Uniformly hyperbolic finite-valued SL(2, ℝ)-cocycles

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Abstract. We consider finite families of $SL(2, \mathbb{R})$ matrices whose products display uniform exponential growth. These form open subsets of $(SL(2, \mathbb{R}))^N$, and we study their components, boundary, and complement. We also consider the more general situation where the allowed products of matrices satisfy a Markovian rule.

Contents

1. Introduction

Let $\pi : E \to X$ be a vector bundle over a compact metric space X and let $f : X \to X$ be a homeomorphism defining a dynamical systems in X. A *linear cocycle* mover f is a vector bundle map $F : E \to E$ which is fibered over F. The most important example occurs when X us a manifold, f is a diffeomorphism, E is the tangent bundle TX, and F is the tangent map Tf. But it is very profitable to consider larger classes of linear cocycles, allowing in particular to separate the base dynamics from the fiber dynamics.

The most powerful tool in the study of linear cocycles is Oseledets' Multiplicative Ergodic Theorem; see e.g. [1]. Given a probability measure on X which is invariant and ergodic under the basic dynamics f, it allows to define Lyapunov exponents and split accordingly the fiber E_x over almost all points of x. In this context, one says that F is hyperbolic if none of the Lyapunov exponents is equal to zero.

There is a stronger notion of hyperbolicity, called uniform hyperbolicity, which is of purely topological nature. One requires that E splits into a continuous direct sum $E^s \oplus E^u$, with both E^s, E^u invariant under F, E^s being contracted under F and E^u contracted under F^{-1} (after suitable choices of norms on E).

The easiest non-commutative setting, and one of the most studied, is when $E = X \times \mathbb{R}^2$ is trivial and 2-dimensional, and F comes from a continuous map $A : X \to SL(2, \mathbb{R})$. In this case, one is led to consider the products

$$A^n(x) := \begin{cases} A(f^{n-1}x) \dots A(x) & \text{for } n \geq 0, \\ A(f^n x)^{-1} \dots A(f^{-1}x)^{-1} & \text{for } n < 0. \end{cases} \tag{1}$$

The case where X is a torus and f is an irrational rotation has attracted a lot of attention in recent years, in particular in connection with the spectral properties of 1-d discrete Schrödinger operators with quasiperiodic potential: see for instance [7], [8], [9] and references therein. The values of the spectral parameter (energy) corresponding to uniform hyperbolicity are those in the resolvent, and the Lyapunov exponent is the main tool to study the spectrum.

The case where the base dynamics are chaotic is obviously also important. Starting from the fundamental work of Furstenberg [10], control of Lyapunov exponents has been obtained in several more general settings: see [12], [11], [5], [6].

In this work, we will consider, after [13], $SL(2, \mathbb{R})$-valued cocycles over chaotic base dynamics from the point of view of uniform hyperbolicity. More precisely, N will be an integer bigger than 1, and the base $X = \Sigma \subset N^{\mathbb{Z}}$ will be a transitive subshift of finite type (also called topological Markov chain), equipped with the shift map $\sigma \colon \Sigma \to \Sigma$. We will only consider cocycles defines by a map $A \colon \Sigma \to SL(2, \mathbb{R})$ depending only on the letter in position zero. The parameter space will be therefore the product $(SL(2, \mathbb{R}))^N$. The parameters $(A_1, \ldots, A_N)$ which correspond to a uniformly hyperbolic cocycle form an open set $\mathcal{H}$ which is the object of our study: we would like to describe its boundary, its connected components, and its complement. Roughly speaking, we will see that this goal is attained for the full shift on two symbols, and that new phenomena appear with at least 3 symbols which make such a complete description much more difficult and complicated.

Let us now review the contents of the following sections.

Associated to a $SL(2, \mathbb{R})$-valued cocycle $A \colon X \to SL(2, \mathbb{R})$ over a base $f \colon X \to X$, we have a fibered map $\bar{A} \colon X \times \mathbb{P}^1 \to X \times \mathbb{P}^1$. The standard cone criterion says that A is uniformly hyperbolic iff one can find an open interval $I(x) \subset \mathbb{P}^1$ depending continuously on x such that $A(x)I(x)$ is compactly contained in $I(f(x))$ for all $x \in X$. In our setting, A depends only on the zero coordinate x_0 of $x \in \Sigma$ and we would like for $I(x)$ to do the same. This is in general not possible but nevertheless a result in this direction exists if one allows several components for $I(x)$, leading to the notion of multicone. In the full shift case the result is as follows:

Theorem (Theorem 2.2). *A parameter $(A_1, \ldots, A_N)$ is uniformly hyperbolic (over the full shift $N^{\mathbb{Z}}$) iff there exists a non-empty open set $M \neq \mathbb{P}^1$ with finitely many components having disjoint closures which satisfies $\overline{A_\alpha M} \subset M$ for $1 \leq \alpha \leq N$.*

There is a similar statement (Theorem 2.3) for general subshifts of finite type.

Section 3 is dedicated to the case where Σ is the full shift on two symbols. We have a rather complete understanding of the hyperbolicity locus $\mathcal{H}$ in this case. The simplest components of $\mathcal{H}$ are the 4 principal components; they consist of parameters for which the multicone M in Theorem 2.2 is connected and are deduced from each other by change of signs of the matrices. Next there are the so-called free components of $\mathcal{H}$ (8 of them), consisting of parameters for which the multicone has two

components. All the other non-principal components of $\mathcal{H}$ are obtained by taking the preimage of one of the free components by a diffeomorphism of $(\mathrm{SL}(2, \mathbb{R}))^2$ belonging to the free monoid generated by

$$F_+(A, B) = (A, AB), \quad F_-(A, B) = (BA, B).$$

Moreover, any two distinct components of $\mathcal{H}$ have disjoint closures, and any compact set in parameter space meets only finitely many components of $\mathcal{H}$. In Subsection 3.8, the combinatorics and dynamics of the multicones are described for each component of $\mathcal{H}$.

Recall that a matrix $A \in \mathrm{SL}(2, \mathbb{R})$ is said to be hyperbolic (resp. parabolic, resp. elliptic) if $|\operatorname{tr} A| > 2$ (resp. $|\operatorname{tr} A| = 2$, resp. $|\operatorname{tr} A| < 2$). Denote by $\mathcal{E}$ the set of parameters for which there exists a periodic point $x \in \Sigma$ (of period k) such that $A^k(x)$ is elliptic. Obviously, $\mathcal{E}$ is an open set disjoint from $\mathcal{H}$. Avila has proved that for a general subshift of finite type, the closure of $\mathcal{E}$ is equal to the complement of $\mathcal{H}$. When Σ is the full shift on two symbols, we prove the stronger statement that $\mathcal{E}$ and $\mathcal{H}$ have the same boundary, the complement of their union.

The main result of Section 4 is the following result (for general subshifts of finite type):

Theorem (Theorem 4.1). *Let $(A_1, \ldots, A_N)$ belong to the boundary of a component of $\mathcal{H}$. Then one of the following possibilities hold:*

- *There exists a periodic point x of Σ, of period k, such that $A^k(x)$ is parabolic.*
- *There exist periodic points x, y of Σ, of respective periods k, ℓ, an integer $n \geq 0$, and a point $z \in W^u_{\mathrm{loc}}(x) \cap \sigma^{-n} W^s_{\mathrm{loc}}(y)$ such that $A^k(x)$, $A^\ell(y)$ are hyperbolic and*
$$A^n(z)u(A^k(x)) = s(A^\ell(y)).$$

We denote here by $u(A)$ or u_A (resp. $s(A)$ or s_A) the unstable (resp. stable) direction of a hyperbolic matrix A. (When A is parabolic and $A \neq \pm\mathrm{id}$, we still write $u_A = s_A$ for the unique invariant direction.) The second case in the statement of the theorem is called a heteroclinic connection. The integers k, ℓ, n occurring in Theorem 4.1 are actually bounded by a constant depending only on the component of $\mathcal{H}$ considered in the statement. It follows easily that:

Corollary (Corollary 4.5). *Every connected component of $\mathcal{H}$ is a semialgebraic set.*

In the full-shift case, for parameters on the boundary of non-principal components, no product of the matrices can be equal to $\pm\mathrm{id}$. The result we prove in Subsection 4.2, together with similar results, is actually stronger.

In Subsections 4.5–4.7, we investigate what happens along parameter families going through a heteroclinic connection. Starting with a single component of $\mathcal{H}$

(for the full shift on 3 symbols), it may happen that the complement of $\mathcal{H} \cup \mathcal{E}$ is locally a smooth hypersurface; but it may also happen that the boundary of the starting component is accumulated by a sequence of distinct components of $\mathcal{H}$.

In Section 5, we consider from a purely combinatorial point of view the dynamics on the components of the multicones for positive and negative iteration: this leads to the concept of combinatorial multicones and monotone correspondences. Necessary conditions on these objects to come from a matrix realization are introduced. It is shown that these conditions are also sufficient in the case of the full-shift on two symbols. An example is provided to show that the conditions are no longer sufficient for full-shifts with more symbols.

Except for the case of the full-shift on two symbols, many questions are still open and are discussed in Section 6.

In Annex A.1, a criterium characterizing relative compactness modulo conjugacy in parameter space is proved: tr A_i and tr $A_i A_j$ have to stay bounded.

There is one part of the study of the components of the hyperbolicity locus $\mathcal{H}$ which is only briefly mentioned in this paper, and deserves further work: this is the group vs monoid question. In the full shift case, a parameter $(A_1, \ldots, A_N)$ is hyperbolic if and only if matrices in the *monoid* generated by $A_1, \ldots, A_N$ grow exponentially with word length. For certain components of $\mathcal{H}$, but not all, it actually implies that the matrices in the (free) group generated by $A_1, \ldots, A_N$ grow exponentially with word length. For instance, for the full-shift on two symbols, this is true for non-principal components, but not true for principal components. In a further paper we plan to characterize which components have this property for the full-shift on 3 or more symbols.

2. Multicones

We recall the following result from [13], that says that uniform exponential growth of the products in (1) guarantees uniform hyperbolicity:

Proposition 2.1. *If $f: X \to X$ is a homeomorphism of a compact space and $A: X \to \mathrm{SL}(2, \mathbb{R})$ is a continuous map, then the cocycle (T, A) is uniformly hyperbolic iff there exist $c > 0$ and $\lambda > 1$ such that $\|A^n(x)\| \geq c\lambda^n$ for all $x \in \Sigma$, $n \geq 0$.*

As explained in the Introduction, we consider a general transitive subshift of finite type $\Sigma \subset N^{\mathbb{Z}}$, where $N \geq 2$. This means that there is a $N \times N$ matrix (ε_{ij}) of zeros and ones so that Σ is formed by the sequences $(x_i)_{i \in \mathbb{Z}}$ so that $\varepsilon_{x_i x_{i+1}} = 1$ for every i, and the shift transformation restricted to Σ is transitive.

Given $A_1, A_2, \ldots, A_N \in \mathrm{SL}(2, \mathbb{R})$, we consider the map $(x_i)_{i \in \mathbb{Z}} \in \Sigma \mapsto A_{x_0} \in \mathrm{SL}(2, \mathbb{R})$. If the associated cocycle is uniformly hyperbolic then we say that *the N-tuple $(A_1, \ldots, A_N)$ is uniformly hyperbolic with respect to the subshift Σ.*

If $A \in \mathrm{SL}(2, \mathbb{R})$, we also indicate by A the induced map $\mathbb{P}^1 \to \mathbb{P}^1$, where $\mathbb{P}^1$ is the projective space of $\mathbb{R}^2$.

Next we describe a geometric condition which is equivalent to uniform hyperbolicity of a N-tuple. Let us begin with full shifts:

Theorem 2.2. *An N-tuple $(A_1, \ldots, A_N)$ is uniformly hyperbolic w.r.t. the full shift $\Sigma = N^{\mathbb{Z}}$ iff there exists a nonempty open subset $M \subset \mathbb{P}^1$ with $\overline{M} \neq \mathbb{P}^1$ such that[1] $A_\alpha(M) \Subset M$ for every $\alpha \in \{1, \ldots, N\}$. We can take M with finitely many connected components, and those components with disjoint closures.[2]*

A set M satisfying all the conditions in the theorem is called a *multicone* for (A_α).

Now let Σ be any subshift of finite type. If α and β are symbols in the alphabet $\{1, \ldots, N\}$, we write $\alpha \to \beta$ to indicate that the symbol α can be followed by the symbol β. The generalization of Theorem 2.2 is:

Theorem 2.3. *An N-tuple $(A_1, \ldots, A_N)$ is uniformly hyperbolic w.r.t. Σ iff there are non-empty open sets $M_\alpha \subset \mathbb{P}^1$, one for each symbol α, with $\overline{M_\alpha} \neq \mathbb{P}^1$, and such that*

$$\alpha \to \beta \quad \textit{implies} \quad A_\beta(M_\alpha) \Subset M_\beta.$$

We can take each M_α with finitely many connected components, and those components with disjoint closures.

A family of sets (M_α) satisfying all the conditions in the theorem is called a *family of multicones* for the N-tuple (A_α).

For any subshift of finite type $\Sigma \subset N^{\mathbb{Z}}$, we can define the *dual subshift* $\Sigma^* \subset N^{\mathbb{Z}}$ as follows: if $\alpha \to \beta$ are the allowed transitions for Σ, then the allowed transitions for Σ^* are $\beta \xrightarrow{*} \alpha$. If (A_α) is a uniformly hyperbolic N-tuple w.r.t. Σ, with a family of multicones (M_α), then the N-tuple (A_α^{-1}) is uniformly hyperbolic w.r.t. Σ^*, with family of multicones $(M_\alpha') = (\mathbb{P}^1 \smallsetminus A_\alpha^{-1}(\overline{M_\alpha}))$.

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[1] $X \Subset Y$ means that the closure $\overline{X}$ of X is contained in the interior of Y.

[2] Added in proof: After this paper was completed, Bochi and Gourmelon [3] obtained generalizations of Proposition 2.1 and Theorem 2.2 to arbitrary dimension, replacing uniform hyperbolicity by the existence of a dominated splitting.

Let us see that Theorem 2.2 is a corollary of Theorem 2.3: If (A_α) is uniformly hyperbolic, and M_α's are given by Theorem 2.3, let $M = \bigcup_\alpha M_\alpha$. Since $A_\alpha \overline{M} \subset \overline{M_\alpha} \neq \mathbb{P}^1$, we have $\overline{M} \neq \mathbb{P}^1$. Conversely, given a multicone M we simply take $M_\alpha = M$ for all α.

2.1. Examples. Let $\Sigma = N^{\mathbb{Z}}$ be the full shift on N symbols. If the matrices $A_1, \ldots,$ A_N have a common strictly invariant interval, then by Theorem 2.2 $(A_1, \ldots, A_N)$ is uniformly hyperbolic. Consider the set of such N-tuples; its connected components are the *principal components* of the hyperbolic locus $\mathcal{H}$. By Proposition 3 from [13], such a component must contain some N-tuple of the form $(\pm A_*, \ldots, \pm A_*)$, where tr $A_* > 2$. Hence there are 2^N principal components.

Let $\Sigma = 2^{\mathbb{Z}}$ be the full shift on 2 symbols. For any $m \geq 2$, let us show that there is a uniformly hyperbolic pair (A, B) which has a multicone M with m components, but no multicone with $m - 1$ components. Take any hyperbolic matrix A. Choose u, $s \in \mathbb{P}^1$ such that

$$s_A < u \leq A^{m-2}u < s < A^{m-1}u < u_A < s_A$$

(for some cyclical order on the circle P^1). Take a hyperbolic matrix B with $u_B = u$, $s_B = s$. If the spectral radius of B is large enough, it is easy to see that (A, B) has a multicone M with m components containing respectively the points u_B, $A(u_B)$, …, $A^{m-2}(u_B)$, u_A. Figure 1 illustrates the case $m = 4$.
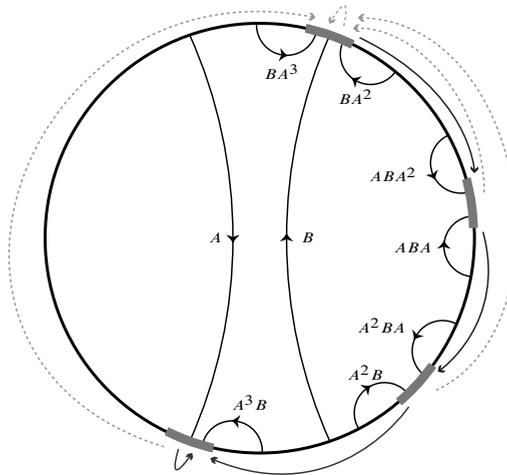
Figure 1. Example of a uniformly hyperbolic pair (A, B) and a multicone. Outer arrows indicate the action of A and B in the components of the multicone. Inner arrows indicate stable and unstable directions of A, B, and some of their products.

The examples just described do not exhaust the possibilities for the full 2-shift. See Figure 2 for a more complicate example. We postpone the description of this and all other possible examples for $\Sigma = 2^{\mathbb{Z}}$ to Section 3.
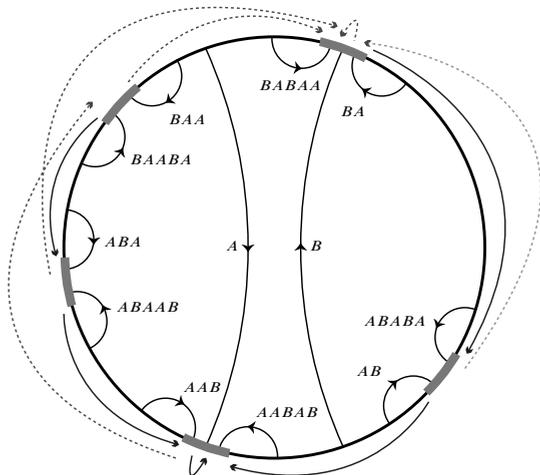
Figure 2. Another example of a uniformly hyperbolic pair (A, B).

Some examples of uniformly hyperbolic 3-tuples are indicated in Figure 3.
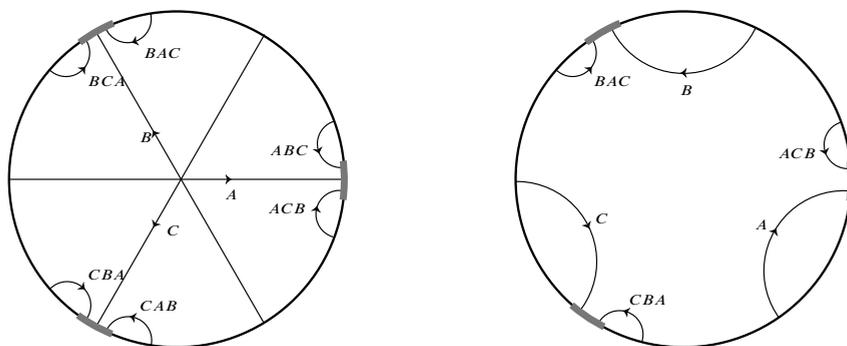
Figure 3. Two examples of uniformly hyperbolic 3-tuples (A, B, C).

An example illustrating the situation of Theorem 2.3, is indicated in Figure 4. (For another example, see §3.3, especially Figure 5.)
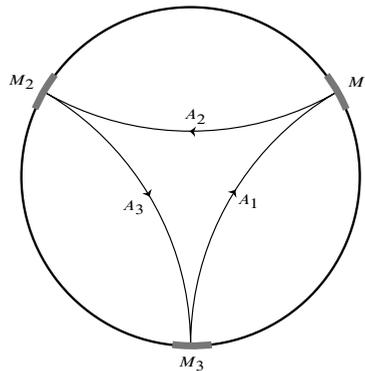
Figure 4. An example of a 3-tuple (A_1, A_2, A_3) that is uniformly hyperbolic with respect to the subshift on the symbols 1, 2, 3 whose only forbidden transitions are $1 \to 2$, $2 \to 3$, and $3 \to 1$. The intervals M_1, M_2, M_3 form a family of multicones.

2.2. Proof of the "if" part of Theorem 2.3.

Let us first establish some notation to be used from now on:

Given an ordered basis $\mathcal{B} = \{v_1, v_2\}$ of $\mathbb{R}^2$, we define a bijection $P_{\mathcal{B}}: \mathbb{P}^1 \to \mathbb{R} \cup \{\infty\}$ by $P_{\mathcal{B}}^{-1}(t) = v_1 + t v_2$, $P_{\mathcal{B}}^{-1}(\infty) = v_2$. The map $P_{\mathcal{B}}$ is called a *projective chart*.

If a, b, c, d are four distinct points in the extended real line $\mathbb{R} \cup \{\infty\}$ then we define their *cross-ratio*

$$[a, b, c, d] = \frac{c - a}{b - a} \cdot \frac{d - b}{d - c} \in \mathbb{R}. \tag{2}$$

If x, y, z, w are distinct points in the circle $\mathbb{P}^1$, we take any projective chart $P: \mathbb{P}^1 \to \mathbb{R} \cup \{\infty\}$ and define the cross-ratio

$$[x, y, z, w] = [P(x), P(y), P(z), P(w)].$$

The definition is good because (2) is invariant under Möbius transformations. Of course, for any $A \in \mathrm{SL}(2, \mathbb{R})$ we have $[x, y, z, w] = [A(x), A(y), A(z), A(w)]$.

A set $I \subset \mathbb{P}^1$ is called an *open interval* if it is non-empty, open, connected, and its complement contains more than one point. A set $I \subset \mathbb{P}^1$ is called a *closed interval* if either it consists of one point or is the complement of an open interval.

An open interval I can be endowed with the *Hilbert metric d_I*, defined as follows: If a, b are the endpoints of I then

$$d_I(x, y) = \big| \log[a, x, y, b] \big| \quad \text{for all distinct } x, \ y \in I.$$

Recall the following properties of the Hilbert metric: If $A \in \mathrm{SL}(2, \mathbb{R})$ satisfies $A(I) = J$ then A takes d_I to d_J. If $J \subsetneqq I$ are open intervals then the metric of J

is greater than the metric of I. If, in addition, $J \Subset I$ then the metric of J is greater than the metric of I by a factor at least $\lambda(I, J) > 1$.

Proof of the "if" part of Theorem 2.3. For each symbol α, let d_α be the Riemannian metric on M_α which coincides with the Hilbert metric in each of its components. Let K_α be the closure of the union of the sets $A_\alpha M_\gamma$, where $\gamma \to \alpha$. We can assume that K_α intersects each connected component of M_α, because otherwise we can take a smaller M_α. Let $L_\alpha \Subset M_\alpha$ be an open set containing K_α and with the same number of connected components as M_α. Then each component $M_{\alpha,i}$ of M_α contains a unique component $L_{\alpha,i}$ of L_α. Let $\lambda = \min_{\alpha,i} \lambda(M_{\alpha,i}, L_{\alpha,i})$.

Take an admissible sequence of symbols $\alpha_0 \to \alpha_1 \to \cdots \to \alpha_n$, and let $A = A_{\alpha_n} \ldots A_{\alpha_1}$. If u, v belong to the same component of M_{α_0} then

$$d_{\alpha_n}(Au, Av) \le \lambda^{-n} d_{\alpha_0}(u, v).$$

The metrics $d_\alpha | L_\alpha$ are comparable to the Euclidean metric d on $\mathbb{P}^1$. So if u, v belong to the same component of L_{α_0} we get $d(Au, Av) \le C\lambda^{-n} d(u, v)$, where $C > 0$ is some constant. This in turn implies that $\|A\| \ge C^{-1/2} \lambda^{n/2}$. By Proposition 2.1, we are done. $\qquad\square$

2.3. Proof of the "only if" part of Theorem 2.3.

Assume the cocycle associated to $(A_1, \ldots, A_N)$ is uniformly hyperbolic. This means that there are continuous functions $e^s, e^u \colon \Sigma \to \mathbb{P}^1$ and constants $C > 0$, $\lambda > 1$ such that for all $x \in \Sigma$:

$$A(x)e^s(x) = e^s(\sigma x); \quad \|A^n(x)v\| \le C\lambda^{-n} \|v\| \quad \text{for all } v \in e^s(x) \text{ and } n \ge 0;$$
$$A(x)e^u(x) = e^u(\sigma x); \quad \|A^{-n}(x)v\| \le C\lambda^{-n} \|v\| \quad \text{for all } v \in e^u(x) \text{ and } n \ge 0.$$

Moreover, $e^s(x)$ and $e^u(x)$ are uniquely determined by those properties, and $e^u(x) \ne e^s(x)$ for every $x \in \Sigma$. Thus, for $x = (x_i)_{i \in \mathbb{Z}}$, $e^u(x)$ depends only on $(\ldots, x_{-2}, x_{-1})$, while $e^s(x)$ depends only on $(x_0, x_1, \ldots)$. (That is, e^u, resp. e^s, is constant on local unstable, resp. stable, manifolds.)

If α is a symbol, we define the following two compact sets:

$$K_\alpha^u = \{e^u(x); \ x_{-1} = \alpha\}, \quad K_\alpha^s = \{e^s(x); \ x_0 = \alpha\}.$$

Notice that if $\alpha \to \beta$ then $K_\alpha^u \cap K_\beta^s = \varnothing$. Also,

$$K_\beta^u = \bigcup_{\alpha; \, \alpha \to \beta} A_\beta K_\alpha^u \quad \text{and} \quad K_\alpha^s = \bigcup_{\beta; \, \alpha \to \beta} A_\alpha^{-1} K_\beta^s.$$

So $K_\alpha^u \cap A_\alpha K_\alpha^s = \varnothing$.

Let us now define two families of sets U_α and S_α, called the *unstable* and *stable families of cores of* $(A_1, \ldots, A_N)$ as follows:

- U_α is the complement of the union of the connected components of $\mathbb{P}^1 \smallsetminus K_\alpha^u$ that intersect $A_\alpha K_\alpha^s$;

- S_α is the complement of the union of the connected components of $\mathbb{P}^1 \smallsetminus K_\alpha^s$ that intersect $A_\alpha^{-1} K_\alpha^u$.

It is straightforward to check that the families of cores satisfy the following properties:

(i) U_α, S_α are non-empty compact sets with finitely many connected components;

(ii) $U_\alpha \cap A_\alpha S_\alpha = \varnothing$;

(iii) every connected component of $\mathbb{P}^1 \smallsetminus A_\alpha S_\alpha$, resp. $\mathbb{P}^1 \smallsetminus A_\alpha^{-1} U_\alpha$, contains a unique connected component of U_α, resp. S_α;

(iv) $U_\beta \supset \displaystyle\bigcup_{\alpha;\; \alpha \to \beta} A_\beta U_\alpha$ and $S_\alpha \supset \displaystyle\bigcup_{\beta;\; \alpha \to \beta} A_\alpha^{-1} S_\beta$.

It follows from these conditions that each U_α has the same number $k(\alpha)$ of connected components as S_α. We define the *rank* of the families as the integer $\sum_\alpha k(\alpha)$.

Lemma 2.4. *Let $(A_1, \ldots, A_N) \in \mathrm{SL}(2, \mathbb{R})^N$. Assume that there exist two families of sets U_α and S_α (where α runs on the symbols) satisfying properties* (i)–(iv) *above, and with rank n_0. Assume also that for every periodic point $x \in \Sigma$ of period $n \le n_0$, the corresponding matrix product $A^n(x)$ is not $\pm\mathrm{id}$. Then $(A_1, \ldots, A_N)$ has a family of multicones (M_α). Moreover, $U_\alpha \subset M_\alpha \Subset \mathbb{P}^1 \smallsetminus A_\alpha S_\alpha$, and each connected component of $\mathbb{P}^1 \smallsetminus A_\alpha S_\alpha$ contains a unique connected component of M_α.*

Clearly, Lemma 2.4 implies the "only if" part of Theorem 2.3. The reason why we stated Lemma 2.4 in this generality is that it gives a criterion for uniform hyperbolicity which will be useful in some other occasions.

Proof of Lemma 2.4. Let $V_\alpha = \mathbb{P}^1 \smallsetminus A_\alpha S_\alpha$. Write each V_α as a disjoint union of open intervals $V_{\alpha,1} \sqcup \cdots \sqcup V_{\alpha,k(\alpha)}$, and write $U_\alpha = U_{\alpha,1} \sqcup \cdots \sqcup U_{\alpha,k(\alpha)}$ with $U_{\alpha,i} = U_\alpha \cap V_{\alpha,i}$.

Define a Riemannian metric d_α on V_α by taking on each component of V_α the corresponding Hilbert metric. For $\varepsilon > 0$, let $U_{\alpha,i}(\varepsilon)$ denote an ε-neighborhood of $U_{\alpha,i}$ with respect to d_α. Also let $U_\alpha(\varepsilon) = \bigcup_{i=1}^{k(\alpha)} U_{\alpha,i}(\varepsilon)$. Notice that if $\alpha \to \beta$ then $A_\beta \cdot V_\alpha \subset V_\beta$ and hence $A_\beta \cdot U_\alpha(\varepsilon) \subset U_\beta(\varepsilon)$.

Let $x \in \Sigma$ be such that $x_{-1} = x_{n-1} = \alpha$ for some n with $1 \le n \le n_0$. Assume that $A^n(x) \cdot V_{\alpha,i} \subset V_{\alpha,i}$ for some i (or, equivalently, $A^n(x) \cdot U_{\alpha,i} \subset U_{\alpha,i}$). We claim that then $A^n(x) \cdot U_{\alpha,i}(\varepsilon) \Subset U_{\alpha,i}(\varepsilon)$, for any $\varepsilon > 0$. Indeed, the matrix $B = A^n(x)$ is not $\pm\mathrm{id}$, by assumption, nor elliptic, because it leaves the interval $V_{\alpha,i}$ invariant. Therefore $u(B)$ and $s(B)$ are defined. We have $u(B) \in U_{\alpha,i}$ and $s(B) \notin V_{\alpha,i}$, so $s(B) \notin \overline{U_{\alpha,i}(\varepsilon)}$. Therefore B is hyperbolic and its restriction to $U_{\alpha,i}(\varepsilon)$ strictly contracts the metric d_α. This proves the claim.

From now on fix some arbitrary $\varepsilon' > 0$. By compactness, there exists a positive $\varepsilon'' < \varepsilon'$ such that if $x \in \Sigma$ and $1 \le n \le n_0$ are such that $x_{-1} = x_{n-1} = \alpha$ and $A^n(x) \cdot V_{\alpha,i} \subset V_{\alpha,i}$ for some α and i, then $A^n(x) \cdot U_{\alpha,i}(\varepsilon') \subset U_{\alpha,i}(\varepsilon'')$.

For $n \ge 0$, let

$$U_\alpha^n(\varepsilon) = \bigcup_{x \in \Sigma; \, x_{n-1}=\alpha} A^n(x) \cdot U_{x_{-1}}(\varepsilon).$$

Notice that $U_\alpha^k(\delta) \subset U_\alpha^n(\varepsilon)$ if $\delta \le \varepsilon$ and $k \ge n$, and also that $A_\beta U_\alpha^n(\varepsilon) \subset U_\beta^{n+1}(\varepsilon)$ if $\alpha \to \beta$.

We claim that $U_\alpha^{n_0}(\varepsilon') \subset U_\alpha(\varepsilon'')$ for any α. Indeed, take $x \in \Sigma$ with $x_{n_0-1} = \alpha$ and $v \in U_{x_{-1}}(\varepsilon')$. By the definition of the rank n_0, there exist $0 \le k < \ell \le n_0$ such that $x_{k-1} = x_{\ell-1}$ and moreover $A^k(x) \cdot v$ and $A^\ell(x) \cdot v$ belong to the same connected component of $U_{x_{k-1}}(\varepsilon')$, say $U_{x_{k-1},i}(\varepsilon')$. Then

$$A^\ell(x) \cdot v \in A^{\ell-k}(\sigma^k x) \cdot U_{x_{k-1},i}(\varepsilon') \subset U_{x_{\ell-1},i}(\varepsilon''),$$

and so $A^{n_0}(x) \cdot v \in U_\alpha(\varepsilon'')$, proving the claim.

At last, take a sequence $\varepsilon'' = \varepsilon_0 < \varepsilon_1 < \cdots < \varepsilon_{n_0} = \varepsilon'$ and let

$$M_\alpha = \bigcup_{n=0}^{n_0-1} U_\alpha^n(\varepsilon_{n+1}),$$

for each α. If $\alpha \to \beta$ then

$$A_\beta M_\alpha \subset \bigcup_{n=0}^{n_0-1} U_\beta^{n+1}(\varepsilon_{n+1}) \subset \bigcup_{n=0}^{n_0-1} U_\beta^n(\varepsilon_n) \Subset M_\beta.$$

So the family of sets M_α has the required properties. $\square$

2.4. The case of full shifts.

Here we will give some additional information about multicones in the specific case of the full shift $\Sigma = N^{\mathbb{Z}}$, which interests us most. In that case, a characterization of uniform hyperbolicity becomes simpler, involving a single multicone (cf. Theorem 2.2), instead of a family of multicones (cf. Theorem 2.3).

2.4.1. Multicones.

Given a uniformly hyperbolic N-tuple $(A_1, \ldots, A_N)$, let e^u, $e^s \colon N^{\mathbb{Z}} \to \mathbb{P}^1$ be the same maps as in §2.3, and let K^u, $K^s \subset \mathbb{P}^1$ be their respective images. Notice that these sets are disjoint, $K^u = \bigcup_\alpha A_\alpha(K^u)$, and $K^s = \bigcup_\alpha A_\alpha^{-1}(K^s)$.

These sets relate with multicones in the following way: If M is any multicone for $(A_1, \ldots, A_N)$ then

$$K^u = \bigcap_{n=0}^{\infty} \bigcup_{i_1, \ldots, i_n} A_{i_n} \ldots A_{i_1}(M), \quad K^s = \bigcap_{n=0}^{\infty} \bigcup_{i_1, \ldots, i_n} (A_{i_n} \ldots A_{i_1})^{-1}(\mathbb{P}^1 \smallsetminus \overline{M}).$$

The proof is left to the reader.

Another fact that is worth to mention is:

Proposition 2.5. *Assume that M is a multicone for a uniformly hyperbolic N-tuple $(A_1, \ldots, A_N)$. Then there exists k such that every product of A_i's of length $\geq k$ sends M into a single connected component of M.*

Proof. Fix a multicone M for $(A_1, \ldots, A_N)$. We have $K^u \subset M$ and $K^s \subset \mathbb{P}^1 \smallsetminus \overline{M}$. In particular, there is $\varepsilon > 0$ such that the 2ε-neighborhood of K^u (resp. K^s) is contained in M (resp. $\mathbb{P}^1 \smallsetminus \overline{M}$). There is $c = c(\varepsilon) > 1$ such that if $B \in \mathrm{SL}(2, \mathbb{R})$ is hyperbolic, the distance between u_B and s_B is at least 4ε, and $\|B\| > c$ then B sends the complement of the ε-neighborhood of s_B into the ε-neighborhood of u_B. Let k be such that every product of A_i's of length $\geq k$ has norm at least c. Then we are done. $\qquad\square$

2.4.2. Cores. As already mentioned, Theorem 2.2 is a corollary of Theorem 2.3. Nevertheless, it is worthwhile to see how the proof in §2.3 could be simplified.

Given the hyperbolic N-tuple $(A_1, \ldots, A_N)$, let K^u, $K^s \subset \mathbb{P}^1$ be as above. Define other sets U and S as follows:

- U is the complement of the union of the connected components of $\mathbb{P}^1 \smallsetminus K^u$ that intersect K^s;
- S is the complement of the union of the connected components of $\mathbb{P}^1 \smallsetminus K^s$ that intersect K^u.

The set U, resp. S, is called the *unstable*, resp. *stable*, *core of* $(A_1, \ldots, A_N)$. The following properties are easily checked:

(i) U, S are non-empty compact sets with finitely many components;

(ii) U and S are disjoint, and moreover each connected component of $\mathbb{P}^1 \smallsetminus S$, resp. $\mathbb{P}^1 \smallsetminus U$, contains a unique connected component of U, resp. S;

(iii) $A_i(U) \subset U$ and $A_i^{-1}(S) \subset S$ for every symbol i.

It follows from these conditions that the sets U and S have the same number of connected components; call this number the *rank* of the sets.

Remark 2.6. The relation between the cores U, S and the families of cores U_α, S_α considered before is simple: $\mathbb{P}^1 \smallsetminus U$ is the union of the connected components of $\mathbb{P}^1 \smallsetminus \bigcup U_\alpha$ that meet $\bigcup S_\alpha$, and analogously for S. In particular, U contains $\bigcup U_\alpha$ and that ∂U is contained in $\bigcup \partial U_\alpha$.

The following is a criterium for uniform hyperbolicity (specific for the full shift):

Lemma 2.7. *Let* $(A_1, \ldots, A_N) \in \mathrm{SL}(2, \mathbb{R})^N$. *Assume that there exist sets* U, $S \subset \mathbb{P}^1$ *satisfying properties* (i)–(iii) *above. Assume also that for every string of* A_i's *of length less of equal to the rank of the sets, the product is different from* $\pm\mathrm{id}$. *Then* $(A_1, \ldots, A_N)$ *has a multicone* M. *Moreover,* $U \subset M \Subset \mathbb{P}^1 \smallsetminus S$, *each connected component of* $\mathbb{P}^1 \smallsetminus S$ *contains a unique connected component of* M.

The proof of Lemma 2.7 is merely a simplification of the proof of Lemma 2.4, and will be left to the reader. Of course, using Lemma 2.7 one can give a direct proof of the "only if" part of Theorem 2.2.

2.4.3. Tightness. A multicone M for the N-tuple $(A_1, \ldots A_N)$ will be called *tight* if the following two conditions hold:

- the set $\bigcup_i A_i(M)$ intersects every connected component of M;
- the set $\bigcup_i A_i^{-1}(\mathbb{P}^1 \smallsetminus \overline{M})$ intersects every connected component of $\mathbb{P}^1 \smallsetminus \overline{M}$.

(Notice that no condition implies the other.)

Tightness has a simple reformulation in terms of the cores:

Proposition 2.8. *A multicone* M *is tight iff every connected component of* M *contains a unique connected component of* U *and every connected component of* $\mathbb{P}^1 \smallsetminus \overline{M}$ *contains a unique connected component of* S.

Proof. Fixed a uniformly hyperbolic N-tuple, let K^u, K^s, U, S be as before. Let M be a multicone, and let $M^* = \mathbb{P}^1 \smallsetminus \overline{M}$.

First, let us prove the "if" part: Assume every connected component of M (resp. M^*) intersects U (resp. S). Since $K^u \subset U$, each component of M intersects K^u. Now, each point in K^u is the image of another point in K^u (and hence in M) by some A_i. So each component of M intersects some $A_i(M)$. With a symmetric argument for M^* and S we conclude that M is tight.

Now let us prove the "only if" part of the proposition. Assume that the multicone M is tight. To conclude, it is sufficient to show that every connected component of M intersects K^u, and that every connected component of M^* intersects K^s. In fact, by symmetry, we only need to prove the first claim.

Fix a connected component of M, say, M_0. By the first condition in the definition of tightness, there exists a connected component M_1 of M such that $A_{i_1}(M_1) \subset M_0$ for some i_1. Continuing by induction, define components M_n and indices i_n for all $n \geq 1$ so that $A_{i_{n+1}}(M_{n+1}) \subset M_n$. The number of connected components is finite, so let $k \geq 1$ be the least index such that $M_k = M_\ell$ for some $\ell < k$. The interval M_ℓ is forward-invariant by $A_{i_{\ell+1}} \ldots A_{i_{k-1}} A_{i_k}$, so it contains the unstable direction of that product. So M_ℓ intersects K^u. The interval M_0 contains $A_{i_1} A_{i_2} \ldots A_{i_\ell}(M_\ell)$, hence it intersects K^u as well. This concludes the proof. $\qquad\square$

Remark 2.9. It follows from Proposition 2.8 that a multicone for a uniformly hyperbolic N-tuple $(A_1, \ldots, A_N)$ is tight iff there is no multicone with a smaller number of connected components.

3. The full 2-shift case

3.1. Statements. Before going into other general results, we study the simplest case: the full shift on two symbols. So in this section we let $\Sigma = 2^{\mathbb{Z}}$ and let $\mathcal{H} \subset SL(2, \mathbb{R})^2$ denote the associated hyperbolicity locus.

By definition, a connected component of $\mathcal{H}$ is called principal if every pair in it has a multicone consisting of a single interval. Recall from §2.1 that there are four such components. Let H_0 indicate their union.

The next simplest case is when a tight multicone consists on two intervals. So let $H_{\mathrm{id}} \subset SL(2, \mathbb{R})^2$ denote the (open) set of pairs (A, B) that do not belong to a principal component, and have a multicone M which is a union of two intervals.

(See Figure 5 for an example of $(A, B) \in H_{\mathrm{id}}$; $M = I_1 \cup I_2$ is a multicone.)

In fact (see Proposition 3.4), we have

$$H_{\mathrm{id}} = \{(A, B) \in SL(2, \mathbb{R})^2; \, |\operatorname{tr} A| > 2, \, |\operatorname{tr} B| > 2,$$
$$|\operatorname{tr} AB| > 2, \, \operatorname{tr} A \operatorname{tr} B \operatorname{tr} AB < 0\},$$

and moreover, H_{id} has eight connected components. Let us call these as the *free* components of $\mathcal{H}$.

Define mappings $F_+, F_- \colon SL(2, \mathbb{R})^2 \to SL(2, \mathbb{R})^2$ by

$$F_+(A, B) = (A, AB) \quad \text{and} \quad F_-(A, B) = (BA, B).$$

These are diffeomorphisms of $SL(2, \mathbb{R})^2$. Let $\mathcal{M}$ be the monoid[3] generated by F_+ and F_-.

Theorem 3.1 (Connected components of $\mathcal{H}$). *Every connected component of $\mathcal{H}$ is one of the following:*

- *either a principal component;*
- *or $F^{-1}(H)$ for some free component $H \subset H_{\mathrm{id}} \subset \mathcal{H}$ and some $F \in \mathcal{M}$.*

Moreover, such components are distinct.

Theorem 3.2 (Boundary of $\mathcal{H}$). *A compact subset of $SL(2, \mathbb{R})^2$ intersects only finitely many components of $\mathcal{H}$.*

The boundary of $\mathcal{H}$ is the disjoint union of the boundaries of its components.

Moreover, if $(A, B) \in \partial\mathcal{H}$ then (at least) one of the following holds:

---

[3] semigroup with identity

(i) *there is a product of A's and B's which is parabolic; or*

(ii) $u_A = s_B$ *or* $u_B = s_A$.

The second possibility can only occur if (A, B) belongs to the boundary of a principal component.

Let $\mathcal{E} \subset \mathrm{SL}(2, \mathbb{R})^2$ be the set of pairs (A, B) such that there exists a product of A's and B's which is elliptic. Of course, $\mathcal{E}$ is an open set, disjoint from $\mathcal{H}$. In fact, $\mathcal{E}$ is the complement of $\overline{\mathcal{H}}$, as a consequence of the following result:

Theorem 3.3 (Relation between $\mathcal{H}$ and $\mathcal{E}$). $\partial \mathcal{H} = \partial \mathcal{E} = (\mathcal{H} \sqcup \mathcal{E})^c$.

We are also able to give a precise description of the multicones for all components of $\mathcal{H}$, see §3.8.

The results above answer all questions of [13] for the full 2-shift. (Namely, the answers are 1: yes, 1′: no, 2: no, 3, 3′, 4: yes.) The solution of Problem 1 can also be given using the description of §3.8.

The proofs of Theorems 3.1, 3.2, and 3.3 occupy the following subsections.

3.2. Plan of proof. First, let us prove the assertions already made about H_{id}:

Proposition 3.4. *We have*

$$H_{\mathrm{id}} = \{(A, B) \in \mathrm{SL}(2, \mathbb{R})^2; \, |\operatorname{tr} A| > 2, \, |\operatorname{tr} B| > 2, \\ |\operatorname{tr} AB| > 2, \operatorname{tr} A \operatorname{tr} B \operatorname{tr} AB < 0\}. \tag{3}$$

The set H_{id} has eight connected components, and these components have disjoint boundaries.

The subset of H_{id} given by

$$\{(A, B); \, \operatorname{tr} A > 2, \operatorname{tr} B > 2, \operatorname{tr} AB < -2\} \tag{4}$$

has two connected components, which are conjugated by an orientation-reversing automorphism of $\mathbb{P}^1$. Fixed a cyclical order on $\mathbb{P}^1$, we have in one of the two components that

$$u_B < u_{BA} < s_{BA} < s_A < u_A < u_{AB} < s_{AB} < s_B < u_B. \tag{5}$$

The component of the set in (4) where (5) holds is called the *positive free component*. (Of course this definition depends on the choice of an orientation in $\mathbb{P}^1$.)

Proof. If $(A, B) \in H_{\mathrm{id}}$ then modulo sign changes (which do not affect being in either side of (3)) we can assume that $\operatorname{tr} A, \operatorname{tr} B > 2$. The fact that (A, B) does not belong to a principal component implies that $u_B < s_A < u_A < s_B < u_B$ for some cyclical

order on $\mathbb{P}^1$. Let M be the multicone for the pair (A, B); write it as union of two intervals $M = I \cup J$. Then one of the intervals, say I, must contain u_A and the other, u_B. So u_{AB} is contained in I, and, as it is easy to see, the associated eigenvalue of AB is negative. This shows that tr $AB < -2$ so (A, B) belongs to the right-hand side of (3).

On the other hand, Proposition 5 in [13] and its proof show that the set in (4) has two connected components with the stated properties. The proof also shows that pairs (A, B) in that set have a multicone consisting in two intervals. Of course, if $u_B < s_A < u_A < s_B < u_B$ for some cyclical order on $\mathbb{P}^1$ then (A, B) cannot be in a principal component of $\mathcal{H}$. So the set in (4) is contained in H_{id}. We conclude that the set in the right-hand side of (3) is also contained in H_{id} and has eight connected components.

To prove that the connected components of H_{id} have disjoint boundaries, it suffices to see that the two components of the set (4) have disjoint boundaries. So assume (A, B) is a boundary point of both components. Then $u_B = s_A = u_A = s_B$. So tr $A =$ tr $B = 2$, and this implies tr $AB = 2$, a contradiction.   □

Given $F \in \mathcal{M}$, let us denote $H_F = F^{-1}(H_{\mathrm{id}})$. Our plan to prove the main results is as follows. In §3.3–3.4 we will show:

Proposition 3.5. *For any $F \in \mathcal{M}$, $H_F \subset \mathcal{H}$.*

Then in §3.5–3.6 we will prove:

Proposition 3.6. $\mathrm{SL}(2, \mathbb{R})^2$ *is the disjoint union of* $\mathcal{E}$, $\overline{H_0}$, *and* $\bigsqcup_{F \in \mathcal{M}} \overline{H_F}$. *Moreover, a compact set in* $\mathrm{SL}(2, \mathbb{R})^2$ *intersects only finitely many of the sets* $\overline{H_F}$.

Putting things together, we will prove Theorems 3.1, 3.2, and 3.3 in §3.7.
In §3.8 we will give an alternative proof of Proposition 3.5, by describing explicitly the multicones.

3.3. Group-hyperbolic pairs. Let $(A, B) \in \mathrm{SL}(2, \mathbb{R})^2$ be given. Let $\Sigma \subset 4^{\mathbb{Z}}$ be the (transitive) subshift of finite type where the only forbidden transitions are $1 \to 3$, $3 \to 1$, $2 \to 4$, and $4 \to 2$. Take the 4-tuple $(A_1, A_2, A_3, A_4) = (A, B, A^{-1}, B^{-1})$, and consider the usual cocycle map over the subshift. If this cocycle is uniformly hyperbolic, then we will say the pair (A, B) is *group-hyperbolic*.

Lemma 3.7. *If (A, B) belongs to a free component then (A, B) is group-hyperbolic.*

Proof. Without loss, we assume that (A, B) belongs to the positive free component (so (5) holds). Take four disjoint (open) intervals I_1, I_2, I_3, I_4 such that $I_1 \cup I_2$ is

a multicone for (A, B) (over the full 2-shift), $I_3 \cup I_4$ is a multicone for (A^{-1}, B^{-1}) (over the full 2-shift), and

$$I_1 \supset [u_A, u_{AB}], \ I_4 \supset [s_{AB}, s_B], \ I_2 \supset [u_B, u_{BA}], \ I_3 \supset [s_{BA}, s_A].$$
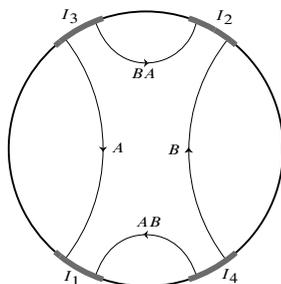

Figure 5. Group-hyperbolicity of the free component.

Since $A(I_1)$, $A(I_2) \Subset I_1$, we see that $A(I_4) \Subset I_1$ as well. In the same manner, we have

$$A(I_1 \cup I_4 \cup I_2) \Subset I_1, \qquad B(I_2 \cup I_3 \cup I_1) \Subset I_2,$$
$$A^{-1}(I_3 \cup I_2 \cup I_4) \Subset I_3, \quad B^{-1}(I_4 \cup I_1 \cup I_3) \Subset I_4.$$

So Theorem 2.3 applies, and our cocycle over the subshift $\Sigma \subset 4^{\mathbb{Z}}$ is uniformly hyperbolic. That is, (A, B) is group-hyperbolic.   □

3.4. Length comparison. Let $\mathbb{F}_2$ be the free group in two generators a, b. Let $|\cdot|$ be the usual length function on $\mathbb{F}_2$, relative to the generators a, b. Let f_+, f_- be the homomorphisms of $\mathbb{F}_2$ such that $f_+(a) = a$, $f_+(b) = ab$, $f_-(a) = ba$, $f_-(b) = b$. Notice $|f_\pm(\omega)| \leq 2|\omega|$ for all $\omega \in \mathbb{F}_2$. Since f_+ and f_- are in fact automorphisms, it follows that $|f_\pm^{-1}(\omega)| \geq \frac{1}{2}|\omega|$ for all $\omega \in \mathbb{F}_2$.

Given $(A, B) \in \mathrm{SL}(2, \mathbb{R})^2$, there is a unique homomorphism $\langle \cdot, (A, B) \rangle \colon \mathbb{F}_2 \to \mathrm{SL}(2, \mathbb{R})$ such that $\langle a, (A, B) \rangle = A$ and $\langle b, (A, B) \rangle = B$. In fact, this gives a bijection between $\mathrm{SL}(2, \mathbb{R})^2$ and the set of homomorphisms $\mathbb{F}_2 \to \mathrm{SL}(2, \mathbb{R})$.

If $f : \mathbb{F}_2 \to \mathbb{F}_2$ is a homomorphism then there is a unique map $f^* \colon \mathrm{SL}(2, \mathbb{R})^2 \to \mathrm{SL}(2, \mathbb{R})^2$ such that $\langle f(\omega), (A, B) \rangle = \langle \omega, f^*(A, B) \rangle$. The functorial properties $\mathrm{id}^* = \mathrm{id}$ and $(g \circ f)^* = f^* \circ g^*$ hold. Also notice that $f_+^* = F_+$ and $f_-^* = F_-$.

Proof of Proposition 3.5. Let $(A, B) \in H_F$, where $F = F_{\varepsilon_k} \circ \cdots \circ F_{\varepsilon_1}$, $\varepsilon_i \in \{+, -\}$. Let $(A_0, B_0) = F(A, B) \in H_{\mathrm{id}}$. By Lemma 3.7, (A_0, B_0) is group-hyperbolic. This means that there exist $c, \tau > 0$ such that for every $\omega \in \mathbb{F}_2$,

$$\|\langle \omega, (A_0, B_0) \rangle\| \geq c \exp(\tau |\omega|).$$

Let $f = f_{\varepsilon_1} \circ \cdots \circ f_{\varepsilon_k}$, so $f^* = F$. For any $\omega \in \mathbb{F}_2$, we have

$$\|\langle \omega, (A, B) \rangle\| = \|\langle f^{-1}(\omega), (A_0, B_0) \rangle\| \geq c \exp\left(\tau |f^{-1}(\omega)|\right) \geq c \exp\left(2^{-k} \tau |\omega|\right).$$

This proves that (A, B) is group-hyperbolic and, in particular, (A, B) is a uniformly hyperbolic pair w.r.t. the full 2-shift. $\square$

3.5. Twisted pairs. Let us say that $(A, B) \in \mathrm{SL}(2, \mathbb{R})^2$ is *straight* if $(A, B) \in \overline{H_0}$, that is, (A, B) belongs to the closure of a principal component.

Notice that if (A, B) is straight then so are $F_+(A, B)$ and $F_-(A, B)$.

It is easy to see that if there is an open interval which is forward-invariant for both A and B then (A, B) is straight. The converse is not true: for example, if $A \neq \pm\mathrm{id}$ is parabolic then (A, A^{-1}) is straight, but there is no invariant open interval.

Let us say that a pair (A, B) is *twisted* if A and B are not elliptic and (A, B) is not straight.

Let A be non-elliptic, and $A \neq \pm\mathrm{id}$, so $u_A, s_A \in \mathbb{P}^1$ are defined. Assume that an orientation is fixed in $\mathbb{P}^1$. Given $p \in \mathbb{P}^1$, we shall write $p < u_A \lesssim s_A < p$ to indicate that $p < Ap < u_A \leq s_A < p$. This means that there exist $\tilde{A}$ arbitrarily close (possibly equal) to A such that $p < u_{\tilde{A}} < s_{\tilde{A}} < p$. In the case A is parabolic we can define $u_A \lesssim s_A$ without mentioning a point p.

Lemma 3.8. *Let $A, B \in \mathrm{SL}(2, \mathbb{R})$ be non-elliptic. Then (A, B) is twisted iff A, $B \neq \pm\mathrm{id}$ and for some cyclical order on $\mathbb{P}^1$ we have*

$$u_A < s_B \lesssim u_B < s_A \lesssim u_A. \tag{6}$$

Proof. If A or B equals $\pm\mathrm{id}$, then (A, B) is easily seen to be straight. So we can assume $A, B \neq \pm\mathrm{id}$.

The rest of the proof is merely a case-by-case inspection. The following list exhausts all possible (mutually exclusive) cases, modulo inverting the cyclical order on $\mathbb{P}^1$, or interchanging A and B, or replacing (A, B) by (A^{-1}, B^{-1}):

1. A and B are hyperbolic:

   1.1. $u_A = u_B$ or $u_A = s_B$,

   1.2. $u_A < u_B < s_B < s_A < u_A$,

   1.3. $u_A < u_B < s_A < s_B < u_A$,

   1.4. $u_A < s_B < u_B < s_A < u_A$.

2. A hyperbolic and B parabolic:

   2.1. $u_A = u_B$,

   2.2. $u_A < u_B \lesssim s_B < s_A < u_A$,

2.3. $u_A < s_B \lesssim u_B < s_A < u_A$.

3. A and B parabolic:

    3.1. $u_A = u_B$ with $u_A \lesssim s_A$ and $s_B \lesssim u_B$,

    3.2. $u_A = u_B$ with $u_A \lesssim s_A$ and $u_B \lesssim s_B$,

    3.3. $u_A \neq u_B$ with $u_A \lesssim s_A$ and $s_B \lesssim u_B$,

    3.4. $u_A \neq u_B$ with $u_A \lesssim s_A$ and $u_B \lesssim s_B$.

The cases 1.1, 1.2, 1.3, 2.1, 2.2, 3.2, and 3.3 are those where there is an invariant open interval, and hence are straight. In the case 3.1, there is no invariant open interval, but it is straight nevertheless. The remaining cases, 1.4, 2.3, and 3.4 are precisely those where condition (6) holds; and none of them can be straight. $\qquad \square$

Lemma 3.9. *Let* (A, B) *satisfy* tr A, tr $B \geq 2$. *Then* (A, B) *is twisted iff there exists a basis* (*called canonical basis for* (A, B)) *where A, B are written as*

$$A = \begin{pmatrix} \mu & \alpha \\ 0 & \mu^{-1} \end{pmatrix}, \quad B = \begin{pmatrix} \nu^{-1} & 0 \\ \beta & \nu \end{pmatrix}, \qquad (7)$$

with $\mu \geq 1$, $\nu \geq 1$ *and* $\alpha\beta < 0$. *Moreover,* $\gamma \equiv \alpha\beta$ *only depends on* (A, B) *and not on the choice of the canonical basis.*

Proof. Let (A, B) be such that tr A, tr $B > 2$. Introduce coordinates so that $u_A = \mathbb{R}(1, 0)$ and $u_B = \mathbb{R}(0, 1)$. Then A and B are in the form (7), with μ, $\nu > 1$. Write the other eigendirections as $s_A = \mathbb{R}(x, 1)$ and $s_B = \mathbb{R}(1, y)$. We have

$$x = \frac{-\alpha}{\mu - \mu^{-1}}, \quad y = \frac{-\beta}{\nu - \nu^{-1}}.$$

Then (6) holds iff $xy < 0$, that is, iff $\alpha\beta < 0$.

We leave the cases where A or B is parabolic as exercises to the reader.

For the last remark, notice that $\alpha\beta$ is a function of tr A, tr B, and tr AB. $\qquad \square$

Let us say that (A, B) is *free* if

$$|\text{tr } A|, |\text{tr } B|, |\text{tr } AB| \geq 2, \quad \text{and} \quad \text{tr } A \text{ tr } B \text{ tr } AB < 0.$$

Lemma 3.10. *Every free pair is twisted. A pair* (A, B) *is free iff it belongs to* $\overline{H_{\text{id}}}$.

Proof. If (A, B) is straight then, replacing A by $-A$ or B by $-B$ if necessary, we have tr A, tr B, tr $AB \geq 2$, so (A, B) cannot be free.

If (A, B) is free, say with tr A, tr $B \geq 2$, and tr $AB \leq -2$, then using a canonical basis we see that there exist $(\tilde{A}, \tilde{B})$ arbitrarily close to (A, B) such that tr $\tilde{A}$, tr $\tilde{B} > 2$, and tr $\tilde{A}\tilde{B} < -2$. $\qquad \square$

Lemma 3.11. *Let* (A, B) *be twisted. Then exactly one of the following holds:*

(i) (A, AB) *is twisted.*

(ii) (BA, B) *is twisted.*

(iii) (A, B) *is free.*

(iv) AB *is elliptic.*

Proof. If (iv) holds then clearly (i), (ii), and (iii) do not hold. It follows from Proposition 3.4 and Lemma 3.10 that if (iii) holds then (i) and (ii) do not hold. Thus we only have to prove that if (A, B) is twisted and not free and if AB is not elliptic then either (i) or (ii) holds.

We can assume that tr A, tr $B \geq 2$. Then tr $AB \geq 2$. By taking a canonical basis for (A, B), we may assume that the expressions (7) hold, where we may choose $\alpha > 0$ and $\beta < 0$. Notice that with that basis u_A corresponds to $(1, 0)$ and u_B to $(0, 1)$. Let us orient $\mathbb{P}^1$ so that $(1, 0) < (1, y) < (0, 1)$ if $y > 0$. For this cyclical order, (6) holds. It is easy to see that $AB \neq \pm$id.

Assume that tr $A = 2$. Then tr $B > 2$, otherwise we would have tr $AB < 2$. First, let us locate the fixed points of the projective action of AB. It is easy to see that there is no fixed point in $[u_B, u_A]$. If there were a fixed point of AB in $[u_A, s_B]$ then the associated eigenvalue would be negative, contradicting tr $AB \geq 2$. So u_{AB}, $s_{AB} \in (s_B, u_B)$. It easily follows that

$$u_A < s_{AB} \lesssim u_{AB} < s_A \lesssim u_A,$$

and so, by Lemma 3.8, (A, AB) is twisted. We have $u_{BA} = Bu_{AB}, s_{BA} = Bs_{AB} \in (s_B, u_B)$. Notice that (u_{BA}, u_B) is an invariant interval for (BA, B) so that (BA, B) is straight. This shows that the lemma holds if tr $A = 2$. The same argument gives the case tr $B = 2$.

We assume from now on that tr A, tr $B > 2$. In this case we have $u_A < s_B < u_B < s_A < u_A$.

Let us locate the eigendirections of AB. None can belong to $\{u_A, u_B, s_A, s_B\}$. It is immediate that AB cannot have a fixed point in the interval (u_B, s_A). Neither can AB have a fixed point in (u_A, s_B), because otherwise the associated eigenvalue would be negative, contrary to the assumptions. So each eigendirection of AB must be in one of the intervals (s_A, u_A) and (s_B, u_B).

Consider the case that u_{AB} belongs to (s_B, u_B). Observe that BA sends s_B into the interval (u_B, s_B). It follows that s_{AB} also belongs to (s_B, u_B), and also

$$u_A < s_{AB} \lesssim u_{AB} < s_A < u_A.$$

So (A, AB) is twisted, by Lemma 3.8. The points $u_{BA} = Bu_{AB}$ and $s_{BA} = Bs_{AB}$ also belong to (s_B, u_B). The interval (u_{BA}, u_B) is invariant for BA and B, so (BA, B) is straight.

In the case that u_{AB} belongs to (s_A, u_A), then $u_{BA} = A^{-1}u_{AB}$ also belongs to the same interval. It follows as in the last case (interchanging the roles of A and B) that (BA, B) is twisted and (A, AB) is straight. $\square$

3.6. Dynamics of the monoid. Let $I(A, B) = (\operatorname{tr} A, \operatorname{tr} B, \operatorname{tr} AB)$ and let

$$\phi_+(x, y, z) = (x, z, xz - y),$$
$$\phi_-(x, y, z) = (z, y, yz - x),$$
$$j(x, y, z) = x^2 + y^2 + z^2 - xyz.$$

Proposition 3.12. *We have $I \circ F_\pm = \phi_\pm \circ I$ and $j \circ \phi_\pm = j$.*

Proof. The first assertion follows from the identity $\operatorname{tr} A^2 B = \operatorname{tr} A \; \operatorname{tr} AB - \operatorname{tr} B$. The second one is straightforward. $\square$

Let $J = j \circ I$.

Let (A, B) be twisted with $\operatorname{tr} A \geq 2$ and $\operatorname{tr} B \geq 2$, so that in a canonical basis

$$A = \begin{pmatrix} \mu & \alpha \\ 0 & \mu^{-1} \end{pmatrix}, \quad B = \begin{pmatrix} \nu^{-1} & 0 \\ \beta & \nu \end{pmatrix},$$

with $\mu \geq 1$, $\nu \geq 1$ and $\gamma \equiv \alpha\beta < 0$. Then $\operatorname{tr} AB = \mu\nu^{-1} + \mu^{-1}\nu + \gamma$. Thus

$$\operatorname{tr} AB \leq \max(\operatorname{tr} A, \operatorname{tr} B) + \gamma < \max(\operatorname{tr} A, \operatorname{tr} B). \tag{8}$$

Moreover, we have

$$J(A, B) = 4 + \gamma^2 - \gamma(\mu - \mu^{-1})(\nu - \nu^{-1}) > 4.$$

Let us say that (A, B) is *almost hyperbolic* if $F(A, B)$ is a pair of non-elliptic matrices for every $F \in \mathcal{M}$. The following is the key fact we need about the action of F:

Lemma 3.13. *Let (A, B) be almost hyperbolic and twisted. Then there exists a unique $F \in \mathcal{M}$ such that the pair $F(A, B)$ is free. Moreover, the length of F in terms of the generators F_+, F_- is $\leq \frac{1}{4}(|\operatorname{tr} A| + |\operatorname{tr} B|) - 1$.*

Proof. We may assume that $\operatorname{tr} A \geq 2$ and $\operatorname{tr} B \geq 2$. Let $(A_0, B_0) = (A, B)$. Assume that it was defined an almost hyperbolic and twisted pair (A_k, B_k), for some $k > 0$. Then, by Lemma 3.11, there are 3 possibilities:

either $F_+(A_k, B_k)$ is twisted, or $F_-(A_k, B_k)$ is twisted, or (A_k, B_k) is free. (9)

In the first, resp. second, alternative we set $\varepsilon_k = +$, resp. $\varepsilon_k = -$, and $(A_{k+1}, B_{k+1}) = F_{\varepsilon_k}(A_k, B_k)$.

We claim that the third alternative in (9) holds for some $k > 0$. If not, we have an (infinite) sequence of twisted pairs (A_k, B_k). Then tr $A_k \geq 2$ and tr $B_k \geq 2$ for all $k \geq 0$. In a canonical basis we have

$$A_k = \begin{pmatrix} \mu_k & \alpha_k \\ 0 & \mu_k^{-1} \end{pmatrix}, \quad B_k = \begin{pmatrix} \nu_k^{-1} & 0 \\ \beta_k & \nu_k \end{pmatrix}.$$

Define sequences

$$M_k = \max(\text{tr } A_k, \text{tr } B_k), \quad m_k = \min(\text{tr } A_k, \text{tr } B_k), \quad \text{and} \quad t_k = \text{tr } A_k + \text{tr } B_k.$$

Since (A_k, B_k) is twisted, $\gamma_k = \alpha_k \beta_k < 0$. So, by (8), $\{M_k\}$ is non-increasing.

Let also

$$\Delta_k = t_{k+1} - 2t_k + t_{k-1}, \quad k > 0.$$

Using Proposition 3.12, one easily checks that

$$\Delta_k = \begin{cases} (\text{tr } A_k - 2) \text{ tr } B_k & \text{if } (\varepsilon_k, \varepsilon_{k+1}) = (+, +), \\ (\text{tr } B_k - 2) \text{ tr } A_k & \text{if } (\varepsilon_k, \varepsilon_{k+1}) = (-, -), \\ \text{tr } A_k \text{ tr } B_k - \text{tr } A_k - \text{tr } B_k & \text{if } (\varepsilon_k, \varepsilon_{k+1}) = (-, +) \text{ or } (+, -). \end{cases} \tag{10}$$

In particular, $\Delta_k \geq (m_k - 2)M_k \geq 0$, so the function $k \mapsto t_k$ is convex. Since $4 \leq t_k \leq 2M_0$, we conclude that $\{t_k\}$ is non-increasing and $\Delta_k \to 0$ (indeed $\sum \Delta_k < \infty$). It follows that $\lim m_k = 2$. The proof now splits in two cases:

First case: $\lim M_k > 2$. Assume $\lim \text{tr } A_k = 2$ and $\lim \text{tr } B_k > 2$ (the other possibility being analogous). We get from (10) that $\varepsilon_k = +$ for all k big enough. Thus $A_{k+1} = A_k$ for all big k and tr $A_k = 2$ for big k. So $\Delta_k = 0$ for big k. Since $\{t_k\}$ is bounded we have, for all big k, that $t_{k+1} = t_k$ and hence tr $B_{k+1} = \text{tr } B_k$. But tr $B_{k+1} = \text{tr } B_k + \gamma_k < \text{tr } B_k$ for big k, contradiction.

Second case: $\lim M_k = 2$. Then tr A_k, tr B_k, tr $A_k B_k \to 2$, so $J(A_k, B_k) \to 4$. This contradicts $J(A_k, B_k) = J(A, B) > 4$.

We conclude that the third alternative in (9) holds for some $k = N$, say. That is, if $F = F_{\varepsilon_{N-1}} \circ \cdots \circ F_{\varepsilon_0}$ then $F(A, B)$ is free. Such $F \in \mathcal{M}$ is unique. Indeed, if $0 \leq j < N$ and $\delta \neq \varepsilon_j$ then $F_\delta \circ F_{\varepsilon_{j-1}} \circ \cdots \circ F_{\varepsilon_0}(A, B)$ is straight. (This follows from uniqueness in Lemma 3.11.) And $F_+(F(A, B))$ and $F_-(F(A, B))$ are also straight.

To complete the proof, we have to bound N. Since tr A_N, tr $B_N \geq 2$, and tr $A_N B_N \leq -2$, we have $t_{N+1} - t_N \leq -4$. For $1 \leq k \leq N$ we have $\Delta_k \geq 0$ and so $t_k - t_{k-1} \leq -4$. Thus $t_0 \geq 4N + t_N \geq 4N + 4$, so $N \leq \frac{1}{4}t_0 - 1$, as claimed. $\qquad \square$

Now we can give the

Proof of Proposition 3.6. First, $\overline{H_0} \cap \overline{H_{\text{id}}} = \varnothing$, and since $F(\overline{H_0}) \subset \overline{H_0}$, we have $\overline{H_0} \cap \overline{H_F} = \varnothing$ for any $F \in \mathcal{M}$. By Proposition 3.5, we have

$$\overline{H_0} \sqcup \bigcup_{F \in \mathcal{M}} \overline{H_F} \subset \overline{\mathcal{H}} \subset \mathcal{E}^c.$$

On the other hand, let $(A, B) \in \mathcal{E}^c$. If the pair (A, B) is straight, then it belongs to $\overline{H_0}$. If it is not, then it is twisted and almost hyperbolic. So Lemma 3.13 gives that there exists $F \in \mathcal{M}$ such that $(A, B) \in \overline{H_F}$. Moreover, F is unique. This shows that the sets $\overline{H_F}$ are disjoint, so the first assertion in the proposition is proved. The second one follows from the length estimate in Lemma 3.13. $\square$

3.7. Conclusion of the proofs

Proof of Theorems 3.1, 3.2, *and* 3.3. First let us see that

$$\mathcal{H} = H_0 \sqcup \bigsqcup_{F \in \mathcal{M}} H_F. \tag{11}$$

The $\supset$ inclusion follows from Proposition 3.5. To show the other inclusion, it suffices, by Proposition 3.6, to show that $\partial H_0, \partial H_F \subset \mathcal{H}^c$ for all $F \in \mathcal{M}$.

The boundary of H_0 is described by Proposition 4 in [13]: if (A, B) belongs to it then either A is parabolic or B is parabolic or $u_A = s_B$ or $u_B = s_A$. In any case, $(A, B) \in \mathcal{H}^c$.

By definition of H_{id}, if (A, B) belongs to its boundary then at least one of A, B, or AB is parabolic. It follows that if $(A, B) \in \partial H_F$ then there is a product of A's and B's which is parabolic. In particular, $(A, B) \in \mathcal{H}^c$.

We have proved equality (11) and hence Theorem 3.1.

Notice that the four principal components have disjoint boundaries, and so do the eight free components (this follows easily from Proposition 3.4.) So, by Proposition 3.6, the boundaries of the components of $\mathcal{H}$ are disjoint, and a compact set in $\mathrm{SL}(2, \mathbb{R})^2$ intersects only a finite number of components. It follows that the union of those boundaries gives all of $\partial \mathcal{H}$. This completes the proof of Theorem 3.2.

We have also shown that $\mathrm{SL}(2, \mathbb{R})^2 = \mathcal{E} \sqcup \overline{\mathcal{H}}$. To complete the proof of Theorem 3.3, it suffices to show that $\mathcal{H}^c \subset \overline{\mathcal{E}}$. That is an immediate consequence of Lemma 2 from [13]. $\square$

Remark 3.14. Our proof of Theorem 3.1 also gave an algorithm to decide whether a pair $(A, B) \in \mathrm{SL}(2, \mathbb{R})^2$ is uniformly hyperbolic or not (w.r.t. the full 2-shift). Namely: first, check if both A and B are hyperbolic; second, compute eigendirections of A, B to see if the pair belongs to a principal component; third, repeat the first step for all pairs $F_{\varepsilon_k} \circ \cdots \circ F_{\varepsilon_1}(A, B)$, with $k \leq \frac{1}{2} \max\{|\operatorname{tr} A|, |\operatorname{tr} B|\} - 1$. (By the way, this third step can be done without actually computing matrix products, if we use Proposition 3.12 instead.) The algorithm ends in "finite time"; moreover, given an upper bound for the size of the matrices, an upper bound for the "running time" of the algorithm can be given explicitly. An example of §4.7 (see Proposition 4.18) shows that the situation for the full 3-shift is much more complicated.

3.8. Description of the multicones. Here we will give another proof of Proposition 3.5, and also obtain an explicit description of the multicones for the twisted hyperbolic components.

3.8.1. Let $\mathcal{M}^*$ be the monoid on the generators F_+, F_- operating on words in A, B by the substitutions

$$F_+ : \quad A \mapsto A, \quad B \mapsto AB,$$
$$F_- : \quad A \mapsto BA, \quad B \mapsto B.$$

(The monoid $\mathcal{M}^*$ is opposite to the previously introduced $\mathcal{M}$.) We identify $\mathcal{M}^*$ with $\mathbb{Q} \cap (0, 1)$ via the canonical bijection j: for $F \in \mathcal{M}^*$, $j(F) = p/q$ if $F(AB)$ has length q and contains p times the letter B. We have $j(\mathrm{id}_{\mathcal{M}^*}) = 1/2$.

3.8.2. For $F \in \mathcal{M}^*$, with $j(F) = p/q$, denote by $O(p/q)$ the set of words of length q deduced from $F(AB)$ by cyclic permutation. This set can also be described in the following way: consider the map $R_{p/q} : [0, 1) \to [0, 1)$, $x \mapsto x + p/q \bmod 1$; set $\theta(x) = A$ if $x \in [0, 1 - p/q)$ and $\theta(x) = B$ if $x \in [1 - p/q, 1)$; set $\Theta(x) = (\theta(R_{p/q}^i(x)))_{0 \leq i < q}$; the image of Θ is $O(p/q)$.

In $O(p/q)$, the first word by lexicographical order is $\Theta(0)$, the second one is $\Theta(1/q)$ and so on until the last word $\Theta(1 - 1/q)$.

3.8.3. Let $F \in \mathcal{M}^*$, with $j(F) = p/q$; let $[p_0/q_0, p_1/q_1]$ be the Farey interval with center p/q. Recall that

$$p_0 + p_1 = p, \quad q_0 + q_1 = q, \quad p_1 q_0 - p_0 q_1 = 1. \tag{12}$$

Then $O(p_0/q_0)$ is the set of words deduced from $F(A)$ by cyclic permutation, and $O(p_1/q_1)$ is similarly the set of words deduced from $F(B)$ by cyclic permutation. Here, we extend the definition of $O(p/q)$ setting $O(0/1) = \{A\}$ and $O(1/1) = \{B\}$.

It follows from (12) that $R_{p/q}^{q_1}(0) = R_{p/q}^{-q_0}(0) = 1 - 1/q$. Set

$$O_1(p/q) = \{\Theta(R_{p/q}^i(0)); \ 0 < i < q_1\},$$
$$O_0(p/q) = \{\Theta(R_{p/q}^{-i}(0); \ 0 < i < q_0\};$$

we have thus defined a partition of $O(p/q) \smallsetminus \{\Theta(0), \Theta(1 - 1/q)\}$.

3.8.4. Let F, p/q, p_0/q_0, p_1/q_1 be as above. We define a cyclical order on $O(p/q) \sqcup O(p_0/q_0) \sqcup O(p_1/q_1)$.

For this cyclical order, the two sets $O(p/q)$ and $O(p_0/q_0) \sqcup O(p_1/q_1)$, both of cardinality q, alternate. The two intervals bounded by $\Theta(0)$ and $\Theta(1 - 1/q)$ are $O(p_1/q_1) \sqcup O_1(p/q)$ and $O(p_0/q_0) \sqcup O_0(p/q)$; moreover the element that succeeds

$\Theta(0)$ is in the former interval. The order induced on $O(p_1/q_1)$ or $O_1(p/q)$ is the lexicographical order, while the order induced on $O(p_0/q_0)$ or $O_o(p/q)$ in the antilexicographical order. See Figure 6 with $p/q = 2/5$.
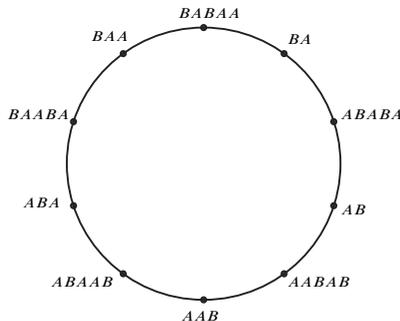

Figure 6. Order on $O(2/5) \sqcup O(1/3) \sqcup O(1/2)$.

Let us give a more explicit description of this cyclical order:

Lemma 3.15. *Let ω be an element in $O(p/q)$, and denote by ω^-, ω^+ the elements (in $O(p_0/q_0) \sqcup O(p_1/q_1)$) which are immediately before and after ω for the cyclical order. Denote by Θ_0, Θ_1 the maps defined as Θ with respect to p_0/q_0, p_1/q_1. Then the following holds:*

- *If $\omega = \Theta(R^i_{p/q}(0))$ with $0 \le i < q_1$ then $\omega^+ = \Theta_1(R^i_{p_1/q_1}(0))$;*
- *if $\omega = \Theta(R^i_{p/q}(1 - 1/q))$ with $0 \le i < q_0$ then $\omega^+ = \Theta_0(R^i_{p_0/q_0}(1 - 1/q_0))$;*
- *if $\omega = \Theta(R^{-i}_{p/q}(0))$ with $0 \le i < q_0$ then $\omega^- = \Theta_0(R^{-i}_{p_0/q_0}(0))$;*
- *if $\omega = \Theta(R^{-i}_{p/q}(1 - 1/q))$ with $0 \le i < q_1$ then $\omega^- = \Theta_1(R^{-i}_{p_1/q_1}(1 - 1/q_1))$.*

Proof. From (12) we obtain $p_1/q_1 - p/q = 1/q_1 q$. It follows that given i, j with $0 \le i, j < q_1$, the point $R^i_{p/q}(0)$ is before $R^j_{p/q}(0)$ (for the usual order in $[0, 1)$) if and only if the point $R^i_{p_1/q_1}(0)$ is before $R^j_{p_1/q_1}(0)$. Therefore the first assertion of the lemma holds. The others are proven similarly. $\square$

Define some special words

$$\omega_A = \Theta(p/q), \quad \omega_B = \Theta((p-1)/q), \quad {}_B\omega = \Theta(1 - p/q), \quad {}_A\omega = \Theta(1 - (p+1)/q).$$

From the description of the cyclical order, we see that the words respectively starting with A, starting with B, ending with A, ending with B form the intervals

$$ {}^A O = [{}_B\omega^+, {}_A\omega], \quad {}^B O = [{}_A\omega^+, {}_B\omega], \quad O^A = [\omega_A, \omega_B^-], \quad O^B = [\omega_B, \omega_A^-].$$

Observe that for $0 < p/q < 1/2$, the union of O^A and ${}^A O$ is the full set $O(p/q) \sqcup O(p_0/q_0) \sqcup O(p_1/q_1)$, and these intervals intersect at both ends.

3.8.5. We assume now that $p/q \neq 1/2$. If $p/q < 1/2$ (resp. $p/q > 1/2$) then we can write $F = F_+ F'$ (resp. $F_- F'$), with $F' \in \mathcal{M}^*$, $j(F') = p/(q-p)$ (resp. $j(F') = (2p-q)/p$).

Assume for instance that $p/q < 1/2$. Write $p'/q' = p/(q-p)$, and let $[p'_0/q'_0, p'_1/q'_1]$ be the Farey interval which has p'/q' as center; we have

$$\frac{p'_0}{q'_0} = \frac{p_0}{q_0 - p_0}, \qquad \frac{p'_1}{q'_1} = \frac{p_1}{q_1 - p_1}.$$

Lemma 3.16. *The image of $O(p'/q') \sqcup O(p'_0/q'_0) \sqcup O(p'_1/q'_1)$ under F_+ is exactly the interval $^A O$; moreover F_+ preserves the cyclic orders.*

Proof. Consider the map induced by $R_{p/q}$ on $[0, 1 - p/q)$; it is equal to

$$x \mapsto x + p/q \qquad \text{if } 0 \leq x < 1 - 2p/q,$$
$$x \mapsto x + 2p/q - 1 \quad \text{if } 1 - 2p/q \leq x < 1 - p/q.$$

Conjugating by the homothety of ratio $(q-p)/q$, we obtain $R_{p'/q'}$ on $[0, 1)$. This shows that the image of $O(p'/q')$ under F_+ is the interval of $O(p/q)$ formed by the words $\Theta(R^i_{p/q}(0))$ such that $R^i_{p/q}(0) \in [0, 1 - p/q)$, i.e., the words that start with A. The other conclusions of the lemma are proved similarly. One should observe that for $\varepsilon = 0, 1$, $F_+(O_\varepsilon(p'/q'))$ is the intersection of $F_+(O(p'/q'))$ with $O_\varepsilon(p/q)$. $\qquad \square$

3.8.6. For $F \in \mathcal{M}^*$, denote by H_F^+ the set of $(A, B) \in SL(2, \mathbb{R})^2$ such that $(F(A), F(B))$ belongs to the positive free component (which is described by Proposition 3.4).

Proposition 3.17. *Let $(A, B) \in H_F^+$. For any $\omega \in O(p/q) \sqcup O(p_0/q_0) \sqcup O(p_1/q_1)$, the corresponding matrix is hyperbolic. Moreover, the stable directions $s(\omega)$ and unstable directions $u(\omega)$ are all distinct and are positioned according to the following rules:*

- *for any $\omega \in O(p/q)$, $s(\omega)$ is immediately after $u(\omega)$;*
- *for any $\omega \in O(p_0/q_0) \sqcup O(p_1/q_1)$, $s(\omega)$ is immediately before $u(\omega)$;*
- *the restriction of the cyclic order to the $u(\omega)$ is the cyclic order considered above.*

(It follows from these three rules that the same is true for the restriction to the $s(\omega)$.)

Proof. The first assertion is clear. If $j(F) = 1/2$, the cyclic order is the one described above. Assume $j(F) = p/q \neq 1/2$, for instance $p/q < 1/2$. We write $F = F_+ F'$, $p'/q' = p/(q-p)$ as above. Let $A' = A$, $B' = AB$. We prove the proposition by induction, thus we may assume that the conclusions are satisfied for $(A', B') \in H_{F'}^+$. This means that the points $\{u(\omega), s(\omega); \omega \in {}^A O\}$ are all distinct

and the restriction of the cyclical order to this set is in accordance with the proposition. Let $a\colon O^A \to {}^A O$ be the bijection which takes the final letter A into first position; this map corresponds to A in the sense that

$$Au(\omega) = u(a\omega), \quad As(\omega) = s(a\omega), \quad \omega \in O^A,$$

and therefore the restriction of the cyclical order to the set $\{u(\omega), s(\omega); \omega \in O^A\}$ is also in accordance with the proposition. As ${}^A O$, O^A are intervals which cover $O(p/q) \sqcup O(p_0/q_0) \sqcup O(p_1/q_1)$ and have non-empty intersection at both ends, the points $\{u(\omega), s(\omega); \omega \in O(p/q) \sqcup O(p_0/q_0) \sqcup O(p_1/q_1)\}$ are all distinct and there is only one cyclical order with the given restrictions, which is the one described in the proposition. $\square$

3.8.7. Now we give the other proof of Proposition 3.5. It is sufficient to show that any $(A, B) \in H_F^+$ is uniformly hyperbolic. We will apply Lemma 2.7 and therefore we will define sets U and S satisfying the required conditions.

For $\omega \in O(p/q)$, we define intervals $I_\omega^u = [u(\omega^-), u(\omega)]$, $I_\omega^s = [s(\omega), s(\omega^+)]$. Let $U = \bigcup_{\omega \in O(p/q)} I_\omega^u$, $S = \bigcup_{\omega \in O(p/q)} I_\omega^s$. Then U, S are disjoint compact subsets with finitely many components which alternate. To apply Lemma 2.7, we need to check that $AU \cup BU \subset U$, $A^{-1}S \cup B^{-1}S \subset S$. Indeed, we have:

- $A(I_\omega^u) = I_{a\omega}^u$ for $\omega_A < \omega < \omega_B$;
- $A(I_\omega^u) \subset I_{\Theta(0)}^u$ for $\omega_B \le \omega \le \omega_A$;
- $B(I_\omega^u) = I_{b\omega}^u$ for $\omega_B < \omega < \omega_A$;
- $B(I_\omega^u) \subset I_{\Theta(1-p/q)}^u$ for $\omega_A \le \omega \le \omega_B$.

(The map $b\colon O^B \to {}^B O$ is defined analogously as a, by switching a letter B from the last to the first place.) This proves that AU and BU are disjoint and contained in U; it also follows that no non-trivial product of A, B is equal to $\pm\mathrm{id}$. Similar formulas hold for A^{-1}, B^{-1} and the intervals I_ω^s. Thus we can apply Lemma 2.7 and conclude that (A, B) is uniformly hyperbolic. The sets U and S are of course the unstable and stable cores, and the formulas above give the action of A, B on the components of the associated multicone. Both U and S have q components, and the set $O(p/q) \sqcup O(p_0/q_0) \sqcup O(p_1/q_1)$ is in canonical correspondence with the connected components of the complement of $U \sqcup S$: see Figure 7.

4. Boundaries of the components

4.1. A general theorem on boundary points. Again, fix any subshift of finite type $\Sigma \subset N^{\mathbb{Z}}$, and let $\mathcal{H} \subset \mathrm{SL}(2, \mathbb{R})^N$ be the associated hyperbolicity locus.
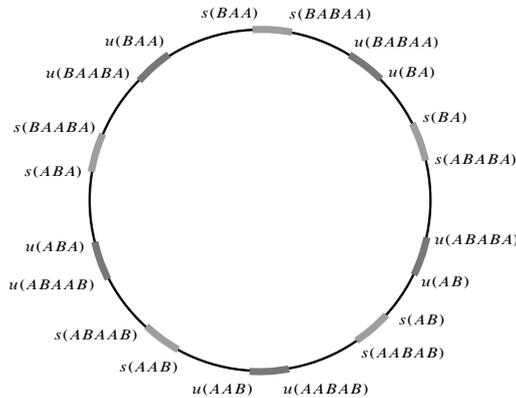
Figure 7. The intervals I_ω^u, I_s^ω for $p/q = 2/5$.

Given $x = (x_i)_{i \in \mathbb{Z}} \in \Sigma$, we denote

$$W_{\text{loc}}^u(x) = \{(z_i) \in \Sigma;\ z_i = x_i \text{ for } i < 0\},$$
$$W_{\text{loc}}^s(x) = \{(z_i) \in \Sigma;\ z_i = x_i \text{ for } i \geq 0\}.$$

The next result describes the boundary points of connected components of $\mathcal{H}$.

Theorem 4.1. *Let* $(A_1, \ldots, A_N)$ *belong to the boundary of a connected component* H *of* $\mathcal{H}$. *Then one of the following possibilities holds:*

(i) *There exists a periodic point* $x \in \Sigma$ *of period* k *such that* $A^k(x) = \pm\text{id}$.

(ii) (*"parabolic periodic"*) *There exists a periodic point* $x \in \Sigma$ *of period* k *such that* $A^k(x) \neq \pm\text{id}$ *is parabolic.*

(iii) (*"heteroclinic connection"*) *There exist periodic points* x *and* $y \in \Sigma$, *of respective periods* k *and* ℓ, *such that the matrices* $A^k(x)$ *and* $A^\ell(y)$ *are hyperbolic and there exist an integer* $n \geq 0$ *and a point* $z \in W_{\text{loc}}^u(x) \cap \sigma^{-n} W_{\text{loc}}^s(y)$ *such that*

$$A^n(z) \cdot u(A^k(x)) = s(A^\ell(y)). \tag{13}$$

Furthermore, for each component H, *one can give uniform bounds to the numbers* k, ℓ, n *that may appear in the alternatives above.*

In alternative (iii), there exists a point $z = (z_i)_{i \in \mathbb{Z}}$ such that $z_{-k-1} = z_{-1}$, $z_{n+\ell} = z_n$, and

$$A_{z_{n-1}} \ldots A_{z_0} \cdot u(A_{z_{-1}} \ldots A_{z_{-k}}) = s(A_{z_{n+\ell-1}} \ldots A_{z_n}).$$

That is what we call a *heteroclinic connection* (provided both $A_{z_{-1}} \ldots A_{z_{-k}}$ and $A_{z_{n+\ell-1}} \ldots A_{z_n}$ are hyperbolic).

Remark 4.2. In alternative (iii), the periodic points x and y cannot belong to the same periodic orbit.

Proof. Assume the contrary, so $k = \ell$ and $x = \sigma^j(y)$ for some j with $0 \leq j < k$. Then

$$s(A^k(y)) = A^n(z) \cdot u(A^k(x)) = A^n(z) \cdot A^j(y) \cdot u(A^k(y))$$
$$= A^{n+j}(\sigma^{-j}z) \cdot u(A^k(y)).$$

So, writing $A = A^k(y)$ and $B = A^{n+j}(\sigma^{-j}z)$, we have that A is hyperbolic and $B \cdot u(A) = s(A)$. A direct calculation shows that $\lim_{m \to +\infty} \operatorname{tr} A^m B = 0$. Therefore there is $m > 0$ such that $A^m B = A^{km+n+j}(\sigma^{-j}z)$ is elliptic. Since $z_{km+n} = z_{-j}$, this contradicts the assumption that the N-tuple belongs to the boundary of $\mathcal{H}$.   □

Remark 4.3. If Σ is the full-shift, and H is a principal component, then by Proposition 4 in [13] one can take $n = 0$, $k = \ell = 1$ in alternative (iii) of Theorem 4.1.

Remark 4.4. We will see later (Proposition 4.9) that in the case of full shifts, alternative (i) in Theorem 4.1 is only possible if H is a principal component.

Theorem 4.1 has the following interesting consequence:

Corollary 4.5. *Every connected component of $\mathcal{H}$ is a semialgebraic set.*

Notice $\mathcal{H}$ itself is not semialgebraic, because it has infinitely many connected components (see Theorem 2.4.5 from [4]).

Proof of the corollary. Of course, $\mathrm{SL}(2, \mathbb{R})^N$ itself is a (semi) algebraic subset of $\mathbb{R}^{4N}$.

Let H be a connected component of $\mathcal{H}$. Let K be the upper bound on the numbers k, ℓ, n that appear in Theorem 4.1. Let S_1, S_2, and S_3 be the subsets of $\mathrm{SL}(2, \mathbb{R})^N$ formed by the N-tuples that satisfy respectively alternatives (i), (ii), and (iii) of the theorem, with k, ℓ, n not greater than K.

The set $S_1 \cup S_2$ is obviously semialgebraic; let us see that S_3 also is. Introduce variables $\lambda, \mu \in \mathbb{R}$, $w_1, w_2 \in \mathbb{R}^2$, and rewrite (13) as

$$\begin{cases} A^k(x) \cdot w_1 = \lambda w_1, & \lambda^2 > 1, \\ A^\ell(y) \cdot w_2 = \mu w_2, & -1 < \mu < 1, \\ A^n(z) \cdot w_1 = w_2, & w_1 \neq (0, 0). \end{cases}$$

Such relations define a semialgebraic set on $\mathrm{SL}(2, \mathbb{R})^N \times \mathbb{R}^6$, which is sent by the obvious projection onto S_3. Therefore S_3 is semialgebraic, by the Tarski–Seidenberg principle (see [4], Theorem 2.2.1).

The set $S = S_1 \cup S_2 \cup S_3$ is closed, disjoint from H, and contains the boundary of H. Thus H is a connected component of the semialgebraic set $\mathrm{SL}(2, \mathbb{R})^N \setminus S$, and hence is semialgebraic, by Theorem 2.4.5 from [4]. □

To prove Theorem 4.1, we first establish two lemmas. In both of them we assume that $(A_1, \ldots, A_N)$ belongs to the hyperbolic locus, and let U_α, S_α be its unstable and stable families of cores (see §2.3).

Lemma 4.6. *Let β be a symbol, and $v \in \partial U_\beta$. Then there exist a symbol α such that $\alpha \to \beta$ and $A_\beta^{-1}(v) \in \partial U_\alpha$.*

Proof. Recalling the definition of U_β, we see that the condition $v \in \partial U_\beta$ is equivalent to the following:

$v \in K_\beta^u$ and there exist a point $w \in A_\beta K_\beta^s$ and an open interval $I \subset \mathbb{P}^1$ such that $\partial I = \{v, w\}$ and $I \cap K_\beta^u = \varnothing$.

Let v, w, and I be as above. Take $x = (x_i)_{i \in \mathbb{Z}} \in \Sigma$ such that $x_{-1} = \beta$ and $e^u(x) = v$. Let $\alpha = x_{-2}$. Set $v' = A_\beta^{-1}(v)$, $w' = A_\beta^{-1}(w)$, and $I' = A_\beta^{-1}(I)$. We have $v' \in K_\alpha^u$, $w' \in A_\alpha K_\alpha^s$, and $I' \cap K_\alpha^u = \varnothing$. We conclude that $v' \in \partial U_\alpha$. □

Lemma 4.7. *Let $v \in \partial U_\alpha$. Then there exist a periodic point $x \in \Sigma$ of period k, a point $z \in W_{\mathrm{loc}}^u(x)$, and an integer $m \geq 0$ such that $z_{m-1} = \alpha$ and*

$$v = A^m(z) \cdot u(A^k(x)).$$

Analogously, if $v' \in \partial S_\alpha$ then there exist a periodic point $y \in \Sigma$ of period ℓ, a point $w \in W_{\mathrm{loc}}^s(x)$, and an integer $p \geq 0$ such that $w_{-p} = \alpha$ and

$$v' = A^{-p}(w) \cdot s(A^\ell(y)).$$

Moreover, k, m, ℓ, and p are less or equal than the rank of the families of cores.

Proof. We will prove one half of the lemma. Take $v \in \partial U_\alpha$. Set $\alpha_0 = \alpha$ and $v_0 = v$. Applying repeatedly Lemma 4.6 we find a sequence $\alpha_0 \leftarrow \alpha_1 \leftarrow \alpha_2 \leftarrow \cdots$ such that

$$v_{n+1} = A_{\alpha_n}^{-1} \ldots A_{\alpha_0}^{-1} v_0 \in \partial U_{\alpha_{n+1}} \quad \text{for every } n \geq 0.$$

Let n_0 be the rank of the family U_α. By the pigeon-hole principle, there exist integers m and k such that $0 \leq m < m + k \leq n_0$ and $\alpha_m = \alpha_{m+k}$ and $v_m = v_{m+k}$. Then v_m is fixed by $A_{\alpha_m} \ldots A_{\alpha_{m+k-1}}$, and so must be the unstable direction of this matrix product. We also have $v_0 = A_{\alpha_0} \ldots A_{\alpha_{m-1}} \cdot v_m$. The lemma follows. □

Proof of Theorem 4.1. Observe that unstable and stable families of cores vary continuously with the N-tuple. So if we restrict ourselves to N-tuples in H, the rank n_0 of the families of cores is constant.

Now take $(A_1, \ldots, A_N)$ in the boundary of H. Assume that there is no periodic point $x \in \Sigma$ of period $n \leq n_0$ for which $A^n(x) = \pm\text{id}$. We will show that then one of the alternatives (ii) or (iii) in the theorem holds.

Consider the following finite subsets of $\mathbb{P}^1$:

$$
\begin{aligned}
U_\alpha^* = \{A^m(z) \cdot u(A^k(x)); \ &1 \leq k \leq n_0, \ 0 \leq m \leq n_0, \\
&x = \sigma^k x, \ z \in W_{\text{loc}}^u(x), \ z_{m-1} = \alpha\}, \\
S_\beta^* = \{A^{-p}(w) \cdot s(A^\ell(y)); \ &1 \leq \ell \leq n_0, \ 0 \leq p \leq n_0, \\
&y = \sigma^\ell y, \ w \in W_{\text{loc}}^s(y), \ w_{-p} = \beta\}.
\end{aligned}
\tag{14}
$$

Notice that

$$
U_\beta^* \subset \bigcup_{\alpha; \, \alpha \to \beta} A_\beta U_\alpha^* \quad \text{and} \quad S_\alpha^* \subset \bigcup_{\beta; \, \alpha \to \beta} A_\alpha^{-1} S_\beta^*.
\tag{15}
$$

(To see this, use for instance that if $x = \sigma^k x$ then $u(A^k(x)) = A_{x_{-1}} \cdot u(A^k(\sigma^{-1}x))$.)

Assume that $U_\alpha^* \cap S_\beta^* \neq \varnothing$ for some α, β with $\alpha \to \beta$. Then, for some m, x etc as in (14), we have an equality $A^m(z) \cdot u(A^k(x)) = A^{-p}(w) \cdot s(A^\ell(y))$. Moreover, we can assume that $w = \sigma^n z$, where $n = m + p$. Then $A^n(z) \cdot u(A^k(x)) = s(A^\ell(y))$, with $z \in W_{\text{loc}}^u(x) \cap \sigma^{-n} W_{\text{loc}}^s(y)$. If $A^k(x)$ or $A^\ell(y)$ is parabolic, we are in alternative (ii) of the theorem. Otherwise, both $A^k(x)$ and $A^\ell(y)$ are hyperbolic and alternative (iii) holds.

In order to complete the proof of the theorem, we will assume by contradiction that $U_\alpha^* \cap S_\beta^* = \varnothing$ for every α, β with $\alpha \to \beta$. It follows from (15) that $U_\alpha^* \cap A_\alpha S_\alpha^* = \varnothing$ for every α.

Take a sequence $(A_1(i), \ldots, A_N(i))$ in H converging to $(A_1, \ldots, A_N)$ as $i \to \infty$. Let $A_\alpha(\infty) = A_\alpha$.

Define sets $U_\alpha^*(i)$, $S_\alpha^*(i)$ in the same way U_α^*, S_α^* were defined, replacing each A_β with $A_\beta(i)$. By continuity of the u and s directions for non-elliptic matrices far from $\pm\text{id}$, we have that for every large i, $U_\alpha^*(i)$ and $S_\alpha^*(i)$, are close to U_α^* and S_α^*, respectively.

For $i \in \mathbb{N} \cup \{\infty\}$, define other sets $U_\alpha(i)$, $S_\alpha(i)$ as follows: $U_\alpha(i)$ is the complement of the union of the connected components of $\mathbb{P}^1 \setminus U_\alpha^*(i)$ that intersect $A_\alpha S_\alpha^*(i)$, and $S_\alpha(i)$ is the complement of the union of the connected components of $\mathbb{P}^1 \setminus S_\alpha^*(i)$ that intersect $A_\alpha^{-1} U_\alpha^*(i)$. If I is large enough then $U_\alpha(i)$ and $S_\alpha(i)$ are respectively close (with respect to the Hausdorff distance) to $U_\alpha(\infty)$ and $S_\alpha(\infty)$.

By Lemma 4.7, if $i < \infty$ then $U_\alpha(i)$ and $S_\alpha(i)$ are precisely the unstable and stable families of cores of the N-tuple $(A_\alpha(i))$. It follows from continuity that the sets

$U_\alpha = U_\alpha(\infty)$, $S_\alpha = S_\alpha(\infty)$ also satisfy properties (i)–(iv) of §2.3. By Lemma 2.4, (A_α) has a family of multicones, that is, $(A_\alpha) \in \mathcal{H}$. Contradiction. $\qquad \square$

From this point until the end of Section 5, we will be interested only in full shifts.

4.2. Non-principal components.

As mentioned in Remark 4.4, we will prove that no ±identity products exist in the boundaries of non-principal components.

Let us begin with a lemma about pairs of matrices. Recall that a uniformly hyperbolic pair induces maps e^u, $e^s \colon 2^{\mathbb{Z}} \to \mathbb{P}^1$ (see §2.3).

Lemma 4.8. *For every $c > 0$ there exists $\delta = \delta(c) > 0$ with the following properties. If (A, B) is a uniformly hyperbolic pair with*

$$\|A\| \leq c \quad and \quad \|B \mp \mathrm{id}\| < \delta \tag{16}$$

*then (A, B) belongs to a principal component. Moreover, the images of the maps e^u, e^s are (disjoint closed) **intervals** I_u, $I_s \subset \mathbb{P}^1$.*

Proof. Our study of the $N = 2$ case shows that the boundary of a non-principal component cannot contain a pair of the form $(A, \pm\mathrm{id})$. If follows that there exists $\delta = \delta(c)$ such that every hyperbolic pair (A, B) satisfying (16) belongs to a principal component.

Let us also assume that $\delta(c)$ is small enough so that (16) implies

$$\inf_{x \in \mathbb{P}^1} |(A^{\pm 1})'(x)| + \inf_{x \in \mathbb{P}^1} |(B^{\pm 1})'(x)| > 1.$$

Now, given a hyperbolic pair (A, B) satisfying (16), let I_u and I_s be disjoint closed intervals such that $\partial I_u = \{u_A, u_B\}$ and $\partial I_s = \{s_A, s_B\}$. By the choice of $\delta > 0$, we have $|A(I_u)| + |B(I_u)| > |I_u|$ (where $|\cdot|$ denotes interval length). Therefore

$$I_u = A(I_u) \cup B(I_u).$$

Let us write $A_1 = A$, $A_2 = B$. Given $z_0 \in I_u$, there exists $x_{-1} \in \{1, 2\}$ and $z_1 \in I_u$ such that $A_{x_{-1}}(z_1) = z_0$. Inductively, we find $x_{-n} \in \{1, 2\}$ and $z_n \in I_u$ such that $A_{x_{-n}}(z_n) = z_{n-1}$. We form a sequence $x = (x_i)_{i \in \mathbb{Z}} \in 2^{\mathbb{Z}}$, choosing arbitrarily x_i for $i \geq 0$. Then it is easy to see that $z_0 = e^u(x)$. This shows that $e^u(2^{\mathbb{Z}}) = I_u$. The proof that $e^s(2^{\mathbb{Z}}) = I_s$ is analogous. $\qquad \square$

Let $\mathcal{H}_{\mathrm{NP}} \subset SL(2, \mathbb{R})^N$ be the union of the non-principal components.

Proposition 4.9. *If an N-tuple is in $\overline{\mathcal{H}_{\mathrm{NP}}}$ then no product of the matrices in the N-tuple equals $\pm\mathrm{id}$.*

Furthermore, for every compact subset K of $SL(2, \mathbb{R})^N$, there exists a neighborhood V of $\{\pm\mathrm{id}\}$ such that if an N-tuple belongs to $K \cap \overline{\mathcal{H}_{\mathrm{NP}}}$ then no product of the matrices in the N-tuple belongs to V.

Proof. Given $c > 1$, let $\delta = \delta(2c)$ be given by Lemma 4.8. For a compact set of the form $K(c) = \{(A_1, \ldots, A_N) \in \mathrm{SL}(2, \mathbb{R})^N; \|A_i\| \leq c\}$, we will take V as the open neighborhood of $\{\pm\mathrm{id}\}$ of size δ.

Fix an N-tuple $\xi_0 \in K(c) \cap \overline{\mathcal{H}_{\mathrm{NP}}}$. By contradiction, assume that there exists a product of the matrices in ξ_0 which is δ-close to $\pm\mathrm{id}$.

Take $\xi = (A_1, \ldots, A_N) \in \mathcal{H}_{\mathrm{NP}}$ close to ξ_0. If ξ is close enough to ξ_0, there exists a product of the A_i's, say B, which is δ-close to $\pm\mathrm{id}$.

Fix some cyclical order on $\mathbb{P}^1$. Since ξ is not in a principal component, there exist $i, j, k, \ell \in \{1, \ldots, N\}$ such that

$$u(A_i) < s(A_j) < u(A_k) < s(A_\ell) < u(A_i).$$

Lemma 4.8 applied to the pair (A_i, B) implies that there is an interval containing $u(A_i)$ and $u(B)$, and disjoint from $\{s(A_j), s(A_\ell)\}$; in particular $u(B)$ must belong to the interval $(s(A_\ell), s(A_j))$. A symmetric argument gives $u(B) \in (s(A_j), s(A_\ell))$. We reached a contradiction. $\square$

Next, let us prove that connected components of cores associated to a N-tuple in a non-principal component are non-degenerate intervals:

Lemma 4.10. *Fix a non-principal component $H \subset \mathrm{SL}(2, \mathbb{R})^N$, and let $K \subset \mathrm{SL}(2, \mathbb{R})^N$ be a compact set. Then there exists $\delta > 0$ such that for any $\xi \in H \cap K$, each interval composing the unstable or stable cores of ξ has length at least δ.*

Proof. Assume that there exists $\xi \in H \cap K$ whose unstable core U has a connected component I which is very small. Recalling Proposition(s) 2.5 (and 2.8), there exists a product B of matrices in ξ such that $B(U) \subset I$. Moreover, we can give an upper bound for $\|B\|$ depending on H and K only. If follows that the diameter of U is small. Consider the shortest closed interval that contains U. That interval is forward-invariant by each matrix in ξ. This implies that ξ is in a principal component, contradiction. $\square$

4.3. Limit cores. The proof of Theorem 4.1 gives some useful information about the families of cores. We will register that information for later use, however we will focus on the case of full shifts, where cores are defined differently (see §2.4.2).

The analogue of Lemmas 4.6 and 4.7 for full shifts are the following:

Lemma 4.11. *Let $(A_1, \ldots, A_N)$ be uniformly hyperbolic w.r.t. the full shift, and let U be the unstable core. For any $v \in \partial U$, then there exists a symbol i such that $A_i^{-1}(v) \in \partial U$.*

The proof is analogue to that of Lemma 4.6, but let us give it for the reader's convenience:

Proof. Let $v \in \partial U$; then $v \in K^u$, so $v = e^u(x)$. Let $v' = A_i^{-1}(v)$ where $i = x_{-1}$; then $v' = e^u(\sigma^{-1}(x)) \in K^u$. Since $v \in \partial U$, there is an open interval I disjoint from K^u with endpoints v and $w \in K^s$. Then the open interval $I' = A_i^{-1}(I)$ is disjoint from K^u, has one endpoint v' in K^u and the other in K^s. This implies that $v' \in \partial U$. $\qquad\square$

From this lemma one easily gets:

Lemma 4.12. *Let $(A_1, \ldots, A_N)$ be uniformly hyperbolic w.r.t. the full shift, and let U and S be the unstable and stable cores. Let $v \in \partial U$. Then*

$$v = A_{i_m} \ldots A_{i_1} \cdot u(A_{j_k} \ldots A_{j_1})$$

for some choice of indices. (m can be zero, meaning that $v = u(A_{j_k} \ldots A_{j_1})$.) Analogously, if $v' \in \partial S$ then

$$v' = A_{i'_1}^{-1} \ldots A_{i'_p}^{-1} \cdot s(A_{j'_\ell} \ldots A_{j'_1})$$

for some choice of indices. (p can be zero.) Moreover, k, m, ℓ, and p are less or equal than the rank of U.

Using the last lemma, one shows:

Proposition 4.13. *Let H be a connected component of the hyperbolic locus relative to the full shift on N symbols. For each $i \in \mathbb{N}$, let $(A_1(i), \ldots, A_N(i)) \in H$ have unstable core $U(i)$ and stable core $S(i)$. Suppose that $(A_1(i), \ldots, A_N(i))$ converges to some $(A_1, \ldots A_N)$ in the boundary of H as $i \to \infty$. Also assume every product of the A_j's of length less or equal than the rank of the cores is different from $\pm \mathrm{id}$. Then the sets $U(i)$ and $S(i)$ converge (with respect to the Hausdorff distance) as $i \to \infty$, say to sets U and S. Moreover, the intersection $U \cap S$ is finite and non-empty.*

We call the sets U and S given by the proposition the *limit cores* of $(A_1, \ldots, A_N)$.

If H is a non-principal component then, by Proposition 4.9, the no $\pm \mathrm{id}$ assumption in Proposition 4.13 is satisfied; hence the limit cores are well-defined for each point in the boundary of H. Moreover, we have:

Proposition 4.14. *If an N-tuple belongs to the boundaries of two different non-principal components, then the respective limit cores are precisely the same.*

However, we do not know if the boundaries of two different components can meet.

Proof of the proposition. Fix an N-tuple $(A_1, \ldots, A_N)$ in the closure of a non-principal component H. Let U and S be the limit cores with respect to H.

Let K_*^u be the set of all points of the form u_P or $Q(u_P)$, where P and Q are products of the A_i's. (Recall that u_P is defined, by Proposition 4.9.) Analogously, let K_*^s be the set of all s_P and $Q^{-1}(s_P)$. Then $K_*^u \subset U$ and $K_*^s \subset S$. Also, by Lemma 4.12, $\partial U \subset K_*^u$ and $\partial S \subset K_*^s$.

We claim that no point in K_*^u is isolated. Indeed, consider a point $x = Q(u_P)$. By Lemma 4.10, ∂U, and hence K_*^u, contains at least 4 points. In particular, we can find $y \in K_*^u$ different from u_P and from s_P. The sequence $QP^n(y)$ is contained in $K_*^u \smallsetminus \{x\}$ and converges to x. This shows that x is not isolated. Symmetrically, no point in K_*^s is isolated.

It follows from these facts that the complement of the union of the connected components of $\mathbb{P}^1 \smallsetminus \overline{K_*^u}$ (resp. $\mathbb{P}^1 \smallsetminus \overline{K_*^s}$) that intersect K_*^s (resp. K_*^u) is precisely U (resp. S). This procedure describes U and S without referring to H, so the proposition follows. $\square$

4.4. An addendum for the full 2-shift.
In the light of the general results about boundaries obtained so far, let us come back to the case of the full *two*-shift and give some additional information complementing Theorem 3.2:

Proposition 4.15. *Let H be a **non-principal** connected component of the hyperbolic locus relative to the full shift on two symbols. Then the following holds:*

(i) *No $\pm$identity products exist for a pair on the boundary of H.*

(ii) *No heteroclinic connection occurs on the boundary of H.*

(iii) *There are only three words (other than their cyclic permutations and powers) that can become parabolic on the boundary of H.*

Proof. Let H be a twisted component. Assertion (i) follows from Proposition 4.9. For $(A_0, A_1) \in H$, the cores U and S are described precisely in §3.8.7 – in particular, we have the following:

(a) The sets ∂U and ∂S are respectively formed by unstable and stable directions of certain "special" products of A_0's and A_1's.

(b) If points $v \in \partial U$ and $w \in \partial S$ are "neighbors" (in the sense that there is an open interval with endpoints v and w that does not meet $U \cup S$) then they are respectively the unstable and stable directions of the *same* "special" product of A_0's and A_1's.

(c) There are three words in the letters A_0 and A_1 which are not powers and that form, together with their cyclic permutations, the full list of special words that need to be considered in (b) and (c).

(d) No connected component of U intersects both $A_0(U)$ and $A_1(U)$.

It follows from (d) and Lemma 4.11 that:

(e) For every $v \in \partial U$ there exist a unique $i \in \{0, 1\}$ such that $A_i^{-1}(v) \in \partial U$.

Repeated application of (e) gives:

(f) For any $v_0 \in \partial U$, there exists a unique sequence $i_1, i_2, \ldots$ in $\{0, 1\}$ such that $v_{j+1} = A_{i_j}^{-1}(v_j) \in \partial U$.

Now it follows from (a) that:

(g) For any $v_0 \in \partial U$, if v_j is the sequence given by (f) and ℓ is the least positive integer such that $v_\ell \in \{v_0, \ldots, v_{\ell-1}\}$ then $v_\ell = v_0$.

Now let (A_0, A_1) be in the boundary of H, and let U and S be the limit cores given by Proposition 4.13 (which are well-defined because H is not principal). By Lemma 4.10, U and S have the same number of components as before taking the limit, and none of these components is a point. It follows that Properties (d) and (e) above are also true for the limit cores. Property (f) follows from (e). So (g) makes sense for the limit cores, and it is true by continuity.

Any $v_0 \in \partial U$ equals $u(P)$ where $P = A_{i_1} \ldots A_{i_\ell}$ and the indices i_j are as in (f) and (g). The word P is not a power, and so is one the special words alluded in (a)–(c). Let $w_0 \in \partial S$ be the neighbor of v_0. (Precisely, we define w_0 as v_0 if $v_0 \in S$, otherwise we let $w_0 \in S$ be so that there is an open interval with endpoints v_0 and w_0 that does not intersect $U \cup S$.) We infer from property (b) that $w_0 = s(P)$. In particular, $v_0 \in S$ implies that P is parabolic.

Now, suppose v_0 is also given by $Ru(Q)$, where Q and R are words in the letters A_0 and A_1, with R allowed to be the empty word (corresponding to id product). It follows from uniqueness in (f) that the infinite words $RQQQ \ldots$ and $PPP \ldots$ must coincide. In particular, Q is (as a word) a power of a cyclic permutation of P. Therefore Q is parabolic (as a matrix) if and only if so is P.

By contradiction, assume there is a heteroclinic connection $Ru(Q) = s(P')$, for some products P', Q, R, of A_0's and A_1's. Then $v_0 = Ru(Q)$ belongs to $U \cap S$. Therefore, as we have seen, Q has to be parabolic. This is forbidden by definition of heteroclinic connection, so assertion (ii) of the theorem is proved. Assertion (iii) follows similarly. $\square$

4.5. An example of heteroclinic connection. In this subsection, we introduce what is probably the simplest example of heteroclinic connection for a principal component. The base dynamics is full-shift on 3 symbols. The component H of the hyperbolicity locus $\mathcal{H}$ is the one that contains triples (A, B, C) such that $(A, B) \in H_{\mathrm{id}}^+$ (the positive free component for the full-shift on two symbols) and $C = -AB$; such triples are indeed obviously uniformly hyperbolic. The associated stable and unstable cores have two components.

Proposition 4.16. *A triple (A, B, C) belongs to H iff the following conditions are satisfied:*

(i) $(A, B) \in H_{\mathrm{id}}^+$;

(ii) $\operatorname{tr} C > 2$;

(iii) *the stable and unstable directions for C satisfy*

$$s_A < u_C < s_{AB}, \quad u_{AB} < s_C < u_B, \quad s_A < u_C < s_C < u_B;$$
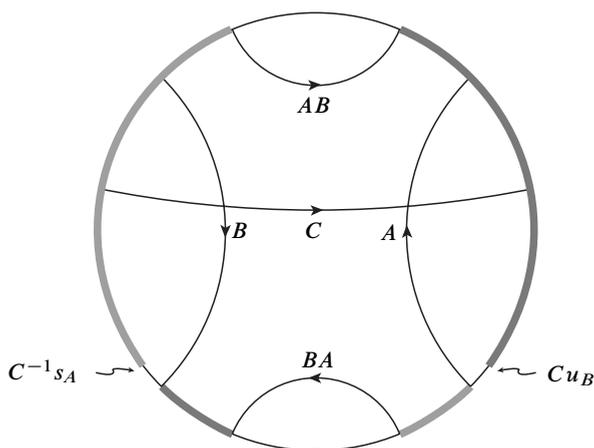
(iv) $s_A < C u_B < u_C$.

Figure 8. A possible situation for $(A, B, C) \in H$ in Proposition 4.16; the cores are indicated.

Proof. Let $\widehat{H}$ be the set of parameters defined by the 4 conditions in the proposition. Clearly, $\widehat{H}$ is open in $(\mathrm{SL}(2, \mathbb{R}))^3$. It is also clear that the boundary of $\widehat{H}$ does not intersect the hyperbolicity locus $\mathcal{H}$, and that $\widehat{H}$ contains any triple $(A, B, -AB)$ with $(A, B) \in H_{\mathrm{id}}^+$. To prove that $\widehat{H} = H$, we prove that $\widehat{H}$ is connected and contained in $\mathcal{H}$.

To see that $\widehat{H}$ is connected, we fix $(A, B) \in H_{\mathrm{id}}^+$ and check that the set of C satisfying (ii), (iii), (iv) is connected. Indeed, the set of positions for (u_c, s_c) in $\mathbb{P}^1 \times \mathbb{P}^1$ determined by (iii) is connected, and for any such position, condition (iv) is equivalent to some condition $\operatorname{tr} C > k \, (> 2)$. This proves that $\widehat{H}$ is connected.

Let $(A, B, C) \in \widehat{H}$. Define

$$U_{AC} = [\min(u_A, C u_B), \max(u_{AB}, u_C)], \quad S_A = [\min(s_{BA}, A^{-1} s_c), s_A],$$
$$U_B = [u_B, \max(u_{BA}, B u_C)], \quad S_{BC} = [\min(s_{AB}, s_C), \max(s_B, C^{-1} s_A)],$$
$$U = U_{AC} \sqcup U_B, \quad S = S_A \sqcup S_{BC}.$$

We have then

$$A(U) \cup C(U) \subset U_{AC}, \qquad B(U) \subset U_B,$$
$$B^{-1}(S) \cup C^{-1}(S) \subset S_{BC}, \quad A^{-1}(S) \subset S_A.$$

It follows from Lemma 2.7 that (A, B, C) is uniformly hyperbolic (with cores U, S). The proof is now complete.   □

    We have seen in the proof of the proposition that for fixed A, B, u_C, s_C satisfying (i), (ii), (iii), the set H is determined by a condition $\operatorname{tr} C > k$ for some $k = k(A, B, u_C, s_C) > 2$. If we take $C = C_0$ we still have a triple (A, B, C_0) such that (i), (ii), (iii) are satisfied and $C_0 u_B = s_A$. In a neighborhood V of (A, B, C_0) in $(\mathrm{SL}(2, \mathbb{R}))^3$, the equation $C u_B = s_A$ determines a smooth hypersurface contained in the boundary of H. This part of the boundary of H corresponds to a heteroclinic connection.

    We will investigate in the next two subsections what happens on the side of the hypersurface not contained in H. We already know from Proposition 6 in [13] that the other side $V \smallsetminus \bar{H}$ intersects the elliptic locus $\mathcal{E}$ (the (open) set of triples that have an elliptic product.) In the sequel we will construct two examples displaying different phenomena near boundary points:

- In one example (Proposition 4.17) we have $V \smallsetminus \bar{H} \subset \mathcal{E}$.

- In another example (Proposition 4.18), any neighborhood V intersects infinitely many hyperbolic components.

For convenience, we will assume that $s_A < u_C < u_A$ and $s_B < s_c < u_B$ (as in Figure 8).

4.6. Heteroclinic connection with elliptic products on the other side. Let $H \subset \mathrm{SL}(2, \mathbb{R})^3$ be the hyperbolic component introduced in §4.5.

Proposition 4.17. *There there exist a point (A_0, B_0, C_0) in the boundary of H, and a neighborhood $V \subset \mathrm{SL}(2, \mathbb{R})^3$ of (A_0, B_0, C_0) such that the following holds:*

- *If $(A, B, C) \in V \cap \partial H$ then $C \cdot u(B) = s(A)$.*

- *If $(A, B, C) \in V \smallsetminus \bar{H}$ then $(A, B, C) \in \mathcal{E}$ (that is, there exists an elliptic product of A, B, and C's).*

    For another example with similar properties, see Proposition 7 in [13].

Proof. Fix numbers λ, θ, and ν such that

$$1 < \lambda < 1 + \sqrt{2}, \quad \frac{\lambda^2 + 1}{\lambda^2 - 1} < \theta < \frac{2}{\lambda - 1}, \quad \nu > \theta. \tag{17}$$

Define three matrices in $SL(2, \mathbb{R})$:

$$A_0 = \begin{pmatrix} \lambda & 0 \\ -\theta(\lambda - \lambda^{-1}) & \lambda^{-1} \end{pmatrix}, \quad B_0 = \begin{pmatrix} \lambda & \theta(\lambda - \lambda^{-1}) \\ 0 & \lambda^{-1} \end{pmatrix}, \quad C_0 = \begin{pmatrix} 0 & -1 \\ 1 & v + v^{-1} \end{pmatrix}.$$

All matrices have traces > 2. The stable and unstable directions are ordered as follows:

$$u(B_0) = \begin{pmatrix} 1 \\ 0 \end{pmatrix} < s(A_0) = \begin{pmatrix} 0 \\ 1 \end{pmatrix} < u(C_0) = \begin{pmatrix} 1 \\ -v \end{pmatrix} < u(A_0) = \begin{pmatrix} 1 \\ -\theta \end{pmatrix}$$

$$< s(B_0) = \begin{pmatrix} 1 \\ -\theta^{-1} \end{pmatrix} < s(C_0) = \begin{pmatrix} 1 \\ -v^{-1} \end{pmatrix} < u(B_0).$$

Also, $C_0(u(B_0)) = s(A_0)$. Finally, due to one inequality in (17) we have

$$\operatorname{tr} A_0 B_0 = \lambda^2 - \theta^2(\lambda - \lambda^{-1})^2 + \lambda^{-2} < -2.$$

We conclude that (A_0, B_0, C_0) belongs to the boundary of the hyperbolic component H described in §4.5. Let V be a small neighborhood of this 3-tuple such that $V \smallsetminus \bar{H} = \{(A, B, C) \in V; \ u(B) < C \cdot u(B) < s(A)\}$. To complete the proof, we will show that this set is contained in $\mathcal{E}$, provided V is small enough.

For any $(A, B, C) \in V \smallsetminus \bar{H}$, take a basis of $\mathbb{R}^2$ close to the canonical basis and formed by vectors collinear to $u(B)$, $s(A)$, so that the matrices of A, B, and C become

$$A = \begin{pmatrix} \lambda_1 & 0 \\ -\theta_1(\lambda_1 - \lambda_1^{-1}) & \lambda_1^{-1} \end{pmatrix}, \quad B = \begin{pmatrix} \lambda_2 & \theta_2(\lambda_2 - \lambda_2^{-1}) \\ 0 & \lambda_2^{-1} \end{pmatrix},$$

and

$$C = \begin{pmatrix} t & -1 + td \\ 1 & d \end{pmatrix},$$

for certain numbers λ_1 and λ_2 close to λ, θ_1 and θ_2 close to θ, d close to $v + v^{-1}$, and t close to zero. Since $u(B) < C(u(B)) < s(A)$, t must be positive.

We are going to look for elliptic products of the form $A^m C B^n$. So we write

$$A^m = \begin{pmatrix} \lambda_1^m & 0 \\ -\xi_1(m) & \lambda_1^{-m} \end{pmatrix}, \quad B^n = \begin{pmatrix} \lambda_2^n & \xi_2(n) \\ 0 & \lambda_2^{-n} \end{pmatrix},$$

with $xi_1(m) = \theta_1(\lambda_1^m - \lambda_1^{-m})$ and $\xi_2(n) = \theta_2(\lambda_2^n - \lambda_2^{-n})$. A computation gives

$$\begin{aligned} \operatorname{tr} A^m C B^n &= \lambda_1^m t \lambda_2^n - \xi_1(m) t \xi_2(n) - \xi_1(m)(-1 + td)\lambda_2^{-n} \\ &\quad + \lambda_1^{-m}\xi_2(n) + \lambda_1^{-m} d \lambda_2^{-n} \\ &= -v(m, n)t + u(m, n), \end{aligned}$$

where

$$v(m, n) = \lambda_1^m \lambda_2^n (\theta_1 \theta_2 - 1)\left(1 + \mathcal{O}(\lambda_1^{-2m} + \lambda_2^{-2n})\right), \qquad (18)$$

$$u(m, n) = \theta_1 \lambda_1^m \lambda_2^{-n} + \theta_2 \lambda_1^{-m} \lambda_2^n + \mathcal{O}(\lambda_1^{-m} \lambda_2^{-n}). \qquad (19)$$

Choose a sequence (m_k, n_k) (depending on λ_1 and λ_2 only) starting at $(m_0, n_0) = (0, 0)$, such that for all k, (m_{k+1}, n_{k+1}) is either $(m_k + 1, n_k)$ or $(m_k, n_k + 1)$, and

$$\lambda_2^{-1} \leq \lambda_1^{m_k} \lambda_2^{-n_k} \leq \lambda_1. \qquad (20)$$

Write $v_k = v(m_k, n_k)$, $u_k = u(m_k, n_k)$. Assuming V is sufficiently small, there is some constant k_0 (not depending on (A, B, C) in V) such that $v_k > 0$ and $u_k > 2$ for every $k \geq k_0$.

Let

$$\delta = \max\left(|\lambda_1 - \lambda|, |\lambda_2 - \lambda|, |\theta_1 - \theta|, |\theta_2 - \theta|, |d - v - v^{-1}|\right).$$

(Notice that t does not appear above.) Let $\mathcal{O}_\delta(1)$ indicate a quantity that goes to zero as $\delta \to 0$. It follows from (18), (19), and (20) that

$$\frac{v_{k+1}}{v_k} = \lambda + \mathcal{O}_\delta(1), \quad 2\theta + \mathcal{O}_\delta(1) < u_k < \theta(\lambda + \lambda^{-1}) + \mathcal{O}_\delta(1). \qquad (21)$$

For $k \geq k_0$, define intervals

$$I_k = (\alpha_k, \beta_k) = \left(\frac{u_k - 2}{v_k}, \frac{u_k + 2}{v_k}\right).$$

Each I_k depends on $\lambda_1, \lambda_2, \theta_1, \theta_2$, and d, *but not on t*. Also,

$$|\operatorname{tr} A^{m_k} C B^{n_k}| < 2 \quad \text{iff} \quad t \in I_k.$$

We claim that if δ is sufficiently small then $I_k \cap I_{k+1} \neq \varnothing$ for all $k \geq k_0$. Indeed, using (21), we get

$$\frac{\alpha_k}{\beta_{k+1}} = \frac{u_k - 2}{u_{k+1} + 2} \cdot \frac{v_{k+1}}{v_k} \leq \frac{\theta(\lambda + \lambda^{-1}) - 2}{2\theta + 2} \cdot \lambda + \mathcal{O}_\delta(1), \qquad (22)$$

$$\frac{\alpha_{k+1}}{\beta_k} = \frac{u_{k+1} - 2}{u_k + 2} \cdot \frac{v_k}{v_{k+1}} \leq \frac{\theta(\lambda + \lambda^{-1}) - 2}{2\theta + 2} \cdot \frac{1}{\lambda} + \mathcal{O}_\delta(1). \qquad (23)$$

From the assumption $\theta < 2/(\lambda - 1)$ in (17), it follows that the right-hand side of (22) is strictly less than $1 + \mathcal{O}_\delta(1)$. The same is true for the (smaller) right-hand side of (23). Thus we have shown that if $k \geq k_0$ and δ is small enough then $\alpha_k < \beta_{k+1}$ and $\alpha_{k+1} < \beta_k$; in particular $I_k \cap I_{k+1} \neq \varnothing$. Hence for small δ, we have

$$\bigcup_{k \geq k_0} I_k = \left(\liminf_{k \to \infty} \alpha_k, \sup_{k \geq k_0} \beta_k\right) \supset (0, \beta_{k_0}).$$

The number β_{k_0} has a positive lower bound on V. Therefore, reducing the neighborhood V of (A_0, B_0, C_0) if necessary, we have that for any $(A, B, C) \in V \smallsetminus \bar{H}$, there exists some $k \geq k_0$ such that the corresponding t belongs to the corresponding I_k. This means that the matrix $A^{m_k} C B^{n_k}$ is elliptic, showing that (A, B, C) belongs to the elliptic locus $\mathcal{E}$. $\square$

4.7. An example of accumulation of components. Again consider the hyperbolic component H for the full shift in three symbols that was introduced in §4.5.

Proposition 4.18. *There exists a path $t \mapsto (A, B, C(t))$ with the following properties:*

(i) *$(A, B, C(t)) \in H$ for $t < 0$.*

(ii) *At the parameter $t = 0$, the heteroclinic connection $C(0) \cdot u_B = s_A$ occurs; in particular, $(A, B, C(0))$ belongs to ∂H.*

(iii) *There exists a sequence of hyperbolic components H_i, all different, and a sequence $t_i > 0$ converging to 0 as $i \to \infty$ such that $(A, B, C(t_i)) \in H_i$ for all i.*

(iv) *There exist a sequence $s_i > 0$ converging to 0 as $i \to \infty$ such that $(A, B, C(s_i))$ belongs to the elliptic locus $\mathcal{E}$ for all i.*

Proof. Take (A, B) in the positive free component of the full 2-shift. Assume that the order in $\mathbb{P}^1$ is so that

$$u_B < s_A < u_A < s_B < u_B.$$

Take points $p, q \in (u_{BA}, s_{BA})$ such that

$$u_{BA} < BA \cdot q < p < q < s_{BA}. \tag{24}$$

Define the following cross-ratios (recall formula (2) from §2.2):

$$\alpha = [u_A, p, q, s_A], \quad \beta = [u_B, BAq, p, s_B]. \tag{25}$$

Then $\alpha, \beta > 1$. We claim that the choices of A, B, p, q can be made so that

$$(\alpha - 1)(\beta - 1) > 1. \tag{26}$$

Indeed, if B is replaced with B^T with $T > 1$ (keeping A fixed) then (A, B) remains in the free component; moreover (24) still holds keeping p, q (and hence α) fixed. If T is large enough then so is β and (26) is satisfied.

If μ, ν are the spectral radii of A, B, respectively, we also assume that

$$\frac{\log \nu}{\log \mu} \notin \mathbb{Q}. \tag{27}$$

Take any smooth path $t \mapsto C(t)$ such that

$$\operatorname{tr} C(t) > 2, \quad s_{C(t)} \in (s_B, u_B), \quad u_{C(t)} \in (s_A, u_A) \quad \text{for all } t,$$

and

$$C(0) \cdot u_B = s_A, \quad \frac{\partial}{\partial t} C(t) \cdot u_B \Big|_{t=0} < 0. \tag{28}$$

(In particular, $C(t) \cdot u_B$ belongs to (s_A, u_C), resp. (s_C, s_A) for small negative, resp. positive t.) By Proposition 4.16, $(A, B, C(t))$ belongs to H for all small $t < 0$. So assertions (i) and (ii) of the statement hold.

Next define (disjoint) intervals

$$I_n = B^n \cdot [Bp, BAq], \quad J_m = A^{-m} \cdot [p, q] \quad \text{for integers } n, m \geq 0.$$

Define also

$$I_n^* = [u_B, B^{n+1} Aq] \quad \text{for } n \geq 0.$$

(See Figure 9.)

In the manifold $\mathbb{P}^1$ we take charts using euclidian angle; these serve to compute derivatives and speak of length of intervals. Let $\kappa > 0$ be the derivative of $C(0) \colon \mathbb{P}^1 \to \mathbb{P}^1$ at u_B. By (26), we can find $\varepsilon > 0$ such that

$$(\alpha - 1)(\beta - 1)(1 - 2\kappa\varepsilon) > 1. \tag{29}$$

We claim that

$$\text{there are sequences } n_i, m_i \uparrow +\infty \text{ such that } \kappa^{-1} - 2\varepsilon < \frac{|I_{n_i}^*|}{|J_{m_i}|} < \kappa^{-1} - \varepsilon. \tag{30}$$

Indeed, there is a projective chart (see §2.2) $P \colon \mathbb{P}^1 \to \mathbb{R} \cup \{\infty\}$ such that $P \circ B \circ P^{-1}(t) = \nu^{-2}t$. It follows that the limit $\lim_{n \to +\infty} \nu^{2n} |I_n^*|$ exists. Analogously, the limit $\lim_{m \to +\infty} \mu^{2m} |J_m|$ exists. By (27), for any N the set $\{\mu^{2m}\nu^{-2n}; \, m, \, n > N\}$ is dense in $\mathbb{R}_+$. So (30) follows.

Define also intervals

$$\tilde{J}_n = [B^{n+1} Aq, B^n p], \quad \tilde{I}_m^* = [A^{-m}q, s_A]. \tag{31}$$

Next we claim that if i is large enough and t is sufficiently close to zero then

$$|C(t) \cdot I_{n_i}^*| < |J_{m_i}|, \tag{32}$$

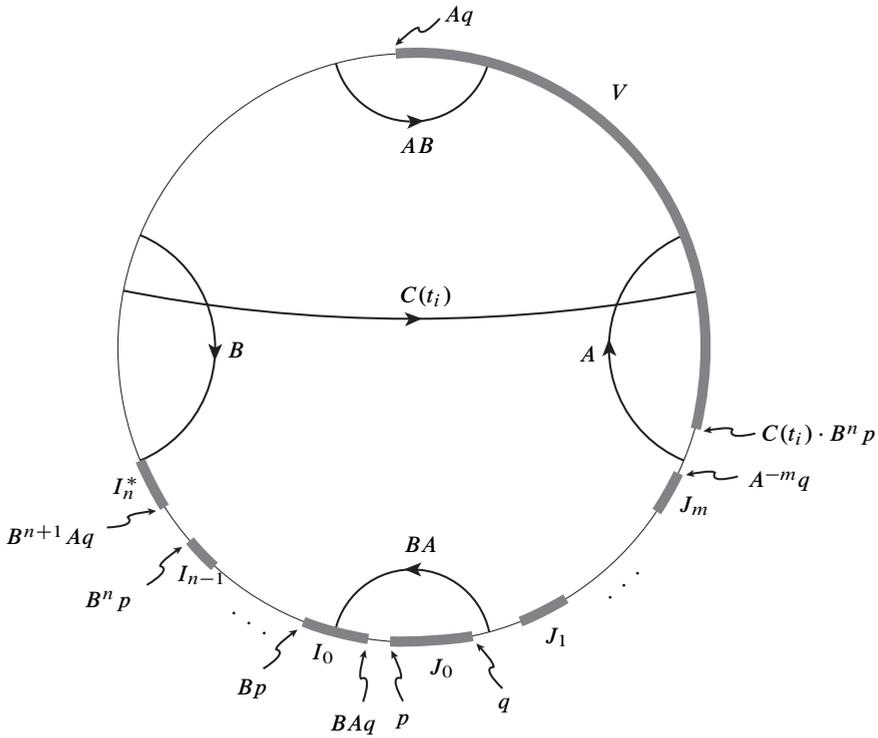$$|C(t) \cdot \tilde{J}_{n_i}| > |\tilde{I}_{m_i}^*|. \tag{33}$$

Figure 9. A "non-strict" multicone for $(A, B, C(t_i))$.

On the one hand, $|C(t) \cdot I^*_{n_i}| / |I^*_{n_i}| \to \kappa$ as $i \to \infty$ and $t \to 0$. So, by (30),

$$\limsup_{i \to \infty, \, t \to 0} \frac{|C(t) \cdot I^*_{n_i}|}{|J_{m_i}|} \le \kappa(\kappa^{-1} - \varepsilon) < 1,$$

proving (32). On the other hand, it is easy to see that

$$\alpha - 1 = \lim_{m \to +\infty} \frac{|J_m|}{|\tilde{I}^*_m|}, \quad \beta - 1 = \lim_{n \to +\infty} \frac{|\tilde{J}_n|}{|I^*_n|}.$$

So we can write

$$\liminf_{i \to \infty, \, t \to 0} \frac{|C(t) \cdot \tilde{J}_{n_i}|}{|\tilde{I}^*_{m_i}|} = \kappa \liminf_{i \to \infty} \frac{|\tilde{J}_{n_i}|}{|\tilde{I}^*_{m_i}|}$$

$$= (\alpha - 1)(\beta - 1)\kappa \liminf_{i \to \infty} \frac{|I^*_{n_i}|}{|J_{m_i}|}$$

$$\geq (\alpha - 1)(\beta - 1)\kappa(\kappa^{-1} - 2\varepsilon) \qquad \text{(by (30))}$$
$$> 1 \qquad\qquad\qquad\qquad\qquad \text{(by (29))},$$

proving (33).

Now, it follows from (28), (32), and (33) that for every sufficiently large i, there exists a small $t_i > 0$ such that

$$C(t_i) \cdot I_{n_i}^* \Subset J_{m_i} \quad \text{and} \quad C(t_i) \cdot I_{n_i-1} \Subset (s_A, u_C). \tag{34}$$

Indeed, it is sufficient to take t_i such that $C(t_i)$ maps the right endpoint of $I_{n_i}^*$ inside the interval J_{m_i} and close to its right endpoint. (See Figure 10.)
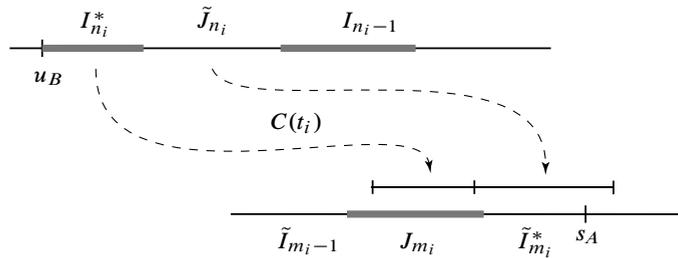

Figure 10. Proof of (34).

Next we claim that for every sufficiently large i, the 3-tuple $(A, B, C(t_i))$ is uniformly hyperbolic. For simplicity of writing, let i be fixed and let $n = n_i$, $m = m_i$, $C = C(t_i)$. Let $V = V_i$ be the interval $[C(t_i) \cdot B^n p, Aq]$. The set (see Figure 9)

$$U_i = I_{n_i}^* \cup I_{n_i-1} \cup \cdots \cup I_0 \cup J_0 \cup \cdots \cup J_{m_i} \cup V_i \tag{35}$$

is mapped inside itself by each of the maps A, B, and C. Indeed, the intervals are mapped into themselves as follows:

	I_n^*	I_{n-1}	I_{n-2}	...	I_0	J_0	J_1	...	J_m	V
A	V	V	V	...	V	V	J_0	...	J_{m-1}	V
B	I_n^*	I_n^*	I_{n-1}	...	I_1	I_0	I_0	...	I_0	I_0
C	J_m	V	V	...	V	V	V	...	V	V

We want to apply Lemma 2.7 with $U = U_i$ given by (35); thus we need to define also a set $S = S_i$. We will make use of the symmetry of the example. Define a new family of triples

$$(\tilde{A}, \tilde{B}, \tilde{C}(t)) = (B^{-1}, A^{-1}, C(t)^{-1}).$$

We claim that the new triples meets all the requirements we imposed on $(A, B, C(t))$, if we consider on $\mathbb{P}^1$ the reverse cyclic order. Indeed, let $\tilde{p} = p$ and $\tilde{q} = BAq$. Define new cross-ratios $\tilde{\alpha}$, $\tilde{\beta}$ as in (25) (but with reversed order); then $\tilde{\alpha} = \beta$ and $\tilde{\beta} = \alpha$, so the new (26) still holds. Other conditions as (27) and (28) are easily checked. Consider the new families of intervals $\tilde{I}_m$, $\tilde{J}_n$, $\tilde{I}_m^*$ (it is convenient to swap the letters in the indices); then $\tilde{I}_m$ is the gap between J_m and J_{m+1} and $\tilde{J}_n$ is the gap between I_n and I_{n+1}. (In particular the notation (31) is coherent.) The relevant condition on m_i, n_i, t_i is (34). Its dual version is:

$$C(t_i)^{-1} \cdot \tilde{I}_{m_i}^* \Subset \tilde{J}_{m_i} \quad \text{and} \quad C(t_i)^{-1} \cdot \tilde{I}_{m_i-1} \Subset (s_C, u_B). \tag{36}$$

An inspection of Figure 10 shows that it is true. Let $\widetilde{V}_i = [Aq, C(t_i)^{-1} A^{-m} p]$. Then the set $S_i = \tilde{I}_{m_i}^* \cup \tilde{I}_{m_i-1} \cup \cdots \cup \tilde{I}_0 \cup \tilde{J}_0 \cup \cdots \cup \tilde{J}_{m_i} \cup \widetilde{V}_i$ is sent inside itself for A^{-1}, B^{-1}, and $C(t_i)^{-1}$.

This still not good if we want to apply Lemma 2.7 because S_i is not disjoint from U_i. To remedy that, it suffices for each i to make $\tilde{J}_0$ slightly smaller (making sure (36) is still satisfied) and modify the definition of S_i accordingly. In this way we can apply the lemma and conclude that $(A, B, C(t_i))$ is hyperbolic.

Next, we claim that:

$$k, \ell \geq 0 \quad \Rightarrow \quad \operatorname{tr} C(t_i) B^\ell A^k \begin{cases} < -2 & \text{if } k \geq m_i + 1 \text{ and } \ell \geq n_i + 1, \\ > 2 & \text{otherwise.} \end{cases} \tag{37}$$

Although the proof is not difficult, we prefer to postpone it to §5.4. Recall from (30) that the sequences (n_i) and (m_i) are strictly increasing. Then it follows from (37) that $(A, B, C(t_i))$ and $(A, B, C(t_j))$ do not belong to the same connected component of $\mathcal{H}$ if $i \neq j$. This proves assertion (iii) of the proposition.

At last, by (37) again, for every i there exists s_i between t_i and t_{i+1} such that $\operatorname{tr} C(s_i) B^{n_i+1} A^{m_i+1} = 0$, so $(A, B, C(s_i))$ belongs to the elliptic locus. This proves the last assertion of the proposition. $\square$

Remark 4.19. With a little additional work, one can find the unstable and stable cores for $(A, B, C(t_i))$; they are given by the subintervals below (again we write $n = n_i$, $m = m_i$, $C = C(t_i)$ for simplicity):

$[u_B, u(B^{n+1} A^{m+1} C)] \subset I_n^*,$ $\qquad [s(B^{n+1} A^{m+1} C), C^{-1} s_A] \subset \tilde{J}_n,$

$[B^n A^m C u_B, u(B^n A^{m+1} CB)] \subset I_{n-1},$ $\quad [s(B^n A^{m+1} CB), B^{-1} C^{-1} s_A] \subset \tilde{J}_{n-1},$

$\qquad \qquad \cdots$ $\qquad \qquad \qquad \cdots$

$[BA^m C u_B, u(BA^{m+1} CB^n)] \subset I_0,$ $\qquad [s(BA^{m+1} CB^n), B^{-n} C^{-1} s_A] \subset \tilde{J}_0,$

$[A^m C u_B, u(A^m CB^{n+1} A)] \subset J_0,$ $\qquad [s(A^m CB^{n+1} A), A^{-1} B^{-n} C^{-1} s_A] \subset \tilde{I}_0,$

$\qquad \qquad \cdots$ $\qquad \qquad \qquad \cdots$

$$[ACu_B, u(ACB^{n+1}A^m)] \subset J_{m-1}, \qquad [s(ACB^{n+1}A^m), A^{-m}B^{-n}C^{-1}s_A] \subset \tilde{I}_{m-1},$$
$$[Cu_B, u(CB^{n+1}A^{m+1})] \subset J_m, \qquad [s(CB^{n+1}A^{m+1}), s_A] \subset \tilde{I}_m^*,$$
$$[CB^n A^m Cu_B, u(A^{m+1}CB^{n+1})] \subset V, \qquad [s(A^{m+1}CB^{n+1}), C^{-1}A^{-m}B^{-n}C^{-1}s_A] \subset \tilde{V}.$$

In particular, the rank of the cores for the component H_i is $m_i + n_i + 3$; so we get another proof that $H_i \neq H_j$ if $i \neq j$.

5. Combinatorial multicone dynamics

5.1. The setting

5.1.1. A *pair of combinatorial multicones* is a finite cyclically ordered set M which is partitioned into 2 disjoint subsets M_s, M_u of the same cardinality which are met alternately according to the cyclic ordering. The subset M_s is the *stable combinatorial multicone*, the subset M_u is the *unstable combinatorial multicone* in the pair. The integer $q = \#M_s = \#M_u = \frac{1}{2}\#M$ is the *rank* of M.

5.1.2. A *correspondence* on M is a subset of $M \times M$.

Given two correspondences C, C' on M, their product $C \circ C'$ is defined by

$$C \circ C' = \{(x, z); \text{ there exists } y \in M \text{ such that } (x, y) \in C, (y, z) \in C'\}.$$

This composition law is obviously *associative*; the diagonal in $M \times M$ is an identity (both left and right). Thus correspondences form a monoid.

5.1.3. Let C be a correspondence on M. We say that C is *monotonic* if the following properties hold:

- $C \subset (M_s \times M_s) \sqcup (M_u \times M_u)$;
- $C \cap (M_s \times M_s)$ is the graph $\{(C_s(x_s), x_s); x_s \in M_s\}$ of a map $C_s \colon M_s \to M_s$;
- $C \cap (M_u \times M_u)$ is the graph $\{(x_u, C_u(x_u)); x_u \in M_u\}$ of a map $C_u \colon M_u \to M_u$;
- C can be endowed with a cyclic ordering such that the element next to (x, y) is either (x^{++}, y) or (x^+, y^+) or (x, y^{++}), where x^+ (resp. y^+, x^{++}, y^{++}) denotes the element next to x (resp. to y, x^+, y^+).

Observe that the cyclic ordering on C is uniquely defined by the latter property: if for instance $(x, y) \in M_u \times M_u$, then either x^+ belongs to the image of C_s and the next element is (x^+, y^+), or it is not the case and the next element is (x^{++}, y). Similarly, if $(x, y) \in M_s \times M_s$ then the next element is (x^+, y^+) if $y^+ \in \operatorname{Im} C_u$, and (x, y^{++}) otherwise.

The last condition (existence of the cyclic ordering) in the definition of monotonicity may be reformulated as follows:

- for $(x_u, y_u) \in C \cap (M_u \times M_u)$ we must have $x_u^+ = C_s(y_u^+)$ if $x_u^+ \in \operatorname{Im} C_s$ and $y_u = C_u(x_u^{++})$ if $x_u^+ \notin \operatorname{Im} C_s$;
- for $(x_s, y_s) \in C \cap (M_s \times M_s)$ we must have $y_s^+ = C_u(x_s^+)$ if $y_s^+ \in \operatorname{Im} C_u$ and $x_s = C_s(y_s^{++})$ if $y_s^+ \notin \operatorname{Im} C_u$.

Obviously, a monotonic correspondence must satisfy

$$\#C = \#M = 2 \operatorname{rk}(M),$$
$$1 \le \# \operatorname{Im} C_s = \# \operatorname{Im} C_u \le \operatorname{rk}(M).$$

5.1.4. Examples

- The diagonal (or identity) correspondence is monotonic.
- Let $a_s \in M_s$, $a_u \in M_u$; set

$$C_{a_s, a_u} = M_u \times \{a_u\} \sqcup \{a_s\} \times M_s$$

(i.e., C_s, C_u are the constant maps with values a_s, a_u respectively). This correspondence is monotonic and is called a *constant* correspondence (with values a_s, a_u). The left or right composition of a monotonic correspondence with any constant correspondence is a constant correspondence.
- See Figures 11 and 12 for more examples.

5.1.5. Elementary properties

5.1.5.1. The composition $C \circ C'$ of monotonic correspondences is monotonic.

Proof. Let C_s, C_u, C_s', C_u' be the maps associated with C, C'. From the definition of the composition law, we see that $C \circ C' \subset (M_s \times M_s) \cup (M_u \times M_u)$ with

$$(C \circ C') \cap (M_s \times M_s) = \{(C_s \circ C_s'(x_s), x_s); \ x_s \in M_s\},$$
$$(C \circ C') \cap (M_u \times M_u) = \{(x_u, C_u' \circ C_u(x_u)); \ x_u \in M_u\}.$$

Let $(x_u, z_u) \in (C \circ C') \cap (M_u \times M_u)$; set $y_u = C_u(x_u)$, so we have $z_u = C_u'(y_u)$.
- If $x_u^+ \notin \operatorname{Im} C_s$, then also $x_u^+ \notin \operatorname{Im} C_s \circ C_s'$ and we have $y_u = C_u(x_u^{++})$, $z_u = C_u' \circ C_u(x_u^{++})$.
- Assume $x_u^+ \in \operatorname{Im} C_s$; then $x_u^+ = C_s(y)$ if and only if $y \in M_s$ is between $y_u = C_u(x_u)$ and $C_u(x_u^{++})$. If no such y belongs to $\operatorname{Im} C_s'$, we must have

$$C_u'(C_u(x_u^{++})) = C_u'(C_u(x_u)).$$

Otherwise, let y_s be the first y in $\operatorname{Im} C_s'$ between y_u and $C_u(x_u^{++})$; we have

$$C_u'(y_s^-) = C_u'(C_u(x_u)) = z_u, \quad y_s = C_s'(z_u^+), \quad x_u^+ = C_s(C_s'(z_u^+)).$$
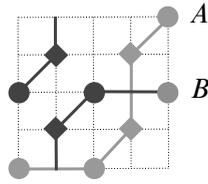
Figure 11. Two (constant) monotonic correspondences A and B (related to a free uniformly hyperbolic pair). The rank of $\boldsymbol{M}$ is 2. The borders of the square should be identified in a torus-like way. Circles and squares denote points in $\boldsymbol{M}_u \times \boldsymbol{M}_u$ and $\boldsymbol{M}_s \times \boldsymbol{M}_s$, respectively.
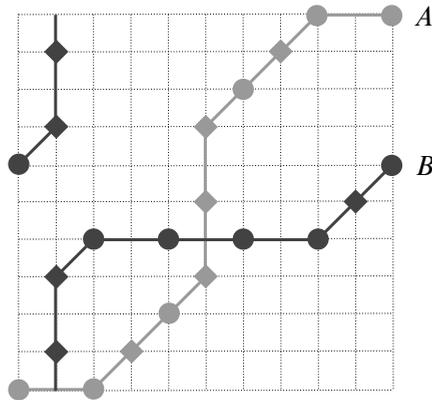

Figure 12. Two monotonic correspondences A and B (related to the situation of Figure 2). The rank of $\boldsymbol{M}$ is 5.

We have checked the first half of the condition for the existence of the cyclic ordering on $C \circ C'$; the other half is checked in a symmetric way. $\qquad \square$

5.1.5.2. We have seen that

$$\# \operatorname{Im} C_s = \# \operatorname{Im} C_u.$$

In particular, C_s is a constant map iff C_u is a constant map; in this case, the values of C_s and C_u are independent.

However, when C_s is *not* a constant map, there is *at most* one monotonic correspondence C such that $C \cap (\boldsymbol{M}_s \times \boldsymbol{M}_s)$ is the graph of C_s. (And similarly when we exchange the roles of C_s and C_u.) More precisely, such a monotonic correspondence exists if and only if the map C_s is *monotonic* (increasing) in the following sense: For any $x_s \in \boldsymbol{M}_s$, either $C_s(x_s^{++}) = C_s(x_s)$ or there is no point of the image of C_s strictly between $C_s(x_s)$ and $C_s(x_s^{++})$; we then have $x_s^+ = C_u(x_u)$ for $x_u \in \boldsymbol{M}_u$ between $C_s(x_s)$ and $C_s(x_s^{++})$.

5.2. Free monoids of monotonic correspondences

5.2.1. We have seen that the monotonic correspondences on a pair of combinatorial multicones $M = M_s \sqcup M_u$ form a monoid that we denote by $\mathcal{C}(M)$.

Let $N \geq 1$ and let $\mathcal{F}_N$ be the free monoid on N generators. Let $\Phi \colon \mathcal{F}_N \to \mathcal{C}(M)$ be a morphism, uniquely determined by the images $C^{(1)}$, ..., $C^{(N)}$ of the canonical generators of $\mathcal{F}_N$.

5.2.2. The morphism is called *hyperbolic* if there exists $\ell \geq 1$ such that the image of any word of length $\geq \ell$ in the generators is a constant correspondence.

5.2.3. The morphism is called *tight* if we have

$$\bigcup_{i=1}^{N} \text{Im } C_u^{(i)} = M_u \quad \text{and} \quad \bigcup_{i=1}^{N} \text{Im } C_s^{(i)} = M_s.$$

A justification for this definition and terminology is the following: assume for instance that some $x'_u \in M_u$ does not belong to any $\text{Im } C_u^{(i)}$, $1 \leq i \leq N$; then we have

$$C_s^{(i)}((x'_u)^-) = C_s^{(i)}((x'_u)^+) \quad \text{for all } 1 \leq i \leq N.$$

Consider the pair of combinatorial multicones $M' = M'_s \sqcup M'_u$ where $M'_u = M_u \smallsetminus \{x'_u\}$ and M'_s is deduced from M_s by identifying $(x'_u)^-$ with $(x'_u)^+$; M' is equipped with the obvious cyclic ordering. One can define in an obvious way correspondences $C^{(i)'}$, $1 \leq i \leq N$ on M', and the study of the morphism $\Phi \colon \mathcal{F}_N \to \mathcal{C}(M)$ reduces to a morphism $\Phi' \colon \mathcal{F}_N \to \mathcal{C}(M')$ with a smaller pair of combinatorial multicones.

5.2.4. We would like to analyze tight hyperbolic morphisms.

For $N = 1$, a morphism is tight iff the correspondence $C^{(1)}$ is invertible, and then it cannot be hyperbolic except in the trivial case where the rank is 1.

In §5.5, we will determine all tight hyperbolic morphisms when $N = 2$.

5.3. Relation with matrices.
Let us see how a uniformly hyperbolic N-tuple of matrices induces a tight hyperbolic morphism.

Let $(A_1, \ldots, A_N) \in \text{SL}(2, \mathbb{R})^N$ be uniformly hyperbolic. Let U and S be respectively the unstable and stable cores. Let M_u, resp. M_s, be the set of connected components of U, resp. S. Give $M = M_u \sqcup M_s$ the cyclic order induced from $\mathbb{P}^1$. Then M is a pair of combinatorial multicones.

For each $i = 1, \ldots, N$, let $C^{(i)}$ be the subset of $(M_u \times M_u) \sqcup (M_s \times M_s)$ formed by the pairs (x, y) such that $A_i(x) \cap y \neq \varnothing$.

Lemma 5.1. *Each $C^{(i)}$ is a monotonic correspondence. Moreover, the morphism $\Phi \colon \mathcal{F}_N \to \mathcal{C}(M)$ determined by $C^{(1)}$, ..., $C^{(N)}$ is tight and hyperbolic.*

(The same Φ could also be obtained from a tight multicone M and its dual $\mathbb{P}^1 \smallsetminus \overline{M}$ in an obvious way, see Proposition 2.8.)

Proof of the lemma. Fix i, and let us show that $C^{(i)}$ is monotonic. First, if $(x_u, y_u) \in C^{(i)} \cap (M_u \times M_u)$ then $A_i(x_u) \subset y_u$, so x_u uniquely determines y_u. Write $y_u = C_u^{(i)}(x_u)$. Analogously, if $(x, y) \in C^{(i)} \cap (M_s \times M_s)$ then $A_i^{-1}(y_s) \subset x_s$, so y_s determines $x_s = C_s^{(i)}(y_s)$.

Next, let $(x_u, y_u) \in C^{(i)} \cap (M_u \times M_u)$. In the case that $x_u^+ \notin \mathrm{Im}\, C_s^{(i)}$ then we must have $C_u^{(i)}(x_u^{++}) = y_u$. (Because if $C_u^{(i)}(x_u^{++}) \neq C_u^{(i)}(x_u)$ then there would exist a point in the unstable core S between the intervals $C_u^{(i)}(x_u)$ and $C_u^{(i)}(x_u^{++})$; this point would be sent by A_i^{-1} into a point in S between x_u and x_u^{++}, and hence in x_u^+, contradicting the fact that $x_u^+ \notin \mathrm{Im}\, C_s^{(i)}$.) And in the case that $x_u^+ \in \mathrm{Im}\, C_s^{(i)}$ then we must have $C_s^{(i)}(y_u^+) = x_u^+$. (Indeed, x_u^+ is the $C_s^{(i)}$ image of some z_s; if $z_s = y_u^+$ we are done; otherwise y_u^+ is between the U-interval y_u and S-interval z_s; then the interval $A_i^{-1}(y_u^+)$ is between $A_i^{-1}(y_u) \supset x_u$, and $A_i^{-1}(z_s) \subset x_u^+$, and so it must be contained in the interval x_u^+, showing that $x_u^+ = C_s^{(i)}(y_u^+)$.) This proves "one half" of the monotonicity of $C^{(i)}$, and the other half is completely analogous.

The induced morphism Φ is clearly tight, while hyperbolicity follows from Proposition 2.5. $\qquad\square$

In view of the lemma, we call Φ the morphism induced by $(A_1, \ldots, A_N)$. Examples from Figures 11 and 12 are induced by matrices.

Sometimes we call these data (that is, the morphism Φ) the *combinatorics* of $(A_1, \ldots, A_N)$. The combinatorics is an *invariant* in the sense that in remains the same inside each connected component of $\mathcal{H}$. (More precisely, if two N-tuples belong to the same connected component then they induce conjugate morphisms.)

Let us very briefly return to the topic of the boundary of the hyperbolic components:

Theorem 5.2. *Non-principal components of $\mathcal{H}$ with different combinatorics have disjoint boundaries.*

Proof. For each N-tuple in the boundary of a non-principal component H, the limit cores are defined (by Propositions 4.9 and 4.13). These limit cores induce a tight hyperbolic morphism Φ in an obvious way. In fact Φ is the same (ie, conjugate to the) morphism determined by the component H itself. Now, if the N-tuple belongs also to the boundary of another component H_1, then the limit cores relative to H_1 are exactly the same as before, by Proposition 4.14. It follows that H and H_1 have the same combinatorics. $\qquad\square$

5.4. Winding numbers

5.4.1. The winding numbers for a uniformly hyperbolic N-tuple. As mentioned above, the combinatorics $\Phi \colon \mathcal{F}_N \to \mathcal{C}(M)$ is an invariant on $\mathcal{H}$. A much more elementary invariant was introduced in [13]; it is the map $\tau \colon \mathcal{F}_N \to \{+1, -1\}$ that gives the signs of the traces.

Here we will introduce another elementary (in the sense that it does not depend on the multicones) invariant called the *winding number*; it is a map $n \colon \mathcal{F}_N \to \mathbb{Z}$.

Fix a cyclic order on $\mathbb{P}^1$, and identify $\mathbb{P}^1$ with $\mathbb{R}/\mathbb{Z}$ via an orientation-preserving homeomorphism. So any $A \in \mathrm{SL}(2, \mathbb{R})$ induces an orientation-preserving homeomorphism $A \colon \mathbb{R}/\mathbb{Z} \to \mathbb{R}/\mathbb{Z}$. Then we can lift A with respect to the covering map $\mathbb{R} \to \mathbb{R}/\mathbb{Z}$ and obtain a homeomorphism $\hat{A} \colon \mathbb{R} \to \mathbb{R}$.

Now, let a uniformly hyperbolic N-tuple $(A_1, \ldots, A_N)$ be given. Since each A_i is hyperbolic, it has a unique lift $\hat{A}_i$ whose graph intersects the diagonal of $\mathbb{R}^2$. Given a word $\omega = A_{i_j} \ldots A_{i_1}$, its winding number $n(\omega)$ is defined as the only integer n such that

$$\hat{A}_{i_j} \circ \cdots \circ \hat{A}_{i_1}(x_0) = x_0 + n \quad \text{for some } x_0 \in \mathbb{R}.$$

It is clear that the winding number map $n \colon \mathcal{F}_N \to \mathbb{Z}$ is an invariant, ie, it depends only on the connected component of $\mathcal{H}$ the hyperbolic N-tuple is in.

Let us see that the trace signs τ essentially depend only on n. More precisely, if $\mathrm{tr}\, A_1$, ..., $\mathrm{tr}\, A_j$ are all positive, then the sign of $\mathrm{tr}\, A_{i_j} \ldots A_{i_1}$ is $(-1)^n$, where n is the winding number of the word. To see this fact, first notice that if we substitute the covering map $\mathbb{R} \to \mathbb{R}/\mathbb{Z} = \mathbb{P}^1$ with the double covering $S^1 \to \mathbb{P}^1$ along the definition of the winding number, then we obtain the invariant $n \bmod 2$. And the relation between that invariant and signs of eigenvalues is transparent.

To give an example, let us compute the winding numbers for the positive free component of $\mathrm{SL}(2, \mathbb{R})^2$. Consider a word ω in the letters A and B that contains both (otherwise the winding number is zero). Notice that the winding number of a word is left invariant by cyclic permutations. (That is a general fact.) So we can assume the word is of the form $\omega = A^{k_1} B^{\ell_1} A^{k_2} B^{\ell_2} \ldots A^{k_m} B^{\ell_m}$, with all k_i, ℓ_i positive. Then the winding number of ω is $-m$. (The winding numbers are opposite for the free component obtained from the positive by conjugation with an orientation-reversing linear map.)

Let us pause our general discussion to give the:

Completion of the proof of Proposition 4.18. We need to prove (37). Let k, $\ell \geq 0$ and consider the matrix $C(t_i) B^\ell A^k$. Notice that its expanding direction is in V if $\ell \leq n_i$ and in J_{m_i} otherwise. Looking at the action of the lifts on that fixed point, we see that if $k \geq m_i + 1$ and $\ell \geq n_i + 1$ then the winding number of is -1, otherwise it is zero. $\qquad\square$

5.4.2. Combinatorial definition of winding numbers. Fix a pair of combinatorial multicones M, and let q be its rank. Identify M with $\mathbb{Z}/2q\mathbb{Z}$ via some bijection that preserves the cyclic orders; such identification will remain fixed in the sequel. Let $x \in \mathbb{Z} \mapsto \bar{x} \in \mathbb{Z}/2q\mathbb{Z}$ be the quotient map.

A subset $\widehat{C}$ of $\mathbb{Z}^2$ is called a *lifted correspondence* if there exists a monotonic correspondence C on M such that the following properties hold:

- if $(x, y) \in \widehat{C}$ then $(\bar{x}, \bar{y}) \in C$;
- there is a bijection between $\mathbb{Z}$ and $\widehat{C}$ such that if we endow $\widehat{C}$ with the order induced from $\mathbb{Z}$ then the element next to (x, y) is $(x + 2, y)$, or $(x + 1, y + 1)$, or $(x, y + 2)$, according to whether the element in C next to $(\bar{x}, \bar{y})$ is $(\bar{x}^{++}, \bar{y})$, or $(\bar{x}^+, \bar{y}^+)$, or $(\bar{x}, \bar{y}^{++})$.

We also say $\widehat{C}$ is a *lift* of C. Notice $\widehat{C}$ is invariant by the translation of $\mathbb{Z}^2$ by $(2q, 2q)$, in other words, $\widehat{C} = \widehat{C} + (2q, 2q)$. Also notice that if $\widehat{C}, \widehat{C}_1$ are two lifts of the same monotonic correspondence C then there is an unique $n \in \mathbb{Z}$ such that $\widehat{C}_1 = \widehat{C} + (2qn, 0)$.

Composition of lifted correspondences is defined in a similar manner as for monotonic correspondences. Associativity holds (the proof is similar). Also, the composition of lifts is a lift of the composition of two monotonic correspondences.

If a monotonic correspondence C is hyperbolic (in the sense that some power of it is a constant) then for every lift $\widehat{C}$ of C there is a unique $n \in \mathbb{Z}$ such that $\widehat{C}$ contains a point of the form $(x, x + 2qn)$; such number n is called the *height* of $\widehat{C}$.

Now let $\Phi \colon \mathcal{F}_N \to \mathcal{C}(M)$ be a tight hyperbolic morphism. Let $a_1, \dots, a_N$ be the canonical generators of $\mathcal{F}_N$, and let the correspondences $C^{(1)}, \dots, C^{(N)}$ be their respective images by Φ. Let $\widehat{C}^{(i)}$ be the unique lift of $C^{(i)}$ of height zero.

The *winding number* $n(\omega)$ of a word $\omega = a_{i_j} \dots a_{i_1}$ in $\mathcal{F}_N$ is the height of the lifted correspondence $\widehat{C}^{(i_j)} \circ \cdots \circ \widehat{C}^{(i_1)}$. The winding number of the empty word is defined as zero.

(Notice that winding numbers do not depend on the identification between M and $\mathbb{Z}/2q\mathbb{Z}$.)

It is easy to see that if the morphism $\Phi \colon \mathcal{F}_N \to \mathcal{C}(M)$ is induced by a hyperbolic N-tuple, then our two definitions of winding numbers give the same results.

5.4.3. A non-vanishing property

Lemma 5.3. *If the rank of M is bigger than 1 then there is a word ω such that* $n(\omega) = \pm 1$.

Proof. It follows immediately from the definition of the winding number that, for any word ω and any letter a_i, one has

$$|n(\omega a_i) - n(\omega)| \le 1, \quad |n(a_i \omega) - n(\omega)| \le 1.$$

On the other hand, let e_1, e_2, e_3, e_4 be elements of M such that

$$e_1, e_3 \in M_s, \quad e_2, e_4 \in M_u, \quad e_1 < e_2 < e_3 < e_4 < e_1.$$

As Φ is hyperbolic and tight, there exist words ω_{12} and ω_{34} such that the image of ω_{12} is the constant correspondence $C_{e_1 e_2}$ and the image of ω_{34} is $C_{e_3 e_4}$. We claim that

$$n(\omega_{12} \omega_{34}) = n(\omega_{12}) + n(\omega_{34}) - 1. \tag{38}$$

Indeed, let $\hat{C}_{e_1 e_2}$ and $\hat{C}_{e_3 e_4}$ be the lifts of $C_{e_1 e_2}$ and $C_{e_3 e_4}$ whose heights are $n(\omega_{12})$, $n(\omega_{34})$, respectively. Take integers $k_1 < k_2 < k_3 < k_4$ such that $\bar{k}_i = e_i$ and $k_4 - k_1 < 2q$. Then

$$\left(k_2, k_4 + 2q(n(\omega_{34}) - 1)\right) \in \hat{C}_{e_3 e_4} \quad \text{and} \quad \left(k_4, k_2 + 2q\,n(\omega_{12})\right) \in \hat{C}_{e_1 e_2}.$$

Therefore

$$\left(k_2, k_2 + 2q(n(\omega_{12}) + n(\omega_{34}) - 1)\right) \in \hat{C}_{e_1 e_2} \circ \hat{C}_{e_3 e_4},$$

proving (38). The lemma now follows at once. $\qquad\square$

Lemma 5.3 has the following consequence: If we restrict ourselves to N-tuples $(A_1, \ldots, A_N)$ with tr A_1, ..., tr A_N all positive, then there is a unique component of $\mathcal{H}$ where all products of A_i's have positive trace, namely the principal component. This answers positively Question $1'$ of [13].

5.5. Tight hyperbolic morphisms for $N = 2$.
The aim of this section is to prove the following result:

Proposition 5.4. *Every tight hyperbolic morphism $\Phi\colon \mathcal{F}_2 \to \mathcal{C}(M)$ is induced by some uniformly hyperbolic pair of matrices.*

5.5.1. When the rank of M is 1, there is only one monotonic correspondence on M, namely the identity (ie, the diagonal in $M \times M$). Therefore, for any $N \geq 1$, there is exactly one morphism $\Phi\colon \mathcal{F}_N \to \mathcal{C}(M)$. It is tight and hyperbolic.

From now on, we assume that the rank q of M is at least 2.

5.5.2. Fix some tight hyperbolic morphism $\Phi\colon \mathcal{F}_2 \to \mathcal{C}(M)$ and write A, B instead of $C^{(1)}$, $C^{(2)}$ for the images of the generators of $\mathcal{F}_2$.

Lemma 5.5. *There exist two distinct points $x_s^{(0)}$, $x_s^{(1)}$ in M_s such that*

$$A_s(x_s^{(0)}) = A_s(x_s^{(1)}), \quad B_s(x_s^{(0)}) = B_s(x_s^{(1)}).$$

Similarly, there exist two distinct points $x_u^{(0)}$, $x_u^{(1)}$ in M_u such that

$$A_u(x_u^{(0)}) = A_u(x_u^{(1)}), \quad B_u(x_u^{(0)}) = B_u(x_u^{(1)}).$$

Remark 5.6. We will see later that $\{x_s^{(0)}, x_s^{(1)}\}$, $\{x_u^{(0)}, x_u^{(1)}\}$ are uniquely determined by these properties.

Proof of the lemma. We prove the first half of the lemma. Take two distinct points x_s, x_s' in $\boldsymbol{M}_s$. If the conclusion of the lemma does not hold, one can construct inductively arbitrarily long words w such that

$$[\Phi(w)]_s(x_s) \neq [\Phi(w)]_s(x_s'),$$

which contradicts hyperbolicity. $\qquad\square$

5.5.3. Let $x_s^{(0)}$, $x_s^{(1)}$, $x_u^{(0)}$, $x_u^{(1)}$ be as in Lemma 5.5. Renaming if necessary $x_s^{(0)}$ and $x_s^{(1)}$, we can assume that the image of A_u contains a point between $x_s^{(0)}$ and $x_s^{(1)}$.

Lemma 5.7. *The image of A_u is the set of points in $\boldsymbol{M}_u$ between $x_s^{(0)}$ and $x_s^{(1)}$. The image of B_u is the set of points in $\boldsymbol{M}_u$ between $x_s^{(1)}$ and $x_s^{(0)}$.*

Proof. As $A_s(x_s^{(0)}) = A_s(x_s^{(1)})$, it follows from the definition of monotonicity that there cannot be any point of the image of A_u between $x_s^{(1)}$ and $x_s^{(0)}$. Therefore, as $\boldsymbol{M}_u = \operatorname{Im} A_u \sqcup \operatorname{Im} B_u$, every point in $\boldsymbol{M}_u$ between $x_s^{(1)}$ and $x_s^{(0)}$ belongs to the image of B_u. Exchanging A_u, B_u we get all the conclusions of the lemma. $\qquad\square$

In the same manner, after renaming if necessary $x_u^{(0)}$, $x_u^{(1)}$, we see that $\operatorname{Im} A_s$ is the set of points in $\boldsymbol{M}_s$ between $x_u^{(1)}$ and $x_u^{(0)}$, while $\operatorname{Im} B_s$ is the set of points in $\boldsymbol{M}_s$ between $x_u^{(0)}$ and $x_u^{(1)}$.

It follows immediately from Lemma 5.7 that $x_s^{(0)}$, $x_s^{(1)}$, $x_u^{(0)}$, $x_u^{(1)}$ are now uniquely defined.

5.5.4. Next we identify invariant intervals for the maps A_u, B_u, A_s, B_s.

Lemma 5.8. *We have $A_u([x_u^{(0)}, x_u^{(1)}]) \subset [x_u^{(0)}, x_u^{(1)}]$ and similarly $B_u([x_u^{(1)}, x_u^{(0)}]) \subset [x_u^{(1)}, x_u^{(0)}]$, $A_s([x_s^{(1)}, x_s^{(0)}]) \subset [x_s^{(1)}, x_s^{(0)}]$, $B_s([x_s^{(0)}, x_s^{(1)}]) \subset [x_s^{(0)}, x_s^{(1)}]$.*

Proof. We prove the first statement. As the image of A_u^{n+1} is contained in the image of A_u^n, we deduce from the hyperbolicity of Φ that there exists $x^* \in \boldsymbol{M}_u$ such that $A_u(x^*) = x^*$ and $\operatorname{Im} A_u^n = \{x^*\}$ for large n. If one had $x^* \notin [x_u^{(0)}, x_u^{(1)}]$ then one would have $A_u^{-1}(x^*) = \{x^*\}$, which is not compatible with $\operatorname{Im} A_u^n = \{x^*\}$. Therefore $x^* \in [x_u^{(0)}, x_u^{(1)}]$ and $A_u([x_u^{(0)}, x_u^{(1)}]) = \{x^*\}$. $\qquad\square$
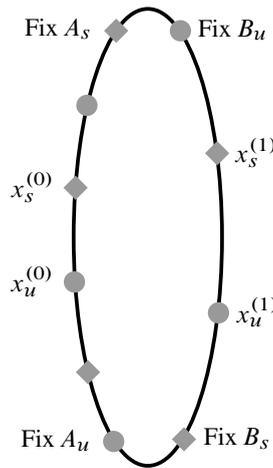
Figure 13. The pair of multicones M for $p/q = 2/5$.

5.5.5. Recall that we have denoted $q = \#M_u = \#M_s$ the rank of M. Let us denote

$$p = \# \operatorname{Im} B_u = \# \operatorname{Im} B_s, \quad \text{hence} \quad q - p = \# \operatorname{Im} A_u = \# \operatorname{Im} A_s.$$

If $q = 2$ then $p = 1$; both A and B are constant correspondences and these are the dynamics associated to the free components. We will therefore assume that $q > 2$. By exchanging A and B we can assume that $p \leq q/2$.

Lemma 5.9. *One has $p < q/2$ and $x_u^{(0)}, x_1^{(1)} \in \operatorname{Im} A_u$.*

Proof. $\operatorname{Im} A_u$ and $[x_u^{(1)}, x_u^{(0)}] \cap M_u$ are intervals in M_u with respective cardinalities $q - p$ and $q - p + 1$; therefore at least one of the two points $x_u^{(0)}, x_u^{(1)}$ belongs to $\operatorname{Im} A_u$, and exactly one if $q = 2p$. Assume that only one of the points $x_u^{(0)}, x_u^{(1)}$ belongs to $\operatorname{Im} A_u$. Starting with $x_0, x_0' \in M_u$ with $x_0 \neq x_0'$, $\{x_0, x_0'\} \neq \{x_u^{(0)}, x_u^{(1)}\}$, we can construct sequences $(x_n)_{n \geq 0}$, $(x_n)_{n \geq 0}$ in M_u such that $x_n \neq x_n'$ and $x_n = C_n x_{n-1}$, $x_n' = C_n x_{n-1}'$ for some $C_n \in \{A, B\}$: indeed one can never have $\{x_{n-1}, x_{n-1}'\} \neq \{x_u^{(0)}, x_u^{(1)}\}$ as both x_{n-1}, x_{n-1}' belong to $\operatorname{Im} C_{n-1}$. Such sequences would contradict hyperbolicity. Therefore the lemma is proved. $\square$

5.5.6. Let us summarize what we know so far about the correspondences A, B. (See Figure 14 for $p/q = 2/5$.)

As a subset of $M \times M$, A is made of

- a horizontal segment from $(x_u^{(0)}, \operatorname{Fix} A_u)$ to $(x_u^{(1)}, \operatorname{Fix} A_u)$;
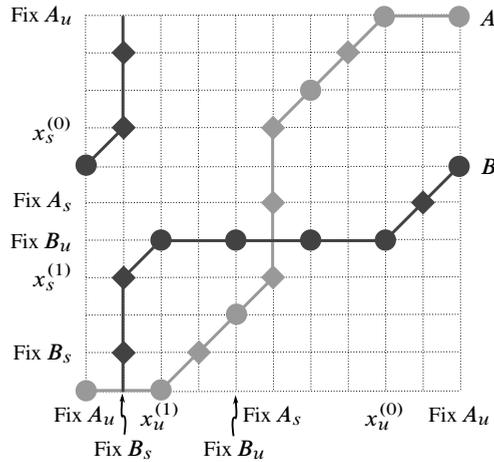
Figure 14. The correspondences A and B for $p/q = 2/5$.

- a vertical segment from $(\text{Fix } A_s, x_s^{(1)})$ to $(\text{Fix } A_s, x_s^{(0)})$;
- two diagonal segments from $(x_u^{(1)}, \text{Fix } A_u)$ to $(\text{Fix } A_s, x_s^{(1)})$ and from $(\text{Fix } A_s, x_s^{(0)})$ to $(x_u^{(0)}, \text{Fix } A_u)$.

  Here we have

$$x_u^{(0)} < x_u^{(1)} < x_s^{(1)} < x_s^{(0)} < x_u^{(0)},$$
$$x_u^{(0)} \leq \text{Fix } A_u \leq x_u^{(1)}, \quad x_s^{(1)} \leq \text{Fix } A_s \leq x_s^{(0)}.$$

Similarly, B is made of

- a horizontal segment from $(x_u^{(1)}, \text{Fix } B_u)$ to $(x_u^{(0)}, \text{Fix } B_u)$;
- a vertical segment from $(\text{Fix } B_s, x_s^{(0)})$ to $(\text{Fix } B_s, x_s^{(1)})$;
- two diagonal segments from $(x_u^{(0)}, \text{Fix } B_u)$ to $(\text{Fix } B_s, x_s^{(0)})$ and from $(\text{Fix } B_s, x_s^{(1)})$ to $(x_u^{(1)}, \text{Fix } B_u)$.

  We also have

$$x_s^{(1)} < \text{Fix } B_u < x_s^{(0)}, \quad x_u^{(0)} < \text{Fix } B_s < x_u^{(1)}.$$

We would like to show that q and p are relatively prime and that (A, B) is obtained from the component described in Subsection 3.8 (or its mirror image). This will be done by induction on q, the case $q = 2$ having been checked already.

Changing the cyclic orientation if necessary, we may also assume that

$$x_u^{(0)} \leq \text{Fix } A_u < \text{Fix } B_s < x_u^{(1)}.$$

Observe that the pair (A, B) is completely determined by the following data (besides p, q):

- the number $\bar{p}_0 := \#(M_u \cap [x_u^{(0)}, \text{Fix } A_u))$;
- the number $\bar{q}_0 := \bar{p}_0 + \#(M_s \cap [x_s^{(0)}, \text{Fix } B_s))$;
- the number $\delta := \#(\text{Fix } A_u, \text{Fix } B_s)$.

Indeed these numbers determine the relative positions of $x_u^{(0)}, x_u^{(1)}, x_s^{(0)}, x_s^{(1)}, \text{Fix } A_u$, $\text{Fix } A_s, \text{Fix } B_u, \text{Fix } B_s$ on M. Setting $\bar{p}_1 = p - \bar{p}_0, \bar{q}_1 = q - \bar{q}_0$, we have

$$\bar{p}_1 = \#(M_u \cap [\text{Fix } A_u, x_u^{(1)})),$$
$$\bar{q}_1 = \bar{p}_1 + \#(M_s \cap [\text{Fix } B_s, x_s^{(1)})).$$

For the component described in Subsection 3.8, one checks that $\bar{p}_0 = p_0, \bar{q}_0 = q_0$, $\bar{p}_1 = p_1, \bar{q}_1 = q_1, \delta = 0$, where p/q is the Farey center of the Farey interval $[p_0/q_0, p_1/q_1]$. We have to prove these relations in our case.

5.5.7. From A, B we will construct a new par of combinatorial multicones $M' = M'_s \sqcup M'_u$ of rank $q' := q - p$, and two monotone correspondences A', B' on M' which generate a tight hyperbolic morphism. Applying the induction hypothesis will allow us to conclude.

We define $M' := M'_s \sqcup M'_u$ where $M'_u := \text{Im } A_u = (x_s^{(0)}, x_s^{(1)}) \cap M_u$ and M'_s is obtained from M_s by collapsing the interval $[x_s^{(1)}, x_s^{(0)}] \cap M_s$ into a point denoted by $\bar{x}'$. We write π for the canonical map from M_s to M'_s. Observe that A_s is constant on $[x_s^{(1)}, x_s^{(0)}] \cap M_s$, with value $\text{Fix } A_s$. Therefore the composition $A_s \circ \pi^{-1}$ is well defined and is a bijection from M'_s to $\text{Im } A_s$. (This shows that the asymmetry of the definition of M' is only apparent.)

We equip M' with the obvious cyclic order inherited from M. We define:

$$A'_u = A_u | M'_u, \qquad A'_s = \pi \circ A_s \circ \pi^{-1},$$
$$B'_u = A_u \circ B_u | M'_u, \qquad B'_s = \pi \circ B_s \circ A_s \circ \pi^{-1}.$$

One checks easily that this defines monotone correspondences A', B' on M'. Let $\Phi' : \mathcal{F}_2 \to \mathcal{C}(M')$ be the morphism generated by A', B'.

Let us check that Φ' is hyperbolic: for any long enough word w' in A', B', the unstable part w'_u is an even longer word in A_u, B_u; as Φ is hyperbolic, the image is reduced to a point. This proves that w' is a constant correspondence.

Let us check that Φ' is tight. Any $x'_u \in M'_u$ can be written as $A_u(x_u)$ with $x_u \in M_u$; as Φ is tight, either $x_u \in M'_u$ and $x'_u \in \text{Im } A'_u$ or $x_u \in \text{Im } B_u$; as $B_u(M'_u) = \text{Im } B_u$, we have $x'_u \in \text{Im } B'_u$ in this case. Similarly, let $x'_s \in M'_s$; if $x'_s \in \pi(\text{Im } A_s)$ then $x'_s \in \text{Im } A'_s$; if $x'_s \in \pi(\text{Im } B_s)$ then, as $\text{Im } B_s = \text{Im } B_s A_s$, we have $x'_s \in \text{Im } B'_s$. Therefore Φ' is tight.

As A_u is injective on $(x_s^{(1)}, x_s^{(0)}) \cap \boldsymbol{M}_u$ and the image of this set is disjoint from $A_u(\mathrm{Im}\, A_u)$, we have

$$\# \,\mathrm{Im}\, A'_u = \# \,\mathrm{Im}\, A_u - p = q - 2p,$$

and therefore (as $\mathrm{Im}\, A'_u \cap \mathrm{Im}\, B'_u = \varnothing$)

$$p' := \mathrm{Im}\, B'_u = p, \quad \# \,\mathrm{Im}\, A'_u = q' - p'.$$

We will apply the induction hypothesis to the tight hyperbolic morphism Φ' and therefore we have to identify the parameters $\bar{p}'_0, \bar{q}'_0, \delta'$ for this morphism.

We have

$$A'_u(x_u^{(0)}) = A'_u(x_u^{(1)}) = \mathrm{Fix}\, A_u,$$
$$B'_u(x_u^{(0)}) = B'_u(x_u^{(1)}),$$

therefore $\mathrm{Fix}\, A'_u = \mathrm{Fix}\, A_u$ and $\bar{p}'_0 = \bar{p}_0$. Let $x_s'^{(0)}, x_s'^{(1)}$ be the points in $\boldsymbol{M}_s$ such that $A_s(x_s'^{(0)}) = x_s^{(0)}$, $A_s(x_s'^{(1)}) = x_s^{(1)}$ (if $\mathrm{Fix}\, A_s \neq x_s^{(i)}$ then $x_s'^{(i)}$ is uniquely determined by this condition; if $\mathrm{Fix}\, A_s = x_s^{(i)}$ then we take $x_s'^{(i)} = \mathrm{Fix}\, A_s$). It is easy to see that $\pi(x_s'^{(0)}) \neq \pi(x_s'^{(1)})$. We have then

$$A'_s(\pi(x_s'^{(0)})) = A'_s(\pi(x_s'^{(1)})) = \bar{x}',$$
$$B'_s(\pi(x_s'^{(0)})) = B'_s(\pi(x_s'^{(1)})) = \pi(\mathrm{Fix}\, B_s).$$

This shows that $\bar{q}'_0 = \bar{q}_0 - \bar{p}_0$, $\mathrm{Fix}\, B'_s = \pi(\mathrm{Fix}\, B_s)$ and so $\delta' = \delta$. From the inductive hypothesis, we must have $\delta' = 0$, $\bar{q}'_0 = q'_0$, $\bar{p}'_0 = p'_0$, where $[p'_0/q'_0, (p'-p'_0)/(q'-q'_0)]$ is the Farey interval with center p'/q'. But then we have also $\delta = 0$, $\bar{p}_0 = p_0$, $\bar{q}_0 = q_0$. This is the end of the proof of Proposition 5.4.

5.6. Non-realizable multicone dynamics. Here we will show that Proposition 5.4 does not extend to every N:

Proposition 5.10. *There exists a tight hyperbolic morphism* $\Phi \colon \mathcal{F}_N \to \mathcal{C}(\boldsymbol{M})$ *which is not induced by any uniformly hyperbolic N-tuple.*

Recall our definition of cross-ratio (2) from §2.2. It may be useful to bear in mind that $1 < [a, b, c, d] < \infty$ if $a < b < c < d$ ($< a$) (where $<$ is the cyclic ordering on $\mathbb{R} \cup \{\infty\}$). The following lemma compares certain cross-ratios:

Lemma 5.11. *Take eight distinct points in $\mathbb{P}^1$ with*

$$a' < a < b < b' < c' < c < d < d' \; (< a').$$

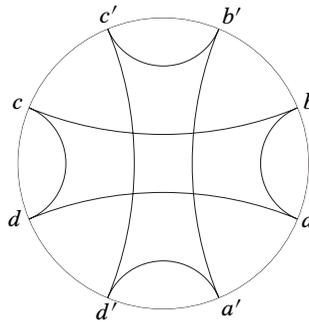*Then* $[a', b', c', d'] < [a, b, c, d]$.

Figure 15. Cross-ratio comparison.

Proof. Using an orientation-preserving projective chart (see §2.2) we can identify $\mathbb{P}^1$ with the extended line $\mathbb{R} \cup \{\infty\}$, and also assume that $d' = \infty$. Then

$$[a, b, c, d] = \frac{c-a}{b-a} \cdot \frac{d-b}{d-c} > \frac{c-a}{b-a} > \frac{c-a'}{b-a'} > \frac{c'-a'}{b'-a'} = [a', b', c', d']. \quad \square$$

Proof of Proposition 5.10. Consider a pair of combinatorial multicones $\boldsymbol{M} = \boldsymbol{M_s} \sqcup \boldsymbol{M_u}$ of order 15. Write the unstable combinatorial multicone as

$$\boldsymbol{M_u} = \{\alpha < a < b < \omega < c < d < \beta < \beta' < d' < o < a' < \omega' < b' < c' < \alpha' < \alpha\}.$$

Let maps $A_u, B_u, C_u \colon \boldsymbol{M_u} \to \boldsymbol{M_u}$ be defined by:

x_u	α	a	b	ω	c	d	β	β'	d'	o	a'	ω'	b'	c'	α'
$A_u(x_u)$	ω	β	α	ω	ω	ω	ω	ω	ω	ω	ω	ω	ω	ω	ω
$B_u(x_u)$	a'	a'	b'	c'	c'	d'	d'	o	o	o	o	o	o	o	o
$C_u(x_u)$	ω'	ω'	ω'	ω'	ω'	ω'	ω'	ω'	ω'	ω'	ω'	ω'	α'	β'	ω'

The maps above are monotonic in the sense of §5.1.5.2. Therefore there exist unique correspondences A, B, C on $\boldsymbol{M}$ whose respective u-maps are A_u, B_u, C_u, respectively.

Choose some constant correspondences $C^{(4)}, \ldots, C^{(N)}$ such that the morphism Φ determined by $A, B, C, C^{(4)}, \ldots, C^{(N)}$ is tight.

Let us see that the morphism is hyperbolic. We only need to consider products of the correspondences A, B, C, because the others are constant. Inspecting the following diagram, one sees that any product of length ≥ 4 of the maps A_u, B_u, C_u
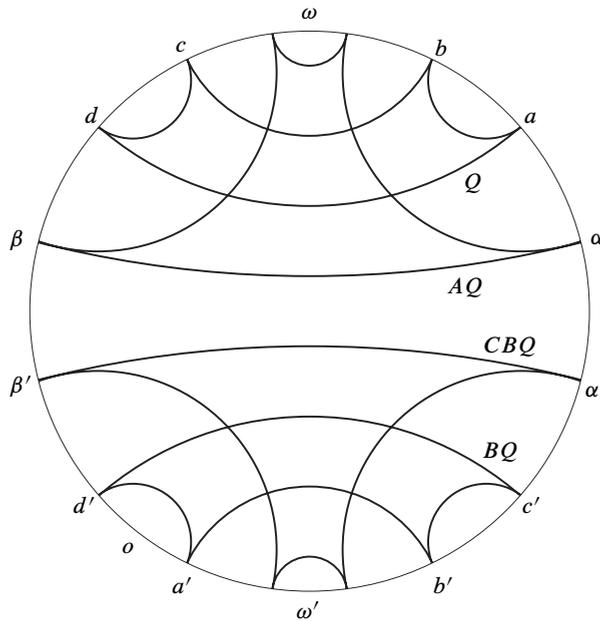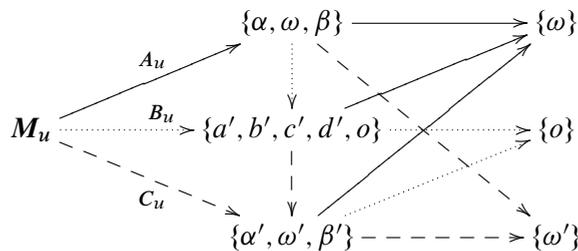
Figure 16. The unstable combinatorial multicone. We will compare cross-ratios of the four rectangles Q, AQ, BQ, CBQ.

is constant:

Since any correspondence is constant iff so is its unstable map, we conclude that the morphism Φ is hyperbolic.

By contradiction, assume that the morphism Φ is induced by some hyperbolic N-tuple. Then there is a tight multicone composed of 15 intervals, and each element $\xi \in \mathbf{M_u}$ corresponds to one of those intervals, say I_ξ.

With abuse of notation, let A, B, C indicate the first three matrices of the N-tuple. Choose four points in the circle: $\boldsymbol{a} \in I_a, \boldsymbol{b} \in I_b, \boldsymbol{c} \in I_c, \boldsymbol{d} \in I_d$. Then their images

by A belong respectively to I_β, I_α, I_ω, I_ω. So

$$A(b) < a < b < A(c) < A(d) < c < d < A(a) \ (< A(b)),$$

and therefore Lemma 5.11 gives

$$[b, c, d, a] = [A(b), A(c), A(d), A(a)] < [a, b, c, d].$$

On the other hand, defining $a' = B(a) \in I_{a'}$, $b' = B(b) \in I_{b'}$, $c' = B(c) \in I_{c'}$, $d' = B(d) \in I_{d'}$, then

$$C(a') < b' < c' < C(b') < C(c') < d' < a' < C(d') \ (< C(a')),$$

and so, using Lemma 5.11 again,

$$[a, b, c, d] = [a', b', c', d'] < [b', c', d', a'] = [b, c, d, a].$$

We have reached a contradiction. $\qquad\square$

5.7. Non-linear realization of multicone dynamics.
We will now see that any combinatorial multicone dynamics has a non-linear realization.

Given N homeomorphisms $f_1, \ldots, f_N \colon \mathbb{P}^1 \to \mathbb{P}^1$, we define a skew-product homeomorphism $F \colon N^{\mathbb{Z}} \times \mathbb{P}^1 \to N^{\mathbb{Z}} \times \mathbb{P}^1$ over the shift $\sigma \colon N^{\mathbb{Z}} \to N^{\mathbb{Z}}$ by $(\omega, x) \mapsto (\sigma(\omega), f_{\omega_0}(x))$.

Proposition 5.12. *Let $C^{(1)}, \ldots, C^{(N)}$ be correspondences on a pair of combinatorial multicones $\boldsymbol{M}$. Then there exist*

- *orientation-preserving diffeomorphisms $f_1, \ldots, f_N \colon \mathbb{P}^1 \to \mathbb{P}^1$,*
- *a family of disjoint closed intervals $I_\xi \subset \mathbb{P}^1$, for $\xi \in \boldsymbol{M}$, such that the order inherited from $\boldsymbol{M}$ is compatible with an orientation of the circle $\mathbb{P}^1$*

with the following properties:

(i) *For each $\xi \in \boldsymbol{M}_u$, we have $f_i(I_\xi) \Subset I_{C_u^{(i)}(\xi)}$ and $f_i'|I_\xi < 1$.*

(ii) *For each $\xi \in \boldsymbol{M}_s$, we have $f_i^{-1}(I_\xi) \Subset I_{C_s^{(i)}(\xi)}$ and $(f_i^{-1})'|I_\xi < 1$.*

(iii) *If $F \colon N^{\mathbb{Z}} \times \mathbb{P}^1 \hookleftarrow$ is the skew-product homeomorphism induced by the f_i's then its non-wandering set $\Omega(F)$ is the union of two disjoint compact F-invariant sets Λ_s and Λ_u, contained respectively in $N^{\mathbb{Z}} \times \bigcup_{\xi \in \boldsymbol{M}_s} I_\xi$ and $N^{\mathbb{Z}} \times \bigcup_{\xi \in \boldsymbol{M}_u} I_\xi$.*

(iv) *If the morphism $\Phi \colon \mathcal{F}_N \to \mathcal{C}(\boldsymbol{M})$ induced by the correspondences $C^{(i)}$'s is hyperbolic then the F-invariant sets Λ_u and Λ_s are topologically transitive.*

(v) *If the morphism Φ is tight then $\Omega(F)$ intersects $N^{\mathbb{Z}} \times I_\xi$ for every $\xi \in \boldsymbol{M}$.*

Proof. Let $C^{(1)}, \ldots, C^{(N)}$ be correspondences on a pair of multicones $\boldsymbol{M}$.

Choose a family I_ξ, indexed by $\xi \in \boldsymbol{M}$, of disjoint closed intervals contained in the circle $\mathbb{P}^1$, all with the same positive length, and such that the order inherited from $\boldsymbol{M}$ is compatible with an orientation of the circle.

Fix some $i = 1, \ldots, N$. For each $\eta \in \mathrm{Im}\, C_u^{(i)}$, there exist a unique connected component $J_\eta^{(i)}$ of $\mathbb{P}^1 \smallsetminus \bigsqcup_{\xi \in \mathrm{Im}\, C_s^{(i)}} I_\xi$ that contains all the intervals I_x such that $(x, \eta) \in C^{(i)} \subset \boldsymbol{M} \times \boldsymbol{M}$. We have

$$\mathbb{P}^1 = \bigsqcup_{\xi \in \mathrm{Im}\, C_s^{(i)}} I_\xi \sqcup \bigsqcup_{\eta \in \mathrm{Im}\, C_u^{(i)}} J_\eta^{(i)}. \tag{39}$$

Analogously, for each $\xi \in \mathrm{Im}\, C_s^{(i)}$, there exist a unique connected component $J_\xi^{(i)}$ of $\mathbb{P}^1 \smallsetminus \bigsqcup_{\eta \in \mathrm{Im}\, C_u^{(i)}} I_\eta$ that contains all the intervals I_y for which $(\xi, y) \in C^{(i)}$. In addition,

$$\mathbb{P}^1 = \bigsqcup_{\xi \in \mathrm{Im}\, C_s^{(i)}} J_\xi^{(i)} \sqcup \bigsqcup_{\eta \in \mathrm{Im}\, C_u^{(i)}} I_\eta. \tag{40}$$

Let $f_i \colon \mathbb{P}^1 \to \mathbb{P}^1$ be an orientation-preserving diffeomorphism such that

$$f_i\big(\mathrm{cl}\, J_\eta^{(i)}\big) = I_\eta \quad \text{for all } \eta \in \mathrm{Im}\, C_u^{(i)}, \quad f_i(I_\xi) = \mathrm{cl}\, J_\xi^{(i)} \quad \text{for all } \xi \in \mathrm{Im}\, C_s^{(i)}.$$

Then for each $\xi \in \boldsymbol{M}_u$, we have $f_i(I_\xi) \subset f_i\big(\mathrm{cl}\, J_{C_u^{(i)}(\xi)}^{(i)}\big) = I_{C_u^{(i)}(\xi)}$. Also, f_i can be chosen to be linear in I_ξ. Analogously, for each $\xi \in \boldsymbol{M}_s$ we have $f_i^{-1}(I_\xi) \subset I_{C_s^{(i)}(\xi)}$, and we can take $f_i^{-1}|I_\xi$ linear. Then the maps f_i satisfy properties (i) and (ii) of the proposition.

Define two disjoint subsets of $\mathbb{P}^1$ by $S = \bigsqcup_{\xi \in \boldsymbol{M}_s} I_\xi$ and $U = \bigsqcup_{\xi \in \boldsymbol{M}_u} I_\xi$. Next we claim that for any i,

$$f_i(\mathbb{P}^1 \smallsetminus S) \subset U, \quad f_i^{-1}(\mathbb{P}^1 \smallsetminus U) \subset S. \tag{41}$$

Indeed, if $x \in \mathbb{P}^1 \smallsetminus S$ then by (39) x belongs to $J_\eta^{(i)}$ for some $\eta \in \mathrm{Im}\, C_u^{(i)}$. In particular, $f_i(x) \in I_\eta$, proving the first part of (41). The second part follows by symmetry.

It follows from (41) that all points in $N^{\mathbb{Z}} \times \big(\mathbb{P}^1 \smallsetminus (U \cup S)\big)$ are wandering. Hence assertion (iii) holds.

Now assume the morphism Φ is hyperbolic. Given symbols $i_0, \ldots, i_{n-1}$, the set $f_{i_{n-1}} \circ \cdots \circ f_{i_0}(\mathbb{P}^1 \smallsetminus S)$ is contained in the union of the intervals I_ξ such that ξ belongs to the image of $C_u^{(n-1)} \circ \cdots \circ C_u^{(0)}$. So ξ becomes uniquely determined if n is large enough. By the contraction property (i), we get that

$$\mathrm{dist}\big(F^n(\omega, x), F^n(\omega, y)\big) \xrightarrow[n \to +\infty]{} 0 \quad \text{uniformly for } \omega \in N^{\mathbb{Z}}, \, x, y \in \mathbb{P}^1 \smallsetminus S.$$

Using this, it is easy to show that the F-invariant set $\bigcap_{n \geq 0} F^n(N^{\mathbb{Z}} \times U)$ is topologically transitive. In particular, this set must be equal to $\Omega(F) \cap (N^{\mathbb{Z}} \times U)$, that is, Λ_u. Analogously, one shows that $\Lambda_s = \bigcap_{n \geq 0} F^{-n}(N^{\mathbb{Z}} \times S)$ is topologically transitive. This proves part (iv).

The simple proof of assertion (v) is left to the reader. $\square$

6. Questions

The questions and problems proposed in [13] are solved for the full 2-shift, but for the general case many questions remain unanswered. To summarize:

Question or Problem from [13]	Full 2-shift	General case
Q1 (trace signs)	yes	unknown
P1 (trace signs)	easy now – use §3.8, §5.4	unknown
Q1' (trace signs × principal)	no	no – see §5.4.3
P2 (principal)	–	unknown
Q2 (boundary)	no	no, if Q3' is "yes" – see Theorem 4.1
Q3 (boundary)	yes	no (in general) – see Proposition 4.18
Q3'(boundary)	yes	unknown
Q4 (elliptic products)	yes	unknown

We will recall and discuss some of those questions, and also propose new ones.

We return to the general situation where Σ is some subshift of finite type, and $\mathcal{H}$ is associated hyperbolic locus.

6.1. Boundaries of the components

Question 1. Are the boundaries of the connected components of $\mathcal{H}$ disjoint?

A result that goes in the direction of answering (positively) Question 1 is Theorem 5.2.

Question 2 (Question 3' in [13]). Is the union of the boundaries of the components equal to the boundary of $\mathcal{H}$?

A positive answer to Question 2 would answer Question 2 from [13] negatively (using Theorem 4.1).

Question 3. If $\gamma \colon [a, b] \to \mathrm{SL}(2, \mathbb{R})^N$ is an analytic curve, does the set $\gamma^{-1}(\partial \mathcal{H})$ necessarily have countably many components?

A negative answer to Question 3 would answer Question 2 negatively (because the components of $\mathcal{H}$ are semialgebraic).

6.2. Elliptic products. Denote by $\mathcal{E} \subset \mathrm{SL}(2, \mathbb{R})^N$ the set of N-tuples such that there exists a periodic point for the subshift over which the corresponding product is an elliptic matrix.

It is shown in [13] that $\overline{\mathcal{E}} = \mathcal{H}^c$.

Question 4 (Question 4 in [13]). Is $\overline{\mathcal{H}} = \mathcal{E}^c$? Equivalently, is $\partial\mathcal{H} = \partial\mathcal{E} = (\mathcal{H} \cup \mathcal{E})^c$?

We remark that $\mathcal{E}$ is connected: see Proposition A.3 in the Appendix.

6.3. Unboundedness of the components. Let us say that a set $Z \subset \mathrm{SL}(2, \mathbb{R})^N$ is *bounded modulo conjugacy* if there exists a compact set $K \subset \mathrm{SL}(2, \mathbb{R})^N$ such that every N-tuple in Z is of the form $(RA_1R^{-1}, \ldots, RA_NR^{-1})$, for some $(A_1, \ldots, A_N) \in K$ and $R \in \mathrm{SL}(2, \mathbb{R})$. Otherwise, we say that Z is *unbounded modulo conjugacy*.

Question 5. Is every connected component of $\mathcal{H}$ unbounded modulo conjugacy?

Theorem A.1 in the Appendix says that a set of N-tuples $(A_1, \ldots, A_N)$ is bounded modulo conjugacy iff the traces of A_i's and A_iA_j's are all bounded. Motivated by it, we pose a stronger version of Question 5:

Question 6 (for full shifts). Are *all* functions $\operatorname{tr} A_i$ and $\operatorname{tr} A_iA_j$ unbounded in each component?

If A is a uniformly hyperbolic N-tuple w.r.t. some subshift Σ, we define its (least) *hyperbolicity rate* as

$$\rho(A) = \liminf_{n \to \infty} \min \left\{ \|A^n(x)\|^{1/n}; \ x \in \Sigma \text{ has period } n \right\}.$$

Of course, $\rho(A) > 1$.

Question 7. Is ρ unbounded in each component?

A positive answer to Question 7 implies positive answers to Questions 5 (because of Theorem A.1) and 6 (because $\rho(A)$ is a lower bound for the modulus of the trace of *any* product of the matrices in the N-tuple A).

It is easy to see that ρ is unbounded in principal components (for full shifts, of course). The case $\Sigma = 2^{\mathbb{Z}}$ is also easily settled:

Proposition 6.1. *For the case of the full 2-shift, the answer to Question 7 is positive.*

Proof. It suffices to see that ρ is unbounded on non-principal components.

First consider a free component H. Let Σ be the subshift on four symbols considered in §3.3. If $(A, B) \in H$, then (A, B, A^{-1}, B^{-1}) is uniformly hyperbolic with respect to Σ; see Lemma 3.7; let $\rho_\Sigma(A, B)$ indicate the hyperbolicity rate of (A, B, A^{-1}, B^{-1}) with respect to Σ. It is easy to see that ρ_Σ (and in particular, ρ) is unbounded in H.

Now, consider any other component $H_F = F^{-1}(H)$, where $F \in \mathcal{M}$. Given $\tau > 1$, take $(A_0, B_0) \in H$ with $\rho_\Sigma(A_0, B_0) > \tau$, and let $(A, B) = F^{-1}(A_0, B_0) \in H_F$. In the notations of §3.4 we have that there exist $c > 0$ such that

$$\|\langle \omega, (A_0, B_0) \rangle\| \geq c \exp(\tau |\omega|) \quad \text{for every } \omega \in \mathbb{F}_2.$$

As in the proof of Proposition 3.5, it follows that

$$\|\langle \omega, (A, B) \rangle\| \geq c \exp\left(2^{-k} \tau |\omega|\right) \quad \text{for any } \omega \in \mathbb{F}_2$$

(where k depends only on F). Therefore

$$\rho(A, B) \geq \liminf_{|\omega| \to \infty} \|\langle \omega, (A, B) \rangle\|^{1/|\omega|} \geq \exp\left(2^{-k} \tau\right).$$

Hence ρ is unbounded on H_F. □

6.4. Topology of the components

Question 8. What are the possible homotopy types of the hyperbolic components? What about the elliptic locus $\mathcal{E}$?

In the case of the full 2-shift, each component has the homotopy type of a circle. In Appendix A.2, we show that $\mathcal{E}$ is connected.

6.5. Combinatorial characterization of the components. Assume the subshift is full in N letters.

An uniformly hyperbolic N-tuple induces a multicone $\boldsymbol{M}$ in the sense of Section 5, and a tight hyperbolic morphism Φ.

Recall that if two uniformly hyperbolic N-tuples belong to the same connected component then they have the same combinatorics, in the sense the respective morphisms Φ are conjugate.

Question 9. Does the combinatorics characterize the connected components of $\mathcal{H}$, modulo reflections $(A_1, \ldots, A_N) \mapsto (\pm A_1, \ldots, \pm A_N)$?

In the case $\Sigma = 2^\mathbb{Z}$, our description of the multicone dynamics (see §3.8) gives a positive answer to Question 9.

Appendices

A.1. A compactness criterion for finite families of matrices in SL(2, ℝ) modulo conjugacy. Let K be a compact subset of $SL(2, \mathbb{R})$. Then there exists $C = C(K) > 0$ such that, for any $A, B \in K$,

$$|\operatorname{tr} A| \leq C, \quad |\operatorname{tr} AB| \leq C.$$

This also holds if A, B belong to some conjugate $R^{-1}KR$, $R \in SL(2, \mathbb{R})$. We prove that the converse is true:

Theorem A.1. *Let $C > 0$. There exists a compact set $K = K(C)$ with the following property: If $A_1, ..., A_N \in SL(2, \mathbb{R})$ satisfy*

$$|\operatorname{tr} A_i| \leq C, \quad 1 \leq i \leq N, \tag{42}$$

$$|\operatorname{tr} A_i A_j| \leq C, \quad 1 \leq i < j \leq N, \tag{43}$$

then there exists $R \in SL(2, \mathbb{R})$ such that $R A_i R^{-1} \in K$ for $1 \leq i \leq N$.

Remark A.2. It follows that if the inequalities (42), (43) are satisfied over a subset Z of $SL(2, \mathbb{R})^N$ then there is a compact set $K \subset SL(2, \mathbb{R})^N$ such that the union of conjugacy classes of elements of K covers Z. This result does *not* hold for infinite families $(A_i)_{i \in \mathbb{N}}$. More precisely, consider in $SL(2, \mathbb{R})^{\mathbb{N}}$ the product topology. If $f : \mathbb{N} \to \mathbb{N}$ is any map, let $A_i^f = \begin{pmatrix} 1 & f(i) \\ 0 & 1 \end{pmatrix}$. Let $Z \subset SL(2, \mathbb{R})^{\mathbb{N}}$ be the set of $A^f = (A_i^f)_{i \in \mathbb{N}}$ for all possible f. We have $\operatorname{tr} A_i^f = \operatorname{tr} A_i^f A_j^f = 2$ for all i, j, f. On the other hand, given any compact set $K \subset SL(2, \mathbb{R})^{\mathbb{N}}$, there exist $c_i > 0$ such that $(B_i) \in K$ implies $\|B_i\| \leq c_i$ for every $i \in \mathbb{N}$. Now, if $f : \mathbb{N} \to \mathbb{N}$ is such that $f(i)/c_i \to \infty$ then $A^f \in Z$ does not belong to any conjugacy class of elements of K.

Proof of Theorem A.1. Write

$$A_i = \begin{pmatrix} x_i & y_i \\ z_i & t_i \end{pmatrix}.$$

We have

$$x_i t_i - y_i z_i = 1 \quad \text{for all } i, \tag{44}$$

$$|x_i + t_i| \leq C \quad \text{for all } i, \tag{45}$$

$$|x_i x_j + t_i t_j + y_i z_j + y_j z_i| \leq C \quad \text{for all } i < j. \tag{46}$$

We want to find a common conjugacy after which all coefficients are bounded by $C_1 = C_1(C)$.

We start with a particular case.

Special case: Assume that we have moreover

$$|x_i| \leq C_2, \quad \text{for all } i, \tag{47}$$

for some C_2 depending only on C. We will then conjugate all A_i by the same diagonal matrix. Observe that from (44), (45), (46), (47), we get (for some $C_3 = C_3(C)$)

$$|t_i| \leq C_3 \quad \text{for all } i, \tag{48}$$

$$|y_i z_i| \leq C_3 \quad \text{for all } i, \tag{49}$$

$$|y_i z_j + y_j z_i| \leq C_3 \quad \text{for all } i < j. \tag{50}$$

From (49), (50) we also get

$$|y_i z_i y_j z_j| \leq C_3^2 \quad \text{for all } i < j, \tag{51}$$

$$|y_i z_j| \leq C_4 \quad \text{for all } i, j. \tag{52}$$

Let

$$R_\lambda = \begin{pmatrix} \lambda & 0 \\ 0 & \lambda^{-1} \end{pmatrix}, \quad A_i' = R_\lambda A_i R_\lambda^{-1} = \begin{pmatrix} x_i & \lambda^2 y_i \\ \lambda^{-2} z_i & t_i \end{pmatrix}.$$

From (52), we have

$$\max_i |y_i| \cdot \max_j |z_j| \leq C_4.$$

Thus we can choose λ such that

$$\max_i |\lambda^2 y_i| \leq C_4^{1/2},$$

$$\max_i |\lambda^{-2} z_i| \leq C_4^{1/2},$$

which concludes the proof in the special case.

Let $S_\theta = \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix}$. Write $S_\theta A_i S_\theta^{-1} = \begin{pmatrix} x_i(\theta) & y_i(\theta) \\ z_i(\theta) & t_i(\theta) \end{pmatrix}$. We have

$$x_i(\theta) = x_i \cos^2\theta + t_i \sin^2\theta + (y_i + z_i) \sin\theta \cos\theta.$$

We want to prove that there exists $C_2 = C_2(C)$ and θ such that

$$|x_i(\theta)| \leq C_2 \quad \text{for all } i. \tag{53}$$

Indeed, in this case we are reduced to the special case above. From (45), we see that (53) is equivalent to

$$\left| x_i \cos 2\theta + \frac{y_i + z_i}{2} \sin 2\theta \right| \leq C_2' \quad \text{for all } i. \tag{54}$$

Observe that

$$x_i^2(\theta) + y_i^2(\theta) + z_i^2(\theta) + t_i^2(\theta) = \text{tr } S_\theta \, A_i \, {}^tA_i \, S_\theta^{-1}$$

does not depend on θ. We can assume that

$$x_1^2 + y_1^2 + z_1^2 + t_1^2 \geq x_i^2 + y_i^2 + z_i^2 + t_i^2 \quad \text{for all } i \geq 1. \tag{55}$$

Choose θ such that

$$x_1 \cos 2\theta + \frac{y_1 + z_1}{2} \sin 2\theta = 0. \tag{56}$$

Replacing A_i by $S_\theta A_i S_\theta^{-1}$, we can assume that

$$|x_1| \leq C. \tag{57}$$

We will show that (44), (45), (46), (56), (57) together imply (47). Actually, we only need (46) for $i = 1$, i.e.,

$$|x_1 x_i + t_1 t_i + y_1 z_i + y_i z_1| \leq C \quad \text{for all } i > 1. \tag{58}$$

Observe first that from (44), (45), (57) we get

$$|t_1| \leq 2C, \tag{59}$$

$$|y_1 z_1| \leq 1 + 4C^2. \tag{60}$$

Replacing if necessary all A_i by ${}^tA_i$, we can assume that

$$|y_1| \geq |z_1| \tag{61}$$

From (57), (59), (60), (61), we have

$$x_1^2 + z_1^2 + t_1^2 \leq C_5 = C_5(C). \tag{62}$$

From (55), we then get

$$\max(|x_i|, |y_i|, |z_i|, |t_i|) \leq |y_1| + C_6. \tag{63}$$

In particular,

$$|y_i z_1| \leq |y_1 z_1| + C|z_1| \leq C_7 \tag{64}$$

and thus, from (58),

$$|x_1 x_i + t_1 t_i + y_1 z_i| \leq C_7 + C. \tag{65}$$

From (45), (59) we also have

$$|t_1 t_i + t_1 x_i| \leq 2C^2 \tag{66}$$

and therefore, using (57), (59), (65),

$$|y_1 z_i| \leq C_8(|x_i| + 1). \tag{67}$$

If $|y_1| \leq C_2'''$, we conclude directly from (63) that $|x_i| \leq C_2$. Assume therefore that $|y_1|$ is large. Then, from (67) we have

$$|z_i| \leq C_8 \frac{1 + |x_i|}{|y_1|}. \tag{68}$$

We have also, from (63), (45),

$$|y_i| \leq |y_1| + C_6,$$
$$|t_i| \geq |x_i| - C.$$

Therefore, from (44),

$$\begin{aligned}
|x_i|(|x_i| - C) &\leq 1 + |y_i z_i| \\
&\leq 1 + C_8 \frac{|y_1| + C_6}{|y_1|}(1 + |x_i|) \\
&\leq C_9|x_i|,
\end{aligned}$$

which gives finally (47). $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\square$

A.2. Connectivity of the elliptic locus.
Recall that in the case of the full shift in N symbols, $\mathcal{E}$ denotes the (open) subset of $\mathrm{SL}(2, \mathbb{R})^N$ formed by the N-tuples which have an elliptic product.

Proposition A.3. $\mathcal{E}$ *is connected.*

Let $R_\theta \in \mathrm{SL}(2, \mathbb{R})$ denote the rotation by angle θ. The proof of connectivity of $\mathcal{E}$ needs the following:

Lemma A.4. *Fix $B_1, \ldots, B_n \in \mathrm{SL}(2, \mathbb{R})$, and let*

$$F(\theta) = \mathrm{tr}\left(B_1 R_\theta B_2 R_\theta \ldots B_n R_\theta\right).$$

Then for every parameter θ for which $|F(\theta)| < 2$ we have $F'(\theta) \neq 0$.

Proof. This lemma is essentially proved in [2]. Complexification gives a rational function $Q(z)$ such that $Q(e^{i\theta}) = F(\theta)$ for real θ. Moreover, $Q(z) = P(z)/z^n$ where $P(z)$ is a polynomial of degree at most $2n$.

First assume that the matrices B_i satisfy

$$B_1 R_\theta B_2 R_\theta \ldots B_n R_\theta \neq \pm\mathrm{id} \quad \text{for all } \theta \in \mathbb{R}. \tag{69}$$

A topological argument then gives that the intersection of $Q^{-1}([-2,2])$ with the unit circle S^1 has at least $2n$ connected components – this is Lemma 10 in [2]. On the other hand, Q restricted to S^1 is real-valued and thus each connected component of $S^1 \smallsetminus Q^{-1}([-2,2])$ contains at least one zero of $Q'(z) = (zP'(z) - nP(z))/z^{n+1}$. It follows that all the zeros of Q' are simple and contained in $S^1 \smallsetminus Q^{-1}([-2,2])$. Moreover, $Q^{-1}([-2,2])$ consists of exactly $2n$ intervals in S^1, each with length at least $4\|Q'|S^1\|_\infty^{-1}$.

Now it follows by a perturbation argument that even if condition (69) is not satisfied, all the zeros of Q' are simple and contained in $S^1 \smallsetminus Q^{-1}((-2,2))$. This concludes the proof of the lemma. $\qquad\square$

Proof of Proposition A.3. First notice that the set of elliptic matrices is connected, that is, the proposition is true for $N = 1$.

Now let $N \geq 2$. Take $(A_1, \ldots, A_N)$ in $\mathcal{E}$, so some product $A_{j_1} \ldots A_{j_m}$ is elliptic. To prove the proposition, it suffices to find a path $t \in [0,1] \mapsto (A_i(t))_i$ in $\mathcal{E}$ starting from (A_i) such that $A_\ell(1)$ is elliptic for *some* ℓ. Let ℓ be any of $j_1, \ldots, j_m$. We can assume some j_i is different from ℓ, because otherwise there is nothing to prove.

Take a path $t \in [0,1] \mapsto A_\ell(t)$ starting at A_ℓ and ending at some elliptic matrix. Let $A_i(t, \theta)$ be equal to $R_\theta A_i$ if $i \neq \ell$, and $A_\ell(t, \theta) = A_\ell(t)$. Also, let $F(t, \theta)$ be the trace of $A_{j_1}(t, \theta) \ldots A_{j_m}(t, \theta)$. Lemma A.4 (together with the assumption that some j_i is different from ℓ) guarantees that $\frac{\partial F}{\partial \theta} \neq 0$ when $|F| < 2$. Therefore the differential equation $\frac{d}{dt} F(t, \theta(t)) = 0$ with initial condition $\theta(0) = 0$ has a solution $\theta(t)$ defined for $t \in [0,1]$. Consider the path $t \mapsto (A_i(t))$ where $A_i(t) = A_i(t, \theta(t))$. The path is contained in $\mathcal{E}$ because the trace of $A_{j_1}(t) \ldots A_{j_m}(t)$ is constant; also, $A_\ell(1)$ is elliptic. So we are done. $\qquad\square$

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