Quantum cohomology of Grassmannians and cyclotomic fields

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For the quantum cohomology of Grassmannians, see [1]. Given integers l, N with $1 \leqslant l < N$, we write $G = G(l, N)$ for the Grassmannian of l-dimensional subspaces of an N-dimensional vector space. Let S denote the tautological subbundle on G and Ω_λ the Schubert class in $H^*(G)$ corresponding to a partition λ. The three-point correlator $\langle \Omega_\lambda, \Omega_\mu, \Omega_\nu \rangle_d$ is defined as the number of rational curves of degree d intersecting general representatives of these classes. The algebra $QH^*(G, \mathbb{Z})$ is a $\mathbb{Z}[q]$-module isomorphic to $H^*(G, Z)_{\mathbb{Z}} \otimes \mathbb{Z}[q]$ and equipped with the following multiplication. Denoting the Schubert classes in this algebra by σ_λ, so that $\sigma_\lambda = \Omega_\lambda \otimes 1$, we put $\sigma_\lambda \cdot \sigma_\mu = \sum_{\nu, d \geqslant 0} \langle \sigma_\lambda, \sigma_\mu, \sigma_{\nu^\vee} \rangle_d \, q^d \sigma_\nu$, where $\nu^\vee$ is the partition dual to ν.

Let ζ be a primitive Nth root of $(-1)^{l+1}$.

Put $K = \mathbb{Q}(\zeta)$, $k = N - l$ and let $\Lambda = \mathbb{Z}[e_1, e_2, \dots]$ be the (graded) ring of symmetric functions, where e_i is the ith elementary symmetric function. Let h_i be the ith complete symmetric function. Putting $E(t) = \sum e_i t^i$, $H(t) = \sum h_i t^i$, we have $E(-t)H(t) = 1$. The result of applying a symmetric function σ to a tuple $(x_1, \dots, x_n)$ of arguments will be denoted by $\sigma(x_1, \dots, x_n)$. Finally, we put $\Lambda' = \Lambda/(e_{l+1}, e_{l+2}, \dots)$, $\Lambda_{\mathbb{Q}} = \Lambda \otimes \mathbb{Q}$, $\Lambda'_{\mathbb{Q}} = \Lambda' \otimes \mathbb{Q}$ and $\Lambda'_K = \Lambda'_{\mathbb{Q}} \otimes_{\mathbb{Q}} K$.

Theorem 1 (Siebert–Tian [2]). *The homomorphism of rings* $ST\colon \Lambda[q] \to QH(G, \mathbb{Z})$ *defined by* $ST(q) = 1 \otimes q$, $ST(e_i) = c_i(S^*) \otimes 1$ *is an epimorphism*, *and* $\operatorname{Ker} ST = (e_{l+1}, e_{l+2}, \dots; h_{N-l+1}, \dots, h_{N-1}, h_N + (-1)^l q)$.

Let $QH(G, \mathbb{Q}) = QH(G, \mathbb{Z}) \otimes \mathbb{Q}$ be the rational quantum cohomology ring and let $QH(G, \mathbb{Q}, 1)$ denote its specialization for $q = 1$. Putting

$$I_1 = (h_{N-l+1}, \dots, h_{N-1}, h_N + (-1)^l),$$

we clearly have $QH(G, \mathbb{Q}, 1) = \Lambda'_{\mathbb{Q}}/I_1$. The K-algebra $QH(G, K, 1) = QH(G, \mathbb{Q}, 1) \otimes_{\mathbb{Q}} K$ and the ideal $I_1^K = I_1 \otimes K$ are defined similarly.

Let $\theta_1, \dots, \theta_N$ denote the distinct Nth roots of unity and let $J = j_1 j_2 \dots j_l$ be the multi-index numbering the distinct lth roots of unity with $j_1 \leqslant \dots \leqslant j_l$, $j_k \in \{1, \dots, N\}$. We shall write θ_J in place of $\{\theta_{j_1}, \dots, \theta_{j_l}\}$ and $\sigma(\zeta\theta_J)$ in place of $\sigma(\zeta\theta_{j_1}, \dots, \zeta\theta_{j_l})$. We put $\theta_{\bar J} = \{\theta_1, \dots, \theta_N\} \setminus \{\theta_{j_1}, \dots, \theta_{j_l}\}$ and let $\phi_J \colon \Lambda'_K \to K$ be the homomorphism given by $\phi_J(\sigma) = \sigma(\zeta\theta_J)$. We put $I_J = \operatorname{Ker} \phi_J$ and $\phi = \bigoplus \phi_J$.

Lemma 2. *If $J \neq J'$, then $I_J \neq I_{J'}$.*

Proof. If the K-points on $\mathbb{A}^l_K = \operatorname{Spec} \Lambda'_K$ corresponding to the ideals I_J and $I_{J'}$ coincide, then so do their coordinates, which are the values of the symmetric functions $\sigma_1, \dots, \sigma_l$ on the tuples θ_J and $\theta_{J'}$. Hence, the tuples themselves also coincide.

Lemma 3. *For a fixed J there is an embedding of ideals $I_1^K \subset I_J$.*

Proof. We have $\bigl(\sum_{i=0}^l e_i(\theta_J)(-t)^i\bigr)\bigl(\sum_{i=0}^k e_i(\theta_{\bar J})(-t)^i\bigr) = 1 - t^N$. On the other hand,

$$\left(\sum_{i=0}^{l} e_i(\theta_J)(-t)^i\right)\left(\sum_{i=0}^{\infty} h_i(\theta_J)t^i\right) = 1.$$

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Hence,

$$\begin{aligned}\sum_{i=0}^{k} e_i(\theta_{\bar J})(-t)^i &= \left(\sum_{i=0}^{l} e_i(\theta_J)(-t)^i\right)\left(\sum_{i=0}^{k} e_i(\theta_{\bar J})(-t)^i\right)\left(\sum_{i=0}^{\infty} h_i(\theta_J)t^i\right)\\ &= (1-t^N)\sum_{i=0}^{\infty} h_i(\theta_J)t^i.\end{aligned}$$

Comparing the coefficients of t^i, we find that $h_i(\theta_J)=0$ for $i=k+1,\dots,N-1$ and $h_N(\theta_J)-1=0$. Hence, $h_i(\zeta\theta_J)=0$ for $i=k+1,\dots,N-1$ and $h_N(\zeta\theta_J)=\zeta^N=(-1)^{l+1}$.

Corollary 4. *There is an embedding of ideals* $I_1^K \subset \operatorname{Ker}\phi$.

Corollary 5. *The representation* ϕ *factors through the homomorphism*

$$\psi\colon QH^*(G,K,1) \to \bigoplus_J K$$

of finite-dimensional commutative algebras:

$$\begin{tikzcd}
\Lambda'_K \arrow[rr, "\phi"] \arrow[dr] & & \bigoplus_J K \\
& QH^*(G,K,1) = \Lambda'_K/I_1^K \arrow[ur, dashed, "\psi"] &
\end{tikzcd}$$

Theorem 6. ψ *is an isomorphism.*

Proof. We must show that $\operatorname{Ker}\phi = I_1^K$. By Lemma 3, we have the embedding $I_1^K \subset \operatorname{Ker}\phi$. It is clear that $\operatorname{Ker}\phi = \bigcap_J I_J$. $\operatorname{Ker}\phi$ is the intersection of the distinct (by Lemma 2) maximal ideals I_J. As a K-vector space in the ring Λ'_K, each of these has codimension 1. Hence, their intersection has codimension over K equal to their number, that is, $\binom{N}{l}$. On the other hand, the codimension of I_1^K is equal to the dimension of the factor-ring Λ'_K/I_1^K. By the Siebert–Tian theorem, this ring is isomorphic to the quantum cohomology ring, which is a free module of rank $\dim_K H^*(G,K,1)=\binom{N}{l}$ over K. Hence, the ideals $\operatorname{Ker}\phi$ and I_1^K coincide.

Corollary 7. *The algebra* Λ'_K/I_1^K *is semisimple. It is the direct sum of its distinct minimal prime ideals, each of which is isomorphic to* K.

Corollary 8. *The algebra* $\Lambda'_{\mathbb{Q}}/I_1$ *is semisimple. The eigenvalues of the operators of multiplication by the Schubert classes lie in* K.

Conjecture 9. *Let* G *be the Grassmannian of maximal isotropic planes in an even-dimensional space equipped with a non-degenerate quadratic form, or the Grassmannian of Lagrangian planes in a space with a symplectic form. Then the eigenvalues of the operator of multiplication by a divisor class, acting on the space* $QH(G,\mathbb{Q},1)$, *are defined over a cyclotomic extension of* $\mathbb{Q}$.

Let G be a generalized Grassmannian, that is, a factor of a (classical) simple group by a maximal parabolic subgroup. We do not expect, in general, that the eigenvalues of the operator of multiplication by a divisor class will be defined over a cyclotomic field.

Bibliography

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