Controllability of Products of Matrices

Beyond Uniform Hyperbolicity – Luminy – 2011

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General Control Systems

A (discrete-time) *control system* is given by a recurrence relation of the form

$$x_{t+1} = F(x_t, u_t), \quad (t = 0, 1, 2, \dots)$$

where $F : \mathcal{X} \times \mathcal{U} \to \mathcal{X}$ is any map. We say that:

- t is the *time*;
- x_t are the *states*;
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The idea is that we have some freedom to choose the u_t's, and we want to induce some desired effect on the x_t's, like:

- stay in some region;
- reach a particular position;
- etc.

Local Controllability Problem

Fix a time N, a consider a length N trajectory $(x_0, \ldots, x_N; u_0, \ldots, u_{N-1})$, with final state

$$x_N = F_{u_{N-1}} \circ \cdots \circ F_{u_0}(x_0), \quad \text{where } F_u = F(\cdot, u).$$

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The trajectory $(x_0, \dots, x_N; u_0, \dots, u_{N-1})$ is called *locally controllable* (or *locally accessible*) if for every $\tilde{x}_N$ sufficiently close to x_N, one can slightly change the controls (keeping fixed the initial state x_0) so that the final state is $\tilde{x}_N$. In symbols:

$$\forall \tilde{x}_N \approx x_N \ \exists (\tilde{u}_i) \approx (u_i) \quad \text{s.t.} \quad F_{\tilde{u}_{N-1}} \circ \cdots \circ F_{\tilde{u}_0}(x_0) = \tilde{x}_N$$

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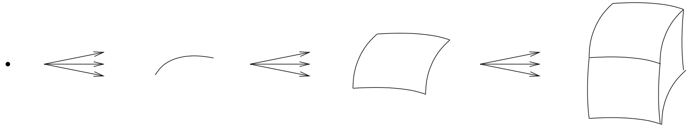
We'll suppose that the state space $\mathcal{X}$ and the control space $\mathcal{U}$ are manifolds, and that F is differentiable.

More interesting situation (for local controllability): $\dim \mathcal{U} < \dim \mathcal{X}$.
We can gain at most $\dim \mathcal{U}$ dimensions at a time.
Example: $\dim \mathcal{U} = 1$.



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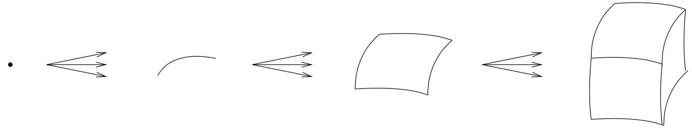

Rem: The analogous local controllability property for *continuous-time* control systems is somewhat easier to obtain: a short time can be enough. . .

Regularity and Universal Regularity

A sufficient condition for local controllability of the trajectory $(x_0; u_0, \ldots, u_{N-1})$ is that the partial derivative " $\dfrac{\partial \phi_N}{\partial u}$ " of the evolution map

$$\begin{aligned} \phi_N : \mathcal{X} \times \mathcal{U}^N &\to \mathcal{X} \\ (x_0; u_0, \ldots, u_{N-1}) &\mapsto x_N = F_{u_{N-1}} \circ \cdots \circ F_{u_0}(x_0) \end{aligned}$$

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An input $(u_0, \dots, u_{N-1})$ is called *universally regular* if the trajectory $(x_0; u_0, \dots, u_{N-1})$ is regular for *every* initial state $x_0 \in \mathcal{X}$.

Semilinear Control Systems

We'll deal with *semilinear control systems*: the state space is $\mathbb{K}^d$ ($\mathbb{K} = \mathbb{R}$ or $\mathbb{C}$), and x_{t+1} depends *linearly* on x_t, that is,

$$x_{t+1} = A(u_t) \cdot x_t, \quad \text{where } A : \mathcal{U} \to \mathrm{Mat}_{d \times d}(\mathbb{K}) \text{ is fixed.}$$

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We're not actually interested in the lengths of the vectors, but only on their directions. So we consider the projectivized system:

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References:

- E. Sontag. Systems Control Lett., 1993.
- F. Wirth. SIAM J. Control and Optimization, 1998.

Problem

Given a (projective) semilinear control system, we want to study the universally regular inputs $(u_0, \ldots, u_{N-1})$.

Rem: If $(u_0, \ldots, u_{N-1})$ is a universally regular input then any longer input of the form $(*, \ldots, *, u_0, \ldots, u_{N-1}, *, \ldots, *)$ is universally regular.

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Aim

Take a "typical" smooth map $A : \mathcal{U} \to \mathrm{GL}(d, \mathbb{K})$, and take N large.

- Show that most inputs in $\mathcal{U}^N$ are good, i.e. universally regular.

More precisely:

- Show that the set of bad inputs is a union of submanifolds;
- and estimate their codimension (large?).

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More precisely:

- Show that the set of bad inputs is a union of submanifolds;
- and estimate their codimension (large?).

General inputs $(u_0, \ldots, u_{N-1})$ are difficult to work with.
In order to approach this kind of problem, we begin considering only *period*-1 inputs, i.e. inputs of the form $(u_0, u_0, \ldots, u_0)$.

Main result (joint with N. Gourmelon)

For today, we'll assume that $\mathcal{U}$ is a *compact* manifold.

Main result (real case) – joint with N. Gourmelon

Given $d \geq 2$ and $m = \dim \mathcal{U}$ there exist:

- $N \in \mathbb{N}$;
- an *open* and (C^∞-)*dense* subset $\mathcal{O} \subset C^2(\mathcal{U}, \mathrm{GL}(d, \mathbb{R}))$

such that if $A \in \mathcal{O}$ then *all but a finite number* of period-1 inputs of length N are universally regular for the system $\xi_{t+1} = A(u_t) \cdot \xi_t$.

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In short: For most maps $A : \mathcal{U} \to \mathrm{GL}(d, \mathbb{R})$ and most choices of $u_0 \in \mathcal{U}$, given any direction $\xi_0 \in \mathbb{R}\mathrm{P}^{d-1}$, the set of directions $A(\tilde{u}_{N-1}) \cdots A(\tilde{u}_0) \cdot \xi_0$ (where each $\tilde{u}_i \approx u_0$) forms an open cone around $[A(u_0)]^N \cdot \xi_0$.

Main Theorem: Remarks

The statement of the theorem is optimal in the following ways:

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- One cannot replace "finite set" by "empty set".

For $\mathbb{K} = \mathbb{C}$, a similar theorem holds: we replace C^2 maps by analytic maps on a domain $\mathcal{U} \subset \mathbb{C}^m$.

Dynamical application

Given $T : \mathcal{U} \to \mathcal{U}$ and $A : \mathcal{U} \to \mathrm{GL}(d, \mathbb{R})$ we define a (skew-product) dynamical system on $\mathcal{U} \times \mathbb{R}\mathrm{P}^{d-1}$ by $(u, \xi) \mapsto (T(u), A(u) \cdot \xi)$. This is called a (projectivized) *linear cocycle*.

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Local Control for Generic Cocycles [BG]

For every A in a residual subset of $C^2(\mathcal{U}, \mathrm{GL}(d, \mathbb{R}))$, and every T in a open and dense subset of $\mathrm{Diff}^r(\mathcal{U})$ the following holds: *Every* segment of orbit $(u, T(u), \dots, T^{N-1}(u))$ of length N is a universally nonsingular input for the control system $\xi_{t+1} = A(u_t) \cdot x_t$.
(As before, N depends only on d and $m = \dim \mathcal{U}$.)

In short: For generic cocycles, any perturbation of directions can appear by taking pseudoorbits.

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In short: For generic cocycles, any perturbation of directions can appear by taking pseudoorbits.

This kind of result has applications to Lyapunov exponents of cocycles (in the style of Bochi–Fayad, Bull. Braz. Math. Soc., 2006).

Proof of local controllability for generic cocycles

Extremely brief indication of the proof:

- The orbits which are harder to control are the fixed points of T. By the Main Theorem, for generic A the set of bad points in $\mathcal{U}$ is finite. So we just need to assure that the fixed points of T are outside this bad set – easy!
- Similar argument for periodic points of low period.
- Other orbits are easier to control.
- Multijet transversality theorem $\rightarrow$ residual set

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- Other orbits are easier to control.
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Remarks:

- To obtain the dynamical "corollary", a weaker version of the Main Theorem would suffice.
- At the end, we were only able to obtain a residual (instead of open and dense) set of A's that are good for generic T. ☹
- Anyway, the "controllable" cocycles (A, T) form a $C^2 \times C^0$-open and $C^\infty \times C^\infty$-dense.

Back to the Main Theorem: Alternative formulation?

Fix $A : \mathcal{U} \to \mathrm{GL}(d, \mathbb{R})$ and consider the system $\boxed{\xi_{t+1} = A(u_t) \cdot \xi_t}$.

Given a period-1 input $(u_0, \dots, u_0)$, let us investigate if it is universally regular or not.

Obviously, this depends only on the *first jet* of $A : \mathcal{U} \to \mathrm{GL}(d, \mathbb{R})$ at the point u_0.

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Take local coordinates where $u_0 = 0 \in \mathbb{R}^m$, and replace the map A by its first-order approximation. Thus we obtain a control system of the form

$$\boxed{\xi_{t+1} = \left(A_0 + \sum_{i=1}^{m} u_{t,i} B_i \right) \xi_t}$$

where $A_0 \in \mathrm{GL}(d, \mathbb{R})$, $B_1, \ldots, B_m \in \mathrm{Mat}_{d \times d}(\mathbb{R})$, $u_{t,1}, \ldots, u_{t,m} \in \mathbb{R}$.

Systems of this form are called *bilinear control systems*.

Ref: D.L. Elliott. *Bilinear Control Systems*. Springer, 2009.

We will show that for "most" bilinear control systems, the input $(0, \ldots, 0)$ is universally regular.

The set of poor jets

Consider the bilinear control system

$$\boxed{\xi_{t+1} = \left(A + \sum_{i=1}^{m} u_{t,i} B_i \right) \xi_t}$$

determined by the data $(A, B_1, \ldots, B_m)$. The data is called:

- *rich* if the input $(0, \ldots, 0)$ (for some appropriate length N) is universally regular;
- *poor* otherwise.

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Rem: If you are old enough (say $\geq d^2$) and still poor then you'll never get rich.
Let $\mathcal{P} = \mathcal{P}_m^{(d,\mathbb{K})} \subset \mathrm{GL}(d, \mathbb{K}) \times [\mathrm{Mat}_{d \times d}(\mathbb{K})]^m$ indicate the set of poor data.
It's not too difficult to show that $\mathcal{P}$ is:

- a semialgebraic set if $\mathbb{K} = \mathbb{R}$;
- an algebraic set if $\mathbb{K} = \mathbb{C}$.

Actual Main Result: Codimension of the set of poor jets

$\mathcal{P}_m^{(d,\mathbb{K})} \subset \mathrm{GL}(d,\mathbb{K}) \times [\mathrm{Mat}_{d\times d}(\mathbb{K})]^m$ = set of poor jets (or poor bilinear control systems if you prefer).

Codimension Theorem

$\mathrm{codim}_{\mathbb{K}}\, \mathcal{P}_m^{(d,\mathbb{K})} = m$, for any $d \geq 2$, $m \geq 1$.

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The "Main Theorem" below is actually a corollary – just apply standard (jet) transversality theorems (for semialgebraic/algebraic sets).

"Main Theorem" (repeated)

Given $d \geq 2$ and $m = \dim \mathcal{U}$ there exist:

- $N \in \mathbb{N}$;
- an open and (C^∞-)dense subset $\mathcal{O} \subset C^2(\mathcal{U}, \mathrm{GL}(d,\mathbb{R}))$

such that if $A \in \mathcal{O}$ then *all but a finite number* of period-1 inputs of length N are universally regular for the system $\xi_{t+1} = A(u_t) \cdot \xi_t$.

Proof of the Codimension Theorem, part 0: (semi)algebraicness of the set of poor jets

Recall that the (jet) data $(A, B_1, \ldots, B_m)$ is poor if $\exists\ \xi_0 \in \mathbb{K}\mathrm{P}^{d-1}$ such that [a certain linear map] is not surjective.

That non-surjectivity condition is algebraic.

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Now use:

- If $\mathbb{K} = \mathbb{C}$: projection of an algebraic set along "compact" (projective) fibers is algebraic.
- If $\mathbb{K} = \mathbb{R}$: projection of an algebraic set is semialgebraic. (Tarski–Seidenberg.)

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Rem: These abstract theorems don't give (at least simple) equations/inequalities for the projected set.
Anyway, those algebraic equations/inequalities for $\mathcal{P}$ would probably be too complicated to work with.

Reduction to the complex case

The easy (and less useful) half of the theorem is $\boxed{\operatorname{codim}_{\mathbb{K}} \mathcal{P}_m^{(\mathbb{K})} \le m}$ for either $\mathbb{K} = \mathbb{R}$ or $\mathbb{C}$ – more about this later. Let's see the other half.

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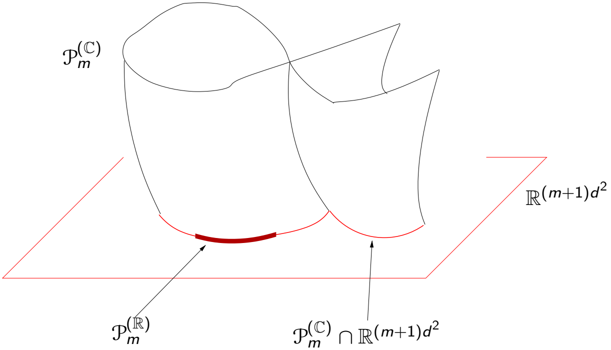

The set $\mathcal{P}_m^{(\mathbb{R})}$ is included in the real section of $\mathcal{P}_m^{(\mathbb{C})}$, and thus $\boxed{\operatorname{codim}_{\mathbb{R}} \mathcal{P}_m^{(\mathbb{R})} \ge \operatorname{codim}_{\mathbb{C}} \mathcal{P}_m^{(\mathbb{C})}}$. So it's sufficient to prove that the latter is $\ge m$.

From now on, we deal only with the complex case, and write $\mathcal{P}_m = \mathcal{P}_m^{(\mathbb{C})}$

Understanding the problem: Easy equivalences

The data $(A, B_1, \ldots, B_m)$ is rich iff:

- For all $\xi_0 \in \mathbb{C}\mathrm{P}^{d-1}$, the derivative of the map

$$(u_{n,i}) \mapsto \left[\prod_{n=N-1}^{0} \left(A + \sum_{i=1}^{m} u_{n,i} B_i \right) \right] \cdot \xi_0$$

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  at zero is surjective.
- $(\Leftrightarrow)$ For all $x_0 \in \mathbb{C}^d_*$, the following set of vectors spans $\mathbb{C}^d$:

$$\{A^N(x_0)\} \cup \{A^{N-n} B_i A^n(x_0);\ 1 \le i \le m,\ 0 \le n \le N-1\}$$

  (The vector $A^N(x_0)$ appeared to account for the projectivization.)

Easy equivalences...

Continuation: $(A, B_1, \dots, B_m)$ is rich $\Leftrightarrow$

- For all $v \in \mathbb{C}^d_*$, the following set of vectors spans $\mathbb{C}^d$:

$$\{v\} \cup \{\underbrace{A^n B_i A^{-n}}_{\mathrm{Ad}_A^n(B_i)} v;\ 1 \le i \le m,\ n \ge 0\}$$

[Recall that Ad_A is the linear operator on $\mathfrak{gl}(d, \mathbb{C}) = \mathrm{Mat}_{d\times d}(\mathbb{C})$ given by $\mathrm{Ad}_A(B) = ABA^{-1}$.]

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- $(\Leftrightarrow)$ The vector space Λ spanned by $\{\mathrm{Id}\} \cup \bigcup_i \{\mathrm{Ad}_A\text{-orbit of } B_i\}$ is transitive.

  [A linear subspace $\Lambda \subset \mathfrak{gl}(d, \mathbb{C}) = \mathrm{Mat}_{d\times d}(\mathbb{C})$ is called *transitive* if it acts transitively on $\mathbb{C}^d_*$, i.e. $\Lambda \cdot v = \mathbb{C}^d\ \forall v \in \mathbb{C}^d_*$.]

Looking for transitive spaces

The prime example of a transitive space $\Lambda \subset \mathfrak{gl}(d, \mathbb{C}) = \mathcal{L}(\mathbb{C}^d, \mathbb{C}^d)$ is the space of *Toeplitz matrices*

$$\begin{pmatrix} t_0 & t_1 & \cdots & t_{d-1} \\ t_{-1} & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & t_1 \\ t_{-d+1} & \cdots & t_{-1} & t_0 \end{pmatrix}$$

Rem: This example has optimal dimension $2d - 1$.

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$$\begin{pmatrix} t_0 & t_1 & \cdots & t_{d-1} \\ t_{-1} & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & t_1 \\ t_{-d+1} & \cdots & t_{-1} & t_0 \end{pmatrix}$$

Rem: This example has optimal dimension $2d - 1$.

We have a **general lemma** (called the *Sudoku Lemma*) that says that certain spaces of "block Toeplitz" matrices are transitive.

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The blocks themselves only need to be transitive.
(Example: replace t_i's above by Toeplitz matrices.)
So the Sudoku Lemma can be applied recursively.
(Actually we'll use up to 3 recursion levels. . .)

Dynamics of $\mathrm{Ad}_A : B \mapsto ABA^{-1}$

Recall the conclusion of a previous slide:

$(A, B_1, \dots, B_m)$ is rich $\Leftrightarrow$ the vector space $\Lambda \subset \mathfrak{gl}(d, \mathbb{C})$ spanned by $\{\mathrm{Id}\} \cup \bigcup_i \{\mathrm{Ad}_A\text{-orbit of } B_i\}$ is transitive.

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Basic fact: $A : \mathbb{C}^d \to \mathbb{C}^d$ has eigenvalues $\lambda_1, \ldots, \lambda_d$ (with multiplicity) $\Rightarrow \mathrm{Ad}_A : \mathfrak{gl}(d, \mathbb{C}) \to \mathfrak{gl}(d, \mathbb{C})$ has eigenvalues $\lambda_i \lambda_j^{-1}$ (with multiplicity).

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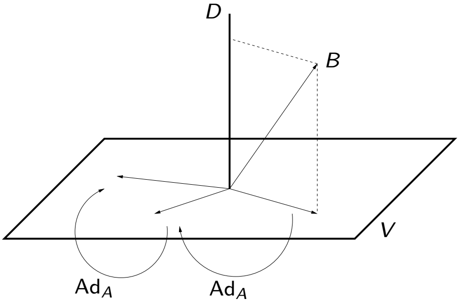
In the generic case (open and dense subset of $\mathrm{GL}(d, \mathbb{C})$) the eigenvalues of A are **unrelated** and so:

- the eigenvalue 1 of Ad_A has multiplicity d, and the eigenspace is the set $\boxed{D}$ of diagonal matrices;
- all other eigenvalues of Ad_A are simple; let $\boxed{V}$ be the sum of the eigenspaces.

Ad_A-invariant splitting: $\boxed{\mathfrak{gl}(d, \mathbb{C}) = D \oplus V}$.

Dynamics of Ad_A (for generic A)



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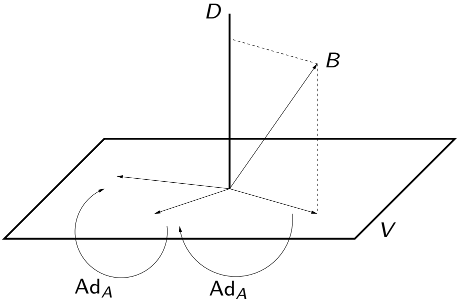

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Actually, in the basis that makes A diagonal, the Ad_A-orbit of B is

$$\left\{ \begin{pmatrix} b_{11}x & & * \\ & \ddots & \\ * & & b_{dd}x \end{pmatrix} ;\ x, *, \dots, * \text{ arbitrary} \right\}$$

A relation along the diagonal $\rightarrow$ (roomy) Toeplitz-like $\rightarrow$ transitivity.

Easy part of the Codimension Theorem: lower estimate

Still assuming A is diagonal with unrelated eigenvalues (i.e. generic):

- The Ad_A orbit of some B is non-transitive iff B has a zero entry outside the diagonal.
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Since eigenvalues and eigenvectors vary analytically in a nghbrhd of A, $\exists$ embedded disks of codimension m inside $\mathcal{P}_m$.

So $\boxed{\operatorname{codim} \mathcal{P}_m \geq m}$. This is the easier half of the Codimension Theorem.

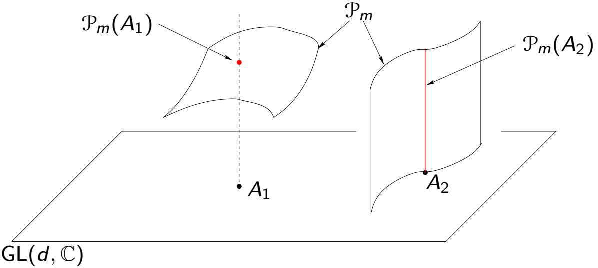
The hard estimate: Fiberwise version

Decompose $\mathcal{P}_m$ into fibers:

$$\boxed{\mathcal{P}_m(A) = \left\{(B_1, \dots, B_m) \in \mathfrak{gl}(d, \mathbb{C})^m;\ (A, B_1, \dots, B_m) \in \mathcal{P}_m\right\}}$$

As we've seen, the codimension of $\mathcal{P}_m(A)$ in $\mathfrak{gl}(d, \mathbb{C})^m$ is m for the generic A in $\mathrm{GL}(d, \mathbb{C})$.

This codimension can be lower for more degenerate matrices A.



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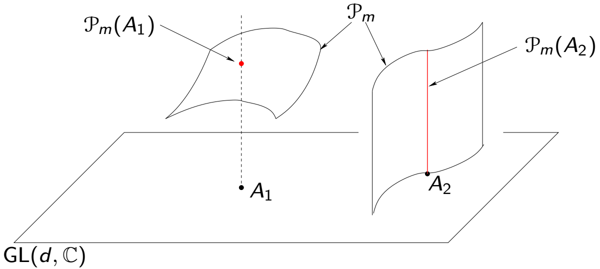

We need to show: $\boxed{\operatorname{codim}\left\{A \in \mathrm{GL}(d, \mathbb{C});\ \operatorname{codim} \mathcal{P}_m(A) \le m - k\right\} \ge k}$, $\forall k \in \{1, \dots, m\}$. (We've already dealt with $k = 1$).

Example: very degenerate matrices

We need to show that A's with "large" fibers form a "small" set, i.e.:

Fibrewise Estimate

$$\operatorname{codim}\left\{A \in \mathrm{GL}(d,\mathbb{C});\ \operatorname{codim}\mathcal{P}_m(A) \le m-k\right\} \ge k, \quad \forall k \in \{1,\dots,m\}$$

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Then Ad_A is the identity; it doesn't help a bit to get transitivity.
So $\operatorname{codim}\mathcal{P}_m(A)$ will be small. (For example it's 0 if $m < 2d-2$.)

Anyway, codim{homotecies} is large ($= d^2-1$) and things compensate (sum $\ge m$).

Strategy to prove the fibrewise estimate

Define the *rigidity* $r = r(A)$ of any $A \in \mathrm{GL}(d, \mathbb{C})$ as the least number of matrices $B_1, \ldots, B_r \in \mathfrak{gl}(d, \mathbb{C})$ such that the jet $(A, B_1, \ldots, B_r)$ is rich.
Example: A generic (unrelated eigenvalues) $\Rightarrow r(A) = 1$ (one good B_1 is enough to get richness).

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More precisely:

1. $\forall\ k \geq 2$, $\operatorname{codim}\{A;\ r(A) \geq k\} \geq k$.
2. $\operatorname{codim} \mathcal{P}_m(A) \geq m + 1 - r(A)$.

With these two pieces the fibrewise estimate follows, and then we are done.

First part: High rigidity is rare

(Lengthy) matrix analysis based on Sudoku Lemma

$\Downarrow$

Lower bounds for $r(A)$ depending on the numbers and sizes of the Jordan blocks of A, and on the occasional algebraic relations between the eigenvalues

☺$\Downarrow$☺

Desired lower estimate for $\operatorname{codim}\{A;\ r(A) \geq k\}$

Second part: $r(A)$ small $\Rightarrow$ fiber $\mathcal{P}(A)$ small

Idea: Prove that an algebraic set is small by showing that its complement contains a large algebraic set.

Knowing $r(A)$ we are able to find a special "large" closed subset of $\mathfrak{gl}(d, \mathbb{C})^m$ that is disjoint from $\mathcal{P}_m(A)$.

By reasons of algebraic geometry, the dimension of $\mathcal{P}_m(A)$ cannot be too large.

Algebraic Geometry: A nice property of projective space $\mathbb{C}P^k$

Let $V \subset \mathbb{C}P^k$ be a (closed) algebraic set, and let V^c be its complement. Then:

- If V^c contains a *point* then $\dim V \le k-1$ [kind of obvious];

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- If V^c contains an (algebraic) *curve* then $\dim V \le k-2$ [not so obvious!]
- and so on ...

  That is:

$$V, W \subset \mathbb{C}\mathrm{P}^k \text{ algebraic and disjoint} \quad \Rightarrow \quad \dim V + \dim W < k.$$

What about Grassmanians?

Grassmanian $G(p,k) = \{k\text{-planes in } \mathbb{C}^k\}$. (It's an algebraic variety.)
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However. . .
Given an algebraic set $V \subset G(p,k)$, it's possible to estimate the maximum dimension of an algebraic set in its complement. ☺

This estimate depends on the homology class of V inside $G(p,k)$ (actually cohomology by Poincaré duality).

Intersection theory in the Grassmannian

Schubert calculus was introduced in the 1870's by Hermann Schubert.

It allows to solve problems like: How many lines in $\mathbb{C}P^3$ simultaneously intersect 4 given (generic) lines? [Answer: 2]

Basically, what it does is to compute $V \cap W$ for V, W (algebraic) in general position, given some (topological) information about V, W.

Putting Schubert's system on a rigorous footing was Hilbert's 15th problem. This was done using modern Algebraic Topology and Algebraic Geometry.

It's not so hard to use. Useful references:

1. Fulton. Young tableaux. Cambridge, 1997.
2. Kleiman, Laksov. Schubert calculus. *Amer. Math. Monthly* (1972).
3. J. Blasiak. Cohomology of the complex Grassmannian. ("An expository paper for the final in Hutchings' algebraic topology class.")