^{-2n}\min\{\alpha_i^{-1}\tau_i^*,\ \alpha_i^{-1}\tau_i';\ 1 \leqslant i \leqslant n\}.$$

By lemma 5,

$$\mathcal{B}_i[\lambda^{-1}a, b] \subset L_i^{-1}(\mathcal{B}_{i-1}[a, b]) \subset \mathcal{B}_i[\lambda a, b], \qquad \text{provided } \frac{b}{a} < \frac{\tau_i'}{\alpha_i}. \tag{4}$$

Therefore

$$\frac{b}{a} < \lambda^n \tau_0 \;\Rightarrow\; \mathcal{B}_i[\lambda^{-n}a, b] \subset (L_i \circ \cdots \circ L_1)^{-1}(\mathcal{B}_0[a, b]) \subset \mathcal{B}_i[\lambda^n a, b]. \tag{5}$$

Now let $0 < \tau < \tau_0$ be fixed, and let U_0, V_0, W_0 be as in the statement of the lemma, that is,

$$U_0 = \mathcal{B}_0[1, \tau], \qquad V_0 = \mathcal{B}_0[(1-\delta), (1-\delta)\tau], \qquad W_0 = \mathcal{B}_0[1, \delta\tau].$$

Let $\tilde{H}_i : \mathbb{R}^d \to \mathbb{R}^d$ be the diffeomorphism supported on $\mathcal{B}_i(1, \lambda^n\tau) = \mathcal{C}_i \times [-\alpha_i\lambda^n\tau, \alpha_i\lambda^n\tau]$ given by lemma 4. We define H_i by

$$H_i(x) = \lambda^{-n} \cdot \tilde{H}_i(\lambda^n x).$$

Then DH_i is ε-close to id. Also, H_i equals the identity outside $\mathcal{B}_i[\lambda^{-n}, \tau] \subset U_i$. And

$$H_i(\mathcal{B}_i[a, b]) = \mathcal{B}_i[a, \kappa b] \qquad \text{if } 0 < a \leqslant \lambda^{-n}\left(1 - \frac{\delta}{2}\right) \quad \text{and} \quad b \leqslant \left(1 - \frac{\delta}{2}\right)\tau. \tag{6}$$

It remains to check that (2) holds; so let $k \leqslant i \leqslant n$. In the following diagram, $X \xrightarrow{F} Y$ means $F(X) \subset Y$. Using repeatedly (3)–(6),

$$\begin{aligned}
&V_i \subset \mathcal{B}_i[\lambda^n(1-\delta), (1-\delta)\tau] \xrightarrow{H_i} \mathcal{B}_i[\lambda^n(1-\delta), \kappa(1-\delta)\tau] \xrightarrow{L_i} \\
&\mathcal{B}_{i-1}[\lambda^{n+1}(1-\delta), \kappa(1-\delta)\tau] \xrightarrow{L_{i-1}\circ H_{i-1}} \cdots \xrightarrow{L_{i-k+1}\circ H_{i-k+1}} \\
&\mathcal{B}_{i-k}[\lambda^{n+k}(1-\delta), \kappa^k(1-\delta)\tau] \subset \mathcal{B}_{i-k}[\lambda^{-n}, \delta\tau] \subset W_{i-k}.
\end{aligned}$$

This proves lemma 3. □

2.5. *Proof of theorem 1*

Define the following (open) subsets of $C^1(M, M)$:

$$\begin{aligned}
\mathcal{V}_\varepsilon = \{f \in C^1(M, M);\ &\text{there exist } K \subset M \text{ compact},\ k \in \mathbb{N} \text{ such that} \\
&m(K) > 1 - \varepsilon \text{ and } m(f^k K) < \varepsilon\}.
\end{aligned}$$

By lemma 1, it suffices to show that each $\mathcal{V}_\varepsilon$ is dense to prove the theorem. So let $f \in C^1(M, M)$ and $\varepsilon > 0$ be fixed; we will explain how to find $g \in \mathcal{V}_{4\varepsilon}$ close to f. For clarity we split the proof into steps.

Step 1. Linearizing f on an open tower. Let P_f be the set of periodic points of f and C_f be the set of critical points of f. We can assume (perturbing f if necessary) that $m(P_f \cup C_f) = 0$. (Indeed, it suffices to take f analytic and Kupka–Smale.)

Let $0 < \delta < \varepsilon$ be such that $(1-\delta)^d > 1 - \varepsilon$. Let $k = k(\varepsilon, \delta)$ be given by lemma 3. Take $n \in \mathbb{N}$ such that $k/(n+1) < \varepsilon$. Now apply theorem 2 (and remark 2) with $\ell = k$, $n_0 = n + 1$, $\varepsilon_0 = \varepsilon/2$, to find an open set $U \subset M$ such that

$$\bar{U}, f^{-1}(\bar{U}), \ldots, f^{-n}(\bar{U}) \text{ are disjoint,}$$

$$\sum_{i=0}^{k-1} m(f^{-i}U) < \varepsilon, \qquad \sum_{i=0}^{n} m(f^{-i}U) > 1 - \varepsilon.$$

It follows easily from lemma 2 that there exist open sets $Q_i \subset f^{-i}(U)$, $0 \leqslant i \leqslant n$ and a C^1 perturbation $\tilde{f}$ of f such that $\tilde{f}(Q_i) = Q_{i-1}$, $1 \leqslant i \leqslant n$, $\tilde{f}|Q_i$ is locally linear and invertible and $\sum_{i=0}^{n} m(Q_i) > 1 - \varepsilon$. We can assume further (by slightly shrinking the Q_i) that each Q_i has only finitely many connected components and $\tilde{f}$ maps each connected component of Q_i onto a connected component of Q_{i-1}. We have

$$\sum_{i=0}^{k-1} m(Q_i) < \varepsilon, \qquad \sum_{i=0}^{n} m(Q_i) > 1 - \varepsilon. \tag{7}$$

To simplify writing, we replace $\tilde{f}$ by f.

To simplify things further, we will assume $\bigcup Q_i$ is a subset of $\mathbb{R}^d$, in order to avoid mentioning the charts.

Step 2. Defining the perturbation g. Let $k = k(\varepsilon, \delta)$ be given by lemma 3. For each sequence $\bar{x} = (x_m, \ldots, x_0)$, $k \leqslant m \leqslant n$ with $f(x_i) = x_{i-1}$ and $x_i \in Q_i$, we apply lemma 3 to the sequence of linear maps $Df(x_i)$, obtaining a certain $\tau_0(\bar{x}) > 0$. There are only finitely many possibilities for the sequence of linear maps $Df(x_i)$, so we can choose $\tau > 0$ such that $\tau < \tau_0(\bar{x})$ for all $\bar{x}$.

For $y \in Q_0$ and small $a, b > 0$, write

$$\mathcal{B}[y, a, b] = y + [-a, a]^{d-1} \times [-b, b].$$

The family of boxes $\mathcal{B}[y, r, \tau r] \subset Q_0$ constitutes a Vitali covering of Q_0. So we can find a finite set $F \subset Q_0$ and numbers $r(y) > 0$, $y \in F$, such that $U_0(y) = \mathcal{B}[y, r(y), \tau r(y)] \subset Q_0$ are disjoint and

$$m\left(Q_0 \setminus \bigsqcup_{y \in F} U_0(y)\right) \leqslant \varepsilon m(Q_0). \tag{8}$$

Let $V_0(y)$, $W_0(y)$ be boxes around y as in lemma 3, namely

$$V_0(y) = \mathcal{B}[y, (1-\delta)r(y), (1-\delta)\tau r(y)],$$
$$W_0(y) = \mathcal{B}[y, r(y), \delta\tau r(y)].$$

Let $\bar{F}$ be the set of the sequences $\bar{y} = (y_m, \ldots, y_0)$ with $k \leqslant m \leqslant n$, $f(y_i) = y_{i-1}$, $y_i \in Q_i$ and $y_0 \in F$. Then $\bar{F}$ is finite. For each $\bar{y} \in \bar{F}$ and $i = 1, \ldots, m$, let $U_i(\bar{y})$ be the image of $U_0(y_0)$ by the branch of f^{-i} that takes y_0 to y_i. Notice that the $U_i(\bar{y})$ are either disjoint or coincide, and if $U_i(\bar{y}) = U_j(\bar{y}')$ then $i = j$ and the last $i+1$ symbols in $\bar{y}$ and $\bar{y}'$ coincide. We define sets $V_i(\bar{y})$ and $W_i(\bar{y})$ analogously.

The choice of δ together with the linearity of f gives

$$\frac{m(V_i(\bar{y}))}{m(U_i(\bar{y}))} > 1 - \varepsilon \qquad \text{and} \qquad \frac{m(W_i(\bar{y}))}{m(U_i(\bar{y}))} < \varepsilon \qquad \text{for all } \bar{y}, i. \tag{9}$$

Let $H_{i,\bar{y}}$ be the diffeomorphisms given by lemma 3. We define $h_{i,\bar{y}}$ as

$$h_{i,\bar{y}}(x) = y_i + r(y_0) \cdot H_{i,\bar{y}}((x - y_i)/r(y_0)), \qquad \text{for } y \in U_i(\bar{y}),$$

and $h_{i,\bar{y}} = \mathrm{id}$ outside $U_i(\bar{y})$. Notice that $h_{i,\bar{y}}$ only depends on $U_i(\bar{y})$. Then $h_{i,\bar{y}}$ is C^1-close to the identity. Moreover, for all $i = k, k+1, \ldots, m$ we have

$$f \circ h_{i-k+1,\bar{y}} \circ \cdots \circ f \circ h_{i-1,\bar{y}} \circ f \circ h_{i,\bar{y}}(V_i(\bar{y})) \subset W_{i-k}(\bar{y}).$$

We define a perturbation $g : M \to M$ of f as follows: $g = f \circ h_{i,\bar{y}}$ on each $U_i(\bar{y})$, and g equals f on

$$M \setminus \bigcup_{\bar{y} \in \bar{F}} \bigcup_{i=0}^{n} U_i(\bar{y}).$$

It follows that $g^k(V_i(\bar{y})) \subset W_{i-k}(\bar{y})$ for all $\bar{y} \in \bar{F}$.

Step 3. Verifications. Define the compact set

$$K = \bigcup_{\bar{y}\in\bar{F}} \bigcup_{i=k}^{n} V_i(\bar{y}).$$

First let us see that K has almost full measure. We have

$$M \setminus K = \overbrace{\left(M \setminus \bigcup_{i=0}^{n} Q_i\right)}^{\text{(I)}} \sqcup \overbrace{\bigcup_{i=0}^{n}\left(Q_i \setminus \bigcup_{\bar{y}\in\bar{F}} U_i(\bar{y})\right)}^{\text{(II)}} \sqcup$$

$$\sqcup \underbrace{\bigcup_{i=0}^{n} \bigcup_{\bar{y}\in\bar{F}} (U_i(\bar{y}) \setminus V_i(\bar{y}))}_{\text{(III)}} \sqcup \underbrace{\bigcup_{i=0}^{k-1} \bigcup_{\bar{y}\in\bar{F}} V_i(\bar{y})}_{\text{(IV)}}.$$

From (7) we get $m(\text{I}) < \varepsilon$. By (8) and linearity of f, we have $m(\text{II}) < \varepsilon$. From (9), $m(\text{III}) \leqslant \varepsilon$. Finally, using (7),

$$m(\text{IV}) \leqslant m\left(\bigcup_{i=0}^{k-1} Q_i\right) < \varepsilon.$$

So we obtain that $m(M \setminus K) < 4\varepsilon$.

Next let us see that $g^k K$ has small measure. We have

$$g^k K = \bigcup_{\bar{y}\in\bar{F}} \bigcup_{i\geqslant k} g^k V_i(\bar{y}) \subset \bigcup_{\bar{y}\in\bar{F}} \bigcup_{i\geqslant 0} W_i(\bar{y}).$$

Using (9),

$$m(g^k K) \leqslant \varepsilon m\left(\bigcup_{\bar{y}\in\bar{F}} \bigcup_{i\geqslant 0} U_i(\bar{y})\right) < \varepsilon.$$

We have shown that $g \in \mathcal{V}_{4\varepsilon}$. This proves theorem 1.

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