# Perturbation of the Lyapunov spectra of periodic orbits

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### Abstract

We describe all Lyapunov spectra that can be obtained by perturbing the derivatives along periodic orbits of a diffeomorphism. The description is expressed in terms of the finest dominated splitting and Lyapunov exponents that appear in the limit of a sequence of periodic orbits, and involves the majorization partial order. Among the applications, we give a simple criterion for the occurrence of universal dynamics.

## 1. Introduction

### 1.1. Perturbing the derivatives along periodic orbits

An important consequence of perturbation results such as Pugh's closing lemma and Hayashi's connecting lemma and its generalizations is that the global dynamics of a $C^1$-generic diffeomorphism is very well approached by its periodic orbits. Understanding the periodic orbits and their behaviour under perturbations is therefore a way for describing the global dynamics of a generic diffeomorphism.

Let $f : M \to M$ be a diffeomorphism of a compact manifold of dimension $d$. The main ingredient for describing the local dynamics in a neighbourhood of a periodic point $p$ is the derivative $Df^{\pi(p)}$ at the period $\pi(p)$, and more specifically the Lyapunov exponents $\lambda_1(p) \leqslant \ldots \leqslant \lambda_d(p)$, which are obtained by applying the function $(1/\pi(p)) \log | \cdot |$ to the eigenvalues. However, the derivative at the period is not sufficient for understanding how the orbit of $p$ reacts under small perturbations. For this purpose, one needs to know the derivatives $Df$ along the whole orbit. Let us give a simple example.

Consider a set of periodic saddles $p_n$ in a compact invariant set $\Lambda$ of a surface diffeomorphism $f$. If $\Lambda$ is a hyperbolic set, then under perturbations of $f$ of $C^1$-size $\varepsilon$, the Lyapunov exponents of the periodic orbits will vary at most by a quantity proportional to $\varepsilon$. Consider now the case where $\Lambda$ is not hyperbolic. This can happen even if the derivatives at the period 'look' uniformly hyperbolic, for example, there may exist arbitrarily large segments of orbits on which the derivative of $f$ is almost an isometry. Then, as Mañé [**18**] noticed, arbitrarily small perturbations of $f$ allow us to mix two Lyapunov exponents and to create a new periodic orbit which is a sink or a source.

More generally, Mañé proved that, in any dimension, if the stable/unstable splitting over a set of periodic orbits is not dominated, then arbitrarily small perturbations of $f$ may create a non-hyperbolic periodic orbit and then change its index. This was an important step in his proof of the $C^1$-stability conjecture. Mañé's simple argument leads to a natural question.

QUESTION 1.1.  Let $f$ be a diffeomorphism. Consider a sequence of periodic orbits $\gamma_n$ converging, in the Hausdorff topology, to a compact set $\Lambda$. By perturbing the derivatives of $f$ along the periodic orbits $\gamma_n$, which derivatives $Df^{\pi(\gamma_n)}$ may we obtain?

Recall that, by Franks Lemma [**13**], every perturbation of the derivatives along a periodic orbit $\gamma_n$ can be realized by a $C^1$-perturbation of $f$ that keeps $\gamma_n$ invariant. So an answer to the question gives information about the $C^1$-nearby diffeomorphisms.

It is clear that the existence of a dominated splitting on $\Lambda$ imposes obstructions. Thus, the answer of the question clearly depends on the finest dominated splitting.

Partial answers of Question 1.1 have already been obtained: It was shown in [BDP] that if the set of periodic orbits homoclinically related to a saddle $p$ has no dominated splitting at all, then the derivative at the period of one of these orbits becomes an homothety after an arbitrarily small perturbation of $f$. Removing the hypothesis that the orbits are all homoclinically related, Bonatti, N. Gourmelon and T. Vivier [**11**] obtain a slightly weaker result: a perturbation gives a periodic orbit with all the Lyapunov exponents equal. Other results along this direction are given in [**17**].

These results lead to the feeling that one can obtain any barycentric combinations of the Lyapunov exponents in a subbundle without dominated splitting. The present paper gives a precise meaning to this intuition, turning it into a theorem.

1.1.1.  *The Lyapunov graph.*  If $p$ is a periodic point with Lyapunov exponents $\lambda_1 \leqslant \ldots \leqslant \lambda_d$, then we associate to $p$ the *Lyapunov graph* $\boldsymbol{\sigma}(p) = (\boldsymbol{\sigma}_0, \boldsymbol{\sigma}_1, \ldots, \boldsymbol{\sigma}_d)$, where $\boldsymbol{\sigma}_0 = 0$ and $\boldsymbol{\sigma}_i = \sum_{j=1}^{i} \lambda_j$ for $i > 0$ (see Figure 1). The fact that the $\lambda_i$ are increasing in $i$ is equivalent to the fact that the Lyapunov graph $\boldsymbol{\sigma}(p) \colon \{0, \ldots, d\} \to \mathbb{R}$ is *convex*:

$$\boldsymbol{\sigma}_j \leqslant \frac{k-j}{k-i} \cdot \boldsymbol{\sigma}_i + \frac{j-i}{k-i} \cdot \boldsymbol{\sigma}_k \quad \text{for every } i < j < k.$$

We denote by $\mathcal{S}_d \subset \{0\} \times \mathbb{R}^d$ the set of convex graphs; its elements are seen as the graphs of convex maps $\sigma \colon \{0, \ldots, d\} \to \mathbb{R}$ with $\sigma_0 = 0$.

1.1.2.  *Mixing Lyapunov exponents, or raising the Lyapunov graph.*  Our first result is given below:

THEOREM 1.  *Let $f$ be a diffeomorphism of a $d$-dimensional compact manifold, and $\gamma_n = \mathrm{orb}(p_n)$ be a sequence of periodic orbits whose periods tend to infinity. Assume that the sequence $\gamma_n$ converges in the Hausdorff topology to a compact set $\Lambda$ that has no dominated splitting. Then given $\varepsilon > 0$, there is $N$ such that*

(1)  *for every $n \geqslant N$ and*
(2)  *for any convex graph $\sigma \in \mathcal{S}_d$ with*
     (i)  $\sigma_d = \boldsymbol{\sigma}_d(\gamma_n)$ *and*
     (ii)  $\sigma_i \geqslant \boldsymbol{\sigma}_i(\gamma_n)$ *for every $i \in \{1, \ldots, d-1\}$,*

*there exists a $\varepsilon$-$C^1$-perturbation $g$ of $f$ with support in an arbitrary neighbourhood of $\gamma_n$, preserving the orbit $\gamma_n$, and such that $\boldsymbol{\sigma}(\gamma_n, g) = \sigma$.*

REMARK 1.2.  The partial order on Lyapunov spectra that appears in the statement of the theorem is called *majorization*.[†] This terminology was introduced by Hardy, Littlewood

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[†] One says that the Lyapunov spectrum $\lambda_1 \leqslant \ldots \leqslant \lambda_d$ *majorizes* the spectrum $\lambda_1' \leqslant \ldots \leqslant \lambda_d'$ if the associated Lyapunov graphs $\sigma, \sigma' \in \mathcal{S}_d$ satisfy $\sigma_d = \sigma_d'$ and $\sigma_i \leqslant \sigma_i'$ for every $i \in \{1, \ldots, d-1\}$. (The disagreement here
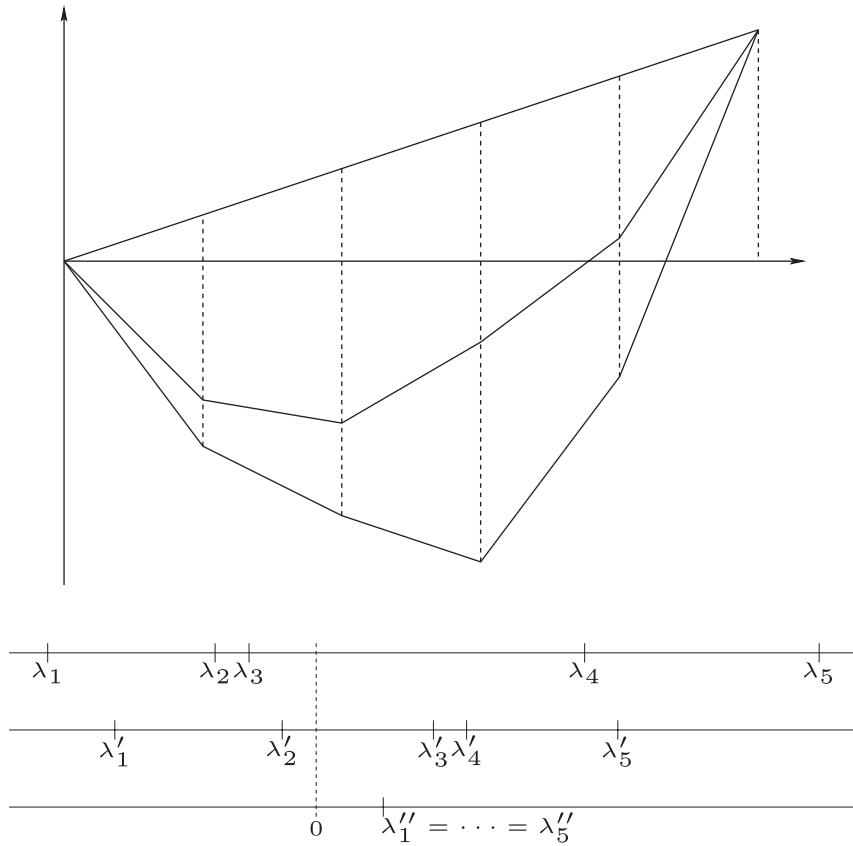
FIGURE 1. *Three graphs $\sigma \leqslant \sigma' \leqslant \sigma''$ in $\mathcal{S}_5$ with $\sigma_5 = \sigma'_5 = \sigma''_5$, and their corresponding Lyapunov spectra.*

and Pólya [**15**]. Majorization is a widely studied subject and has application in many different contexts (see [**19**]).

We present now a more complete version of Theorem 1 including the case where $\Lambda$ admits a dominating splitting.

THEOREM 2.   *Let $f$ be a diffeomorphism of a $d$-dimensional compact manifold, and $\gamma_n = \mathrm{orb}(p_n)$ be a sequence of periodic orbits whose periods tend to infinity and that converges in the Hausdorff topology to a compact set $\Lambda$. Let*

$$E_1 \oplus_{\lneqq} E_2 \oplus_{\lneqq} \ldots \oplus_{\lneqq} E_m$$

*be the finest dominated splitting over $\Lambda$, and denote*

$$i_j = \dim(E_1 \oplus \ldots \oplus E_j) \quad for \ j \in \{1, \ldots, m\}.$$

*Then given $\varepsilon > 0$, there is $N$ such that:*

  (1) *for every $n \geqslant N$ and*

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between $\sigma \leqslant \sigma'$ and the word 'majorize' is due to the fact that we ordered the spectrum in increasing order, while in the literature on majorization the decreasing order is preferred.)

    (2) *for any convex graph $\sigma \in \mathcal{S}_d$ with*
        (i) $\sigma_{i_j} = \boldsymbol{\sigma}_{i_j}(\gamma_n)$ *for every* $j \in \{1, \ldots, m\}$ *and*
        (ii) $\sigma_i \geqslant \boldsymbol{\sigma}_i(\gamma_n)$ *for every* $i \in \{1, \ldots, d\}$,

*there exists a $\varepsilon$-$C^1$-perturbation $g$ of $f$ with support in an arbitrary neighbourhood of $\gamma_n$, preserving the orbit $\gamma_n$ and such that $\boldsymbol{\sigma}(\gamma_n, g) = \sigma$.*

    The case where the finest dominated splitting is *trivial* (that is, $m = 1$) corresponds to Theorem 1.

    In this statement, the requirement that the graph $\sigma$ touches $\boldsymbol{\sigma}(\gamma_n)$ at the dimensions $i_j$ corresponding to the finest dominated splitting cannot be significantly weakened: we could at most move these points by a quantity proportional to $\varepsilon$. Indeed, for large $n$, the orbit $\gamma_n$ (with respect to $f$ or the perturbation $g$) has a dominated splitting close to that on $\Lambda$. So $\boldsymbol{\sigma}_{i_j}(\gamma_n)$ is the average of the logarithm of the determinant of the derivative restricted to the sum of the first $i$ bundles of this splitting and cannot vary much.

    On the other hand, we can improve the conclusions of Theorem 2 along other directions (see Section 4 for precise statements).

    (1) The diffeomorphism $g$ is isotopic to $f$ by an isotopy $g_t$ and all the $g_t$ are close to $f$.
    (2) The $\boldsymbol{\sigma}_i(\gamma_n, g_t)$ vary monotonically along the isotopy.
    (3) If $\gamma_n$ is a hyperbolic periodic orbit, then we may require that along the isotopy $\gamma_n$ remains hyperbolic, provided of course that the index of $\gamma_n$ (that is, the dimension of the contracting bundle) is the same as the index of $\sigma$.
    (4) Theorems 1 and 2 are expressed in terms of dominated splittings on the Hausdorff limit $\Lambda$ of the periodic orbits $\gamma_n$ and not on the orbits themselves. This is because every periodic orbit automatically has a dominated splitting separating different Lyapunov exponents. However, such splittings may be very weak. We give individual (hence stronger) versions of Theorems 1 and 2, expressed in terms of the weakness of the dominated splitting and of the period of a periodic orbit: given $\varepsilon > 0$, there is $N$ and $\ell$ such that for every periodic orbit $\gamma$ with period larger than $N$, we can make any perturbation of the Lyapunov spectrum of $\gamma$ that is compatible with its finest $\ell$-dominated splitting.
    (5) In particular, the conclusions of Theorems 1 and 2 still hold if $\gamma_n$ are periodic orbits of diffeomorphisms $f_n$ such that $\gamma_n$ converge to $\Lambda$ in the Hausdorff topology, and $f_n$ converge to $f$ in the $C^1$-topology.

    1.1.3. *Separating Lyapunov exponents, or lowering the Lyapunov graph.* Theorem 2 explains the Lyapunov graphs $\sigma$ above $\boldsymbol{\sigma}(\gamma_n)$ (that is, $\sigma_i \geqslant \boldsymbol{\sigma}_i(\gamma_n)$) that can be obtained by small $C^1$-perturbations of the derivative $Df$ along $\gamma_n$. To obtain a complete answer to Question 1.1, we need to remove the hypothesis that $\sigma$ is above $\boldsymbol{\sigma}(\gamma_n)$. At first, let us remark that there is a natural lower bound for the possible perturbations of $\boldsymbol{\sigma}(\gamma_n)$ in terms of limit measures

    Consider a sequence of periodic orbits $\gamma_n$ such that the invariant probability $\mu_n$ supported on $\gamma_n$ converges weakly to a (not necessarily ergodic) measure $\mu$. Let $\lambda_1(\mu) \leqslant \ldots \leqslant \lambda_d(\mu)$ be the integrated Lyapunov exponents of $\mu$ and $\boldsymbol{\sigma}(\mu) = \boldsymbol{\sigma}_0(\mu), \ldots, \boldsymbol{\sigma}_d(\mu)$ the associated Lyapunov graph, where $\boldsymbol{\sigma}_i(\mu) = \sum_{j=1}^i \lambda_j(\mu)$. Recall that the map $(f, \mu) \mapsto \boldsymbol{\sigma}(f, \mu)$ is lower semicontinuous (see §2.8). As a consequence, for any $\delta > 0$, there is $\varepsilon > 0$ and $N \in \mathbb{N}$ such that, for any $n \geqslant N$ and any $\varepsilon$-perturbation $g_n$ of the derivative along the orbits $\gamma_n$, one has

$$\boldsymbol{\sigma}_i(g_n, \gamma_n) \geqslant \boldsymbol{\sigma}_i(f, \mu) - \delta \quad \text{for any } i = 1, \ldots, d.$$

Hence the Lyapunov graph of the limit measure $\mu$ appears as being a lower bound of the Lyapunov graph of perturbations of the derivative along $\gamma_n$. The result below asserts that this bound can be attained.

THEOREM 3.  *Let $f$ be a diffeomorphism of a $d$-dimensional compact manifold, and let $f_n$ be a sequence of diffeomorphisms converging to $f$ in the $C^1$-topology. Let $\gamma_n = \mathrm{orb}(p_n, f_n)$ be a sequence of periodic orbits of $f_n$ whose periods tend to infinity. Suppose that the $f_n$-invariant probabilities $\mu_n$ associated to $\gamma_n$ converge in the weak-star topology to an $f$-invariant measure $\mu$. Then there is a sequence of diffeomorphisms $g_n$ such that*

(1) *the $C^1$-distance between $g_n$ and $f_n$ tends to 0;*
(2) *the diffeomorphism $g_n$ preserves $\gamma_n$ and coincides with $f_n$ out of an arbitrarily small neighbourhood of $\gamma_n$;*
(3) *$\boldsymbol{\sigma}(g_n, \gamma_n) = \boldsymbol{\sigma}(f, \mu)$ for all $n$.*

REMARK 1.3.  Section 6 from [**1**] contains some related results (see also § 1.2.2). Theorem 3 is finer than the results in [**1**], because

(1) it allows $\mu$ to be non-ergodic;
(2) the periodic orbits with the desired Lyapunov exponents produced by Theorem 3 are also periodic orbits of the unperturbed map $f$, while in [**1**] these periodic orbits arise from the ergodic closing lemma.

1.1.4. *Answer to Question* 1.1.   Next we combine Theorems 2 and 3, in order to obtain a complete description of what is possible to get as a Lyapunov graph by perturbing the derivative along periodic orbits. Let us first introduce some notation.

Given a compact invariant set $\Lambda$ for a diffeomorphism $f$, let $E_1 \oplus E_2 \oplus \ldots \oplus E_m$ be the finest dominated splitting over $\Lambda$, and denote $i_j = \dim(E_1 \oplus \ldots \oplus E_j)$ for $j \in \{1, \ldots, m\}$. Given an $f$-invariant probability measure $\mu$ whose support is contained in $\Lambda$, let $\mathcal{G}(\mu, \Lambda)$ indicate the set of convex graphs $\sigma \in \mathcal{S}_d$ with

(1) $\sigma_{i_j} = \boldsymbol{\sigma}_{i_j}(f, \mu)$ for every $j \in \{1, \ldots, m\}$ and
(2) $\sigma_i \geqslant \boldsymbol{\sigma}_i(f, \mu)$ for every $i \in \{1, \ldots, d\}$.

THEOREM 4.  *Let $f$ be a diffeomorphism of a $d$-dimensional compact manifold, and $f_n$ a sequence of diffeomorphisms converging to $f$ in the $C^1$-topology. Let $\gamma_n = \mathrm{orb}(p_n, f_n)$ be a sequence of periodic orbits of diffeomorphisms $f_n$ whose periods tend to infinity. Suppose that the $f_n$-invariant probabilities $\mu_n$ associated to $\gamma_n$ converge in the weak-star topology to an $f$-invariant measure $\mu$ and that the sets $\gamma_n$ converge in the Hausdorff topology to an $f$-invariant compact set $\Lambda$.*

*Then $\mathcal{G}(\mu, \Lambda)$ is precisely the set of the limits of Lyapunov graphs $\boldsymbol{\sigma}(g_n, \gamma_n)$ where $\{g_n\}$ runs over the set of all sequences of diffeomorphisms $g_n$ preserving $\gamma_n$ whose $C^1$-distance to $f_n$ tends to 0 as $n \to \infty$.*

Notice that $\Lambda \supset \mathrm{supp}\,\mu$ in the statement above and this inclusion can be strict (see Lemma 7.1, for example).

1.2. *Consequences*

We now explore some consequences of our perturbation theorems.

1.2.1. *Index changes.*   Recall that the index (or stable index) of a hyperbolic periodic orbit is the dimension of its contracting bundle. The first corollary describes explicitly what are the indices we can create by perturbing a set of periodic orbits.
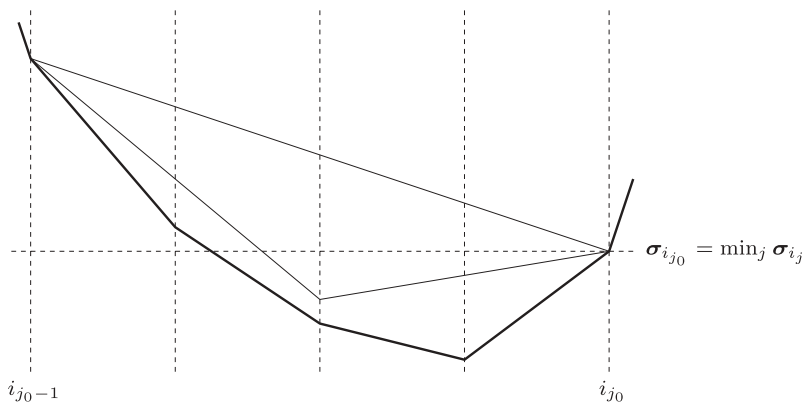
$$\boldsymbol{\sigma}_{i_{j_0}} = \min_j \boldsymbol{\sigma}_{i_j}$$

$i_{j_0-1}$                                    $i_{j_0}$

FIGURE 2. *An example in the situation of Corollary 1.4 : the lower graph is $\boldsymbol{\sigma}(\mu)$. In this case, the minimum of $\boldsymbol{\sigma}_{i_j}(\mu)$ is attained at a unique $j = j_0$. The numbers $k$ that satisfy condition* (1.1) *are $i_{j_0} - 2$, $i_{j_0} - 1$ and $i_{j_0}$; any of these is the index of a periodic point of a perturbation of $f$. Some possibilities for the corresponding Lyapunov graphs are pictured.*

COROLLARY 1.4.   *Let $f$ be a diffeomorphism of a $d$-dimensional compact manifold, and let $\gamma_n = \mathrm{orb}(p_n)$ be a sequence of periodic orbits whose periods tend to infinity. Suppose that the invariant probabilities $\mu_n$ associated to $\gamma_n$ converge in the weak-star topology to a measure $\mu$ and that the sets $\gamma_n$ converge in the Hausdorff topology to an $f$-invariant compact set $\Lambda$. Let $E_1 \oplus E_2 \oplus \ldots \oplus E_m$ be the finest dominated splitting over $\Lambda$ and denote $i_j = \dim(E_1 \oplus \ldots \oplus E_j)$ for $j \in \{1, \ldots, m\}$, and $i_0 = 0$.*

*Assume that $k \in \{0, \ldots, d\}$ satisfies*

$$\boldsymbol{\sigma}_k(\mu) \leqslant \min_{j \in \{0, \ldots, m\}} \boldsymbol{\sigma}_{i_j}(\mu). \tag{1.1}$$

*Then for every sufficiently large $n$, there exists a perturbation $g$ of $f$ such that $\gamma_n$ is preserved by $g$ and is hyperbolic with index $k$.*

An example is shown in Figure 2. Notice that the indices $k$ that satisfy relation (1.1) form an interval in $\mathbb{Z}$, which is contained in an interval of the form $[i_{j-1}, i_j]$.

It was shown in [2] that for any homoclinic class $H$ of a $C^1$-generic diffeomorphism, the indices of the periodic points in $H$ form an interval in $\mathbb{Z}$. This result is a motivation for the so-called *index completeness problems*.

QUESTION 1.5.   Fix an homoclinic class $H$ of a $C^1$-generic diffeomorphism $f$.

(1) *Inner completeness problem:* Does the interval $I_{\mathrm{in}}$ of the indices of periodic orbits contained in $H$ coincide with the set $I_{\mathrm{erg}}$ of indices of ergodic measures supported in $H$?

(2) *Outer completeness problem:* Consider the set $I_{\mathrm{out}}$ of indices of sequences of periodic orbits accumulating in $H$ (which contains $I_{\mathrm{erg}}$, by Abdenur, Bonatti and Crovisier [1, Theorem 3.8]). Does it form an interval? Does it coincide with the interval $I_{\mathrm{in}}$?

As a step towards the solution of the first problem above, we pose the following question:

QUESTION 1.6.   In the situation of Corollary 1.4, suppose additionally that all the periodic orbits $\gamma_n$ belong to the same homoclinic class $H$. Let $k$ satisfy condition (1.1). Can we find a perturbation of $f$ such that the continuation of the class $H$ contains a periodic orbit of index $k$?

We expect the answer to be positive; Gourmelon's [**14**] result (a Franks' lemma that controls the position of the invariant manifolds of a periodic point) should be useful here.

1.2.2. *Lyapunov spectra of periodic orbits for generic diffeomorphisms.*   A general principle is that generic diffeomorphisms already display any robust property that we can get by small perturbations. Theorem 4 explains what Lyapunov exponents are possible to get by perturbing the derivative of a diffeomorphism along a sequence of periodic orbits. Applying the general principle to Theorem 4, we obtain Corollary 1.7. It states that generically the closure of the set of Lyapunov graphs associated to a sequence of periodic orbits is exactly the set of graphs that are greater than the Lyapunov graph of the limit measure and that respect the constraints due to the dominated splittings on the limit support.

In order to be more precise, let us introduce some notation. Let $X$ be a compact metric space and let $\mathcal{P}(X)$ be the set of Borel probability measures on $K$, endowed with the weak-star topology. Also, let $\mathcal{K}(X)$ be the set of compact subsets of $X$ endowed with the Hausdorff distance. Both $\mathcal{P}(X)$ and $\mathcal{K}(X)$ are compact sets.

If $f$ is a diffeomorphism of a compact manifold and $\gamma \in \mathcal{K}(M)$ is a periodic orbit, then $\mu_\gamma$ indicate the only $f$-invariant probability supported on $\gamma$. Let $\mathcal{X}(f)$ be the closure in $\mathcal{P}(M) \times \mathcal{K}(M)$ of the set of pairs $(\mu_\gamma, \gamma)$, where $\gamma$ runs over the set of hyperbolic periodic orbits of $f$. Since hyperbolic periodic orbits persist under perturbations, the map $f \mapsto \mathcal{X}(f)$ is lower semicontinuous.[†]

According to Mañé ergodic closing lemma [**18**] (see also [**1**, Theorem 4.2]), for $C^1$-generic diffeomorphisms $f$, the set $\mathcal{X}(f)$ contains the pair $(\mu, \operatorname{supp}\mu)$ for every ergodic measure $\mu$. For another way of finding elements of $\mathcal{X}(f)$, see Lemma 7.1.

Recall that if $\mu \in \mathcal{P}(M)$ is an $f$-invariant measure and $\Lambda \in \mathcal{K}(M)$ is an $f$-invariant set containing $\operatorname{supp}\mu$, then $\mathcal{G}(\mu, \Lambda)$ indicates the set of Lyapunov graphs that are compatible with $\boldsymbol{\sigma}(\mu)$ and with the finest dominated splitting on $\Lambda$.

Now we can state the following consequence of Theorem 4, which simultaneously improves Abdenur, Bonatti and Crovisier [**1**, Theorem 3.8 and Corollary 3.9]:

COROLLARY 1.7.   *For $C^1$-generic diffeomorphisms $f$, for every $(\mu, \Lambda) \in \mathcal{X}(f)$ and for every $\sigma \in \mathcal{G}(\mu, \Lambda)$, there is a sequence $\gamma_n$ of periodic orbits converging to $\Lambda$ for the Hausdorff topology, with $\mu_{\gamma_n}$ converging to $\mu$ in the weak-star topology, and $\boldsymbol{\sigma}(f, \gamma_n)$ converging to $\sigma$.*

1.2.3. *Universal dynamical systems.*   Among other applications of Theorem 4, we will obtain (in Theorem 5) a criterion for a diffeomorphism to be approximated by wild diffeomorphisms. Let us begin with the relevant definitions.

Let $f$ be a diffeomorphism of a compact manifold $M$. The *chain recurrent set $R(f)$* has a natural partition into *chain recurrence classes*: two points $x$ and $y$ are equivalent, if for any $\varepsilon > 0$, there is an $\varepsilon$-pseudo-orbit starting at $x$, passing by $y$, and coming back to $x$.

We say that the diffeomorphism $f$ is *tame* if each chain recurrence class $C$ is robustly isolated: for every $g$ in a $C^1$-neighbourhood of $f$, there is a unique chain recurrent class contained in a small neighbourhood of $C$. In an equivalent way, a diffeomorphism $f$ is tame if the number of chain recurrence classes is finite and constant in a $C^1$-neighbourhood of $f$.

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[†]Recall that if $Y$ is a compact metric space and $Z$ is a topological space then a map $\Phi : Z \to \mathcal{K}(Y)$ is called *lower* (or *inner*) *semicontinuous* if for every $z \in Z$ and every open $V \subset Y$ with $V \cap \Phi(z) \neq \emptyset$ there is a neighbourhood $U$ of $z$ in $Z$ such that $\Phi(z') \cap V \neq \emptyset$ for all $z' \in U$. Also, $\Phi$ is called *upper* (or *outer*) *semicontinuous* if for every $z \in Z$ and every open $V \subset Y$ with $V \supset \Phi(z)$ there is a neighbourhood $U$ of $z$ in $Z$ such that $\Phi(z') \subset V$ for all $z' \in U$.

The set of tame diffeomorphisms is indicated by $\mathcal{T}(M)$, and the set of *wild diffeomorphisms* is defined as

$$\mathcal{W}(M) = \mathrm{Diff}^1(M) \smallsetminus \overline{\mathcal{T}(M)}.$$

A generic diffeomorphism is tame (resp. wild) if and only if it has finitely (resp. infinitely) many chain recurrence classes. (Proof: If a diffeomorphism $f$ satisfies the generic properties of Remark 1.12 and Corollary 1.13 from Bonatti and Crovisier [6], and if it has finitely many chain recurrence classes, then this number is locally constant.)

It is shown in Bonatti and Díaz [7] that $\mathcal{W}(M)$ is non-empty for every compact manifold $M$ with $\dim M \geqslant 3$.

A stronger notion of wildness called *universal dynamics* was introduced in the paper [8]. Let us define this notion.

Let $\mathbb{D}^k$ be the closed $k$-dimensional disc. Let $\mathrm{Diff}^1_{\#}(\mathbb{D}^k)$ be the set of diffeomorphisms from $\mathbb{D}^k$ to a subset of $\mathrm{Int}\,\mathbb{D}^k$ that are diffeotopic to the identity map.

For any $k \in \{1, \ldots, d\}$, we say that a diffeomorphism $f$ is *$k$-universal* (or has *$k$-universal dynamics*) if there is a collection $\{D_n\}$ of embedded $k$-dimensional closed discs, with embeddings $\phi_n \colon \mathbb{D}^k \to D_n$, such that the following properties hold:

(1) For each $n$ there is $\pi_n$ such that $D_n$, $f(D_n)$, ..., $f^{\pi_n - 1}(D_n)$ are pairwise disjoint and $f^{\pi_n}(D_n)$ is contained in the (relative) interior of $D_n$.
(2) The orbits of the discs are pairwise disjoint.
(3) The discs are normally hyperbolic.
(4) Let $F_n$ indicate the restriction of $f^{\pi_n}$ to $D_n$. Then the maps $\phi_n^{-1} \circ F_n \circ \phi_n \colon \mathbb{D}^k \to \mathrm{Int}\,\mathbb{D}^k$ form a dense family in the set $\mathrm{Diff}^1_{\#}(\mathbb{D}^k)$.

The *$d$-universal* diffeomorphisms are simply called *universal*.

Some easy observations about this definition follow .

(1) If $f$ is $k$-universal, then so is $f^{-1}$ (take a suitable family of smaller discs).
(2) $k+1$-universal dynamics implies $k$-universal dynamics.
(3) The property of 1-universal dynamics is very weak: it is generically satisfied in $\mathrm{Diff}^1(M) \smallsetminus \overline{\mathrm{Hyp}(M)}$, where $\mathrm{Hyp}(M)$ is the set of Axiom A diffeomorphisms without cycles.
(4) If $\mathcal{R}$ is a locally residual set (that is, a set that is residual on an open set) of $d$-universal diffeomorphisms then every $f \in \mathcal{R}$ is wild.

It was shown by Bonatti and Díaz[8] that (non-empty) locally residual sets of diffeomorphisms with $d$-universal (and hence wild) dynamics indeed exist. These diffeomorphisms generically have any robust or locally generic dynamical property that appears in $\mathrm{Diff}^1_{\#}(\mathbb{D}^d)$ (for example, existence of a sink, a source, aperiodic maximal transitive Cantor sets, etc.), and this property is displayed in infinitely many periodic discs.

However, $k$-universal dynamics does not imply wildness, for $k \leqslant d-2$: It is not too hard to modify Shub example of a non-hyperbolic robustly transitive diffeomorphism (see, for example, [10, § 7.1.1]) in order to find open sets of robustly transitive diffeomorphisms that are generically $d-2$-universal.

For this reason, we will introduce a stronger notion. We will say that $f$ is *freely $k$-universal* if one may choose the discs $D_n$ in the definition of $k$-universal dynamics so that they are pairwise separated by a filtration: for any $n \neq m$, there is an attracting region $U$ of $f$ (that is, $f(\bar{U}) \subset \mathrm{Int}\,U$) containing $D_n$ and disjoint from $D_m$, or vice versa.

Some easy observations are:

(1) If $f$ has free $k$-universal dynamics, then so does $f^{-1}$ (replace $U$ by $M \smallsetminus \bar{U}$.)

(2) $k$-universal dynamics is always freely $k$-universal if $k = d$ or $d - 1$. For $k = d$ this is immediate. If $k = d - 1$, then by definition, each disc $D_n$ is either normally contracting or normally expanding, and so there is a small neighbourhood $V$ of $D_n$ (which can be chosen disjoint from the orbit of any other disc $D_m$) such that either $V$ or $M \smallsetminus V$ is an attracting region.

Let us say that $f$ has *normally contracting* (resp. *normally expanding*) $k$-universal dynamics if all discs in the definition of $k$-universality can be taken normally contracting (resp. normally expanding). (More generally, it could be interesting, for a global study of a wild dynamics, to distinguish other types of universal dynamics according to the kind of normal hyperbolicity.) In any case, $f$ is freely $k$-universal.

A direct consequence of results from Bonatti and Díaz [**8**] (see § 7.4 for details) is the following criterion for free $k$-universality.

THEOREM 1.8.   *Let $f$ be a diffeomorphism having a periodic point $p$ such that $Df^{\pi(p)}(p)$ satisfies*:

(1) *there is an invariant subspace $E \subset T_p M$ restricted to which $Df^{\pi(p)}(p)$ is the identity map;*
(2) $\dim E = k \geqslant 3$;
(3) *the other $d - k$ eigenvalues all have modulus bigger than 1.*

*Then there are arbitrarily small $C^1$ perturbations of $f$, supported in arbitrarily small neighbourhoods of $p$, that belong to a locally generic set formed by normally expanding $k$-universal diffeomorphisms.*

Obviously, there is a similar criterion for normally contracting universal dynamics.

REMARK 1.9.   The criterion given by the theorem is certainly wrong if $k = 1$. If $k = 2$, it is unknown and is related with Smale conjecture on the denseness of Axiom A diffeomorphisms on surfaces. However, if $\mathcal{U}$ is a $C^1$-open set such that diffeomorphisms $f$ in a dense subset of $\mathcal{U}$ have a periodic point satisfying the hypotheses of the theorem, with $k = 2$, then generic diffeomorphisms in $\mathcal{U}$ have free 2-universal dynamics.

Theorem 1.8 gives us a hint that the control of Lyapunov exponents can be useful to get free $k$-universality (at least if $k \geqslant 3$): one needs $k$ vanishing exponents, and all the others having the same sign. In fact, using Theorem 4, we can show the following simple criterion for a $C^1$-generic diffeomorphism to generate free $k$-universal dynamics, for any $k < d$.

THEOREM 5.   *Let $f$ be a $C^1$-generic diffeomorphism having a periodic point $p$ of index $k \in \{1, \ldots, d - 1\}$. Let $E_1 \oplus \ldots \oplus E_m$ be the finest dominated splitting on the homoclinic class $H(p)$. Suppose that $|\det Df^{\pi(p)} \restriction E_1(p)| > 1$. Then generic diffeomorphisms in a neighbourhood of $f$ have the normally expanding $k$-universal dynamics.*

Here the interest is not to provide an example of a locally residual set with free $k$-universal dynamics: this could be done without difficulty using the arguments of Bonatti and Díaz [**8**]. We can in fact strengthen Theorem 5 and obtain the following result, which, in particular, characterizes the diffeomorphisms which are far from normally expanding $k$-universal dynamics.

THEOREM 6.    *For any $k \in \{1, \ldots, d-1\}$, if $f$ is a generic diffeomorphism then $f$ has (at least) one of the following properties*:

   (i) *$f$ is normally expanding $k$-universal or*
   (ii) *let $p$ be any periodic saddle of index $k$ and let $E_1 \oplus \ldots \oplus E_m$ be the finest dominated splitting on the homoclinic class $H(p)$. Then $f$ contracts the volume in $E_1$ in the following sense: There is $\alpha = \alpha(p) > 0$ such that, for any $q$ homoclinically related with $p$,*

$$\frac{1}{\pi(q)} \log |\det Df^{\pi(q)} \restriction E_1(q)| < -\alpha.$$

Coming back to $k$-universal dynamics, we also obtain a criterion for it similar to Theorem 5 (see Theorem 7.10).

Let us mention that Bonatti, Li and Yang [**12**] provide, in dimension 3, an example of a $C^1$-open set $\mathcal{O}$ of diffeomorphisms $f$ having a robust quasi-attractor $\Lambda_f$, satisfying the hypotheses of Theorem 5: $\Lambda_f$ admits a dominated splitting $E^{cs} \oplus E^u$ with dim $E^{cs} = 2$, and a periodic point $p_f \in \Lambda_f$ of index 1 with $|\det Df^{\pi(p_f)} \restriction E^{cs}(p_f)| > 1$. Hence, for generic $f$ in $\mathcal{O}$, the local dynamics in a neighbourhood of the quasi-attractor $\Lambda_f$ is freely (normally expanding) 2-universal.

1.2.4.    *Other consequences?*   We expect our results to be useful for other applications. Bearing this in mind, we proved results that are actually stronger than those stated in § 1. For example, a strengthened version of Theorem 2 gives us a whole *path* of perturbations along which we have fine control of the Lyapunov graph (see § 4). This information can be useful if one wants to apply the improved version of Franks lemma proved by Gourmelon [**14**], for instance.

1.3.    *Other comments and organization of the paper*

Actually most of our results are expressed in terms of *linear cocycles*. In fact, since this paper concerns *periodic* orbits, we are mainly interested in cocycles over *cyclic* dynamical systems. The results for diffeomorphisms explained above follow immediately, using Franks lemma.

Thus, some of our results fit into the perturbation theory of matrix eigenvalues. However, the literature in this area usually considers a single matrix or operator, while here we consider a finite product of them. Of course, the key concept of domination is uninteresting for a single matrix.

The paper is organized as follows. Section 2 introduces cocycles, and also contains other definitions, notation and basic facts to be used throughout the paper. In Section 3, we establish a central proposition that permits us to mix two Lyapunov exponents while keeping the others fixed. In Section 4, we obtain cocycle versions of Theorems 1 and 2 that also incorporate the improvements mentioned above. In Section 5, we obtain stronger versions of Theorems 3 and 4. Section 6 contains the proofs of Corollaries 1.4 and 1.7. In Section 7, we give the applications to universal dynamics.

## 2.    *Definitions and notation*

### 2.1.    *Linear cocycles*

A *linear cocycle* is a vector bundle automorphism. Let us be more precise and fix some notation. Let $X$ be a compact metric space, and let $E$ be a vector bundle over $X$ of dimension $d$, endowed with a Euclidean metric $\|\cdot\|$. The fibre over a point $x \in X$ is denoted by $E_x$ or $E(x)$. Then a linear cocycle $A$ on $E$ is completely determined by a homeomorphism $T : X \to X$ and a

continuous map that associates to each $x \in X$ an invertible linear map $A(x) : E(x) \to E(Tx)$. We then say that *A is a cocycle on E over T*, or more precisely that $(X, T, E, A)$ *is a cocycle*.

The *n*-iterate of a cocycle is the cocycle over $T^n$ whose fibre maps are $A^n(x) = A(T^{n-1}x) \ldots A(x)$ if $n > 0$ and $A^n(x) = A(T^{-1}x) \ldots A(T^n x)$ if $n < 0$.

If $K > 1$, then we say that a cocycle as above is *bounded by K* if $K^{-1} \leqslant \mathfrak{m}(A(x)) \leqslant \|A(x)\| \leqslant K$ for every $x \in X$. Here, $\mathfrak{m}(B)$ indicates the minimum expansion factor of the linear map $B$, that is, $\mathfrak{m}(B) = \inf_{\|v\|=1} \|Bv\|$, or $\mathfrak{m}(B) = \|B^{-1}\|^{-1}$ when $B$ is invertible.

A cocycle $\tilde{A}$ is called an *$\varepsilon$-perturbation* of a cocycle $A$ if $\|\tilde{A} - A\| < \varepsilon$.

A *path of cocycles* is a family of cocycles $(X, T, E, A_t)$, where $t$ runs on an interval $[t_0, t_1] \subset \mathbb{R}$, such that $A_t(x)$ depends continuously on $(t, x)$. We say that a path of cocycles $A_t$, where $t \in [t_0, t_1]$, is *$\varepsilon$-short* if each $A_t$ is an $\varepsilon$-perturbation of $A_{t_0}$.

REMARK 2.1.

(1) If $A$ is bounded by $K$, and $\tilde{A}$ is an $\varepsilon$-perturbation of $A$, then $\tilde{A}^{-1}$ is a $K^2 \varepsilon$-perturbation of $A^{-1}$.

(2) For any $K > 1$, there is $\varepsilon > 0$ such that any $\varepsilon$-perturbation of a cocycle bounded by $K$ is also bounded by $2K$.

## 2.2. *Restricted and quotient cocycles*

We say that a subbundle $F$ of $E$ (whose fibres by definition have a constant dimension) is *invariant* if $A(x) \cdot F(x) = F(Tx)$ for each $x \in X$. In this case, we define two new cocycles:

(1) the *restricted cocycle* $A \restriction F$ on the bundle $F$;
(2) the *quotient cocycle* $A/F$ on the quotient bundle $E/F$ (where the norm of an element of $E(x)/F(x)$ is defined as the norm of its unique representative that is orthogonal to $F(x)$).

Notice that if $A$ is bounded by $K$ then $A \restriction F$ and $A/F$ are also bounded by $K$.

Let us recall Lemma 4.1 from [BDP], which gives two useful procedures for extending perturbations of cocycles. Let $A$ be a cocycle on a bundle $E$ with an invariant subbundle $F$. With respect to the splitting $E = F \oplus F^\perp$, we can write

$$A = \begin{pmatrix} A \restriction F & D \\ 0 & A/F \end{pmatrix}.$$

Given any cocycle $B$ on $F$ we can define a cocycle $\hat{B}$ on $E$ that preserves $F$ and satisfies $\hat{B} \restriction F = B$ and $\hat{B}/F = A/F$, namely

$$\hat{B} = \begin{pmatrix} B & D \\ 0 & A/F \end{pmatrix}.$$

Moreover, $\hat{B}$ depends continuously on $A$ and $B$, and is bounded by $K$ if so are $A$ and $B$. Similarly, given any cocycle $C$ on $E/F$, we can define a cocycle $\bar{C}$ on $E$ that preserves $F$, and satisfies $\bar{C} \restriction F = A$ and $\bar{C}/F = C$, namely

$$\bar{C} = \begin{pmatrix} A \restriction F & D \\ 0 & C \end{pmatrix}.$$

Moreover, $\bar{C}$ depends continuously on $A$ and $C$, and is bounded by $K$ if so are $A$ and $C$.

## 2.3. *Domination*

Assume given a coycle $(X, T, E, A)$ and two invariant non-zero subbundles $F$ and $G$ of constant dimensions. Take $\ell$ in the set $2 \wedge \mathbb{N} = \{2^0, 2^1, 2^2, \ldots\}$. We say that $F$ is $\ell$-*dominated by* $G$ if

$$\frac{\|A^\ell \upharpoonright F(x)\|}{\mathfrak{m}(A^\ell \upharpoonright G(x))} < \frac{1}{2} \quad \text{for every } x \in X.$$

This is denoted by $F <_\ell G$. If, in addition, $E = F \oplus G$ then we say that $E = F \underset{<}{\oplus} G$ is an $\ell$-*dominated splitting*. The symbol '$<$' under '$\oplus$' is necessary because the order matters. Notice that with respect to the inverse cocycle, we have the reverse domination, that is, $G <_\ell F$.

A splitting is *dominated* if it is $\ell$-dominated for some $\ell \in 2 \wedge \mathbb{N}$. The *index* of the dominated splitting is the number $\dim F$.

REMARK 2.2.    Most references do not require that the domination parameter $\ell$ must be a power of 2. As it is trivial to see, this gives us the same concept of dominated splitting. An advantage of our powers-of-2 convention is that $\ell$-dominated splittings are $L$-dominated for $L > \ell$.

Given a cocycle and $\ell \in 2 \wedge \mathbb{N}$, an (ordered) invariant splitting $E = F_1 \underset{<}{\oplus} \ldots \underset{<}{\oplus} F_m$ into an arbitrary number of subbundles is called $\ell$-*dominated* if $F_1 \oplus \ldots \oplus F_i$ is $\ell$-dominated by $F_{i+1} \oplus \ldots \oplus F_m$, for each $i = 1, 2, \ldots, m-1$. The *indices* of the splitting are the numbers $i_j = \dim F_1 \oplus \ldots \oplus F_j$ for $1 \leqslant j \leqslant m - 1$.

PROPOSITION/DEFINITION 2.3.    *Given any cocycle and any* $\ell \in 2 \wedge \mathbb{N}$, *there is an unique finest $\ell$-dominated splitting* $F_1 \underset{<}{\oplus} \ldots \underset{<}{\oplus} F_m$, *that is, an $\ell$-dominated splitting such that if* $G_1 \underset{<}{\oplus} \ldots \underset{<}{\oplus} G_k$ *is an $\ell$-dominated splitting, then each $G_j$ is the sum of some of the $F_i$ (in particular, $k \leqslant m$). (Here we must allow the possibility of a trivial splitting, that is, $m = 1$.)*

*Proof.*    If $E = F' \underset{<}{\oplus} F''$ and $E = G' \underset{<}{\oplus} G''$ are $\ell$-dominated splittings with $\dim F' \leqslant \dim G'$ and $\dim F'' \geqslant \dim G''$, respectively, then $F' \subset G'$ and $F'' \supset G''$ (see [10, p. 291]). If these inequalities are strict, we define $H = F'' \cap G'$. Then, by dimension counting, $G' = F' \oplus H$ and $F'' = H \oplus G''$ and, in particular, $F' \underset{<}{\oplus} H \underset{<}{\oplus} G''$ is an $\ell$-dominated splitting into three bundles. Existence and uniqueness of the finest $\ell$-dominated splitting follows easily from these remarks.                                                                        $\square$

REMARK 2.4.    Beware that the bundles of the finest $\ell$-dominated splitting can admit non-trivial $\ell$-dominated splitting themselves. On the other hand, for the more standard notion of *finest dominated splitting* (with no fixed $\ell$), the bundles are indecomposable.

## 2.4. *Lyapunov exponents*

Given a cocycle as above, Oseledets theorem assures the existence of a full probability set $R \subset X$ (that is, a set that has full measure with respect to any $T$-invariant Borel probability measure), called the *set of regular points*, such that for each $x \in R$, we have well-defined *Lyapunov exponents*

$$\lambda_1(A, x) \leqslant \ldots \leqslant \lambda_d(A, x)$$

(repeated according to multiplicity). Define

$$\boldsymbol{\sigma}_i(A, x) = \sum_{j=1}^{i} \lambda_j(A, x).$$

For each $x \in X$, the vector

$$\boldsymbol{\sigma}(A, x) = (0, \boldsymbol{\sigma}_1(A, x), \ldots, \boldsymbol{\sigma}_d(A, x)) \in \mathbb{R}^{d+1}$$

is called the *Lyapunov graph* of $A$ at $x$; the reason for the name is that we think of it as the graph of a map $\{0, 1, \ldots, d\} \to \mathbb{R}$. Lyapunov graphs are always *convex*, that is, they belong to the set

$$\mathcal{S}_d = \{(\sigma_0, \ldots, \sigma_d) \in \mathbb{R}^{d+1}; \ \sigma_0 = 0, \ \sigma_i - \sigma_{i-1} \leqslant \sigma_{i+1} - \sigma_i \text{ for } 0 < i < d\}.$$

If $\tilde{\sigma}$, $\sigma \in \mathcal{S}_d$, then we write $\tilde{\sigma} \geqslant \sigma$ to indicate that

$$\tilde{\sigma}_i \geqslant \sigma_i \quad \text{for } i = 1, 2, \ldots, d.$$

We say that a continuous path of graphs $\sigma(t) \in \mathcal{S}_d$, where $t \in [t_0, t_1]$, is *non-decreasing* if $t > t'$ implies $\sigma(t) \geqslant \sigma(t')$.

### 2.5. *Cocycles over cyclic dynamical systems*

We will be specially concerned with cocycles $(X, T, E, A)$ where the dynamical system $T : X \to X$ is *cyclic*, that is, the set $X$ is finite, say with cardinality $n$, and $T$ is a cyclic permutation. In this case, we will say that $T$ (or $A$) has *period* $n$ and that the cocycle is *cyclic*.

The *eigenvalues* of the cocycle are the eigenvalues of $A^n(x)$, where $x$ is any point of $X$. The Lyapunov exponents are the logarithms of the moduli of the eigenvalues (repeated according to multiplicity) divided by $n$. The Lyapunov graph does not depend on the point $x$ and is written as $\boldsymbol{\sigma}(A)$.

### 2.6. *Difference operator notation and convexity*

(This material will be used in Sections 4 and 5.)

Given a finite sequence of real numbers $y = (y_0, \ldots, y_k)$, we define another sequence $\Delta y = (\Delta y_0, \ldots, \Delta y_{k-1})$ by

$$\Delta y_i = y_{i+1} - y_i. \tag{2.1}$$

A more precise notation would be $(\Delta y)_i$, but we will follow custom and drop the parentheses.

Recursively, we define another sequence $\Delta^2 y = \Delta(\Delta y)$, that is, $\Delta^2 y_i = y_{i+2} - 2y_{i+1} + y_i$, for $i = 0, \ldots, k - 2$. If the numbers $\Delta^2 y_i$ are always non-negative, then the graph of $y$ is convex. The next lemma says that if, in addition, these numbers are always small, then the graph of $y$ is close to affine.

LEMMA 2.5.  *If a sequence $y_0, \ldots, y_k$ satisfies $0 \leqslant \Delta^2 y_i \leqslant \gamma$ for $0 \leqslant i \leqslant k - 2$, then*

$$0 \leqslant \frac{k-i}{k} y_0 + \frac{i}{k} y_k - y_i \leqslant \frac{k^2}{4}\gamma \quad \text{for } 0 \leqslant i \leqslant k.$$

*Proof.*  Since $\Delta^2 y_i \geqslant 0$, the graph of the sequence $y$ is convex and so the first asserted inequality holds. Also by convexity,

$$y_i \geqslant \max(y_0 + i\Delta y_0, y_k - (k - i)\Delta y_{k-1}) \tag{2.2}$$

for any $i \in (0, k)$. Now,

$$y_k = y_0 + \sum_{j=0}^{k-1} \Delta y_j = y_0 + \sum_{j=0}^{k-1} \left( \Delta y_0 + \sum_{i=0}^{j-1} \Delta^2 y_i \right)$$

$$= y_0 + k\Delta y_0 + \sum_{i=0}^{k-2} (k-1-i)\Delta^2 y_i \leqslant y_0 + k\Delta y_0 + \frac{k^2}{2}\gamma.$$

That is, $\Delta y_0 \geqslant (y_k - y_0)/k - k\gamma/2$. Symmetrically, $\Delta y_k \leqslant (y_k - y_0)/k + k\gamma/2$. Using these estimates in (2.2) we obtain that for any $i \in (0, k)$,

$$y_i \geqslant \frac{k-i}{k} y_0 + \frac{i}{k} y_k - \frac{k\gamma}{2} \min(i, k-i),$$

which immediately implies the lemma.                                          $\square$

### 2.7. *Singular values and exterior powers*

(This material will be used in Section 5 only.)

Let $E$ and $E'$ be Euclidean spaces (that is, real vector spaces endowed with inner products) of the same dimension $d$. Let $M : E \to E'$ be a linear map. We denote by jac $M$ the modulus of the determinant of the matrix of $M$ with respect to an arbitrary pair of orthonormal bases. Let us indicate by $\mathbf{s}_1(M) \geqslant \ldots \geqslant \mathbf{s}_d(M)$ the singular values of $M$ (that is, the eigenvalues of $\sqrt{M^*M}$, or equivalently the semi-axes of the ellipsoid $M(\mathbb{S}^{d-1})$) repeated according to multiplicity. (Notice that the singular values are non-increasingly ordered, opposite to our convention for the Lyapunov exponents.) Thus, $\|M\| = \mathbf{s}_1(M)$, $\mathfrak{m}(M) = \mathbf{s}_d(M)$ and jac $M = \prod_{i=1}^{d} \mathbf{s}_i(M)$.

If $E' = E$, then we indicate by $\mathbf{r}(M)$ the spectral radius of $M$.

We will need a few facts about exterior powers (see, for example, [3] for details). If $E$ is a vector space of dimension $d$, then $\wedge^i E$ indicate its $i$th exterior power; this is a vector space of dimension $\binom{d}{i}$ whose elements are called $i$-vectors. Moreover, an inner product on $E$ induces an inner product on $\wedge^i E$ with the following properties:

(1) The norm of a decomposable $i$-vector $v_1 \wedge \ldots \wedge v_i$ equals the $i$-volume of the parallelepiped with edges $v_1, \ldots, v_i$.

(2) If $\{e_1, \ldots, e_d\}$ is an orthonormal basis for $E$, then $\{e_{j_1} \wedge \ldots \wedge e_{j_i}; \ j_1 < \ldots < j_i\}$ is an orthonormal basis for $\wedge^i E$.

Any linear map $M : E \to E'$ induces a linear map $\wedge^i M : \wedge^i E \to \wedge^i E'$ such that the image of a decomposable $i$-vector $v_1 \wedge \ldots \wedge v_i$ is $Mv_1 \wedge \ldots \wedge Mv_i$. Moreover, the singular values of $\wedge^i M$ are obtained by taking all possible products of $i$ singular values of $M$, in particular,

$$\|\wedge^i M\| = \mathbf{s}_1(M)\mathbf{s}_2(M)\ldots\mathbf{s}_i(M),$$
$$\mathbf{s}_2(\wedge^i M) = \mathbf{s}_1(M)\mathbf{s}_2(M)\ldots\mathbf{s}_{i-1}(M)\mathbf{s}_{i+1}(M),$$
$$\mathfrak{m}(\wedge^i M) = \mathbf{s}_{d-i+1}(M)\mathbf{s}_{d-i+2}(M)\ldots\mathbf{s}_d(M).$$

Analogously, if $E' = E$, then the eigenvalues of $\wedge^i M$ are obtained by taking all possible products of $i$ eigenvalues of $M$.

### 2.8. *Semicontinuity of the Lyapunov spectrum*

(This material will be used in Section 5 only.)

Let $(X, T, E, A)$ be a cocycle of dimension $d$. Suppose $\mu$ is a (not necessarily ergodic) $T$-invariant probability measure. We denote

$$\boldsymbol{\sigma}_i(A, \mu) = \int \boldsymbol{\sigma}_i(A, x) \, d\mu(x) \quad \text{and} \quad \boldsymbol{\sigma}(A, \mu) = (\boldsymbol{\sigma}_0(A, \mu), \ldots, \boldsymbol{\sigma}_d(A, \mu)).$$

Then $\boldsymbol{\sigma}(A, \mu)$ is a convex graph, that is, an element of $\mathcal{S}_d$.

It is sometimes more convenient to deal with the integrated sum of the $i$ *biggest* Lyapunov exponents:

$$L_i(A, \mu) = \int (\lambda_d(A, x) + \lambda_{d-1}(A, x) + \ldots + \lambda_{d-i+1}(A, x)) \, d\mu(x).$$

That is, $L_i(A, \mu) = \boldsymbol{\sigma}_d(A, \mu) - \boldsymbol{\sigma}_{d-i}(A, \mu)$. These numbers are also expressed by

$$L_i(A, \mu) = \lim_{m \to \infty} \frac{1}{m} \log \|\wedge^i A^m\| \, d\mu = \inf_m \frac{1}{m} \log \|\wedge^i A^m\| \, d\mu.$$

As an immediate consequence of this formula, the numbers $L_i(A, \mu)$ are upper-semicontinuous with respect to $A$ and $\mu$ (where, in the space of measures, we use the weak-star topology). Of course, if $i = d$, then the function is continuous, because it is given by $\int \log \operatorname{jac} A \, d\mu$.

Thus $\boldsymbol{\sigma}_i$ is lower-semicontinuous and $\boldsymbol{\sigma}_d$ is continuous. In other words, if $A$ is the limit of a sequence of cocycles $B_k$, and $\mu$ is the weak-star limit of a sequence of invariant probabilities $\mu_k$, then every accumulation point $\sigma$ of the sequence $\boldsymbol{\sigma}(B_k, \mu_k)$ satisfies $\sigma \geqslant \boldsymbol{\sigma}(A, \mu)$ and $\sigma_d = \boldsymbol{\sigma}_d(A, \mu)$.

## 3.  *Mixing only two exponents*

The proofs of our results on raising Lyapunov graphs rely on the central Proposition 3.1, which says how to perturb two 'neighbour' Lyapunov exponents, while keeping the others fixed. This section is devoted to prove it.

PROPOSITION 3.1.   *For any $d \geqslant 2$, $K > 1$, $\varepsilon > 0$, there exists $\ell \in 2\wedge\mathbb{N}$ such that the following holds: Let $(X, T, E, A)$ be a $d$-dimensional cyclic cocycle bounded by $K$ and of period at least $\ell$. Assume that $A$ has only real eigenvalues and has no $\ell$-dominated splitting of index $i$.*

*Then there exists an $\varepsilon$-short path of cocycles $A_t$, where $t \in [0, 1]$, starting at $A$, all of them with only real eigenvalues, such that the path of graphs $\boldsymbol{\sigma}(A_t)$ is non-decreasing, $\boldsymbol{\sigma}_j(A_t) = \boldsymbol{\sigma}_j(A)$ for all $j \neq i$, and $\boldsymbol{\sigma}_i(A_1) = (\boldsymbol{\sigma}_{i-1}(A) + \boldsymbol{\sigma}_{i+1}(A))/2$.*

The assertions about $\boldsymbol{\sigma}(A_t)$ can be reread as follows:

(1) the functions $\lambda_1(A_t)$, $\ldots$, $\lambda_{i-1}(A_t)$, $\lambda_i(A_t) + \lambda_{i+1}(A_t)$, $\lambda_{i+1}(A_t)$, $\ldots$, $\lambda_d(A_t)$ are constant;
(2) $t > t'$ implies $\lambda_i(A_{t'}) \leqslant \lambda_i(A_t) \leqslant \lambda_{i+1}(A_t) \leqslant \lambda_{i+1}(A_{t'})$;
(3) $\lambda_i(A_1) = \lambda_{i+1}(A_1)$.

REMARK 3.2.   Such a path of graphs corresponds to what is known in majorization theory as an 'elementary Robin Hood operation' or 'elementary $T$-transform' (see [**19**, p. 82]). In the interpretation where $\lambda_j$ is the wealth of the individual $j$ (see [**19**, pp. 5–8]), such operation consists of transferring part of the wealth of the individual $i + 1$ to its neighbour $i$ in such a way that $i + 1$ stays at least as rich as $i$. (In the case of Proposition 3.1, the two individuals become equally rich.) A general 'Robin Hood operation' is a transfer of wealth between two individuals that are not necessarily neighbours.

Here is a extremely brief indication of the proof. The case $d = 2$ is easy: A preliminary perturbation makes the angle between the two bundles small over some point, and then the exponents are mixed by composing with rotations at this point. The general case would be a trivial consequence of the two-dimensional case if it were true that subbundles of a bundle

without (strong) domination have no (strong) domination as well. (It is false!) To deal with the general case, we show that if a cocycle has no domination, then after a perturbation this non-dominance appears on a subbundle or on a quotient bundle. This permits us to prove Proposition 3.1 by induction on the dimension $d$.

Before going into the proof of Proposition 3.1 itself, we need some auxiliary results.

### 3.1. *Converting non-dominance into small angles*

Assume that a cocycle $A$ has an invariant splitting $F \oplus H$ such that the eigenvalues of the restricted cocycle $A \restriction F$ are all different from those of the restricted cocycle $A \restriction H$. Suppose that $A_t$ is a path of cocycles, all of them with the same eigenvalues. Then $A_t$ has an invariant splitting $F_t \oplus H_t$ that depends continuously on $t$ and coincides with $F \oplus H$ for $t = 0$. The bundles $F_t$ and $H_t$ are called the *continuations* of $F$ and $H$.

The following lemma is based on an argument by Mañé.

LEMMA 3.3.   *For any $d \geqslant 2$, $K > 1$, $\varepsilon > 0$ and $\alpha > 0$, there exists $\ell \in 2 \wedge \mathbb{N}$ such that the following holds: Let $(X, T, E, A)$ be a $d$-dimensional cyclic cocycle bounded by $K$ and of period $n \geqslant \ell$. Assume that $E = F \oplus H$ is an invariant splitting such that*

$$\frac{\|A^n \restriction F(x)\|}{\mathfrak{m}(A^n \restriction H(x))} < 1 \quad \text{for any } x \in X. \tag{3.1}$$

*(In particular, the Lyapunov exponents along $F$ are smaller than those along $H$.) Assume also that $F$ is not $\ell$-dominated by $H$, and that $F$ or $H$ is one-dimensional.*

*Then there exists an $\varepsilon$-short path of cocycles $A_t$, where $t \in [0, 1]$, starting at $A$, all of them with the same eigenvalues, such that if $F_t$ and $H_t$ denote the continuations of $F$ and $H$, then*

$$\measuredangle(F_1(x_0), H_1(x_0)) < \alpha \quad \text{for some point } x_0 \in X.$$

*Proof.*   Let $d \geqslant 2$, $K > 1$, $\varepsilon > 0$ and $\alpha > 0$ be given. Let $\ell$ be large (how large it needs to be will become clear later).

Take the cocycle $(X, T, E, A)$ and the splitting $E = F \oplus H$ as in the statement of the lemma. At least one of the bundles $F$ or $H$ is one-dimensional. Let us suppose it is $H$; the other case will follow by considering the inverse cocycle.

We assume that

$$\measuredangle(F(x), H(x)) \geqslant \alpha \quad \text{for every } x \in X, \tag{3.2}$$

otherwise there is nothing to prove.

Take a large number $\Lambda > 1$ (how large it needs to be will become clear later). We split the proof into two cases:

*Case* 1: We suppose that a strong form of non-dominance is present: there is $z \in X$ and $0 < k < n = \#X$ such that

$$\frac{\|A^k \restriction F(z)\|}{\mathfrak{m}(A^k \restriction H(z))} > \Lambda. \tag{3.3}$$

Then let $f$ be the unit vector in $F(z)$ most expanded by $A^k(z)$, and let $h$ be an unit vector in $H(z)$. For $\tau \in [0, 1]$, we define a linear map $S_\tau : E(z) \to E(z)$ by

$$S_\tau \cdot h = \tau f + h, \quad S_\tau \restriction F(z) \text{ is the identity.}$$

It follows from (3.2) that there exists $C_1 > 0$ depending only on $\alpha$ such that $\|S_\tau - \mathrm{Id}\| \leqslant C_1 \tau$. On the other hand, by (3.1) the vector $f$ is less expanded than $h$ by the map $A^n(z) : E(z) \to E(z)$. Using (3.2) again, we see that if $\tau \in [0, 1]$, then the angle that the vector $A^n(z) \cdot (\tau f + h)$ makes with $F(z)$ cannot be too small. Therefore, defining another linear map

$U_\tau : E(z) \to E(z)$ by

$$U_\tau \cdot [A^n(z) \cdot (\tau f + h)] = A^n(z) \cdot h, \quad U_\tau \upharpoonright F(z) \text{ is the identity,}$$

we have $\|U_\tau - \mathrm{Id}\| \leqslant C_2 \tau$, for some $C_2 > 0$ that depends only on $\alpha$.

Take $0 < \beta < 1$ with $C_1 \beta K, C_2 \beta K < \varepsilon/2$. Define a family of cocycles $A_t$, where $t \in [0, 1]$, by

$$A_t(z) = A(z) \circ S_{\beta t}, \quad A_t(T^{-1}z) = U_{\beta t} \circ A(T^{-1}z), \quad A_t(x) = A(x) \text{ if } x \neq z, T^{-1}z.$$

Then $A_0 = A$ and $\|A_t - A\| \leqslant \varepsilon/2$ for every $t \in [0, 1]$. The cocycle $A_t$ has an invariant splitting $F_t \oplus H_t$, where $F_t = F$ and $H_t(T^j z)$ is spanned by $A_t^j(z) \cdot h$. Also, the linear maps $A_t^n(z)$ and $A^n(z)$ have the same eigenvalues.

If we choose $\Lambda$ large enough (depending on $\alpha$ and $\beta$), then it follows from (3.3) that the angle between $A_1^k(z) \cdot h$ and $A^k(z) \cdot f$ is less than $\alpha$. Thus, $\angle(F_1(T^k z), H_1(T^k z)) < \alpha$, as desired.

*Case* 2: Assume that we are not in the previous case, that is,

$$\frac{\|A^k \upharpoonright F(x)\|}{\mathsf{m}(A^k \upharpoonright H(x))} \leqslant \Lambda \quad \text{for every } x \in X \text{ and } 0 < k < n.$$

By assumption, the splitting $F \oplus H$ is not $\ell$-dominated, so there is $z \in X$ such that

$$\frac{\|A^\ell \upharpoonright F(z)\|}{\mathsf{m}(A^\ell \upharpoonright H(z))} \geqslant \frac{1}{2}.$$

In particular,

$$\frac{\|A^{\ell/2} \upharpoonright F(z)\|}{\mathsf{m}(A^{\ell/2} \upharpoonright H(z))} \geqslant \frac{\mathsf{m}(A^{\ell/2} \upharpoonright H(T^{\ell/2}z))}{\|A^{\ell/2} \upharpoonright F(T^{\ell/2}z)\|} \cdot \frac{\|A^\ell \upharpoonright F(z)\|}{\mathsf{m}(A^\ell \upharpoonright H(z))} \geqslant \frac{1}{2\Lambda}.$$

Let $\gamma > 0$ and define a family of cocycles $A_t$, where $t \in [0, 1]$, by:

  (1) $A_t(x)$ equals $e^{\gamma t} A(x)$ over $F(x)$ and $e^{-\gamma t} A(x)$ over $H(x)$ if $x = T^i z$ with $0 \leqslant i \leqslant \ell/2$;
  (2) $A_t(x)$ equals $e^{-\gamma t} A(x)$ over $F(x)$ and $e^{\gamma t} A(x)$ over $H(x)$ if $x = T^i z$ with $\ell/2 \leqslant i \leqslant \ell$;
  (3) $A_t(x) = A(x)$ otherwise.

We choose $\gamma$ depending only on $\alpha$, $K$ and $\varepsilon$ as large as possible so that $\|A_t - A\| \leqslant \varepsilon/2$. The cocycles $A_t$ have the same invariant subbundles and Lyapunov spectrum as $A$. Also,

$$\frac{\|A_1^{\ell/2} \upharpoonright F(z)\|}{\mathsf{m}(A_1^{\ell/2} \upharpoonright H(z))} = e^{\gamma \ell} \frac{\|A^{\ell/2} \upharpoonright F(z)\|}{\mathsf{m}(A^{\ell/2} \upharpoonright H(z))} \geqslant \frac{e^{\gamma \ell}}{2\Lambda}.$$

We assume that $\ell$ is large enough so that the right-hand side is bigger than $\Lambda$. Thus, the cocycle $A_1$ (which still satisfies (3.1)) falls in case 1 of the proof, with $k = \ell/2$. The desired path of cocycles from $A$ to a cocycle with a small angle is thus obtained by concatenating the path just described with a path given by case 1. $\qquad\square$

### 3.2. *The effect of rotations in dimension 2*

Recall that we indicate the spectral radius of a square matrix $B$ by $\mathbf{r}(B)$. Let $R_\theta$ indicate the rotation of angle $\theta$ in $\mathbb{R}^2$.

LEMMA 3.4. *Suppose $B \in \mathrm{GL}(2, \mathbb{R})$ has eigenvalues of different moduli. Let $\alpha \in (0, \pi/2]$ be the angle between the eigenspaces. Then there exists $\beta \in (0, \alpha)$ and $s \in \{+1, -1\}$ such that:*

  (1) *the function $\theta \in [0, \beta] \mapsto \mathbf{r}(R_{s\theta} B)$ is decreasing;*
  (2) *the matrix $R_{s\beta} B$ has two eigenvalues of the same moduli.*

*Proof.* Begin noticing the following facts: If $B \in \mathrm{GL}(2, \mathbb{R})$ has determinants $\pm 1$, then $B$ has eigenvalues of different moduli if and only if either $\det B = 1$ and $|\operatorname{tr} B| > 2$, or $\det B = -1$ and
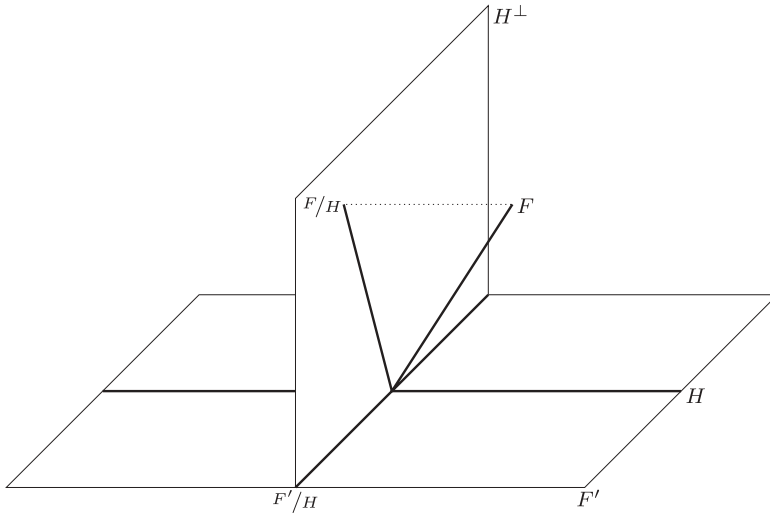
FIGURE 3. *Lemma 3.5.*

$\operatorname{tr} B \neq 0$. Also, the spectral radius $\mathbf{r}(B)$ increases with $\operatorname{tr} B$ on the set $\{\det B = 1 \text{ and } \operatorname{tr} B > 2\} \cup \{\det B = -1 \text{ and } \operatorname{tr} B > 0\}$.

Now let $B$ be any matrix with eigenvalues of different moduli. By multiplying it by a non-zero number, we can assume that it has determinants $\sigma = \pm 1$ and has $\lambda = \mathbf{r}(B) > 1$ as an eigenvalue. By conjugating with a rotation, we can further assume that

$$B = \begin{pmatrix} \lambda & -c \\ 0 & \sigma\lambda^{-1} \end{pmatrix}.$$

Let $s \in \{+1, -1\}$ be the sign of $c$ (if $c = 0$, then either choice works). Then $\operatorname{tr} R_{s\theta} B = (\lambda + \sigma\lambda^{-1})\cos\theta - |c|\sin\theta$ is a decreasing function of $\theta$ on the interval $[0, \pi/2]$. Let $\alpha \in (0, \pi/2]$ be the angle between the eigenvalues. We have $\tan\alpha = |c|^{-1}(\lambda - \sigma\lambda^{-1})$. So $\operatorname{tr} R_{s\alpha} B = 2\sigma\lambda^{-1}\cos\alpha$, which is less than 2 if $\sigma = +1$, and negative if $\sigma = -1$. In either case, it follows that there exists a least $\beta \in (0, \alpha)$ such that $R_{s\beta} A$ has eigenvalues of same moduli. $\qquad\square$

### 3.3. *Some dominance and angle relations between subbundles and quotient bundles*

Lemma 2.6 of Bochi and Viana [**5**] is an useful angle relation; it says that for any splitting of Euclidean space into three non-zero subspaces $U$, $V$ and $W$,

$$\sin\angle(W, U \oplus V) \geqslant \sin\angle(W, U) \cdot \sin\angle(U \oplus W, V). \tag{3.4}$$

The following is a generalization of Bonatti, Díaz and Pujals [**9**, Lemma 4.4] (see Figure 3).

LEMMA 3.5. *For any $d, K$ and $\ell \in 2\wedge\mathbb{N}$, there exists $L = L(d, K, \ell) \in 2\wedge\mathbb{N}$ with the following properties: Let $(X, T, E, A)$ be a cyclic $d$-dimensional cocycle bounded by $K$, with an invariant splitting $F \oplus F'$ into non-zero bundles. Assume that $F'$ has an invariant subbundle $H$ such that $F <_\ell H$ and $F/H <_\ell F'/H$. Then $F <_L F'$.*

*Proof.* It is simple to adapt the proof of Bonatti, Díaz and Pujals [**9**, Lemma 4.4] to our more general lemma, but let us spell out the details for the reader's convenience.

To simplify writing, in what follows $C$ indicates some positive number depending only on $d$, $K$ and $\ell$, whose value may change from one line to another.

Take a cocycle $A$ preserving the subbundles $F$, $F' \supset H$ as in the statement of the lemma. It is slightly more convenient to work with the inverse cocycle $B = A^{-1}$. So the domination relations are reversed, that is, $F >_\ell H$ and $F/H >_\ell F'/H$ with respect to $B$.

We claim that at each point

$$\sin \measuredangle(F, F') \geqslant \sin \measuredangle(F, H) \cdot \sin \measuredangle(F/H, F'/H);$$

and, in particular, since $\measuredangle(F, H)$ and $\measuredangle(F/H, F'/H)$ are not very small, $\measuredangle(F, F')$ is not very small either. The proof consists on applying relation (3.4) with $U = H$, $V = F' \cap H^\perp$ (which we naturally identify with $F'/H$), $W = F$:

$$\sin \measuredangle(F, \underbrace{F'/H \oplus H}_{F'}) \geqslant \sin \measuredangle(F, H) \cdot \sin \measuredangle(F \oplus H, F'/H),$$

and noting that $\measuredangle(F \oplus H, F'/H) = \measuredangle(F/H \oplus H, F'/H) = \measuredangle(F/H, F'/H)$.

At each point $x$, take a basis $\{e_1(x), \ldots, e_d(x)\}$ of $E(x)$ with the following properties. The first $\dim F$ vectors form an orthonormal basis of $F(x)$, and the following $\dim F'$ vectors form an orthonormal basis of $F'(x)$, so that the last $\dim H$ vectors belong to $H(x)$. Since $\measuredangle(F, F')$ is not very small, the changes of bases between $\{e_j(x)\}$ and any orthonormal basis of $E(x)$ are bounded by some constant $C$.

With respect to the bases we chose, we can write

$$B(x) = \begin{pmatrix} M(x) & 0 \\ 0 & N(x) \end{pmatrix}, \quad \text{where } N(x) = \begin{pmatrix} P(x) & 0 \\ Q(x) & R(x) \end{pmatrix}.$$

As usual, we denote $M^k(x) = M(T^{k-1}x) \ldots M(x)$, etc. By the dominance assumptions, there exists $0 < \lambda < 1$ depending only on $K$ and $\ell$ such that

$$\max(\|P^k(x)\|, \|R^k(x)\|) \leqslant C\lambda^k \mathfrak{m}(M^k(x)) \quad \text{for all } k > 0 \text{ and } x \in X. \tag{3.5}$$

We have

$$N^k(x) = \begin{pmatrix} P^k(x) & 0 \\ Q_k(x) & R^k(x) \end{pmatrix}, \quad \text{where } Q_k(x) = \sum_{j=0}^{k-1} R^{k-j-1}(T^{j+1}x)Q(T^j x)P^j(x).$$

We estimate

$$\|Q_k(x)\| \leqslant C\lambda^{k-1} \sum_{j=0}^{k-1} \mathfrak{m}(M^{k-j-1}(T^{j+1}x))\mathfrak{m}(M^j(x)) \quad \text{(using (3.5))}$$

$$\leqslant Ck\lambda^k \mathfrak{m}(M^k(x)) \quad \text{(since } \mathfrak{m}(XY) \geqslant \mathfrak{m}(X)\mathfrak{m}(Y))$$

$$\leqslant C\lambda^{k/2} \mathfrak{m}(M^k(x)).$$

The last inequality together with (3.5) implies that

$$\|N^k(x)\| \leqslant C\lambda^{k/2} \mathfrak{m}(M^k(x)) \quad \text{for all } k > 0 \text{ and } x \in X.$$

Coming back to the original norms, we have $\|(B \restriction F')^k(x)\| \leqslant C\lambda^{k/2} \mathfrak{m}((B \restriction F)^k(x))$. This means that $F >_L F'$ with respect to $B$, or $F <_L F'$ with respect to $A$, for some $L$ that depends only on $d$, $K$ and $\ell$. $\qquad\square$

Another angle relation we will need is as follows:

LEMMA 3.6.   There is $c_0 > 0$ such that for any splitting of Euclidean space into three non-zero subspaces $F, G$ and $H$, we have

$$\measuredangle(H, F \oplus G) > c_0 \cdot \measuredangle(H, F) \cdot \measuredangle(H, G) \cdot \measuredangle(F/H, G/H).$$

*Proof.* Apply (3.4) with $U = F$, $V = G$ and $W = H$:

$$\sin \angle(H, F \oplus G) \geqslant \sin \angle(H, F) \cdot \sin \angle(F \oplus H, G).$$

Now apply (3.4) with $U = H$, $V = (F \oplus H) \cap H^{\perp}$ (which we naturally identify with $F/H$) and $W = G$:

$$\sin \angle(G, \underbrace{F/H \oplus H}_{F \oplus H}) \geqslant \sin \angle(G, H) \cdot \sin \underbrace{\angle(G \oplus H, F/H)}_{\angle(G/H, F/H)}.$$

Therefore

$$\sin \angle(H, F \oplus G) \geqslant \sin \angle(H, F) \cdot \sin \angle(H, G) \cdot \sin \angle(F/H, G/H),$$

which implies the lemma.                                                                                      $\square$

### 3.4. *Proof of Proposition* 3.1

*Proof.* As already mentioned, the proof is by induction in the dimension $d$.

*Starting the induction:* Given $K$ and $\varepsilon$, take a positive $\alpha \ll \varepsilon/K$. Apply Lemma 3.3 with $d = 2$ and $\varepsilon/2$ in the place of $\varepsilon$ to obtain $\ell$. Now take a cyclic dynamical system of period at least $\ell$, and a two-dimensional cocycle $A$ over it that is bounded by $K$, has real eigenvalues, but no $\ell$-dominated splitting. We can assume that the eigenvalues are different, otherwise there is nothing to prove. So consider the two invariant one-dimensional bundles. Using Lemma 3.3, we find a path of cocycles starting at $A$, always $\varepsilon/2$-close to $A$, with the same spectrum as $A$, and such that for the final cocycle $A'$, the two bundles form an angle less than $\alpha$ over at least one point $x_0 \in X$. (Notice that the hypothesis (3.1) in the lemma is automatically satisfied here, because $d = 2$.) Now we consider another path of cocycles, starting at $A'$, ending at $A''$ and always $\varepsilon/2$-close to $A'$, such that the bundle maps are not altered except over the point $x_0$, where the perturbations consist on composing with small rotations. By Lemma 3.4, this can be done so that the upper Lyapunov exponent strictly decreases along the path, and at the endpoint $A''$ the two exponents are equal. The concatenation of the two paths described above is a path with all the required properties. This shows that the proposition is true when $d = 2$.

*The induction step:* Take $d \geqslant 3$, and assume the proposition is true for all dimensions $d'$ between $2$ and $d - 1$. Let $K > 1$ and $\varepsilon > 0$ be given. Reducing $\varepsilon$ if necessary (recall Remark 2.1), we assume that any $\varepsilon$-perturbation of a cocycle bounded by $K$ is bounded by $2K$ and that we are allowed to $\varepsilon$-perturb not only a given cocycle $A$ but also the inverse cocycle $A^{-1}$. Let $\ell_0 \in 2 \wedge \mathbb{N}$ be the maximum of $\ell(2K, \varepsilon/2, d')$, where $2 \leqslant d' \leqslant d - 1$.

Let $\ell \gg \ell_0$ (how large is necessary will become clear along the proof).

Take a cyclic $d$-dimensional cocycle $(X, T, E, A)$ bounded by $K$ and of period at least $\ell$. Also suppose that $A$ has only real eigenvalues and has no $\ell$-dominated splitting of index $i$. We will complete the proof assuming $i \leqslant d - 2$. Then the remaining case $i = d - 1$ will follow from the case $i = 1$ applied to the inverse cocycle. Write for simplicity $\lambda_j = \lambda_j(A)$ for the Lyapunov exponents.

We assume that $\lambda_i < \lambda_{i+1}$, because otherwise we simply take a constant path of cocycles. Let $F \oplus F'$ be the invariant splitting so that the exponents along $F$ are $\lambda_1, \ldots, \lambda_i$, and the exponents along $F'$ are $\lambda_{i+1}, \ldots, \lambda_d$. By assumption,

$$F \not\prec_\ell F'.$$

We consider separately two cases according to the multiplicity of the upper exponent $\lambda_d$.

*The case $\lambda_{d-1} = \lambda_d$:* By assumption, the cocycle has only real eigenvalues. Thus, we can choose an one-dimensional invariant subbundle $H$ with an exponent $\lambda_d$.

In the case that $F/H \not\prec_{\ell_0} F'/H$, we apply the induction assumption to the cocycle $A/H$ (which satisfy the real eigenvalues hypothesis), obtaining a path of cocycles mixing its $i$th and $(i+1)$th exponents, without changing the others. Then, using the second procedure explained

in § 2.2, we extend these cocycles to the whole bundle $E$. The direction $H$ is kept invariant and so the upper exponent $\lambda_d$ does not change.

Thus assume that $F/H <_{\ell_0} F'/H$. Recall that $F \not<_\ell F'$ and $\ell \gg \ell_0$; thus by Lemma 3.5, we have $F \not<_{\ell_0} H$. Let $G$ be the invariant subbundle associated to the exponents bigger than $\lambda_i$ and smaller than $\lambda_d$ (so $G$ is zero if $\lambda_{i+1} = \lambda_d$). Then $F \not<_{\ell_0} G \oplus H$. We apply the induction assumption to the cocycle $A$ restricted to the smaller bundle $F \oplus G \oplus H$, obtaining a path of cocycles mixing its $i$th and $(i+1)$th exponents, without changing the others. Then, using the first procedure from § 2.2, we extend these cocycles to the whole bundle $E$.

*The case $\lambda_{d-1} < \lambda_d$:* We split invariantly $F' = G \oplus H$, where the Lyapunov exponents along $G$ are $\lambda_{i+1}, \ldots, \lambda_{d-1}$, and $H$ is one-dimensional with Lyapunov exponent $\lambda_d$. Then we are in a position to apply the following lemma.

LEMMA 3.7.    *Given $d, K, \ell_0$ and $\varepsilon$, there exists $\ell$ with the following properties: Let $(X, T, E, A)$ be a cyclic $d$-dimensional cocycle bounded by $K$ and of period at least $\ell$. Assume that $A$ has an invariant splitting $E = F \oplus G \oplus H$ such that $\dim H = 1$, the exponents on $F$ are smaller than the exponents on $G$, which in turn are smaller than the exponent on $H$. Also assume that*

$$F \not<_\ell G \oplus H.$$

*Then there exists an $\varepsilon$-short path of cocycles $A_t$, where $t \in [0, 1]$, starting at $A$, all of them with the same eigenvalues, such that, denoting by $F_t, G_t$ and $H_t$ the continuations of the bundles $F, G$ and $H$, we have*

$$F_1 \not<_{\ell_0} G_1 \quad or \quad F_1/H_1 \not<_{\ell_0} G_1/H_1. \tag{3.6}$$

We postpone the proof of the lemma to the next section, and see how it permits us to conclude.

(1) In the case $F_1 \not<_{\ell_0} G_1$, we apply the induction assumption to the cocycle $A_1$ restricted to the bundle $F_1 \oplus G_1$ (which has only real eigenvalues), obtaining a path of cocycles mixing the $i$th and $(i+1)$th exponents, without changing the others. Then, using the first procedure explained in § 2.2, we extend these cocycles to the whole bundle $E$. This does not alter the upper exponent $\lambda_d$ (although the $H_1$ direction is not preserved). The desired path of cocycles is obtained by concatenation.

(2) The case $F_1/H_1 \not<_{\ell_0} G_1/H_1$ is similar: we apply the induction assumption to the quotient cocycle $A_1/H_1$ and then we use the second extension procedure from § 2.2.

So we have proved Proposition 3.1 modulo Lemma 3.7.                                      □

### 3.5.  *Proof of Lemma 3.7*

Let us begin with an informal outline of the proof. We assume that $F < G$ and $F/H < G/H$, otherwise there is nothing to show. By Lemma 3.6, this implies that each time that $\measuredangle(H, F \oplus G)$ is small, then either $\measuredangle(H, F)$ or $\measuredangle(H, G)$ is small, but not both. Thus, the objective is to perturb the cocycle without changing the eigenvalues so that at some point of the orbit, the new space $H$ (that is, the continuation of $H$) is close to the new $F \oplus G$ but far from the new $F \cup G$; this breaks a dominance relation and we are done. Coming back to the unperturbed cocycle, by Lemma 3.5, we have $F \not< H$ (otherwise we would get $F < G \oplus H$). Using Lemma 3.3, we can perturb the cocycle (but not the eigenvalues) so to make $\measuredangle(H, F)$ extremely small at some point $x_0$. We can still assume that $F < G$ and $F/H < G/H$ because otherwise we are done. Iterating negatively from $x_0$, these angles must remain small for a long time, but not forever. So we take a point $x_1$ where $\measuredangle(H, F)$ is small, but not extremely small, and remains small for
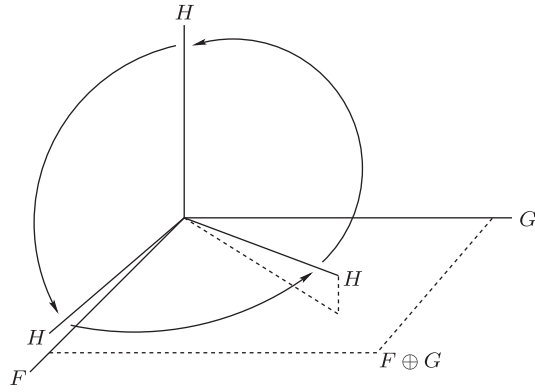
FIGURE 4. *The proof of Lemma 3.7: For the perturbed cocycle, the spaces $F$ and $G$ are always apart. The figure shows the position of the one-dimensional space $H$ at three different times. At one of these, we see a small $\measuredangle(F/H, G/H)$.*

many positive iterates, until we reach $x_0$. Choose eigenvectors $h \in H(x_1)$ and $g \in G(x_1)$ such that $\|g\|$ is much (but not extremely) smaller than $\|h\|$. We follow the iterates of the vector $\tilde{h} = h + g$: while the iterates of $h$ remain close to $F$ (and this happens for a long period of time), the iterates of $\tilde{f}$ remain close to $F \oplus G$. On the other hand, since $F < G$, an iterate of $\tilde{h}$ gets far from $F$ while it is still close to $F \oplus G$ and not yet close to $G$ – see Figure 4. Sometime later (and before we come back to $x_1$), the iterates of $\tilde{h}$ get again close to $H$: that happens because the expansion rate of the eigenvector $g$ until the period is less than the expansion rate of $h$. At that moment, it is easy to perturb the cocycle without changing $F$, $G$ nor the eigenvalues so that the new $H$ follows the route of $\tilde{h}$.

*Proof of Lemma* 3.7.   For clarity, we divide the proof into parts.
*Part 0: Fixing several constants.* Let $d$, $K$, $\ell_0$ and $\varepsilon$ be given. Reducing $\varepsilon$ if necessary (recall Remark 2.1), we assume that any $\varepsilon$-perturbation of a cocycle bounded by $K$ is bounded by $2K$.

Fix numbers $\theta_1 > \theta_0 > 0$ depending only on $K$ and $\ell_0$ such that the following holds: For any cocycle $A$ bounded by $2K$ that has invariant subbundles $F$ and $G$ with $F <_{\ell_0} G$, we have $\measuredangle(F, G) > \theta_0$ at each point. Moreover,

$$x \in X, \quad u \in F^{\theta_1}(x), \quad g \in G(x) \implies \frac{\|A^{\ell_0}(x) \cdot u\|/\|u\|}{\|A^{\ell_0}(x) \cdot g\|/\|g\|} < 0.6, \qquad (3.7)$$

where $F^{\theta_1}(x)$ indicates the cone of size $\theta_1$ around $F(x)$, that is, the set of $v \in E(x)$ such that $\measuredangle(v, F(x)) < \theta_1$. We will keep using this cone notation in what follows.

In the rest of the proof, $c_1$, $c_2$, ..., $c_8$ will indicate certain positive numbers that we have not bothered to calculate explicitly, but depend only on $K$ and $\ell_0$.

We choose a positive $\beta$ satisfying

$$\beta < \min(\theta_1, \theta_0/2, \theta_0/c_7),$$

and a positive $\delta$ such that

$$c_3\delta/\beta < \min(1/2, \varepsilon/(8K)). \qquad (3.8)$$

Fix an integer $m_0 \geqslant 1$, we have $\delta^{-1}0.6^{m_0} < 1$. Then choose $\gamma > 0$ with the following property: If two vectors form an angle less than $\gamma$, then their images by any linear map with norm less than $(2K)^{\ell_0 m_0}$ form an angle less than $\beta$.

Let $\ell_1$ be the number given by Lemma 3.3, applied with $\gamma$ in the place of $\alpha$, $\varepsilon/2$ in the place of $\varepsilon$ and all smaller $d$. Take $\ell = L(d, 2K, \ell_1)$, where the function $L$ is given by Lemma 3.5.

Increasing $\ell$, if necessary, we assume that

$$\ell > \ell_0 m_0 \quad \text{and} \quad K^{\ell_0} 0.6^{\ell/\ell_0} < 1. \tag{3.9}$$

*Part 1: Preliminary perturbation.* Let $(X, T, E, A)$ be a cocycle as in the statement of the lemma: $X$ has cardinality $n \geqslant \ell$; $E$ has dimension $d$; the cocycle is bounded by $K$; there is an invariant splitting $F \oplus G \oplus H$ with exponents along $F$ less than those along $G$, which are less the exponent along the one-dimensional bundle $H$; and $F \not<_\ell G \oplus H$.

We can assume that

$$F <_{\ell_0} G \quad \text{and} \quad F/H <_{\ell_0} G/H,$$

because otherwise there is nothing to show (just take a constant path). So applying Lemma 3.5 (with $F' = G \oplus H$), we conclude that

$$F \not<_{\ell_1} H.$$

Notice that for any $x \in X$, choosing an eigenvector $g \in G(x)$ of $A^n(x)$ (where $n = \#X$), we can write

$$\mathfrak{m}(A^n(x) \restriction H) > \|A^n(x) \cdot g\|/\|g\| \quad \text{(since the exponent along } H \text{ is the biggest)}$$
$$> K^{-\ell_0} 2^{\lfloor n/\ell_0 \rfloor} \|A^n(x) \restriction F\| \quad \text{(because } F <_{\ell_0} G)$$
$$> \|A^n(x) \restriction F\| \quad \text{(by (3.9) and } n \geqslant \ell).$$

Hence the cocycle $A \restriction F \oplus H$ satisfies requirement (3.1) from Lemma 3.3. Applying the lemma together with the first extension procedure from § 2.2, we obtain a $(\varepsilon/2)$-short path of cocycles, all with the same eigenvalues, so that at the end the angle between the continuation of $F$ and the continuation of $H$ is smaller than $\gamma$ at some point $x_0$.

Let $B$ indicate the endpoint of this path of cocycles. To simplify writing, the invariant bundles will still be indicated by $F$, $G$ and $H$. We can assume $F <_{\ell_0} G$ and $F/H <_{\ell_0} G/H$ with respect to $B$, because otherwise there is nothing to prove.

*Part 2: Second perturbation.* Recall that $H(x_0) \subset F^\gamma(x_0)$. Let $k_0$ be the least positive integer such that $H(T^{-\ell_0 k_0} x_0) \not\subset F^\beta(T^{-\ell_0 k_0} x_0)$. Notice that $k_0$ indeed exists and satisfies $k_0 \leqslant \lfloor n/\ell_0 \rfloor$: in the opposite case, using (3.7) and (3.9) we obtain

$$\mathfrak{m}(B^n(x_0) \restriction H) < K^{\ell_0} 0.6^{\lfloor n/\ell_0 \rfloor} \|B^n(x_0) \restriction G\| < \|B^n(x_0) \restriction G\|,$$

which contradicts the assumption that the Lyapunov exponent along $H$ is bigger than those along $G$. Moreover, $k_0 > m_0$ due to the choice of $\gamma$.

Let $x_1 = T^{-\ell_0 (k_0 - 1)} x_0$. Then $\measuredangle(H(x_1), F(x_1)) < \beta < \theta_0/2$, and since $\measuredangle(F, G) \geqslant \theta_0$, we have $\measuredangle(H(x_1), G(x_1)) > \theta_0/2$. On the other hand, by the minimality of $k_0$, we have $\measuredangle(H(T^{-\ell_0} x_1), F(T^{-\ell_0} x_1))$ is at least $\beta$, and since the cocycle $B$ is bounded by $2K$, we obtain

$$\measuredangle(H(x_1), F(x_1)) > c_1 \beta.$$

Using also that $\measuredangle(F/H, G/H) > \theta_0$, Lemma 3.6 gives

$$\measuredangle(H(x_1), F(x_1) \oplus G(x_1)) > c_0 \cdot (c_1 \beta) \cdot (\theta_0/2) \cdot \theta_0 = c_2 \beta. \tag{3.10}$$
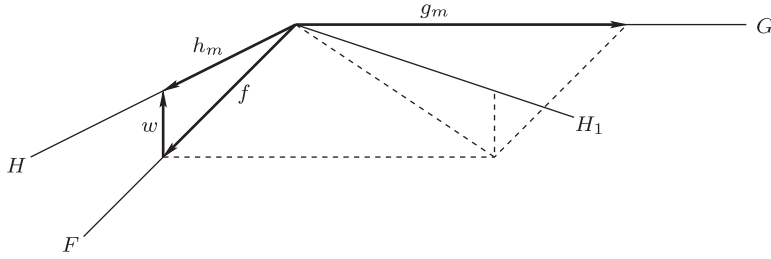
Let $h$ be a unit vector in the direction of $H(x_1)$, and let $g \in G(x_1)$ be an eigenvector for $B^n(x_1)$ with $\|g\| = \delta$.

Define linear maps $S_t : E(x_1) \to E(x_1)$, where $t \in [0, 1]$, by

$$S_t \restriction F(x_1) \oplus G(x_1) = \mathrm{Id}, \quad S_t : h \mapsto h + tg.$$

It follows easily from (3.10) that

$$\|S_t - \mathrm{Id}\| < \frac{1}{\sin(c_1 \beta)} \frac{\|g\|}{\|h\|} < c_3 \delta/\beta \quad \text{for all } t \in [0, 1].$$

FIGURE 5. *The spaces at the point $y$.*

Define other linear maps $U_t : E(x_1) \to E(x_1)$, where $t \in [0, 1]$, by

$$U_t \upharpoonright F(x_1) \oplus G(x_1) = \mathrm{Id}, \quad U_t := B^n(x_1) \cdot h \mapsto B^n(x_1) \cdot (h + tg).$$

Since $g$ and $h$ are eigenvectors of $B^n(x_1)$, and the eigenvalue associated to $h$ has bigger modulus, we have $U_t(h) = h + \rho t g$, where $|\rho| < 1$. Thus, we can estimate exactly as before:

$$\|U_t - \mathrm{Id}\| < c_3 \delta/\beta < \tfrac{1}{2} \quad \text{for all } t \in [0, 1].$$

In particular (using the formula $(\mathrm{Id} - X)^{-1} = \mathrm{Id} + X + X^2 + \ldots$), we have

$$\|U_t^{-1} - \mathrm{Id}\| \leqslant 2\|U_t - \mathrm{Id}\| < 2c_3 \delta/\beta.$$

Define a path of cocycles $B_t$, where $t \in [0, 1]$, starting at $B_0 = B$, as follows:

$$B_t(x_1) = B(x_1) \circ S_t, \quad B_t(T^{-1}x_1) = U_t^{-1} \circ B(T^{-1}x_1),$$

and $B_t(x) = B(x)$ for $x \notin \{x_1, T^{-1}x_1\}$. By the estimates above and (3.8), the path is $(\varepsilon/2)$-short, that is, $\|B_t - B\| < \varepsilon/2$. we see that all cocycles $B_t$ have the same spectrum as $B$. The continuation of the $B$-invariant splitting $F \oplus G \oplus H$ is $F \oplus G \oplus H_t$, where $H_t(x_1)$ is spanned by $h$, and $H_t(T^j x_1)$ is spanned by $B^j(x_1) \cdot (h + tg)$ for $0 < j < n$.

*Part 3: Finding a point that breaks dominance.* We will show that there is a point over which $\angle(F/H_1, G/H_1)$ is small.

Define $h_j = B^{\ell_0 j}(x_1) \cdot h$ and $g_j = B^{\ell_0 j}(x_1) \cdot g$. For $0 \leqslant j \leqslant m_0$, we have $h_j \in F^\beta(T^{\ell_0 j} x_1)$. Therefore, by (3.7), we have

$$\frac{\|h_j\|}{\|g_j\|} < 0.6^j \frac{\|h\|}{\|g\|} = \delta^{-1} 0.6^j.$$

By definition of $m_0$, we have $\delta^{-1} 0.6^{m_0} < 1$; so let $m \geqslant 1$ be the least number so that $\|h_m\|/\|g_m\| < 1$. By minimality of $m$, we have $\|h_m\|/\|g_m\| \geqslant K^{-2\ell_0}$. That is, the norms of the vectors $h_m$ and $g_m$ are far from $0$ and $\infty$. Moreover, since

$$\angle(h_m, g_m) \geqslant \angle(F, G) - \angle(h_m, F) > \theta_0 - \beta > \theta_0/2,$$

the vector $h_m + g_m$ cannot be much smaller than $h_m$ or $g_m$. Thus, we can say that

$$\|h_m\|, \|g_m\|, \|h_m + g_m\| \text{ are between } c_4^{-1} \text{ and } c_4.$$

Let $y = T^{\ell_0 m} x_1$; then $h_m + g_m$ spans $H_1(y)$. We will show that at the point $y$, the space $H_1$ makes a small angle with $F \oplus G$, but not with $F$ or $G$ (see Figure 5).

The *distance* between a vector $u$ and a space $V$ is

$$d(u, V) = \inf_{v \in V} \|u - v\| = \|u\| \sin \angle(u, V).$$

Let $f$ be the vector in $F(y)$ that is closer to $h_m$. Since $h_m \in H(y) \subset F^\beta(y)$, the vector $w = h_m - f$ has a norm $\|w\| = d(h_m, F(y)) < c_4 \beta$. The vectors $h_m + g_m$ and $f + g_m$ have

comparable norms, and their difference is $w$. We conclude that they form an angle less than $c_5\beta$. In particular, $\measuredangle(H_1, F \oplus G) < c_6\beta$ at the point $y$.

On the other hand,

$$d(g_m, F(y)) = \|g_m\| \sin \measuredangle(F, G) > \|g_m\| \sin \theta_0,$$

so the distance from $h_m + g_m = f + g_m + w$ to $F(y)$ is at least $\|g_m\| \sin \theta_0 - \|w\|$. Thus we obtain that $\measuredangle(H_1(y), F(y)) > c_6$. An entirely analogous reasoning gives $\measuredangle(H_1(y), G(y)) > c_6$.

It follows from Lemma 3.6 and the previous estimates that, at the point $y$,

$$\measuredangle(F/H_1, G/H_1) < c_0^{-1} \cdot \frac{\measuredangle(H_1, F \oplus G)}{\measuredangle(H_1, F) \cdot \measuredangle(H_1, G)} < c_7\beta.$$

Since $c_7\beta < \theta_0$, we have $F/H_1 \not\prec_{\ell_0} G/H_1$. So concatenating the paths from $A$ to $B$ and from $B$ to $B_1$, we obtain a path of cocycles that falls into the second alternative in (3.6).

This proves Lemma 3.7 and therefore Proposition 3.1. $\qquad\square$

## 4. Mixing Lyapunov exponents: general results

### 4.1. The first main result

THEOREM 4.1. *For any $d \geqslant 2$, $K > 1$ and $\varepsilon > 0$, there exists $L \in 2 \wedge \mathbb{N}$ such that the following holds: Let $(X, T, E, A)$ be a cyclic $d$-dimensional cocycle bounded by $K$ and of period at least $L$ that admits no $L$-dominated splitting. Let $\sigma \in \mathcal{S}_d$ be a graph such that $\sigma \geqslant \boldsymbol{\sigma}(A)$ and $\sigma_d = \boldsymbol{\sigma}_d(A)$.*

*Then there exists an $\varepsilon$-short path of cocycles $A_t$, where $t \in [0, 1]$, starting at $A$ such that the path of graphs $\boldsymbol{\sigma}(A_t)$ is non-decreasing and $\boldsymbol{\sigma}(A_1) = \sigma$.*

Theorem 1 from § 1 follows directly using Franks lemma and basic properties of dominated splittings.

REMARK 4.2. A consequence of Theorem 4.1 is that with a perturbation $A_1$, we can make all Lyapunov exponents equal. However, it may be impossible to make $A_1^n(x)$ into a homothety at a given point $x$ (where $n$ is the period) (see the footnote on [**11**, p. 1308]). So it is not possible to have an statement similar to Theorem 4.1 where instead of 'mixing' the Lyapunov exponents, one mixes the logarithms of the singular values of $A^n(x)$.

The proof of Theorem 4.1 goes roughly as follows. Starting with a cocycle without $L$-dominated splitting, where $L \gg 1$, we first perturb it to make all eigenvalues real. Then we want to apply Proposition 3.1 a certain number $N$ of times until the spectrum gets very close to $\sigma$, and perturb a last time to get spectrum exactly $\sigma$. To make all this work, each time we want to apply Proposition 3.1 to a cocycle, it must be weakly dominated, or not dominated at all. Thus the perturbations must be sufficiently small so that no strong domination is created. This is done with a careful choice of the quantifiers, where it is important to have an *a priori* bound for the number $N$ of times that we will have to apply Proposition 3.1.

Now we give the detailed proof. The first auxiliary result says how we can get rid of complex eigenvalues.

PROPOSITION 4.3 (Getting real eigenvalues). *Given $d, K$ and $\varepsilon$, there exists $m = m(d, K, \varepsilon)$ such that if $(X, T, E, A)$ is a cyclic $d$-dimensional cocycle bounded by $K$ and of period at least $m$ then there is an $\varepsilon$-short path of cocycles $A_t$, where $t \in [0, 1]$, starting at $A$, all of them with the same Lyapunov spectrum, such that $A_1$ has only real eigenvalues.*

This proposition is essentially contained in [**11**], but not in the exact form we need, so we give a proof. The essential fact is the following lemma, where $R_\theta$ denotes the rotation of angle $\theta$ in $\mathbb{R}^2$.

LEMMA 4.4 [**6**, Lemma 6.6]. *For every $\varepsilon > 0$, there exists $m$ with the following property: For any finite sequence of matrices $A_1, \ldots, A_n$ in $\mathrm{SL}(2, \mathbb{R})$ of length $n \geqslant m$, there exist numbers $\theta_1, \ldots, \theta_n$ in the interval $(-\varepsilon, \varepsilon)$ such that the matrix $R_{\theta_n} A_n \ldots R_{\theta_1} A_1$ has real eigenvalues.*

REMARK 4.5. In fact, it is possible to take all $\theta_i$ equal in Lemma 4.4; this follows immediately from [**4**, Lemma C.2].

*Proof of Proposition* 4.3. The proof is by induction on the dimension $d$. For $d = 1$, the result is true. Next consider a two-dimensional cocycle $(X, T, E, A)$, where $T$ is a cyclic dynamical system of large period $n$. We can assume that the linear map $A^n(x)$ has complex eigenvalues and, in particular, preserves orientation. So with appropriate choices of bases over each point, the matrices of the cocycle have positive determinants. Normalizing the determinants, we obtain matrices $A_1, \ldots, A_n$ in $\mathrm{SL}(2, \mathbb{R})$. By Lemma 4.4, $R_{\theta_n} A_n \ldots R_{\theta_1} A_1$ has real eigenvalues for some choice of small numbers $\theta_i$. Consider the path of matrices $M_t = R_{\theta_n t} A_n \ldots R_{\theta_1 t} A_1$. Take the smallest $t_0 \in [0, 1]$ such that $M_{t_0}$ has real eigenvalues. Then by perturbing the cocycle itself by composing with rotations $R_{\theta_i t_0 t}$, where $t \in [0, 1]$, we obtain the desired path of cocycles.

Now let $d > 2$ be arbitrary and assume the proposition is true for any dimension less than $d$. Any cocycle over a cyclic dynamical system has a two-dimensional invariant subbundle $F$. Apply the induction assumption to the cocycles $A \upharpoonright F$ and $A/F$. We obtain paths of cocycles on $F$ and $E/F$ with constant Lyapunov spectra and only real eigenvalues at the endpoints. By the extension procedures of § 2.2, we obtain the desired path. $\qquad\square$

The next result says that once all eigenvalues are real (which is the case when the spectrum is simple, for example), we can always perturb them a little:

LEMMA 4.6 (Arbitrary perturbation of the spectrum). *Given $d, K$ and $\varepsilon$, there exists $\delta = \delta(d, K, \varepsilon) > 0$ with the following properties. Let $(X, T, E, A)$ be a cyclic $d$-dimensional cocycle bounded by $K$ with only real eigenvalues. Then for any path $\sigma : [0, 1] \to \mathcal{S}_d$ satisfying $\sigma(0) = \boldsymbol{\sigma}(A)$ and $|\sigma(t) - \sigma(0)| < \delta$, there exists an $\varepsilon$-short path of cocycles $A_t$, where $t \in [0, 1]$, starting at $A$, such that $\boldsymbol{\sigma}(A_t) = \sigma(t)$ for each $t$.*

*Proof.* Let $(X, T, E, A)$ be a $d$-dimensional cocycle. Assuming it has only real eigenvalues, it is possible to find invariant subbundles $F_1 \subset F_2 \subset \ldots \subset F_{d-1}$ with $\dim F_i = i$ (that is, an invariant flag). For each $x \in X$, let $\{e_1(x), \ldots, e_d(x)\}$ be an orthonormal basis of $E(x)$ such that $e_i(x) \in F_i(x)$ for each $i$. With respect to those bases, the cocycle is expressed by triangular matrices. By multiplying them with appropriate diagonal matrices (close to the identity), we can perform any prescribed sufficiently small perturbation of the Lyapunov exponents, keeping the eigenvalues real. $\qquad\square$

With the next lemma, we control the effect of our perturbations on the non-dominance of the cocycle.

LEMMA 4.7 (Stability of non-domination). *Given $d, K$ and $\ell$, there exists $\eta = \eta(d, K, \ell) > 0$ such that if $A$ is a $d$-dimensional cocycle bounded by $K$ that has no $2\ell$-dominated splitting of index $p$, then no $\eta$-perturbation of $A$ has an $\ell$-dominated splitting of index $p$.*
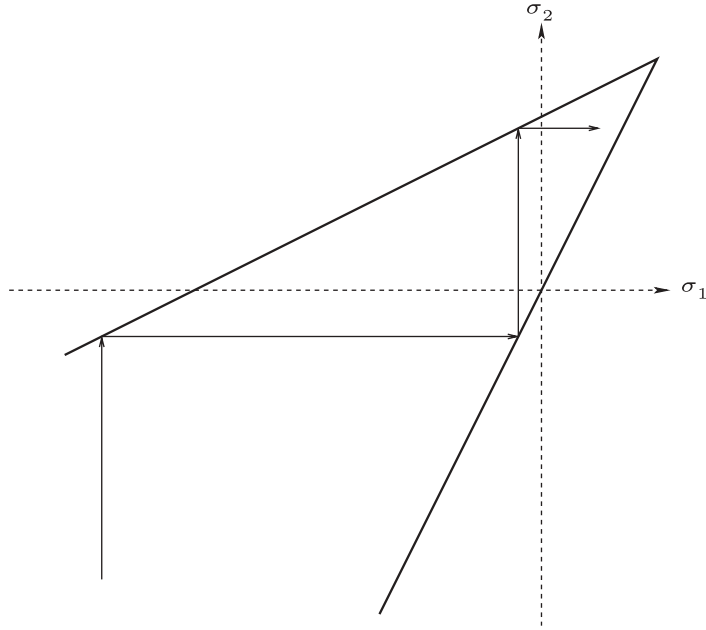
FIGURE 6. *The region between the two lines is a slice of $\mathcal{S}_3$ of constant $\sigma_3 > 0$. A path of spectra as in Lemma 4.8 is represented, with arrows joining points $\sigma^{(j)}$ and $\sigma^{(j+1)}$.*

*Proof.* This is merely the contraposition of the well-known fact that dominated splittings persist under perturbations. (See, for example, [**10**, Appendix B.1].) □

In the proof of Theorem 4.1, it is necessary to have an *a priori* bound on the number of times we will have to apply Proposition 3.1. We will show that given two convex graphs $\sigma' \leqslant \sigma''$ in $\mathcal{S}_d$ that agree at the endpoints, there exists a non-decreasing path of graphs inside $\mathcal{S}_d$ starting at $\sigma'$ and ending close to $\sigma''$; this path is the concatenation of a certain number $N$ of paths of graphs, each of these consisting of moving a single vertex of the graph (see Figure 6).

Moreover, the number $N$ can be bounded in terms of the sizes of $\sigma'$, $\sigma''$ and the desired accuracy of the approximation to $\sigma''$. Precisely speaking, we have the following lemma.

LEMMA 4.8 (Path of spectra). *Given $d \geqslant 2$, $c > 0$ and $\delta > 0$, there exists $N = N(d, c, \delta)$ with the following properties. Let $\sigma', \sigma'' \in \mathcal{S}_d$ be such that $|\sigma'_i| \leqslant ci$ for each $i$, $\sigma'_d = \sigma''_d$ and $\sigma' \leqslant \sigma''$. Then there exists a sequence of points in $\mathcal{S}_d$:*

$$\sigma^{(0)} \leqslant \sigma^{(1)} \leqslant \ldots \leqslant \sigma^{(N)}$$

*such that $\sigma^{(0)} = \sigma'$, $\sigma^{(N)} \leqslant \sigma''$, $\sigma^{(N)}$ is $\delta$-close to $\sigma''$, and there are $i_1, \ldots, i_N \in \{1, \ldots, d-1\}$ such that*

$$1 \leqslant j \leqslant N, \ 0 \leqslant i \leqslant d, \ i \neq i_j \ \Rightarrow \ \sigma^{(j-1)}_i = \sigma^{(j)}_i.$$

REMARK 4.9. It is known in majorization theory that if $\sigma'$ majorizes $\sigma''$ then one can generate $\sigma''$ from $\sigma'$ with a countable sequence of elementary Robin Hood operations (recall Remark 3.2) (see, for example, [**19**, p. 82]. Lemma 4.8) is a finer version of this fact: it says that the number of operations $N$ needed to achieve a certain precision $\delta$ has a uniform bound depending on the 'size' of the graphs (measured by $d$ and $c$).

*Proof of Lemma* 4.8.   Consider any two vectors $\sigma' \leqslant \sigma''$ in $\mathbb{R}^{d+1}$ that are convex, that is, $\Delta^2 \sigma'$, $\Delta^2 \sigma'' \geqslant 0$ in the difference operator notation from § 2.6. Also assume that the two graphs agree at the endpoints, that is, $\sigma'_0 = \sigma''_0$ (it is convenient not to assume these are zero) and $\sigma'_d = \sigma''_d$. We will find a non-decreasing sequence $(\sigma^{(j)})_{j \geqslant 0}$ of convex vectors in $\mathbb{R}^{d+1}$ starting at $\sigma'$ such that any two consecutive elements differ at most one coordinate, which is neither the zeroth nor the $d$th. We will show that the sequence converges exponentially fast to $\sigma''$ with some speed that depends only on $d$. Once this is done, it is trivial to write a formula for $N = N(d, c, \delta)$ that meets the requirements of the lemma.

The case $d = 2$ is immediate: we can reach $\sigma''$ exactly with a single step. Fix $d > 2$ and assume that the procedure known for smaller dimensions.

Let $\sigma^{(0)} = \sigma'$. Let $j \geqslant 0$ and assume that $\sigma^{(j)}$ was already defined. Let $A_j$ be the area between the graphs $\sigma^{(j)}$ and $\sigma''$; this quantity is uniformly comparable with the distance between the two graphs. Let $D_j = \max_{0 \leqslant i \leqslant d-2} \Delta^2 \sigma_i^{(j)}$. Take $i_j \in \{1, \ldots, d-1\}$ such that $\Delta^2 \sigma_{i_j-1}^{(j)} = D_j$. Define $\sigma_i^{(j+1)} = \sigma_i^{(j)}$ for $i \neq i_j$ and

$$\sigma_{i_j}^{(j+1)} = \min \left( \sigma''_{i_j}, \frac{\sigma_{i_j-1}^{(j)} + \sigma_{i_j+1}^{(j)}}{2} \right).$$

If $\sigma_{i_j}^{(j+1)} = \sigma''_{i_j}$, then we break the graph into two parts and define the rest of sequence using the induction hypothesis. In the remaining case, we have $A_{j+1} = A_j - D_j/2$. It follows from Lemma 2.5 that $A_j \leqslant (d^3/4)D_j$. Thus, $A_{j+1}/A_j \leqslant 1 - 2/d^3$. This gives the exponential convergence. □

*Proof of Theorem* 4.1.   Let $d$, $K$ and $\varepsilon$ be given. Reducing $\varepsilon$ if necessary (recall Remark 2.1), we assume that any $\varepsilon$-perturbation of a cocycle bounded by $K$ is bounded by $2K$. Let $\varepsilon_0 = \varepsilon/2$ and let $\delta = \delta(d, 2K, \varepsilon_0)$ be given by Lemma 4.6. Let $N = N(d, \log K, \delta)$ be given by Lemma 4.8.

We recursively define numbers $\varepsilon_1, \ell_1, \ldots, \varepsilon_{N+1}, \ell_{N+1}$ (in this order) by the formulas:

$$\varepsilon_j = \begin{cases} \varepsilon/4 & \text{if } j = 1, \\ \min(\varepsilon/2^{j+2}, \eta(d, 2K, \ell_{j-1})) & \text{if } j \geqslant 2, \end{cases}$$

$$\ell_j = \begin{cases} \ell(d, 2K, \varepsilon_1) & \text{if } j = 1, \\ \max(2\ell_{j-1}, \ell(d, 2K, \varepsilon_j)) & \text{if } 2 \leqslant j \leqslant N, \\ \max(2\ell_N, m(d, 2K, \varepsilon_{N+1})) & \text{if } j = N+1, \end{cases}$$

where the functions $\eta$, $\ell$ and $m$ come, respectively, from Lemma 4.7, Proposition 3.1, and Proposition 4.3. Let $L = \ell_{N+1}$.

Now consider any $d$-dimensional cocycle $(X, T, E, A)$ bounded by $K$ and of period at least $L$ that admits no $L$-dominated splitting. Take $\sigma = (\sigma_0, \ldots, \sigma_d) \in \mathcal{S}_d$ with $\sigma \geqslant \boldsymbol{\sigma}(A)$ and $\sigma_d = \boldsymbol{\sigma}_d(A)$.

By Lemma 4.8, there is a sequence $\boldsymbol{\sigma}(A) = \sigma^{(0)} \leqslant \sigma^{(1)} \leqslant \ldots \leqslant \sigma^{(N)} \leqslant \sigma$ such that $\sigma^{(N)}$ is $\delta$-close to $\sigma$, and there are $i_0, \ldots, i_{N-1} \in \{1, \ldots, d-1\}$ such that

$$1 \leqslant j \leqslant N, \ 0 \leqslant i \leqslant d, \ i \neq i_j \quad \Rightarrow \quad \sigma_i^{(j)} = \sigma_i^{(j+1)}.$$

Since the cardinality of $X$ is at least $L = \ell_{N+1} \geqslant m(d, 2K, \varepsilon_{N+1})$, we can use Proposition 4.3 to obtain a path of cocycles $A_t$, where $t \in [0, 1]$, starting at $A_0 = A$, all $\varepsilon_{N+1}$-close to $A$ and with the same Lyapunov spectrum, so that $A_1$ has only real eigenvalues. Since $L \geqslant 2\ell_N$ and $\varepsilon_{N+1} \leqslant \eta(d, 2K, \ell_N)$, the cocycle $A_1$ has no $\ell_N$-dominated splitting. Moreover, $\ell_N \geqslant \ell(d, 2K, \varepsilon_N)$. Therefore, using Proposition 3.1 we can find a path of cocycles $A_t$, where $t \in [1, 2]$, such that:

(1)  $\|A_t - A_1\| < \varepsilon_N$ for all $t \in [1, 2]$;

(2) $\boldsymbol{\sigma}_i(A_t) = \boldsymbol{\sigma}_i(A_1)$ if $t \in [1, 2]$, $i \neq i_1$;

(3) $\boldsymbol{\sigma}_{i_0}(A_t)$ is non-decreasing for $t$ on $[1, 2]$;

(4) $\boldsymbol{\sigma}_{i_0}(A_2) = \sigma_{i_0}^{(1)}$ (we stop the path given by the proposition when we reach this point).

Since $\ell_N \geqslant 2\ell_{N-1}$ and $\varepsilon_N \leqslant \eta(d, 2K, \ell_{N-1})$, the cocycle $A_2$ has no $\ell_{N-1}$-dominated splitting.

Continuing in this way, we obtain a path of cocycles $A_t$, where $t \in [1, N+1]$, such that for all $j \in \{1, \ldots, N\}$:

(1) $\|A_t - A_j\| < \varepsilon_{N-j+1}$ for all $t \in [j, j+1]$;

(2) $\boldsymbol{\sigma}_i(A_t) = \boldsymbol{\sigma}_i(A_j)$ if $t \in [j, j+1]$, $i \neq i_{j-1}$;

(3) $\boldsymbol{\sigma}_{i_{j-1}}(A_t)$ is non-decreasing for $t$ on $[j, j+1]$;

(4) $\boldsymbol{\sigma}_{i_{j-1}}(A_{j+1}) = \sigma_{i_{j-1}}^{(j)}$.

In particular, the path of spectra $\boldsymbol{\sigma}(A_t)$ is monotonic with respect to the partial order $\leqslant$, the function $\boldsymbol{\sigma}_d(A_t)$ is constant and $\boldsymbol{\sigma}(A_{N+1}) = \sigma^{(N)}$.

Since $\sigma^{(N)} \leqslant \sigma$ is $\delta$-close to $\sigma$, using Lemma 4.6 we can find an $\varepsilon_0$-short path of cocycles $A_t$, where $t \in [N+1, N+2]$, such that the path of spectra $\boldsymbol{\sigma}(A_t)$ is monotonic with respect to the partial order $\leqslant$, the function $\boldsymbol{\sigma}_d(A_t)$ is constant and $\boldsymbol{\sigma}(A_{N+2}) = \sigma$.

Concatenating everything, we obtain a path $A_t$, where $t \in [0, N+2]$. We have $\|A_t - A_0\| \leqslant \sum_0^{N+1} \varepsilon_j \leqslant \varepsilon$. Reparametrizing to the interval $[0, 1]$, we obtain a path of cocycles with all the desired properties. $\square$

While in this paper, we are mainly interested in cocycles over cyclic dynamical systems, it is natural to consider more general cocycles. For example, we ask the following question:

QUESTION 4.10. Let $T$ be a homeomorphism of a compact space $X$, admitting a fully supported ergodic measure $\mu$. Does Theorem 4.1 hold in this setting (taking Lyapunov graphs with respect to $\mu$)?

One can ask for similar extensions of other theorems below.

### 4.2. *Restriction to subbundles*

Let us see some generalizations of Theorem 4.1. First we consider cocycles that have an invariant subbundle with no (strong) dominated splitting:

THEOREM 4.11. *For any $d \geqslant 2$, $K > 1$, and $\varepsilon > 0$, there exists $L \in 2\wedge\mathbb{N}$ such that the following holds: Let $(X, T, E, A)$ be a cyclic $d$-dimensional cocycle bounded by $K$ and of period at least $L$. Let $F \subset E$ be a subbundle of positive dimension $k \leqslant d$ such that the restricted cocycle $A \restriction F$ has no $L$-dominated splitting. Let $\sigma \in \mathcal{S}_k$ be a graph such that $\sigma \geqslant \boldsymbol{\sigma}(A \restriction F)$ and $\sigma_k = \boldsymbol{\sigma}_k(A \restriction F)$.*

*Then there exists an $\varepsilon$-short path of cocycles $A_t$, where $t \in [0, 1]$ starting at $A$, all of them preserving the bundle $F$, such that the path of graphs $\boldsymbol{\sigma}(A_t \restriction F)$ is non-decreasing, $\boldsymbol{\sigma}(A_1 \restriction F) = \sigma$ and $A_t/F = A/F$ for each $t$.*

*Proof.* This follows directly from Theorem 4.1 applied to restricted cocycles, together with the first extension procedure from § 2.2. $\square$

THEOREM 4.12. *For any $d \geqslant 2$, $K > 1$ and $\varepsilon > 0$, there exists $\ell \in 2\wedge\mathbb{N}$ such that the following holds: Let $(X, T, E, A)$ be a cyclic $d$-dimensional cocycle bounded by $K$ and of period*

at least $\ell$. Let $i_1 < \ldots < i_{m-1}$ be the indices of its finest $\ell$-dominated splitting. Let $\sigma \in \mathcal{S}_d$ be a graph such that $\sigma \geqslant \boldsymbol{\sigma}(A)$ and $\sigma_{i_j} = \boldsymbol{\sigma}_{i_j}(A)$ for each $j$, and $\sigma_d = \boldsymbol{\sigma}_d(A)$.

Then there exists an $\varepsilon$-short path of cocycles $A_t$, where $t \in [0, 1]$ starting at $A$, such that the path of graphs $\boldsymbol{\sigma}(A_t)$ is non-decreasing and $\boldsymbol{\sigma}(A_1) = \sigma$.

Notice that this theorem extends Proposition 3.1.

Theorem 2 from § 1 follows directly using Franks lemma and basic properties of dominated splittings.

*Proof of Theorem 4.12.* Let $d$, $K$ and $\varepsilon$ be given. Reducing $\varepsilon$ if necessary (recall Remark 2.1), we assume that any $\varepsilon$-perturbation of a cocycle bounded by $K$ is bounded by $2K$. We recursively define numbers $\varepsilon_1, \ell_1, \ldots, \varepsilon_{d-1}, \ell_{d-1}$ (in this order) by the formulas:

$$\varepsilon_j = \begin{cases} \varepsilon/2 & \text{if } j = 1, \\ \min(\varepsilon/2^j, \eta(d, 2K, \ell_{j-1})) & \text{if } j \geqslant 2, \end{cases}$$

$$\ell_j = \begin{cases} \ell(d, 2K, \varepsilon_1) & \text{if } j = 1, \\ \max(2\ell_{j-1}, L(d, 2K, \varepsilon_j)) & \text{if } j \geqslant 2, \end{cases}$$

where the functions $\eta$ and $L$ come, respectively, from Lemma 4.7 and Theorem 4.11. Let $\ell = \ell_{d-1}$.

Now consider any $d$-dimensional cocycle $(X, T, E, A)$ bounded by $K$ and of period at least $L$. Let $E = F_1 \oplus \ldots \oplus F_m$ be its finest $\ell$-dominated splitting, and let $i_1 < i_2 < \ldots < i_{m-1}$ be its indices. Take a graph $\sigma \in \mathcal{S}_d$ such that $\sigma \geqslant \boldsymbol{\sigma}(A)$ and $\sigma_{i_j} = \boldsymbol{\sigma}_{i_j}(A)$ for each $j$. We can assume that $m > 1$, otherwise there is nothing to show.

Since $A \restriction F_1$ has no $\ell$-dominated splitting, by Theorem 4.11 there is an $\varepsilon_{d-1}$-short path of cocycles $A_t$, where $t \in [0, 1]$ starting at $A$, such that $\boldsymbol{\sigma}_i(A_2) = \sigma_i$ for $i \leqslant i_1$ and $\boldsymbol{\sigma}_i(A_2) = \boldsymbol{\sigma}_i(A)$ for $i > i_1$.

Now let $E = F_1^{(1)} \oplus \ldots \oplus F_{m_1}^{(1)}$ be the finest $\ell_{d-2}$-dominated splitting of $A_1$. By Lemma 4.7, its set of indices $\{i_1^{(1)} < i_2^{(1)} < \ldots < i_{m_1-1}^{(1)}\}$ is contained in $\{i_1 < i_2 < \ldots < i_{m-1}\}$. We consider two cases: if $i_1^{(1)} = i_1$, then we apply Theorem 4.11 to the subbundle $F_2^{(1)}$; if $i_1^{(1)} > i_1$, then we apply the theorem to $F_1^{(1)}$. In either case, we find an $\varepsilon_{d-2}$-short path of cocycles $A_t$, where $t \in [1, 2]$, such that $\boldsymbol{\sigma}(A_t)$ is non-decreasing, $\boldsymbol{\sigma}_i(A_2) = \sigma_i$ for $i \leqslant p$ and $\boldsymbol{\sigma}_i(A_2) = \boldsymbol{\sigma}_i(A)$ for $i > p$, where $p = i_1^{(2)}$ in the first case and $p = i_1^{(1)}$ in the second. Thus $p \geqslant i_2$ Lyapunov exponents are already adjusted.

Continuing in this way, we will adjust all exponents in a number $k \leqslant m$ of steps. That is, by concatenating we obtain a path of cocycles $A_t$, where $t \in [0, k]$, such that $\boldsymbol{\sigma}(A_t)$ is non-decreasing and $\boldsymbol{\sigma}_i(A_k) = \sigma_i$ for all $i$. For each $t$, we have $\|A_t - A\| \leqslant \varepsilon_{d-1} + \ldots + \varepsilon_{d-k} \leqslant \varepsilon$. Hence a reparametrized path has all the desired properties. $\qquad\square$

### 4.3. Perturbing the spectrum with constraints

As we have seen, the path of spectra in the proof of Theorem 4.1 is, except for its final part, a zigzag like in Figure 6. We could ask whether it is possible to prescribe any path of graphs. Strictly speaking, the answer is no, because if $A$ has a pair of non-real eigenvalues, then there are two Lyapunov exponents that we cannot immediately separate. However, we ask whether the *trace* of the path of graphs can be prescribed.

QUESTION 4.13. In the assumptions of Theorem 4.1, and given any non-decreasing path of graphs $\sigma_s \in \mathcal{S}_d$, $s \in [0, 1]$ with $\sigma_0 = \boldsymbol{\sigma}(A)$, is it possible to choose the path of cocycles $A_t$ so that $\{\boldsymbol{\sigma}(A_t);\ t \in [0, 1]\} = \{\sigma_s;\ s \in [0, 1]\}$?

At least some partial control of the path of graphs is possible: for example, we can choose it with a constant index. This is the content of Theorem 4.12, which would follow immediately from a positive answer to Question 4.13.

We say that $\sigma \in \mathcal{S}_d$ is a graph of *index* $p$ if $\sigma_i > \sigma_p$ for all $i \in \{0, \ldots, d\} \smallsetminus \{p\}$. In terms of the Lyapunov exponents $\lambda_i = \sigma_i - \sigma_{i-1}$, this means that $\lambda_p < 0 < \lambda_{p+1}$ (disregard the first inequality if $p = 0$ and the second if $p = d$); in particular, there are no zero Lyapunov exponents.

THEOREM 4.14.   *The number $L = L(d, K, \varepsilon)$ in Theorem 4.1 can be chosen with the following additional property*: *if $\boldsymbol{\sigma}(A)$ and $\sigma$ have the same index, then the path of cocycles $A_t$ can be chosen so that all $\boldsymbol{\sigma}(A_t)$ have the same index.*

Using this theorem, we can repeat the arguments from §4.2 and obtain index-preserving versions of Theorems 4.11 and 4.12.

The proof of Theorem 4.12 is exactly the same as Theorem 4.1, except that we need an index-preserving version of Lemma 4.8.

LEMMA 4.15.   *The number $N = N(d, c, \delta)$ in Lemma 4.8 can be chosen with the following additional property*: *If $\sigma', \sigma'' \in \mathcal{S}_d$ are graphs of the same index $p$, then the graphs $\sigma^{(1)}, \ldots, \sigma^{(d)}$ can be chosen of index $p$ as well.*

*Proof.*   The $d = 2$ situation is trivial. So assume $d > 2$ and assume by induction that the lemma is already known for smaller dimensions. Take two graphs $\sigma' \leqslant \sigma''$ in $\mathcal{S}_d$ with the same endpoints, no zero exponents and the same index $p$. We do not need to consider indices $p = 0$ and $p = d$ because these cases are covered by Lemma 4.8.

Given any graph $\sigma \in \mathbb{R}^{d+1}$, we split it into two graphs: $\sigma^{(L)} = (\sigma_0, \ldots, \sigma_p)$ and $\sigma^{(R)} = (\sigma_p, \ldots, \sigma_d)$.

Let $\sigma^{(0)} = \sigma'$. Let $\bar{\sigma}^{(0)}$ be the maximal convex graph $\leqslant \sigma''$ satisfying $\bar{\sigma}_p^{(0)} = \sigma_p^{(0)}$. The two graphs $\sigma^{(0L)} \leqslant \bar{\sigma}^{(0L)}$ have the same endpoints, and so do $\sigma^{(0R)} \leqslant \bar{\sigma}^{(0R)}$. Applying Lemma 4.8 to each side and gluing sides together, we can find a path of graphs $\sigma^{(0)} \leqslant \ldots \leqslant \sigma^{(j)}$ such that at most one coordinate changes at a time, and the graphs $\sigma^{(j)} \leqslant \bar{\sigma}^{(0)}$ are very close. Notice that the indices do not change along the path.

Assume that there exists $\ell$ with $p < \ell < d$ such that $\sigma_\ell^{(j)} = \sigma_\ell''$. Then the subgraphs $(\sigma_\ell^{(j)}, \ldots, \sigma_d^{(j)}) \leqslant (\sigma_\ell'', \ldots, \sigma_d'')$ are very close. We apply the induction hypotheses to the pair of graphs $(\sigma_0^{(j)}, \ldots, \sigma_\ell^{(j)}) \leqslant (\sigma_0'', \ldots, \sigma_\ell'')$, obtaining a zigzag path of graphs of dimension $\ell$ of index $p$. So we extend this to a path of graphs of dimension $d$, and we are done. Analogously, if there exists $\ell$ with $0 < \ell < p$ such that $\sigma_\ell^{(j)} = \sigma_\ell''$, then we split the graph at this point; the left part is already ok, and using the induction hypotheses on the right part, we are done.

Therefore, we can assume that $\sigma_\ell^{(j)} < \sigma_\ell''$ for every $\ell$ with $0 < \ell < p$ or $p < \ell < d$. This implies that both $\sigma^{(jL)}$ and $\sigma^{(jR)}$ are nearly flat; more precisely, $\Delta^2 \sigma_\ell^{(j)}$ is small for every $\ell \neq p - 1$. (recall notation from §2.6).

Now consider the *gap* $G_j = \sigma_p'' - \sigma_p^{(j)}$. Let $E_j = \min(-\Delta\sigma_{p-1}^{(j)}, \Delta\sigma_p^{(j)})$. It follows from nearly flatness that $\max(\sigma_0^{(j)}, \sigma_d^{(j)}) - \sigma_p^{(j)}$, and hence $G_j$, cannot be much bigger than $E_j$. More precisely, we have $G_j \leqslant 1.1 d E_j$ provided that $\sigma^{(jL)}$ and $\sigma^{(jR)}$ are sufficiently flat.

Define a graph $\sigma^{(j+1)}$ of index $p$ by taking

$$\sigma_p^{(j+1)} = \min(\sigma_p'', \sigma_p^{(j)} + 0.9 E_j)$$

and all the other coordinates equal to those of $\sigma^{(j)}$. If $\sigma_p^{(j+1)} = \sigma_p''$, then the graphs $\sigma^{(j+1)}$ and $\sigma''$ are very close and we are done. Otherwise, the new gap is $G_{j+1} = G_j - 0.9 E_j$. Therefore,

$G_{j+1}/G_j$ is less than 1. So we restart the procedure; after a finite number of steps the gap will be small and we are done. $\qquad\square$

*Proof of Theorem 4.12.* Repeat the proof of Theorem 4.1, using Lemma 4.15 in the place of Lemma 4.8 and taking, for example, an affine path in Lemma 4.6 (so that the index is preserved). $\qquad\square$

REMARK 4.16. More generally, let $\mathcal{P}$ be a property about Lyapunov graphs of dimension $d$ (as having some prescribed index, for example), corresponding to a set $C \subset \mathcal{G}_d$. Suppose that given $\sigma'$, $\sigma'' \in C$, the points $\sigma^{(1)}, \ldots, \sigma^{(N)}$ in Lemma 4.8 can all be chosen in $C$, and moreover the path with those vertices (as in Figure 6) is wholly contained in $C$. Then a $\mathcal{P}$-preserving version of Theorem 4.1 holds: if $\boldsymbol{\sigma}(A)$ and $\sigma$ belong to $C$, then the path of cocycles $A_t$ can be chosen so that $\boldsymbol{\sigma}(A)$ is contained in $C$ for all $t$.

## 5. Separating Lyapunov exponents

In this section, we give the (stronger) cocycle versions of Theorems 3 and 4.

### 5.1. Statements and sketch of the proofs

THEOREM 5.1. *Let $(X, T, E, A)$ be a cocycle. Let $\mu_k$ be a sequence of $T$-invariant probabilities converging to some $\mu$ in the weak-star topology. Assume that each $\mu_k$ is supported on a periodic orbit, whose period tends to infinity with $k$. Then there exist a sequence of cocycles $B_k \to A$ such that $\boldsymbol{\sigma}(B_k, \mu_k) = \boldsymbol{\sigma}(A, \mu)$ for every $k$.*

REMARK 5.2. Let us mention a non-perturbative result that also concerns the approximation of Lyapunov exponents using periodic orbits: [**16**, Theorem 1.4] asserts that if a cocycle is Hölder continuous and the base dynamics has certain hyperbolicity properties (satisfied by basic hyperbolic sets or subshifts of finite type, for example), then the Lyapunov exponents of every ergodic measure can be approximated by the Lyapunov exponents at periodic points.

*Proof of Theorem 3.* Given a sequence of diffeomorphisms $f_n : M \to M$ converging to a diffeomorphism $f = f_\infty$, we define a cocycle $(X, T, E, A)$ as follows: Let $\bar{\mathbb{N}} = \mathbb{N} \cup \{\infty\}$ be the one-point compactification of $\mathbb{N}$, and take

$$X = \bar{\mathbb{N}} \times M, \quad T(n, x) = (n, f_n(x)), \quad E = \bar{\mathbb{N}} \times TM, \quad A(n, x) = Df_n(x).$$

Applying Theorem 5.1 to this cocycle and then using Franks lemma, it is easy to reach the conclusions of Theorem 3. $\qquad\square$

THEOREM 5.3. *Let $(X, T, E, A)$ be a cocycle. Let $\mu_k$ be a sequence of $T$-invariant probabilities converging to some $\mu$ in the weak-star topology. Assume that each $\mu_k$ is supported on a periodic orbit, whose period tends to infinity with $k$. Also assume that the support of $\mu_k$ converges in the Hausdorff topology to an (invariant compact) set $\Lambda$. Let $F_1 \oplus \ldots \oplus F_m$ be the finest dominated splitting for the cocycle $A$ restricted to $\Lambda$. Let $i_j = \dim F_1 \oplus \ldots \oplus F_j$. Let $\sigma \in \mathcal{S}_d$ be any convex graph such that $\sigma \geqslant \boldsymbol{\sigma}(A, \mu)$ and $\sigma_{i_j} = \boldsymbol{\sigma}_{i_j}(A, \mu)$ for each $j$. Then there exists $B$ arbitrarily close to $A$ such that $\boldsymbol{\sigma}(B, \mu_k) = \sigma$ for some $k$.*

*Proof of Theorem 4.* One-half of the theorem is trivial: if a graph $\sigma$ is the limit of graphs $\boldsymbol{\sigma}(g_n, \gamma_n)$ as in the statement, then by semicontinuity, $\sigma$ must be above $\boldsymbol{\sigma}(f, \mu)$, and by basic

properties of dominated splittings, $\sigma$ must touch $\boldsymbol{\sigma}(f, \mu)$ at the points corresponding to the indices of the finest dominated splitting. That is, we necessarily have $\sigma \in \mathcal{G}(\mu, \Lambda)$.

The non-trivial half of the theorem follows from Theorem 5.3, arguing analogously as in the proof of Theorem 3.                                                                                    □

The proofs of Theorems 5.1 and 5.3 occupy the rest of this section. Let us first give an informal exposition of the main ideas of the proof of Theorem 5.1.

It is sufficient to show that for every sufficiently large $k$, there is a perturbation $\tilde{A}$ with $\boldsymbol{\sigma}(\tilde{A}, \mu_k)$ approximately equal to $\boldsymbol{\sigma}(A, \mu)$, because it is always easy to adjust the spectrum a little.

First consider the case $d = 2$. Take a large time-scale $m$ so that $\int m^{-1} \log \|A^m\| \, d\mu$ is close to $L_1 = L_1(A, \mu)$ (recall the notation from § 2.8). Then fix any sufficiently large $k$ so that the period $n = n_k$ of the orbit that supports $\mu_k$ is much bigger than $m$, and the integral above (and thus $L_1$) is approximated by $\int m^{-1} \log \|A^m\| \, d\mu_k$. The latter is of course the average of $m^{-1} \log \|A^m(T^i y)\|$, where $i$ runs from 0 to $n-1$, and $y$ is a point in the periodic orbit. The enemy here are 'cancellations' that can make the value $n^{-1} \log \|A^n(y)\|$ (and thus the Lyapunov exponent with respect to $\mu_k$) significantly smaller than $L_1$. To fix this, for $y$ in the periodic orbit, let $Z_1(y)$ indicate the average of the function $m^{-1} \log \|A^m\|$ over the points $y$, $T^m(y)$, $T^{2m}(y)$, ..., $T^{(\lfloor n/m \rfloor - 1)m}(y)$. We say that $y$ is *good* if $Z_1(y)$ is close to $L_1$. It is easy to see that a good $y$ exists. Then we multiply the cocycle matrices at the points $y$, $T^m(y)$, ..., $T^{(\lfloor n/m \rfloor - 1)m}(y)$ by small rotations in order to remove cancellations. More precisely, we choose a vector that we want to make expanding and we choose the rotations so that the iterates of this vector at times $m$, $2m$, ..., $(\lfloor n/m \rfloor - 1)m$ do not fall into any contracting cones. At the period, another small rotation makes the eigenvalue comparable to the norm of the iterated vector. Now the top eigenvalue for the periodic orbit is approximately $e^{nL_1}$, as desired. Since rotations do not change the determinants, the other eigenvalue is also ok. (Similar techniques of *avoiding cancellations* are used in [**1**, **18**].)

Before considering the case $d > 2$, we remark that *most* points in the periodic orbit are good. To see this, first notice that there is no $y$ in the periodic orbit for which $Z_1(y)$ is significantly *bigger* than $L_1$, because in that case we would perturb the cocycle and realize this exponent, thus contradicting semicontinuity. Since the average of the $Z_1(y)$ over $y$ is close to $L_1$, we get that $Z_1(y)$ is close to $L_1$ for most $y$.

In the higher dimensional case, we have to look norms of exterior powers to see the other exponents. To avoid cancellations, there are $d - 1$ angles that we need to keep away from zero. This is not hard, but requires some technical lemmas. The key part of the argument is to find a point $y$ that is $i$-good for all intermediate dimensions $i$. Here, a point $y$ is $i$-*good* if the average $Z_i(y)$ of the functions $m^{-1} \log \| \wedge^i A^m \|$ over the points $y$, $T^m(y)$, ..., $T^{(\lfloor n/m \rfloor - 1)m}(y)$ is close to $L_i(A, \mu)$. The remark above also applies: most points are $i$-good. In particular, there is at least one point that is $i$-good for every $i$, and we are able to conclude the proof of Theorem 5.1.

## 5.2. *Some geometric lemmas*

In this section, we establish some lemmas that will be used in the proof of Theorem 5.1.

LEMMA 5.4.    *Given $\alpha > 0$ and $d \geqslant 2$, there exists $C_1 = C_1(\alpha, d) > 1$ with the following properties. If $M : E \to E'$ is a linear map between Euclidean spaces of dimension $d$ and $E = F \oplus G$ is a splitting with $\measuredangle(F, G) > \alpha$, then*

$$\mathrm{jac}\, M \leqslant C_1 (\mathrm{jac}\, M \upharpoonright F)(\mathrm{jac}\, M \upharpoonright G).$$

*Proof.* Let $M : E \to E'$, $F$, $G$ and $\alpha$ be as in the statement. Let $\pi : E \to F^\perp$ and $\pi' : E' \to (MF)^\perp$ be the orthogonal projections along $F$ and $MF$, respectively. Take sets $S_1 \subset F$ and $S_2 \subset G$ of positive volume (in the respective dimensions). Then

$$\operatorname{jac} M = \frac{\operatorname{vol} M(S_1 \oplus S_2)}{\operatorname{vol} S_1 \oplus S_2} = \frac{\operatorname{vol} M(S_1) \cdot \operatorname{vol} \pi'(M(S_2))}{\operatorname{vol} S_1 \cdot \operatorname{vol} \pi(S_2)}.$$

Since $\mathfrak{m}(\pi) \geqslant \sin \alpha$ and $\|\pi'\| \leqslant 1$, the lemma holds with $C_1 = (\sin \alpha)^{-d}$. □

A *flag* on a vector space $E$ of dimension $d$ is a nested sequence of vector subspaces $F_1 \subset \ldots \subset F_{d-1}$ such that $\dim F_i = i$ for each $i$.

LEMMA 5.5.  *Given $\varepsilon > 0$ and $d \geqslant 2$ there exists $\alpha = \alpha(\varepsilon, d) > 0$ with the following properties. For any pair of flags $F_1 \subset \ldots \subset F_{d-1}$ and $G_1 \subset \ldots \subset G_{d-1}$ of a Euclidean space $E$ of dimension $d$, there exists an orthogonal map $R : E \to E$ with $\|R - \operatorname{Id}\| < \varepsilon$ such that $\measuredangle(RF_i, G_{d-i}) > \alpha$.*

*Proof.* Apply [**1**, Claim 6.4] to the spaces $F_i$ and $G_{d-i}^\perp$. □

LEMMA 5.6.  *Given $\varepsilon > 0$ and $d \geqslant 2$ there exists $C_2 = C_2(\varepsilon, d) > 0$ with the following properties. Let $M : E \to E$ be a linear map on an Euclidean space of dimension $d$. Then there exists an orthogonal map $R : E \to E$ with $\|R - \operatorname{Id}\| < \varepsilon$ such that*

$$\mathbf{r}(\wedge^i RM) > C_2^{-1}\|\wedge^i M\| \quad \text{for each } i.$$

The proof of Lemma 5.6 will require a few other lemmas (which will not be used directly in the proof of Theorem 5.1).

LEMMA 5.7.  *For every $d$ and $\beta > 0$, there exists $C_3 = C_3(d, \beta) > 1$ with the following properties. Let $M : E \to E$ be a linear on a space of dimension $d$ and let $v$ be a unit vector with $\|Mv\| = \|M\|$. Assume that $\measuredangle(Mv, v) < \pi/2 - \beta$. Then*

$$\max(\mathbf{r}(M), \mathbf{s}_2(M)) \geqslant C_3^{-1}\|M\|.$$

*Proof.* Take the linear map $M : E \to E$ and the unit vector $v$ with $\|Mv\| = \|v\|$. Let $\theta = \measuredangle(Mv, v)$, and assume that $\theta < \frac{\pi}{2} - \beta$. Write the matrix of $M$ with respect to the splitting $(\mathbb{R}v) \oplus v^\perp$:

$$M = \begin{pmatrix} \pm\|M\|\cos\theta & * \\ * & N \end{pmatrix}.$$

Since $\mathbf{s}_2(M) = \|M \restriction v^\perp\|$, the norm of $N$ is less than $\mathbf{s}_2(M)$. Hence there exists $\varepsilon_0 > 0$ depending only on $d$ and $\beta$ such that if $\mathbf{s}_2(M) \leqslant \varepsilon_0\|M\|$, then

$$|\operatorname{tr} M| \geqslant \frac{\sin\beta}{2}\|M\| \quad \text{and in particular } \mathbf{r}(M) \geqslant \frac{\sin\beta}{2d}\|M\|.$$

So taking $C_3^{-1} = \min(\varepsilon_0, (2d)^{-1}\sin\beta)$, the conclusions of the lemma are satisfied. □

LEMMA 5.8.  *For every $d$ and $\alpha > 0$, there exists $\beta_1 = \beta_1(d, \alpha) > 0$ with the following properties. Let $E$ be a space of dimension $d$, let $i \in \{1, \ldots, d-1\}$, and let $\{v_1, \ldots, v_i\}$ and $\{w_1, \ldots, w_i\}$ be linearly independent subsets of $E$ spanning subspaces $F$ and $G$, respectively.*

Consider the $i$-vectors $v = v_1 \wedge \ldots \wedge v_i$ and $w = w_1 \wedge \ldots \wedge w_i$. Then $\measuredangle(F^\perp, G) > \alpha$ implies $\measuredangle(v, w) < \pi/2 - \beta_1$.

*Proof.*   The non-zero decomposable $i$-vectors $v$ and $w$ uniquely determine the spaces $F$ and $G$. Thus the quantity $\measuredangle(F^\perp, G)$ is a function $f(v, w)$. We can assume that $v$ and $w$ have unit norms, so the domain of $f$ becomes compact. Notice that $f(v, w) = 0$ if and only if $v \perp w$. (This follows from the description of the inner product on $\wedge^i E$ explained before.) By continuity, if $\measuredangle(v, w)$ is sufficiently close to $\pi/2$, then $f(v, w)$ is small. This gives us the desired result.   $\square$

LEMMA 5.9.   *For every $\alpha > 0$ and $d \geqslant 2$, there exists $C_4 = C_4(\alpha, d) > 1$ with the following properties. Let $M : E \to E$ be a linear map on an Euclidean space of dimension $d$. Let $F_1 \subset \ldots \subset F_{d-1}$ be a flag such that for each $i$, the quantity $\mathrm{jac}\, M \restriction F_i$ is as big as possible (that is, $\|\wedge^i M\|$). Assume that*

$$\measuredangle(MF_i, F_i^\perp) > \alpha \quad \text{for each } i.$$

*Then*

$$\mathbf{r}(\wedge^i M) > C_4^{-1} \|\wedge^i M\| \quad \text{for each } i.$$

*Proof.*   Given $d$, we define

$$\bar{\beta} = \min_{0 < i < d} \beta_1\left( \binom{d}{i}, \alpha \right), \quad \bar{C} = \max_{0 < i < d} C_3\left( \binom{d}{i}, \bar{\beta} \right),$$

where the functions $\beta$ and $C_3$ and $\beta$ are given by Lemmas 5.7 and 5.8, respectively.

Take $M : E \to E$ with the flag $F_1 \subset \ldots \subset F_{d-1}$ as in the statement, so $\measuredangle(MF_i, F_i^\perp) > \alpha$. Let $e_1, \ldots, e_d$ be a orthonormal basis of $E$ such that $\{e_1, \ldots, e_i\}$ spans $F_i$. Let $v_i$ be the $i$-vector $e_1 \wedge \ldots \wedge e_i$. By Lemma 5.8, we have

$$\measuredangle(v_i, (\wedge^i M) v_i) < \frac{\pi}{2} - \beta.$$

By Lemma 5.7, this implies that

$$\frac{\max(\mathbf{r}(\wedge^i M), \mathbf{s}_2(\wedge^i M))}{\|\wedge^i M\|} \geqslant \bar{C}^{-1}.$$

Now notice that

$$\frac{\mathbf{s}_2(\wedge^i M)}{\|\wedge^i M\|} = \frac{\mathbf{s}_2(\wedge^i M)}{\mathbf{s}_1(\wedge^i M)} = \frac{\mathbf{s}_1(M) \ldots \mathbf{s}_{i-1}(M) \mathbf{s}_{i+1}(M)}{\mathbf{s}_1(M) \ldots \mathbf{s}_{i-1}(M) \mathbf{s}_i(M)} = \frac{\mathbf{s}_{i+1}(M)}{\mathbf{s}_i(M)}.$$

Thus, for each $i$,

$$\mathbf{r}(\wedge^i M) \geqslant \bar{C}^{-1} \|\wedge^i M\| \text{ or } \mathbf{s}_{i+1}(M) \geqslant \bar{C}^{-1} \mathbf{s}_i(M). \tag{5.1}$$

Let us say that $i$ is of the first type if the first alternative holds, and of second type instead. The $i$'s of first type are already controlled. We need a convexity argument to control the $i$'s of the second type.

Define numbers $x_i = \log \mathbf{r}(\wedge^i M)$ and $y_i = \log \|\wedge^i M\|$ for $1 \leqslant i \leqslant d$, and also $x_0 = y_0$. Then the graphs of the functions $x_i$ and $y_i$ over $\{0, 1, \ldots, d\}$, have the same endpoints (in particular 0 and $d$ are of first type), and $x_i \leqslant y_i$. Using the difference operator notation from § 2.6, concavity means that $\Delta^2 x_i$ and $\Delta^2 y_i$ are non-positive. Notice that $\Delta[\log \mathbf{s}_i(M)] = y_{i-1}$. Letting $\gamma = \log \bar{C}$, (5.1) translates to

$$\text{for each } i, \quad x_i \geqslant y_i - \gamma \text{ or } \Delta^2 y_{i-1} \geqslant -\gamma.$$

Let $0 \leqslant i_0 < i_1 \leqslant d$ be any two consecutive indices of first type. Then every $i \in (i_0, i_1)$ is of first type and so $\Delta^2 y_{i-1} \geqslant -\gamma$. It follows from Lemma 2.5 applied to $-y_i$ that

$$0 \leqslant y_i - \left( \frac{i_1 - i}{i_1 - i_0} y_{i_0} + \frac{i - i_0}{i_1 - i_0} y_{i_1} \right) \leqslant \frac{(i_1 - i_0)^2}{4} \gamma. \tag{5.2}$$

So for all $i \in (i_0, i_1)$, we have

$$
\begin{aligned}
x_i &\geqslant \frac{i_1 - i}{i_1 - i_0} x_{i_0} + \frac{i - i_0}{i_1 - i_0} x_{i_1} \quad \text{(by concavity)} \\
&\geqslant \frac{i_1 - i}{i_1 - i_0} (y_{i_0} - \gamma) + \frac{i - i_0}{i_1 - i_0} (y_{i_1} - \gamma) \quad \text{(since } i_0 \text{ and } i_1 \text{ are of first type)} \\
&\geqslant y_i - \left( 1 + \frac{(i_1 - i_0)^2}{4} \right) \gamma \quad \text{(by (5.2))}.
\end{aligned}
$$

Therefore, the lemma holds with $C_4 = \bar{C}^{1 + d^2/4}$. $\qquad\qquad\square$

*Proof of Lemma 5.6.* Given $\varepsilon$ and $d$, let $\alpha = \alpha(\varepsilon, d)$ be given by Lemma 5.5, and let $C_2 = C_4(\alpha, d)$ be given by Lemma 5.9. Now, given a linear map $M$ of a space of dimension $d$, let $F_1 \subset \ldots \subset F_{d-1}$ be a flag such that $\mathrm{jac}\, M \restriction F_i$ is as big as possible. By Lemma 5.5, there is an orthogonal map $R$ with $\|R - \mathrm{Id}\| < \varepsilon$ such that $\measuredangle(RMF_i, F_i^\perp) > \alpha$ for each $i$. Then, by Lemma 5.9,

$$\mathbf{r}(\wedge^i RM) > C_2^{-1} \|\wedge^i RM\| = C_2^{-1} \|\wedge^i M\|. \qquad\qquad\square$$

### 5.3. *Proofs of Theorems 5.1 and 5.3*

Let us begin with a simple extension result.

LEMMA 5.10. *Given* $d \geqslant 2$, $K > 1$ *and* $\varepsilon > 0$, *there exists* $\varepsilon' = \varepsilon'(d, K, \varepsilon) > 0$ *with the following property. Let* $(X, T, E, A)$ *be a cocycle bounded by* $K$. *Assume* $X_0 \subset X$ *is a finite subset, and for each* $x \in X_0$, *it is given a linear map* $B_0(x) : E(x) \to E(Tx)$ *with* $\|B_0(x) - A(x)\| < \varepsilon'$. *Then there exists a (continuous) linear cocycle* $B$ *that is* $\varepsilon$-*close to* $A$ *such that* $B(x) = B_0(x)$ *for each* $x \in X_0$.

*Proof.* Use Tietze extension theorem. $\qquad\qquad\square$

*Proof of Theorem 5.1.* Let $(X, T, E, A)$ be a cocycle. Assume that $\mu_k$ is a sequence of invariant probability measures converging to some $\mu$, with each $\mu_k$ supported on a periodic orbit of period $n_k$. We also assume that $n_k \to \infty$ as $k \to \infty$.

First, let us notice that it is sufficient to prove that *there exists a sequence of cocycles* $\tilde{A}_k$ *converging to* $A$ *such that* $\boldsymbol{\sigma}(A_k, \mu_k)$ *converges to* $\boldsymbol{\sigma}(A, \mu)$. Indeed, if $n_k$ is large enough then by Proposition 4.3 (and Lemma 5.10), we can perturb $\tilde{A}_k$ without changing $\boldsymbol{\sigma}(\tilde{A}_k, \mu_k)$ so that its restriction to $\mathrm{supp}\,\mu_k$ has only real eigenvalues; then using Lemma 4.6, we find another perturbation $B_k$ so that $\boldsymbol{\sigma}(B_k, \mu_k) = \boldsymbol{\sigma}(A, \mu)$.

Now, the assertion above is equivalent to the following: *for every* $\delta > 0$ *and every sufficiently large* $k$ *(depending on* $\delta$*), there exists a* $\delta$-*perturbation* $\tilde{A}$ *of* $A$ *such that the graphs* $\boldsymbol{\sigma}(\tilde{A}, \mu_k)$ *and* $\boldsymbol{\sigma}(A, \mu)$ *are* $\delta$-*close.* So let us prove this assertion instead.

Fix $\delta > 0$. By semicontinuity, there exists a positive $\varepsilon < \delta$ such that

$$L_i(B, \mu_k) < L_i(A, \mu) + \frac{\delta}{4d} \quad \text{for all } i, \text{ all } k > \varepsilon^{-1} \text{ and all } B \ \varepsilon\text{-close to } A. \tag{5.3}$$

Let $K > 1$ be a bound for all cocycles that are $\varepsilon$-close to $A$. Let $\varepsilon' = \varepsilon'(d, K, \varepsilon)$ be given by Lemma 5.10, and $\varepsilon'' = K^{-1}\varepsilon'$. Let $\alpha = \alpha(\varepsilon'', d)$ be given by Lemma 5.5. Let $C_1 = C_1(\alpha, d)$ and $C_2 = C_2(\varepsilon'', d)$ be given by Lemmas 5.4 and 5.6, respectively.

Now let $\eta > 0$ be small; the precise requirements will appear later. Fix an integer $m$ such that

$$m > \eta^{-1} \quad \text{and} \quad L_i(A, \mu) \leqslant \int \frac{1}{m} \log \|\wedge^i A^m\| \, d\mu < L_i(A, \mu) + \eta \quad \forall i.$$

If $k$ is sufficiently large, then

$$\frac{m}{n_k} < \eta \quad \text{and} \quad \left| \int \frac{1}{m} \log \|\wedge^i A^m\| \, d\mu_k - \int \frac{1}{m} \log \|\wedge^i A^m\| \, d\mu \right| < \eta.$$

Fix any $k > \varepsilon^{-1}$ with the properties above. Write $n = n_k$ and let $q = \lfloor n/m \rfloor$. For each $y \in \operatorname{supp} \mu_k$, define

$$Z_i(y) = \frac{1}{n} \sum_{p=0}^{q-1} \log \|\wedge^i A^m (T^{pm} y)\|.$$

We claim that for each $y \in \operatorname{supp} \mu_k$, there is a $\varepsilon$-perturbation $\tilde{A}_y$ of $A$ such that

$$L_i(\tilde{A}_y, \mu_k) > Z_i(y) - C_3 \eta \quad \text{for each } i, \tag{5.4}$$

where $C_3 > 0$ does not depend on $\eta$.

(At the end, the perturbation $\tilde{A}$ that we are looking for will be $\tilde{A}_y$ for an appropriate choice of $y$, but we cannot say a priori which $y$ works.)

*Proof of the claim.* Fix $y \in \operatorname{supp} \mu_k$. For each $p = 0, 1, \ldots, q-1$, we will define two flags of $E(T^{pm} y)$,

$$F_1^{(p)} \subset \ldots \subset F_{d-1}^{(p)} \quad \text{and} \quad G_1^{(p)} \subset \ldots \subset G_{d-1}^{(p)}.$$

The first family of flags is chosen so that $\operatorname{jac} A^m (T^{pm} y) \upharpoonright F_i^{(p)}$ is as small as possible, that is, $\mathfrak{m}(\wedge^i A^m (T^{pm} y))$. The second family of flags is defined recursively. Let $G_i^{(0)} = [F_{d-i}^{(0)}]^\perp$. Once the $(p-1)$th flag is defined ($p > 0$), we apply Lemma 5.5 to choose an orthogonal map $R^{(p)}$ of $E(T^{pm} y)$ with $\|R^{(p)} - \operatorname{Id}\| < \varepsilon''$ such that defining

$$G_i^{(p)} = R^{(p)} A^m (T^{(p-1)m} y) G_i^{(p-1)},$$

we have

$$\angle(G_i^{(p)}, F_{d-i}^{(p)}) > \alpha.$$

By Lemma 5.4, the lower bounds on the angles between the flags imply that, for each $p = 0, \ldots, q-1$,

$$\operatorname{jac} A^m (T^{pm} y) \upharpoonright G_i^{(p)} \geqslant C_1^{-1} \frac{\operatorname{jac} A^m (T^{pm} y)}{\operatorname{jac} A^m (T^{pm} y) \upharpoonright F_{d-i}^{(p)}} = C_1^{-1} \|\wedge^i A^m (T^{pm} y)\|.$$

Consider the linear map $M : E(y) \to E(y)$ given by

$$M = A^{n-mq}(T^{qm} y) R^{(q)} A^m (T^{(q-1)m} y) \ldots R^{(2)} A^m (T^m y) R^{(1)} A^m (y).$$

Then

$$\|\wedge^i M\| \geqslant \operatorname{jac} M \upharpoonright G_i^{(0)} \geqslant K^{-i(n-mq)} C_1^{-q} \prod_{p=0}^{q-1} \|\wedge^i A^m (T^{pm} y)\|$$

$$\geqslant K^{-dm} C_1^{-2n/m} e^{n Z_i(y)}.$$

By Lemma 5.6, there exists an orthogonal map $R^{(0)}$ of $E(y)$ such that $\|R^{(0)} - \mathrm{Id}\| < \varepsilon''$ and $\mathbf{r}(\wedge^i R^{(0)} M) \geqslant C_2^{-1} \|\wedge^i M\|$ for each $i$. Let $\tilde{A}_y$ be a continuous cocycle $\varepsilon$-close to $A$ that equals $A$ along the orbit of $y$ except at the points specified below:

$$\tilde{A}_y(y) = A(y)R^{(0)}, \quad \tilde{A}_y(T^{pm-1}y) = R^{(p)}A^m(T^{pm-1}y) \text{ for } p \in \{1, \dots, q\}.$$

Then

$$L_i(\tilde{A}_y, \mu_k) = \frac{1}{n} \log \mathbf{r}(\wedge^i \tilde{A}_y^n(y)) = \frac{1}{n} \log \mathbf{r}(\wedge^i M R^{(0)}) \geqslant Z_i(y) - C_3\eta,$$

where $C_3 = \log(K^d C_1^2 C_2)$. That is, $\tilde{A}_y$ has property (5.4).   □

It follows from (5.4) and (5.3) that

$$Z_i(y) < L_i(A, \mu) + C_3\eta + (4d)^{-1}\delta \quad \text{for all } y \in \operatorname{supp}\mu_k. \tag{5.5}$$

Fix any $y_0 \in \operatorname{supp}\mu_k$, and denote $y_\ell = T^\ell y_0$ for $0 \leqslant \ell < m$. Then

$$\frac{1}{m}\sum_{\ell=0}^{m-1} Z_i(y_\ell) = \frac{1}{mn}\sum_{j=0}^{qm-1} \log \|\wedge^i A^m(T^j y_0)\|$$

$$\geqslant \int \frac{1}{m} \log \|\wedge^i A^m\| \, d\mu_k - \frac{\log K^{im}}{n}$$

$$\geqslant L_i(A, \mu) - C_4\eta, \tag{5.6}$$

where $C_4 = 2 + d \log K$. Using (5.5) and (5.6), we will show that for each $i$, the number $Z_i(y_\ell)$ is close to $L_i(A, \mu)$ for 'most' $\ell \in \{0, \dots, m-1\}$, and, in particular, we can find some $\ell$ that works for every $i$.

More precisely, let

$$\rho_i = \frac{1}{m}\#\{\ell \in \{0, \dots, m-1\};\ Z_i(y_\ell) < L_i(A, \mu) - \delta/2\}.$$

Then it follows from (5.6) and (5.5) that

$$L_i(A, \mu) \leqslant C_4\eta + \frac{1}{m}\sum_{\ell=0}^{m-1} Z_i(y_\ell)$$

$$\leqslant C_4\eta + \rho_i\left(L_i(A, \mu) - \frac{\delta}{2}\right) + (1 - \rho_i)\left(L_i(A, \mu) + C_3\eta + \frac{\delta}{4d}\right)$$

$$\leqslant L_i(A, \mu) + \left(\frac{1}{4d} - \frac{\rho_i}{2}\right)\delta + (C_3 + C_4)\eta$$

$$< L_i(A, \mu) + \left(\frac{1}{2d} - \frac{\rho_i}{2}\right)\delta, \quad \text{provided we choose } \eta \text{ small enough.}$$

So $\rho_i < 1/d$ and in particular $\sum_{i=1}^d \rho_i < 1$. Thus, there is some $\ell \in \{0, \dots, m-1\}$ such that $Z_i(y_\ell) \geqslant L_i(A, \mu) + \delta/2$ for all $i \in \{1, \dots, d\}$. Now let $\tilde{A} = \tilde{A}_{y_\ell}$; this is a $\delta$-perturbation of $A$ such that for each $i$ we have

$$L_i(\tilde{A}, \mu_k) > Z_i(y_\ell) - C_3\eta > L_i(A, \mu) - \delta$$

(again because $\eta$ is small). By (5.3), $|L_i(\tilde{A}, \mu_k) - L_i(A, \mu)| < \delta$, as we wanted.   □

*Proof of Theorem* 5.3.   Let $(X, T, E, A)$, $\mu$, $\mu_k$, $\Lambda$, $F_i$, $i_j$ and $\sigma$ be as in the statement of the theorem. Let $n_k$ be the period of the orbit that supports $\mu_k$. By assumption, $n_k \to \infty$.

Given a small $\varepsilon > 0$, let $K > 1$ be a bound for all $\varepsilon$-perturbations of $A$. Let $\varepsilon' = \varepsilon'(d, K, \varepsilon/2)$ be given by Lemma 5.10. Let $L = L(d, K, \varepsilon')$ be given by Theorem 4.12. Let $\eta = \eta(d, K, L)$ be given by Lemma 4.7. We can assume $\eta < \varepsilon/2$.

There is $k_0$ such that if $k > k_0$ then $n_k > 2L$ and the restriction of $A$ to supp $\mu_k$ has no $2L$-dominated splittings of indices $i_j$. Using Theorem 5.1, find some $k > k_0$ such that such that $\boldsymbol{\sigma}(\tilde{A}, \mu_k) = \boldsymbol{\sigma}(A, \mu)$ for some $\eta$-perturbation $\tilde{A}$ of $A$. The restriction of $\tilde{A}$ to supp $\mu_k$ has no $L$-dominated splitting of index $i_j$. By Theorem 4.12, there is an $\varepsilon'$-perturbation of $\tilde{A}$ along the orbit that supports $\mu_k$ such that the Lyapunov graph becomes exactly $\sigma$. Using Lemma 5.10 we extend this to a global $\varepsilon/2$-perturbation of $\tilde{A}$, which is the desired $\varepsilon$-perturbation of $A$. $\qquad\square$

## 6.   Immediate applications

Here we show Corollaries 1.4 and 1.7.

### 6.1.   Changing the index of a periodic point

*Proof of Corollary 1.4.*   Consider a sequence $\gamma_n = \mathrm{orb}(p_n)$ of periodic orbits whose period tends to infinity and let $\mu_n$ be the corresponding measures. Assume that $(\mu_n, \gamma_n)$ converges (in the weak-star times Hausdorff topology) to a pair $(\mu, \Lambda)$. Let $i_j = \dim(E_1 \oplus \ldots \oplus E_j)$ for $j \in \{0, \ldots, m\}$, where $E_1 \oplus E_2 \oplus \ldots \oplus E_m$ is the finest dominated splitting over $\Lambda$. Domination implies strict convexity of the graph $\boldsymbol{\sigma}(\mu)$ at the points $i_j$; more precisely, for each $j \in \{1, \ldots, m-1\}$ we have

$$\lambda_{i_j}(\mu) < \lambda_{i_j+1}(\mu), \quad \text{that is, } \boldsymbol{\sigma}_{i_j}(\mu) < \frac{\boldsymbol{\sigma}_{i_j-1}(\mu) + \boldsymbol{\sigma}_{i_j+1}(\mu)}{2}.$$

Let

$$s = \min_{j \in \{0, \ldots, m\}} \boldsymbol{\sigma}_{i_j}(\mu) \quad \text{and} \quad K = \{k \in \{0, \ldots, d\}; \ \boldsymbol{\sigma}_k(\mu) \leqslant s\}.$$

By the convexity properties, $K$ is an interval, and there is a unique $j \in \{1, \ldots, m\}$ such that $K \subset \{i_{j-1}, \ldots, i_j\}$.

Now fix $k \in K$ and define a convex graph $\sigma = (\sigma_0, \ldots, \sigma_d)$ as follows:

$$\sigma_i = \begin{cases} \boldsymbol{\sigma}_i(\mu) & \text{if } i \leqslant i_{j-1} \text{ or } i \geqslant i_j, \\[2mm] \dfrac{k-i}{k-i_{j-1}} \cdot \boldsymbol{\sigma}_{i_{j-1}}(\mu) + \dfrac{i-i_{j-1}}{k-i_{j-1}} \cdot \boldsymbol{\sigma}_k(\mu) & \text{if } i_{j-1} < i \leqslant k, \\[2mm] \dfrac{i_j-i}{i_j-k} \cdot \boldsymbol{\sigma}_k(\mu) + \dfrac{i-k}{i_j-k} \cdot \boldsymbol{\sigma}_{i_j}(\mu) & \text{if } k \leqslant i < i_j. \end{cases}$$

Then $\sigma$ is above $\boldsymbol{\sigma}(\mu)$ and is compatible with the dominated splitting on $\Lambda$. That is, $\sigma$ belongs to the set $\mathcal{G}(\mu, \Lambda)$ (defined in §1.1.4). By Theorem 4, if $n$ is sufficiently large then there is a perturbation $g$ of $f$ preserving the orbit $\gamma_n$ and such that the Lyapunov graph $\boldsymbol{\sigma}(\gamma_n, g)$ is equal to $\sigma$. If $\boldsymbol{\sigma}_k(\mu, f)$ is strictly smaller than $s$, then the index of $\gamma_n$ for $g$ is precisely $k$, concluding the proof in that case. In the remaining case, some of the Lyapunov exponents of $\gamma_n$ for $g$ are zero. Then an arbitrarily small perturbation of $g$ along the orbit of $\gamma_n$ (using Lemma 4.6 and Proposition 4.3) allows us to change these vanishing exponents in order to get any prescribed signs, thus concluding the proof. $\qquad\square$

### 6.2.   Lyapunov spectra of periodic orbits for generic diffeomorphisms

Let us rephrase Corollary 1.7. If $f$ is a diffeomorphism of the compact $d$-dimensional manifold $M$, then $\mathcal{Z}(f)$ indicates the closure in $\mathcal{P}(M) \times \mathcal{K}(M) \times \mathbb{R}^{d+1}$ of the set of triples $(\mu_\gamma, \gamma, \boldsymbol{\sigma}(f, \gamma))$, where $\gamma$ runs on all hyperbolic periodic orbits of $f$. (Here we write

$\boldsymbol{\sigma}(f, \gamma) = \boldsymbol{\sigma}(f, \mu_\gamma)$ for simplicity.) Then Corollary 1.7 states that for generic $f$ we have

$$\mathcal{Z}(f) = \bigcup_{(\mu, \Lambda) \in \mathcal{X}(f)} \{(\mu, \Lambda)\} \times \mathcal{G}(\mu, \Lambda). \tag{6.1}$$

Now we prove it.

*Proof.* The '$\subset$' inclusion is the easy one and works for every $f$. By semicontinuity of the Lyapunov graph (see § 2.8),

$$(\mu_{\gamma_n}, \gamma_n, \boldsymbol{\sigma}(f, \gamma_n)) \to (\mu, \Lambda, \sigma) \quad \Rightarrow \quad \sigma \geqslant \boldsymbol{\sigma}(f, \mu).$$

Moreover, by persistence and continuity of dominated splittings, for each $i \in \{1, \dots, d-1\}$ such that there is a dominated splitting $T_\Lambda M = E \oplus F$ with $\dim E = i$, we must have $\sigma_i = \boldsymbol{\sigma}_i(f, \mu)$. That is, we have $\sigma \in \mathcal{G}(\mu, \Lambda)$, proving one inclusion in (6.1).

CLAIM.    The map $f \mapsto \mathcal{Z}(f)$ is lower semicontinuous.

*Proof.* The set $\mathcal{Z}(f)$ may be approached from inside by a finite set of triples $(\mu_\gamma, \gamma, \boldsymbol{\sigma}(f, \gamma))$, where $\gamma$'s are hyperbolic periodic orbits. Each hyperbolic periodic orbit of $f$ persists and varies continuously in a small neighbourhood of $f$, and its Lyapunov graph varies continuously on this neighbourhood. Thus, the finite set of triples varies continuously on that neighbourhood, giving the lower semicontinuity. $\square$

It follows from a well-known result from general topology that the points of continuity of the map $f \mapsto \mathcal{Z}(f)$ form a residual subset $\mathcal{R}$ of $\mathrm{Diff}^1(M)$. Fix any $f \in \mathcal{R}$; we now claim that (6.1) holds for $f$. Take $(\mu, \Lambda) \in \mathcal{X}(f)$ and $\sigma \in \mathcal{G}(\mu, \Lambda)$. Let us show that $(\mu, \Lambda, \sigma) \in \mathcal{Z}(f)$. By definition of $\mathcal{X}(f)$, there exists a sequence of hyperbolic periodic orbits $\gamma_n$ such that $(\mu_{\gamma_n}, \gamma_n) \to (\mu, \Lambda)$. By Theorem 4, there are diffeomorphisms $g_n$ preserving, respectively, the orbits $\gamma_n$ such that $g_n \to f$ and $\boldsymbol{\sigma}(g_n, \gamma_n) \to \sigma$. In addition, we can assume that $\gamma_n$ is hyperbolic with respect to $g_n$. Since $f \in \mathcal{R}$, the sequence of sets $\mathcal{Z}(g_n)$ converges to $\mathcal{Z}(f)$ in the Hausdorff topology. Each element of the sequence $(\mu_{\gamma_n}, \gamma_n, \boldsymbol{\sigma}(g_n, \gamma_n))$ belongs to the respective $\mathcal{Z}(g_n)$ and therefore the limit of the sequence, which is $(\mu, \Lambda, \sigma)$, belongs to $\mathcal{Z}(f)$. Thus (6.1) is true for any $f \in \mathcal{R}$, concluding the proof of Corollary 1.7. $\square$

## 7.    *Consequences to universal dynamics*

Here, we will give the applications to universal dynamics explained in § 1.2.3.

### 7.1.    *Proof of Theorem 5*

We will need a number of lemmas.

The following lemma is useful when we want to apply Theorem 4 to homoclinic classes. Recall that $\mu_\gamma$ indicates the unique invariant probability measure supported on a periodic orbit $\gamma$.

LEMMA 7.1.    *Let $H$ be the homoclinic class of a periodic orbit $\gamma$. Then there is a sequence of periodic orbits $\gamma_n$ homoclinically related to $\gamma$ such that:*

   (1)  *the sets $\gamma_n$ converge to $H$ in the Hausdorff topology;*
   (2)  *the measures $\mu_{\gamma_n}$ converge to the measure $\mu_\gamma$ in the weak-star topology.*

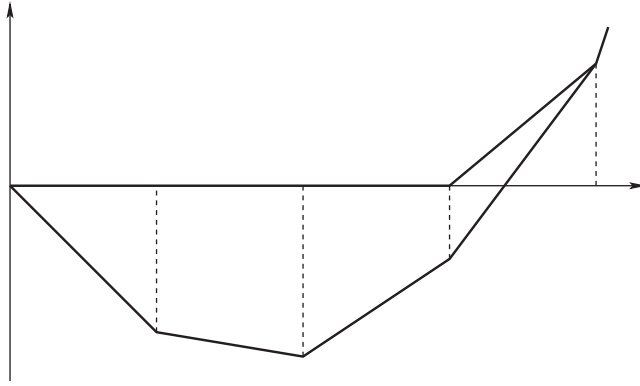In the notation of § 1.2.2, the lemma says that $(\mu_\gamma, H) \in \mathcal{X}(f)$.

FIGURE 7. *An example in the situation of Lemma 7.2 with $K = 4$ and $k = 3$. The lower graph is $\boldsymbol{\sigma}(f, \mu)$ and the upper one is $\boldsymbol{\sigma}(g_n, \gamma_n)$.*

*Proof.* Using Markov partitions, we see that the lemma holds true in the case that $H$ is a horseshoe (that is, a locally maximal hyperbolic set). In the general case, we can take a sequence of horseshoes $H_n$ contained in the homoclinic class $H$ and containing $\gamma$ such that $H_n \to H$ in the Hausdorff topology. Then the lemma follows immediately from the previous case.  □

LEMMA 7.2 (Creating zero exponents).  *Let $f \in \mathrm{Diff}^1(M)$. Let $\gamma_n$ be a sequence of periodic orbits whose periods tend to infinity. Suppose that the invariant probabilities $\mu_{\gamma_n}$ converge in the weak-star topology to some $\mu$ and that the sets $\gamma_n$ converge in the Hausdorff topology to an $f$-invariant compact set $\Lambda$. Let $E_1 \oplus \ldots \oplus E_m$ be the finest dominated splitting over $\Lambda$. Assume that*

$$\boldsymbol{\sigma}_k(\mu) \leqslant 0 \leqslant \boldsymbol{\sigma}_K(\mu), \quad \text{where } k < K = \dim E_1.$$

*Let also $r_n$ be a sequence of positive numbers. Then there is a sequence of diffeomorphisms $g_n$ converging to $f$ such that for each $n$:*

(1) *$\gamma_n$ is a periodic orbit of $g_n$, which equals $f$ outside the $r_n$-neighbourhood of $\gamma_n$;*
(2) *under $g_n$, the orbit $\gamma_n$ has exactly $k$ vanishing Lyapunov exponents and $d - k$ positive Lyapunov exponents (see Figure 7).*

*Proof.* By Franks Lemma, it is sufficient to find out how to perturb the derivatives along the periodic orbits. Define a graph $\sigma = (\sigma_0, \ldots, \sigma_d)$ by

$$\sigma_i = \begin{cases} 0 & \text{if } i \leqslant k, \\ \dfrac{i-k}{K-k} \boldsymbol{\sigma}_K(f, \mu) & \text{if } k \leqslant i \leqslant K, \\ \boldsymbol{\sigma}_i(f, \mu) & \text{if } i \geqslant K. \end{cases}$$

Then $\sigma$ belongs to the set $\mathcal{G}(\mu, \Lambda)$ (defined in §1.1.4). By applying Theorem 4, we find a perturbation $g$ of $f$ preserving some $\gamma = \gamma_n$ so that $\boldsymbol{\sigma}(g, \gamma) = \sigma$. If $\boldsymbol{\sigma}_K(f, \mu) > 0$, then we are done: $\gamma$ has exactly $k$ vanishing Lyapunov exponents and $d - k$ positive Lyapunov exponents under $g$. If $\boldsymbol{\sigma}_K(f, \mu) = 0$ then $\gamma$ has exactly $K$ vanishing Lyapunov exponents and $d - K$ positive Lyapunov exponents under $g$ (because, by domination, $\boldsymbol{\sigma}_{K+1}(f, \mu) > 0$). Then (using Lemma 4.6 and Proposition 4.3) we make another perturbation to make $K - k$ of these exponents positive.  □

LEMMA 7.3 (From zero exponents to identity map). *Let $f$ be a diffeomorphism with a periodic orbit $\gamma$ of period $\pi(\gamma)$. Assume $\gamma$ has exactly $k$ vanishing Lyapunov exponents. Then, for any $C^1$-neighbourhood $\mathcal{U}$ of $f$ and any neighbourhood $V$ of $\gamma$, there exists a diffeomorphism $g \in \mathcal{U}$ that equals $f$ outside $V$, and there exists a $g$-periodic orbit $\tilde{\gamma}$ of period $\pi(\tilde{\gamma}) \geqslant \pi(\gamma)$ contained in $V$, such that the following properties hold:*

*(1) For any $q \in \tilde{\gamma}$, there is a subspace $E \subset T_q M$ of dimension $k$ such that the restriction of $Dg^{\pi(\tilde{\gamma})}$ to $E$ is the identity map.*
*(2) The orbits $\gamma$ and $\tilde{\gamma}$ have the same Lyapunov spectra , and thus the same Lyapunov graphs: $\boldsymbol{\sigma}(f, \gamma) = \boldsymbol{\sigma}(g, \tilde{\gamma})$.*

*Proof.* Take local coordinates around each point in $\gamma$. Up to performing an arbitrarily $C^1$ small perturbation of $f$, one may assume that it is linear in those coordinates. Fix some $p \in \gamma$ and let $E$ denote the subspace, in the coordinates around $p$, corresponding to the vanishing Lyapunov exponents. So $E$ is a locally $f$-invariant submanifold, and the restriction of $f^{\pi(p)}$ to $E$ is a linear map whose Lyapunov exponents are all zero. Now consider the following.

CLAIM. Let $L : \mathbb{R}^k \to \mathbb{R}^k$ be a linear map whose eigenvalues lie in the unit circle. Then there is an arbitrarily small perturbation $\tilde{L}$ of $L$ that is diagonalizable over $\mathbb{C}$ and has all its eigenvalues in the unit circle.

Assume the Claim for a while and take the perturbation $\tilde{L} : E \to E$. Notice that a power $\tilde{L}^n$ of it is the identity; fix the least as $n \geqslant 1$. By a procedure from §2.2, we can extend $\tilde{L}$ to a linear map $\hat{L}$ on the whole space whose Lyapunov spectrum is the same as $Df^{\pi(p)}(p)$. We can find a locally linear $C^1$-perturbation $g$ of $f$ such that the maps $g^{\pi(p)}$ and $\hat{L}$ coincide in a neighbourhood of $p$. Now we take any point $q$ on $E$ close to $p$ that has period $n$ under $\tilde{L}$, and take $\tilde{\gamma}$ as $g$-orbit of $q$. This shows the lemma, modulo proving the Claim.

The proof of the Claim is by induction on the dimension $k$. The case $k = 1$ is trivial: there is no need to perturb $L$. Next consider the case $k = 2$. If $L$ is diagonalizable, but its eigenvalues are not roots of unity, then we can perturb $L$ to make them so. On the other hand, if $L$ is not diagonalizable, then either $L$ or $-L$ has Jordan form

$$\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}.$$

Perturb this to

$$\begin{pmatrix} 1 - \varepsilon & 1 \\ -\varepsilon & 1 \end{pmatrix}$$

for a suitably chosen small $\varepsilon > 0$, the eigenvalues are non-real roots of unity, and we are done.

Now take $k \geqslant 3$ and assume that the Claim has been proved for every dimension between 1 and $k - 1$. Let $L : \mathbb{R}^k \to \mathbb{R}^k$ have all its eigenvalues in the unit circle. First assume that $L$ has an invariant splitting $\mathbb{R}^k = F \oplus G$ into non-zero bundles. Then we apply the induction hypothesis to the restrictions of $L$ to each subspace $F$ and $G$, and take $\tilde{L}$ as the product perturbation. In the remaining case where no such invariant splitting exists, the real Jordan normal form of $L$ has a single block. Take the invariant subspace $F$ of dimension 2 corresponding to the upper left corner of the Jordan matrix. Then (repeating a previous reasoning) we can perturb $L \restriction F$ to make its eigenvalues non-real of modulus 1, and different from the eigenvalues of $L/F$. By a procedure from §2.2, we can extend the perturbation on $F$ to a perturbation on the whole space, without altering eigenvalues. The new linear map has an invariant splitting into non-zero bundles, and we are reduced to the previous case. This concludes the proof of the Claim and hence of the lemma. $\square$

LEMMA 7.4 (From identity to any map).   *Let $f$ be a diffeomorphism with a periodic point $p$ of period $\pi(p)$. Assume that there is a subspace $E \subset T_pM$ of dimension $k$ such that the restriction of $Df^{\pi(p)}$ to $E$ is the identity map. Then, for any $C^1$-neighbourhood $\mathcal{U}$ of $f$, any neighbourhood $V$ of the orbit of $p$ and any $\phi \in \mathrm{Diff}^1_\#(\mathbb{D}^k)$, there exists a diffeomorphism $g \in \mathcal{U}$ that equals $f$ outside $V$, an embedded $k$-disc $D$, and $\pi \geqslant \pi(p)$ such that:*

(1) *$D, g(D), \ldots, g^{\pi-1}(D)$ are pairwise disjoint, and $g^\pi(D)$ is contained in the (relative) interior of $D$;*
(2) *$D \cup g(D) \cup \ldots \cup g^{\pi-1}(D) \subset V$;*
(3) *$D$ is normally hyperbolic for $g^\pi$;*
(4) *the restriction of $g^\pi$ to $D$ is differentiably conjugate to $\phi$.*

*Proof.*   This is basically a reformulation of [**8**, Proposition 3.1].                □

LEMMA 7.5 (Stably finest dominated splittings).   *Let $\mathcal{V}$ be a $C^1$-open set of diffeomorphisms $f$ having a hyperbolic periodic point $p_f$ varying continuously with $f$. Let $E_{1,f} \oplus_< \ldots \oplus_< E_{m_f,f}$ indicate the finest dominated splitting over the chain recurrence class $C(p_f)$, for each $f \in \mathcal{V}$. Then there is an open and dense subset $\mathcal{U} \subset \mathcal{V}$ where the functions $f \mapsto m_f$ and $f \mapsto \dim E_{i,f}$ are locally constant.*

*Proof.*   As a consequence of Conley theorem, the map $f \in \mathcal{V} \mapsto C(p_f)$ is upper semicontinuous. So, given any dominated splitting on $C(p_f)$, for every $g$ sufficiently close to $f$, the class $C(p_g)$ has a dominated splitting with the same number of bundles. Hence the number of bundles $m_f$ in the finest dominated splitting on $C(p_f)$ is a lower semicontinuous function of $f$. So this number is locally constant on a dense open subset $\mathcal{U}$ of $\mathcal{V}$. In this set, the dimensions of the bundles are also locally constant.                □

The following lemma is a version of Theorem 5 for individual periodic orbits.

LEMMA 7.6.   *Let $\mathcal{U}$ be a $C^1$-open set of diffeomorphisms $f$ having a hyperbolic periodic point $p_f$ of index $k$, varying continuously with $f$, so that the dimensions of the bundles of the finest dominated splitting $E_1 \oplus_< \ldots \oplus_< E_m$ over the chain recurrence class $C(p_f)$ do not depend on $f \in \mathcal{U}$.*
*Assume that :*

$$|\det Df^{\pi(p_f)} \restriction E_1(p_f)| > 1, \quad \text{for each } f \in \mathcal{U}. \tag{7.1}$$

*Then there is a residual subset $\mathcal{R}$ of $\mathcal{U}$ such that every $f \in \mathcal{R}$ has normally expanding $k$-universal dynamics.*

*Proof.*   If $n \in \mathbb{N}$ and $\mathcal{O}$ is an open non-empty subset of $\mathrm{Diff}^1_\#(\mathbb{D}^k)$, then $\mathcal{V}(n, \mathcal{O})$ indicates the set of $f \in \mathcal{U}$ such that there is an embedded closed $k$-disc $D$ and $\pi \in \mathbb{N}$ such that:

(i) *$D, f(D), \ldots, f^{\pi-1}(D)$ are pairwise disjoint, and $f^\pi(D)$ is contained in the (relative) interior of $D$;*
(ii) *$D$ is normally expanding for $f^\pi$;*
(iii) *the restriction of $f^\pi$ to $D$ is differentiably conjugate to a map in $\mathcal{O}$;*
(iv) *$D$ is contained in the $1/n$-neighbourhood of $p_f$;*
(v) *$p_f \notin \bigcup_{j=0}^{\pi-1} f^j(D)$.*

We claim that $\mathcal{V}(n, \mathcal{O})$ is open and dense in $\mathcal{U}$. Openness is obvious. To show denseness, take any $f \in \mathcal{U}$, and successively perturb it as follows.

(i) For every diffeomorphism in a residual subset $\mathcal{R}_0 \subset \mathrm{Diff}^1(M)$, the homoclinic classes are chain recurrence classes. Perturb $f$ so that $f \in \mathcal{R}_0$ and thus $H(p_f) = C(p_f)$. Let $\mu$ be the invariant probability measure on the orbit of $p_f$. By Lemma 7.1, $(\mu, H(p_f)) \in \mathcal{X}(f)$. Since $p_f$ has index $k$, $\boldsymbol{\sigma}_k(\mu) < 0$, and by assumption (7.1), $\boldsymbol{\sigma}_K(\mu) \geqslant 0$, where $K = \dim E_1$.

(ii) Having the measure $\mu$ at our disposal, we use Lemma 7.2 to perturb $f$ again so that there is a periodic orbit (obviously different from that of $p_f$) with $k$ vanishing exponents and $d - k$ positive exponents. Moreover, we can take this orbit $1/2n$-close to $H(p_f)$ in the Hausdorff distance.

(iii) Using Lemma 7.3 and then Lemma 7.4, we perturb $f$ again and create the disc $D$ with properties (i)–(v) so that $f \in \mathcal{V}(n, \mathcal{O})$.

Now consider a countable base of (non-empty) open sets $\mathcal{O}_n$ for $\mathrm{Diff}^1_{\#}(\mathbb{D}^k)$.

A first attempt to conclude the proof would be to define $\mathcal{R}$ as $\bigcap \mathcal{V}(n, \mathcal{O}_n)$. Then for any diffeomorphism in this set, we would be able to find a countable family of discs satisfying all the requirements of normally expanding $k$-universal dynamics, *except* for the disjointness between the discs. To fix this problem, we proceed as follows.

Let $\hat{\mathcal{O}}_n$ be the subset of $\mathrm{Diff}^1_{\#}(\mathbb{D}^k)$ formed by the maps $\phi$ such that for each $i = 1, \ldots, n$, there exist a subdisc $D_i \subset \mathrm{Int}\, \mathbb{D}_k$ and an integer $\pi_i > 0$ with the following properties:

(1) $D_i, \phi(D_i), \ldots, \phi^{\pi_i - 1}(D_i)$ are pairwise disjoint, and $\phi^{\pi_i}(D_i) \subset \mathrm{Int}\, D_i$;
(2) the restriction of $\phi^{\pi_i}$ to $D_i$ is differentiably conjugate to a map in $\mathcal{O}_i$.

Obviously, $\hat{\mathcal{O}}_n$ is non-empty and open. Define the following residual subset of $\mathcal{U}$:

$$\mathcal{R} = \bigcap_{n \in \mathbb{N}} \mathcal{V}(n, \hat{\mathcal{O}}_n).$$

Take any $f \in \mathcal{R}$. Let us show that $f$ has normally expanding $k$-universal dynamics. For each $n$, since $f \in \mathcal{V}(n, \hat{\mathcal{O}}_n)$, there is a disc $\hat{D}_n$ so that properties (i)–(v) hold with $D = \hat{D}_n$, $\pi$ equal to some $\hat{\pi}_n$ and $\mathcal{O} = \hat{\mathcal{O}}_n$. Let $\delta_n$ be the distance between $p_f$ and the orbit of $\hat{D}_n$, which is positive by condition (v). Define a subsequence $\{n_i\}$ recursively by taking $n_1 = 1$ and choosing $n_{i+1} > n_i$ so that

$$\frac{1}{n_{i+1}} < \min(\delta_{n_1}, \delta_{n_2}, \ldots, \delta_{n_i}).$$

This guarantees that the orbit of $\hat{D}_{n_{i+1}}$ is disjoint from the orbits of $\hat{D}_{n_1}, \ldots, \hat{D}_{n_i}$.

For each $i$, the restriction of $f^{\hat{\pi}_n}$ to $\hat{D}_{n_i}$ is differentiably conjugate to a map in $\hat{\mathcal{O}}_{n_i}$. Since $i \leqslant n_i$, we can find a periodic subdisc $D_i \subset \hat{D}_{n_i}$ whose first return is differentiably conjugate to an element of $\mathcal{O}_i$. Thus, the family of discs $\{D_i\}$ has all the properties required for normally expanding $k$-universal dynamics, concluding the proof. $\qquad\square$

Let us make a remark that will be useful later (in §7.2): Assumption (7.1) in Lemma 7.6 can be replaced by the following weaker condition:

$$\text{For each } f \text{ in a residual subset } \mathcal{S} \subset \mathcal{U}, \text{ there is a measure } \mu \text{ such that} \tag{7.2}$$
$$(\mu, H(p_f)) \in \mathcal{X}(f) \text{ and } \boldsymbol{\sigma}_k(\mu) \leqslant 0 \leqslant \boldsymbol{\sigma}_K(\mu), \text{ where } K = \dim E_1.$$

Indeed, the only part of the proof that requires modification is step (i) in the proof of denseness of $\mathcal{V}(n, \mathcal{O})$: Here we perturb $f$ so that $f \in \mathcal{R}_0 \cap \mathcal{S}$, and now the measure $\mu$ is given *a priori*.

We need the following lemma from point-set topology:

LEMMA 7.7.   *Let $B$ be a Baire space. Let $B = \bigcup_n V_n$ be a countable pointwise finite cover of $B$ by open sets. Suppose that $R_n$ is a residual subset of $V_n$, for each $n$. Then $\bigcup_n R_n$ is a residual subset of $B$.*

*Proof.*   Write $R_n = \bigcap_{i \in \mathbb{N}} U_{n,i}$, where each $U_{n,i}$ is open and dense in $V_n$. Pointwise finiteness implies that

$$\bigcup_n R_n = \bigcap_i \bigcup_n \bigcap_{j=1}^i U_{n,j},$$

which is clearly a residual subset of $B$.    □

*Proof of Theorem 5.*   Fix $k$ throughout the proof. If $f$ is a diffeomorphism and $p$ is a hyperbolic periodic point, let us say that the pair $(f, p)$ has *property X* if at least one of the following properties hold:

(i)  $f$ has normally expanding $k$-universal dynamics;
(ii)  $|\det Df^{\pi(p)}(p) \restriction E_1(p)| \leqslant 1$, where $E_1$ is the first bundle on the finest dominated splitting on $C(p)$.

For each $n$, consider the set $\mathcal{V}_n$ of diffeomorphisms such that all periodic points of period $n$ are hyperbolic. This is a open and dense set.

For each $f \in \mathcal{V}_n$, we can find an open set $\mathcal{V}_n^f$, an integer $r_f$ and continuous maps $p_1$, ..., $p_{r_f} : \mathcal{V}_n^f \to M$ such that the periodic points of period $n$ of each $g \in \mathcal{V}_n^f$ are precisely $p_1(g)$, ..., $p_{r_f}(g)$. Consider the cover of $\mathcal{V}_n$ by the sets $\mathcal{V}_n^f$. Since $\mathrm{Diff}^1(M)$ is paracompact (as every metric space) and Lindelöf, we can take a countable locally finite subcover, say $\mathcal{V}_n = \bigcup_i \mathcal{V}_{n,i}$.

Apply Lemma 7.5 to each $\mathcal{V}_{n,i}$ and each periodic point $p_j$ obtaining an open dense subset $\mathcal{U}_{n,i,j} \subset \mathcal{V}_{n,i}$ where the dimensions of the bundles on the finest dominated splitting on the chain recurrence class of $p_j$ are locally constant. Obviously, $\mathcal{U}_{n,i,j}$ is the (disjoint) union of a finite family of sets $\mathcal{U}_{n,i,j,\ell}$, where in each of these sets the dimensions are constant.

It follows from Lemma 7.6 that for every $f$ in a residual subset $\mathcal{R}_{n,i,j,\ell}$ of $\mathcal{U}_{n,i,j,\ell}$, the pair $(f, p_j(f))$ has property $X$.

Now define

$$\mathcal{R} = \bigcap_n \bigcup_i \bigcap_j \bigcup_\ell \mathcal{R}_{n,i,j,\ell}.$$

Using Lemma 7.7, we see that $\mathcal{R}$ is a residual subset of $\mathrm{Diff}^1(M)$. If $f \in \mathcal{R}$ then every periodic point $p$ is hyperbolic and $(f, p)$ has property $X$. The theorem follows.    □

### 7.2.   Proof of Theorem 6

Let us begin with a lemma.

LEMMA 7.8.   *Let $\mathcal{U}$ be a $C^1$-open set of diffeomorphisms $f$ having a hyperbolic periodic point $p_f$ of index $k$, varying continuously with $f$, so that the dimensions of the bundles of the finest dominated splitting $E_1 \oplus \ldots \oplus E_m$ over the chain recurrence class $C(p_f)$ do not depend on $f \in \mathcal{U}$.*

*Then there is a residual subset $\mathcal{S}$ of $\mathcal{U}$ such that every $f \in \mathcal{S}$ has (at least) one of the following properties.*

(i)  *There are periodic points $q_n$ homoclinically related to $p_f$ such that*

$$\liminf_{n \to \infty} \frac{1}{\pi(q_n)} \log|\det Df^{\pi(q_n)} \restriction E_1(q_n)| \geqslant 0.$$

(ii) *There is $\alpha > 0$ and there is a neighbourhood $\mathcal{V}$ of $f$ contained in $\mathcal{U}$ such that for every $g \in \mathcal{V}$ and every periodic point $q$ homoclinically related to $p_g$, we have*

$$\frac{1}{\pi(q)} \log |\det Dg^{\pi(q)} \restriction E_1(q)| \leqslant -\alpha.$$

*Proof.* For each $n \in \mathbb{N}$, let $\mathcal{A}_n$ be the set of $f \in \mathcal{U}$ such that there exists a periodic point $q$ homoclinically related to $p_f$ such that

$$\frac{1}{\pi(q)} \log |\det Df^{\pi(q)} \restriction E_1(q)| > -\frac{1}{n}.$$

This is evidently an open set. Let $\mathcal{B}_n = \mathrm{Int}(\mathcal{U} \smallsetminus \mathcal{A}_n)$. Then $\mathcal{A}_n \cup \mathcal{B}_n$ is open and dense in $\mathcal{U}$. Taking the intersection over $n$, we obtain a residual subset $\mathcal{S}$ of $\mathcal{U}$.

Now take $f \in \mathcal{S}$. If $f \in \mathcal{B}_n$ for some $n$, then $f$ has property (ii) with $\alpha = 1/n$. If, on the contrary, $f \notin \bigcup_n \mathcal{B}_n$, then $f \in \bigcap_n \mathcal{A}_n$ and so $f$ has property (i). $\qquad\square$

In order to prove Theorem 6, we first obtain a version of it for individual periodic orbits:

LEMMA 7.9. *Let $\mathcal{U}$ be a $C^1$-open set of diffeomorphisms $f$ having a hyperbolic periodic point $p_f$ of index $k$, varying continuously with $f$, so that the dimensions of the bundles of the finest dominated splitting $E_1 \oplus \ldots \oplus E_m$ over the chain recurrence class $C(p_f)$ do not depend on $f \in \mathcal{U}$.*

*Then there is a residual subset $\mathcal{R}$ of $\mathcal{U}$ such that every $f \in \mathcal{R}$ has (at least) one of the following properties.*

   (i) *$f$ is normally expanding $k$-universal or*
   (ii) *there is $\alpha > 0$ such that for any $q$ homoclinically related with $p$,*

$$\frac{1}{\pi(q)} \log |\det Df^{\pi(q)} \restriction E_1(q)| < -\alpha.$$

*Proof.* Take a set $\mathcal{U}$ as in the statement. Let $\mathcal{A}$ (resp. $\mathcal{B}$) be the set of $f \in \mathcal{U}$ that have property (i) (resp. (ii)) from Lemma 7.8. Then $\mathcal{B}$ is open, and $\mathcal{A} \cup \mathcal{B}$ contains a residual subset $\mathcal{S}$ of $\mathcal{U}$.

CLAIM. *If $f \in \mathcal{A}$, then there is a measure $\mu$ such that $(\mu, H(p_f)) \in \mathcal{X}(f)$ and $\boldsymbol{\sigma}_k(\mu) \leqslant 0 \leqslant \boldsymbol{\sigma}_K(\mu)$, where $K = \dim E_1$.*

*Proof.* We know that there are periodic points $q_n$ homoclinically related to $p_f$ such that $\liminf \boldsymbol{\sigma}_K(q_n) \geqslant 0$. For each $n$, we use Lemma 7.1 and find a periodic point $\hat{q}_n$ homoclinically related to $q_n$ and hence to $p_f$ such that $\boldsymbol{\sigma}_K(\hat{q}_n) > \boldsymbol{\sigma}_K(q_n) - 1/n$ and the Hausdorff distance between the orbit of $\hat{q}_n$ and $H(p_f)$ is less than $1/n$. Let $\mu_n$ be the $g$-invariant probability measure supported on the orbit of $\hat{q}_n$, and let $\mu$ be a accumulation point of this sequence of measures. Since $\boldsymbol{\sigma}_K$ is continuous, we have $\boldsymbol{\sigma}_K(\mu) \geqslant 0$. Each $\hat{q}_n$ has index $k$, and $\boldsymbol{\sigma}_k$ is lower semicontinuous, therefore $\boldsymbol{\sigma}_k(\mu) \leqslant 0$, proving the claim. $\qquad\square$

Let $\mathcal{A}^* = \mathcal{U} \smallsetminus \bar{\mathcal{B}}$. Then $\mathcal{A}$ is residual in the open set $\mathcal{A}^*$. Applying Lemma 7.6 (with assumption (7.1) replaced by (7.2)) to $\mathcal{A}^*$, we conclude that there is a residual subset $\mathcal{R}^*$ of $\mathcal{A}^*$ formed by normally expanding $k$-universal diffeomorphisms. Therefore $\mathcal{R} = \mathcal{R}^* \cup \mathcal{B}$ is the residual set we were looking for. $\qquad\square$

*Proof of Theorem* 6.    Fix $k$. If $f$ is a diffeomorphism and $p$ is a hyperbolic periodic point, let us say that the pair $(f, p)$ has *property Y* if at least one of properties (i) or (ii) from Lemma 7.9 holds.

Then we follow word by word the proof of Theorem 5, just replacing property $X$ by property $Y$, and using Lemma 7.9 instead of Lemma 7.6.                                    □

### 7.3.  *Criterion for k-universality*

Let us give a criterion for $k$-universal dynamics somewhat similar to Theorem 5.

THEOREM 7.10.    *Let $f$ be a $C^1$ generic diffeomorphism. Let $p$ be a periodic point, and let $E_1 \oplus \ldots \oplus E_m$ be the finest dominated splitting on the homoclinic class $H(p)$. Denote $i_j = \dim(E_1 \oplus \ldots \oplus E_j)$ for $j \in \{1, \ldots, m\}$, and $i_0 = 0$. Let*

$$k = \#\{i \in \{1, \ldots, d\}; \ \boldsymbol{\sigma}_i(p) \leqslant s\}, \quad \text{where } s = \min_{j \in \{0, \ldots, m\}} \boldsymbol{\sigma}_{i_j}(p). \tag{7.3}$$

*Then generic diffeomorphisms in a neighbourhood of $f$ have $k$-universal dynamics.*

*Proof* (*sketch*).    First notice that if $k$ is given by (7.3), then it is possible to perturb $f$ to create periodic orbits with exactly $k$ vanishing Lyapunov exponents.

As in the proof of Theorem 5, it is sufficient to prove a version of the theorem for individual orbits. This is done making minor adaptations in the proof of Lemma 7.6.                    □

QUESTION 7.11.    Can one find similar criteria for free, but neither normally expanding nor normally contracting, $k$-universality?

### 7.4.  *Proof of Theorem 1.8*

For completeness, we now explain how Theorem 1.8 follows from [8]. Since we have not used this theorem, this part is independent from anything else in this paper.

The main result of Bonatti and Díaz [8] says that if a diffeomorphism $g$ has a homoclinic class $H$ that is robustly without dominated splitting, and $H$ contains two homoclinically related points, one with jacobian bigger than 1 and the other with jacobian less than 1, then $g$ is in the closure of a locally generic set formed by diffeomorphisms with universal dynamics. Examples of such diffeomorphisms $g$ can be constructed in any manifold of dimension $k \geqslant 3$; in fact they can be constructed in a $k$-disc and be taken close to the identity.

*Proof of Theorem 1.8.*    Let $p$ be a periodic point for $f$ such that $Df^{\pi(p)}(p)$ is the identity on a subspace $E \subset T_p M$ of dimension $k \geqslant 3$, and the other eigenvalues have modulus bigger than 1. With a $C^1$ perturbation supported on a small neighbourhood of the orbit of $p$, we can create a normally expanding periodic $k$-disc $D$ such that $g = f^{\pi(p)} \restriction D$ is the identity. With a new perturbation, $g$ satisfies the conditions from [8] explained above. It follows that the perturbed $f$ belongs to the closure of a locally generic set of diffeomorphisms with normally expanding $k$-universal dynamics.                                    □

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