Lecture 7: The Birkhoff–Grothendieck thm

1. Vector bundles on $\mathbb{P}^1$.

Thm 1. [B.–G.] Every holomorphic vector bundle E (of rank p) on $\mathbb{P}^1$ has the form:

$$E \cong \mathcal{O}(k_1) \oplus \cdots \oplus \mathcal{O}(k_p)$$

for some integers $k_1 \geq \cdots \geq k_p$. ☑

Def 1: The integers $k_1 \geq \cdots \geq k_p$ are called splitting numbers (integers).

Corollary 1 [Lemma 10.1 in Bolibrukh].

$$E := \bigoplus_{i=1}^{p} \mathcal{O}(k_i) \cong \bigoplus_{i=1}^{p} \mathcal{O}(k_i') =: E' \text{ iff } k_i = k_i' \text{ for all } i = 1, 2, \ldots, p$$

Pf. Cover $\mathbb{P}^1$ by $U_0 = \mathbb{C}$ and $U_\infty = \mathbb{P}^1 \setminus \{0\}$

put $K = \mathrm{diag}[k_1, \ldots, k_p]$, $K' = \mathrm{diag}[k_1', \ldots, k_p']$

then E is glued from $U_0 \times \mathbb{C}^p$ and $U_\infty \times \mathbb{C}^p$ via

$$g_{0\infty}(z) = z^K : \underset{\substack{\cap \\ U_\infty}}{(U_0 \cap U_\infty)} \times \mathbb{C}^p \to \underset{\substack{\cap \\ U_0}}{(U_0 \cap U_\infty)} \times \mathbb{C}^p$$

Similarly E' is glued via $g_{0\infty}'(z) = z^{K'}$.

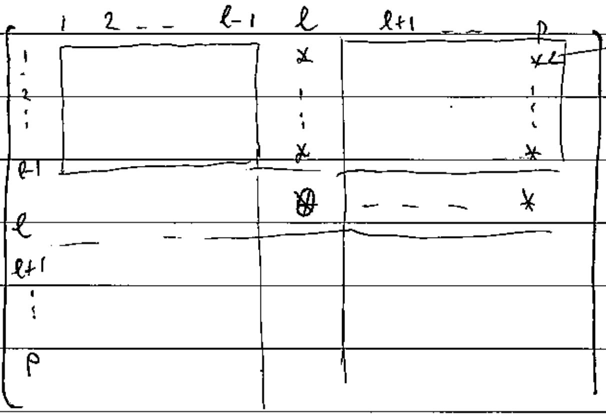
$\exists$ holom. h_α invertible in U_α ($\alpha \in \{0, \infty\}$) s.t.

$$h_0(z) \cdot \underbrace{g_{0\infty}'(z)}_{z^{K'}} = \underbrace{g_{0\infty}(z)}_{z^K} h_\infty(z), \text{ i.e.}$$

$$z^{k_j' - k_i} h_0^{ij}(z) = h_\infty^{ij}(z).$$

Assume $k_i = k_i'$ for $i = 1, 2, \dots l-1$ and $k_l' > k_l \geq k_{l+1} \geq \dots \geq k_p$

$h_\infty(z) =$



these entries are holomorphic in $\mathbb{P}^1$ and vanish at $z=0$ $\Rightarrow$ must be 0.

$\Rightarrow \det\left(h_\infty(z)\right) = 0$. ☑

Corollary 2 [Prop. 10.2 in Bolibrukh]. E : holomorphic bundle on $\mathbb{P}^1$ and $b \in \mathbb{P}^1$ $\Rightarrow$ we can find a basis of meromorphic sections : that are holomorphic on $\mathbb{P}^1 \setminus \{a\}$.

Pf. Assume $E = \mathcal{O}(k_1) \oplus \dots \oplus \mathcal{O}(k_p)$; $(U_0 \setminus \{b\}) \times \mathbb{C}^p$, $(e_1^0, \dots, e_p^0)$ $\downarrow z^{k}$ $(U_\infty \setminus \{b\}) \times \mathbb{C}^p$, $(e_1^\infty, \dots, e_p^\infty)$

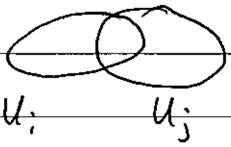
$s_i^\infty = \left(\frac{z-b}{z}\right)^{k_i} e_i^\infty$, $s_i^0 = (z-b)^{k_i} e_i^0$

$(S^\infty) = (s_1^\infty, \dots, s_p^\infty)$ sections over $\mathbb{P}^1 \setminus \{\infty, b\}$, $(S^0) = (s_1^0, \dots, s_p^0)$ sections over $\mathbb{C} \setminus \{b\}$

Since $(s^0) = (s^\infty)$, we get the desired sections by gluing (s^0) and (s^∞). ☑

Remark about trivializ. of v.b.:

$$\begin{tikzcd} E \arrow[d] \\ X \end{tikzcd}$$



$(s) = (s_1, \dots, s_p)$ sections over U_i $\rightsquigarrow$ trivial. $E|_{U_i} \cong U_i \times \mathbb{C}$

$(s') = (s_1', \dots, s_p')$: $E|_{U_j} \cong U_j \times \mathbb{C}^p$

$y = \begin{bmatrix} y_1(z) \\ \vdots \\ y_p(z) \end{bmatrix}$ gives section over U_i : $y = \sum_{i=1}^{p} y_a s_a$

$y' = \begin{bmatrix} y_1' \\ \vdots \\ y_p' \end{bmatrix}$ —"— section over U_j $y' = \sum y_b' s_b'$; then $\boxed{\begin{array}{c} y' = g_{ji} \cdot y \\ \Updownarrow \\ (s) = (s') \cdot g_{ji} \end{array}}$

connection matrices transform as:

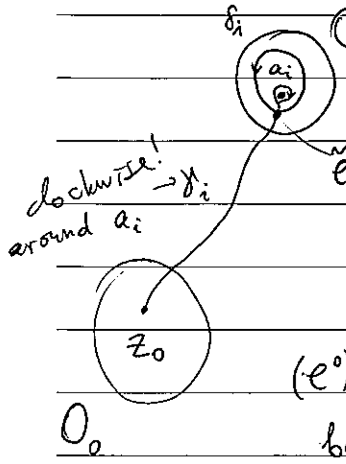
$$\boxed{\underset{\overset{\shortparallel}{B_i}}{B'} = \partial_z g_{ji} \, g_{ji}^{-1} + g_{ji} \underset{\overset{\shortparallel}{B_i}}{B} g_{ji}^{-1}}$$ ← connection matrices over U_i and U_j

$s_a = \sum_{b=1}^{p} (g_{ji})_{ba} s_b'$

2. Applications to the RH problem.

Assume we are given (E,∇)-bundle in the class $\mathcal{F}$.

$\chi : \pi_1(\mathbb{P}^1 \setminus \{a_1, \ldots, a_n\}) \longrightarrow GL(p; \mathbb{C})$



$\tilde{e}^i = (\tilde{e}^i_1, \ldots, \tilde{e}^i_p)$

analyt. contin.

$(e^0) = (e^0_1, \ldots, e^0_p)$

basis of flat sections over O_0

Given a path $[\gamma]$ we get by analytical continuoction along γ a new basis:

$$\gamma \cdot (e^0) = (e^0) \cdot M_\gamma$$

Since $\gamma_1 \cdot \gamma_2$ means 1-st γ_1 then γ_2 we must define

$$\chi(\gamma) := M_{\gamma^{-1}}$$

Pick a matrix S_i s.t. in $(e^i) = (\tilde{e}^i) \cdot S_i$ the monodromy along δ_i: $(e^i) \longmapsto (e^i) \cdot G_i$, G_i - upper triang.

$$\Big[\tilde{e}^i \xrightarrow{\gamma_i^{-1}} \tilde{e}^i M_{\gamma_i^{-1}} = \tilde{e}^i \chi(\gamma_i)$$

$$\downarrow$$

$$(e^i) \cdot G_i \cdot S_i^{-1} = (\tilde{e}^i) S_i G_i S_i^{-1} \quad \Longrightarrow \quad \boxed{\chi(\gamma_i) = S_i G_i S_i^{-1}}$$

Let $\Lambda_i = \{\lambda_i^1 \geq \lambda_i^2 \geq \ldots \geq \lambda_i^p\} \in \mathbb{Z}$, $1 \leq i \leq n$; then

$$\xi^{\Lambda_i} = (\xi_1^{\Lambda_i}, \ldots, \xi_p^{\Lambda_i}) = (e^i) \cdot \underbrace{(z-a_i)^{-\tilde{E}_i} (z-a_i)^{-\Lambda_i}}_{g_{a_i}}$$

where $\tilde{E}_i = \frac{1}{2\pi\sqrt{-1}} \ln G_i$

give a trivialization of $E|_{\overset{\circ}{O}_i}$ and ξ^{Λ_i}, by defin. to extend holomorphic sections of E on O_i!

the connection matrix in the frame ξ^{Λ_i} becomes:

$$\left(\Lambda_i + (z-a_i)^{\Lambda_i}\,\tilde{E}_i\,(z-a_i)^{-\Lambda_i}\right)\frac{dz}{z-a_i}.$$

Theorem 2. [Plemel] $\exists$ a connection ∇ on the trivial bundle on $\mathbb{P}^1$, s.t.,

1) ∇ has a regular singular point at 1 of the points (say, a_i);

2) ∇ has logarithmic (Fuchsian) singularities at the other points

3) The monodromy representation of ∇ coincides w/ the given one χ.

Pf. $(E,\nabla)\in\mathcal{F}$; pick $b=a_i$ $\Rightarrow$ we can identify $E\cong\mathcal{O}(k_1)\oplus\dots\oplus\mathcal{O}(k_n)$ $\Rightarrow$ choose sections

$e=(e_1,\dots,e_p)$: holomorphic basis (trivializ.) of E over $\mathbb{P}^1\setminus\{a_i\}$ meromorphic at $z=a_i$

Let $(e^{a_i})=(e_1^{a_i}\dots e_p^{a_i})$ be trivializ. of $\bigoplus\mathcal{O}(k_i)$ at $z=a_i$ as in Corollary 2.

$$(\xi^{\Lambda_i}) = (e^{a_i})\cdot\bar{V}(z)$$ $\bar{V}$ - invertible holom. at $z=a_i$

Since $(e):=(e^{a_i})\cdot(z-a_i)^{K}$ extends to global sections of E

$$\Rightarrow (\xi^{\Lambda_i}) = (e)\,(z-a_i)^{-K}\,V(z)$$

In the frame (e) the connection becomes:

$$-\frac{K}{z-a_i} + (z-a_i)^{-K}\,\omega_i\,(z-a_i)^{K}$$

where $\omega_i = \partial_z V\cdot V^{-1} + V\left(\Lambda_i + (z-a_i)^{\Lambda_i}\tilde{E}_i\,(z-a_i)^{-\Lambda_i}\right)V^{-1}$

and the $\overset{\text{corresp.}}{\smile}$ fundamental matrix is

$$\boxed{Y_e(z) = (z-a_i)^{-K}\,V(z)\,(z-a_i)^{\Lambda_i}\,(z-a_i)^{\tilde{E}_i}}$$

$\Rightarrow$ (1) a_i is a regular singular point;

(2) in the other points, the connection remains Fuchsian;

(3) the monodromy repres. remains the same as for $\bar{E}$, i.e. χ ☑

Corollary 3: The degree $c_1(E)$ (as introduced by Sergey) is $k_1+\dots+k_p$. ☑

Thm 3. [Plemel] Given χ, s.t. one of the matrices is diagonalizable then the RH problem has a solution.

The proof is easy if we use the above discussion and the following Lemma:

Lemma 1. [10.2 in Bolibrukh]. Assume:

$U(z)$: holomorphically invertible at $z=0$

$K = \mathrm{diag}(k_1 I^{m_1}, \dots, k_t I^{m_t})$, $k_1 > \dots > k_t \in \mathbb{Z}$

Then we have

$$z^K U(z) = \Gamma^{-1}(1/z)\cdot\tilde{U}(z)\cdot z^D,$$

for some matrices: $\Gamma(1/z)$ – polynomial in $1/z$ and invertible in $\mathbb{P}^1 \setminus 0$

$D = \mathrm{diag}[d_1, \dots, d_p] \leftarrow$ perm. of $[k_1 I^{m_1}, \dots, k_t I^{m_t}]$, and $\tilde{U}(z)$ holom. invert. at 0.

Assume all principal minors of $U(0)$ are invertible.

Pf. Induction on t. For $t=1$: trivial.

$$z^K\, U(z) = z^{K'}\, U'(z)\, z^{K''} , \qquad \begin{aligned} K &= K'+K'' \\ U' &= z^{K''}\, U\, z^{-K''} \end{aligned}$$

$$K' = \begin{bmatrix} (k_1-k_{t-1})I^{m_1} & & & 0 \\ & \ddots & & \\ & & (k_{t-2}-k_{t-1})I^{m_{t-2}} & \\ & & & 0 \\ 0 & & & & 0 \end{bmatrix}, \quad K'' = \begin{bmatrix} k_{t-1}I^{n_1} & 0 \\ 0 & k_t I^{m_t} \end{bmatrix}, \quad n_1 = p - m_t$$

$$U' = \begin{array}{c|c|c} & n_1 & m_t \\ \hline n_1 & V & T \\ m_t & W & * \end{array}$$

V is a principal minor of U

T is holom. at $z=0$ w/ order of vanishing $\geq m := k_{t-1} - k_t$

$*$: same as corresp. block in U, so holom. at $z=0$

~~W : has a pole of order $\leq m$~~

$$\Gamma_t\left(\tfrac{1}{z}\right) = \begin{bmatrix} I_{n_1} & 0 \\ z^{-m}R(z) & I_{m_t} \end{bmatrix}, \quad \Gamma_t \cdot U' = \begin{bmatrix} V & T \\ z^{-m}R(z)V + W & z^{-m}R(z)\cdot T + * \end{bmatrix} =: U''(z)$$

$R(z) = R_0 + R_1 z + \cdots + R_m z^m$

$V = V_0 + V_1 z + \cdots + V_m z + O(z^{m+1})$

$W = z^{-m}(W_0 + W_1 z + \cdots + W_m z^m) + O(z)$

choose R_i, $(0 \leq i \leq m)$ so that

$z^{-m}R(z)V(z) + W(z) = O(z)$ (Note V_0 is an invertible matrix!)

all principal minors of $U''(0)$ are invertible $\xrightarrow{\text{by induction}}$

$$\Gamma'\left(\tfrac{1}{z}\right) z^{K'}\, U''(z) = \tilde{U}(z)\, z^{K'} \ ; \qquad \text{Put } \Gamma\left(\tfrac{1}{z}\right) = \Gamma'\left(\tfrac{1}{z}\right)\left(z^{K'}\Gamma_t\left(\tfrac{1}{z}\right)z^{-K'}\right)$$

$$\Gamma\left(\tfrac{1}{z}\right)\cdot z^K U(z) = \boxed{\Gamma\left(\tfrac{1}{z}\right) z^{K'}}\, U'(z)\, z^{K''} = \Gamma'\left(\tfrac{1}{z}\right) z^{K'}\, \underbrace{\boxed{\Gamma_t\left(\tfrac{1}{z}\right) U'(z)}}_{U''}\, z^{K''} = \tilde{U}(z)\, z^{K'+K''=K} .$$

General case is reduced to this one by conjugating $\overset{U''}{U(z)}$ w/ a const. matrix. ☑

Irreducible monodromy

Lecture 8: Another solutions to the RH problem

Prop. 1 [11.1 in Bol.] The degree of any subbundle $F \subset \underline{\mathbb{C}}^p$ trivial bundle

$\Rightarrow c_1(F) \leq 0$.

Corollary. Every subbundle of $\deg = 0$ of a trivial v.b. / $\overline{\mathbb{C}} = \mathbb{P}^1$ is trivial.

Def1: If $E \to \mathbb{P}^1$ is a v.b. ; then $k(E) = \dfrac{c_1(E)}{rk(E)}$ (slope of E)

Def2: E is called stable if $k(F) \lneq k(E)$ $\forall F \subset E$

semi-stable if $k(F) < k(E)$

Claim. There are no stable bundles on $\mathbb{P}^1$ of $rk > 1$.

$E = \mathcal{O}(k_1) \oplus \dots \oplus \mathcal{O}(k_p)$, $k(E) = \dfrac{k_1 + \dots + k_p}{p} > k_i$ $\forall i$

$\Rightarrow k_1 + \dots + k_p > \sum k_i$ contr.

Prop 2. [11.2 in Bol.] E is semi-stable if $k_1 = k_2 = \dots = k_p = k$

(E, ∇) holom. v.b. / $\mathbb{P}^1$ w/ a logarithmic connection.

Def3: A subbundle $F \subset E$ is stabilized by ∇ if

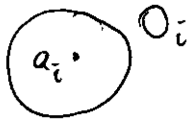
$\nabla(\Gamma(F)) \subset \Gamma(\tau_B^* \otimes F)$ (i.e. $\mathcal{F}$ is ∇-invariant)

Def4: (E, ∇) is stable if: ([illegible] $k(F) < k(E)$ $\forall F \subset E$ s.t. $\nabla F \subset F$

semi-stable if: $k(F) \leq k(E)$

Thm 1. [11.1 in Bol.] Let $(E,\nabla)\in\mathcal{F}$ be semi-stable; then $k_i-k_{i+1}\le n-2$, where $(k_1,\dots,k_p)$ are the splitting numbers of E and n = # of singular pts of ∇.

Pf: Λ_i, S_i, $1\le i\le p$ just like before



(ξ^{Λ_i}) holomorphic frame of E over O_i s.t. the connection matrix of ∇ is

$$\omega^{\Lambda_i}=\Big(\Lambda_i+\underset{\underset{\text{upper triangular}}{\uparrow}}{(z-a_i)^{\Lambda_i}E_i(z-a_i)^{-\Lambda_i}}\Big)\frac{dz}{z-a_i}.$$

$\dot{z}_0$

If $(\xi^{\Lambda_i})=(e^i)\cdot V(z)$, (e^i) some other frame on O_i

then the connection matrix is:

$$\omega_i = dV\cdot V^{-1}+V\cancel{\phi}\cdot\omega^{\Lambda_i}\cdot V^{-1}.$$

Fix i; then E admits a chart $(\mathbb{C}, \overline{\mathbb{C}}\setminus\{0,a_i\}, (z-a_i)^K)$

gluing $\mathbb{C}\times\mathbb{C}^p\ni(x,v)\sim(x,(z-a_i)^K\cdot v)\in(\overline{\mathbb{C}}\setminus\{0,a_i\})\times\mathbb{C}^p$

$\Rightarrow$ in the global basis $(e^i)(z-a_i)^K$ holom. on $\mathbb{C}\setminus\{a_i\}$

in the frame (e^i) the connection matrix

is $$\omega' = -\frac{K}{z-a_i}dz+(z-a_i)^{-K}\omega_i(z-a_i)^K,$$

Suppose (E,∇) is semistable but $\exists\, l$, s.t.

$$k_l-k_{l+1}>n-2$$

$\omega'=(\omega_{mj})_{1\le m,j\le p}$, $\omega:=\omega_i=(u_{mj})_{1\le m,j\le p}$

$\boxed{k_j - k_m \geq k_\ell - k_{\ell+1} > n-2}$

$\begin{pmatrix} k_j - k_m > n-2 \\ \ell \geq j \;, m > \ell \end{pmatrix}$

$\omega_{mj} = u_{mj}(z)\,(z-a_i)^{-k_m+k_j}$ for $m \neq j$

$\Rightarrow$ multipl. of the zero at $z = a_i$ of $\omega_{mj} \geq n-3$.

$\Rightarrow$ ω' at $z = a_i$ has a zero of order $> n-3$

ω' at $z \neq a_j$ has at most pole of order ≤ 1

$\Rightarrow$ ω' has zero $\geq n-4$ and poles of order $\leq n-1$

$$\Rightarrow \quad \omega' = \begin{array}{c|c|c} & \overbrace{\quad}^{\ell} & \overbrace{\quad}^{p-\ell} \\ \hline \ell\{ & \omega^1 & * \\ \hline p-\ell\{ & \underbrace{0}_{\ell} & \omega^2 \end{array} \qquad \omega' \in T^*_{\mathbb{P}^1} \otimes \mathcal{O}(-1)$$

$$\overset{\wr\wr}{\mathcal{O}(-3)}$$

conn. matrix ω^1

$\Rightarrow$ we can choose a rank ℓ subbundle (F^1, ∇^1)

~~Note~~ Note that $F^1 \cong \mathcal{O}(k_1) \oplus \cdots \oplus \mathcal{O}(k_\ell)$, $k_1 \geq \cdots \geq k_\ell$

$\Rightarrow \dfrac{k_1 + \cdots + k_\ell}{\ell} > \dfrac{k_1 + \cdots + k_p}{p}$ - contradiction. □

because $k_\ell > k_{\ell+1}$!

Thm 2. [10.4 in Bol.] Every irreducible repr. $\chi : \pi_1(\mathbb{P}^1 \setminus \{a_1, \dots, a_n\}) \to GL(p; \mathbb{C})$ can be constr. as a monodr. repr. of a Fuchsian system.

Pf. Take $E \in \mathcal{F}$ using $\Lambda_1 = \mathrm{diag}\{\lambda_1, \dots, \lambda_p\}$, $\Lambda_2 = \cdots = \Lambda_n = 0$

λ_j any integers s.t. $\lambda_j - \lambda_{j+1} > (n-2)(p-1)$ $\forall j$.

Choose a meromorphic frame of E which is holom. outside of a_1 and s.t. ~~[illegible]~~ the system

$$dy = \omega \cdot y$$

has a fundam. matrix $Y_1(z)$ whose expansion near $z=a_1$ has the form:

$$Y_1(z) = (z-a_1)^{-K} \underset{\substack{\uparrow \\ \text{holom. invertible at } a_1}}{V(z)} (z-a_1)^{\Lambda_1} (z-a_1)^{E_1}$$

Using Lemma 10.2, $\exists\ \Gamma(z)$ holom. invertible on $\mathbb{P}^1 \setminus \{a_1\}$

$U(z)$: holom. invertible at $z=a_1$, and

D : diagonal w/ entries permut. of $\{-k_1, \ldots, -k_p\}$

$$\Gamma(z)\,(z-a_1)^{-K}\,V(z) = U(z)\,(z-a_1)^{D}$$

χ is irreducible $\Rightarrow$ $k_\ell - k_{\ell+1} \leq n-2$ $\Rightarrow |k_i - k_j| \leq (n-2)(j-i)$

$\Rightarrow |d_i - d_j| \leq (n-2)(p-1)$

$\Rightarrow$ $H_1 = D + \Lambda_1$ is still an admissible matrix.

$\tilde{Y}_1(z) = \Gamma(z)\, Y_1(z)$ gauge transformation (global one)
holom. on $\mathbb{P}^1 \setminus \{a_1\}$
merom. on O_1.

$$\tilde{Y}_1(z) = U(z)\,(z-a_1)^{H_1}\,(z-a_1)^{E_1}$$

$$\Rightarrow \partial_z \tilde{Y}_1 = \left(\partial_z U \cdot U^{-1} + \frac{U \cdot H_1 U^{-1}}{z-a_1} + \frac{U(z)\overbrace{(z-a_1)^{H_1} E_1 (z-a_1)^{-H_1}}^{\text{holomorphic at } z=a_1} U^{-1}}{z-a_1}\right)\tilde{Y}_1$$

$\Rightarrow$ $\tilde{Y}_1$ is the fund. matrix of a Fuchsian system.

☑

Example 11.1

$$G_1 = \begin{bmatrix} 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \end{bmatrix}, \quad G_2 = \begin{bmatrix} 3 & 1 & 1 & -1 \\ -4 & -1 & 1 & 2 \\ 0 & 0 & 3 & 1 \\ 0 & 0 & -4 & -1 \end{bmatrix}, \quad G_3 = \begin{bmatrix} -1 & 0 & 2 & -1 \\ 4 & -1 & 0 & 1 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 4 & -1 \end{bmatrix}$$

$$G_1 \cdot G_2 \cdot G_3 = 1 \qquad S_2 = \frac{1}{3}\begin{bmatrix} 3 & 0 & 0 & 0 \\ -6 & 3 & -3 & 4 \\ 0 & 0 & 1 & -1 \\ 0 & 0 & -2 & 3 \end{bmatrix}, \quad S_3 = \frac{1}{64}\begin{bmatrix} 0 & 16 & 4 & 3 \\ 64 & 0 & 0 & 0 \\ 0 & 0 & 0 & -4 \\ 0 & 0 & -16 & -12 \end{bmatrix}$$

$$S_2^{-1} G_2 S_2 = G_1, \quad S_3^{-1} G_3 S_3 = \begin{bmatrix} -1 & 1 & 0 & 0 \\ 0 & -1 & 1 & 0 \\ 0 & 0 & -1 & 1 \\ 0 & 0 & 0 & -1 \end{bmatrix}$$

The repr. $\chi : \underset{\overset{\shortparallel}{\mathbb{Z} * \mathbb{Z}}}{\pi_1(\mathbb{P}^1 \setminus \{a_1, a_2, a_3\})} \to GL(4; \mathbb{C})$ is reducible

Def. A repres. χ is a B-representation if

(1) all G_i have 1 Jordan block

(2) it is reducible

Thm 11.2. B-repr. ~~of~~ is the monodromy of some Fuchsian system iff $\underset{\underset{\Lambda_i = 0 \ \forall i}{\uparrow}}{F^0}$ is semi-~~simple~~ stable.

In particular $c_1(F^{(0)}) = k \cdot p$ for some integer k.

For the example from above $c_1(F^{(0)}) = 2$

Pf of Thm 11.2 . $\Rightarrow$) If $F^{(0)}$ is semi-stable

Assume all sing. are Fuchsian except for 1 of them

$$Y_i(z) = (z-a_i)^{-K}\, V(z)\, (z-a_i)^{\boxed{\Lambda_i}=0} (z-a_i)^{E_i}$$

fund. solution at the remaining one point a_i.

if $F^{(0)}$ is semi-stable, then $K = \kappa \cdot I^p$ $\Rightarrow$ commutes w/ everything $\Rightarrow$ Y^{a_i} is Fuchsian.

$\Leftarrow$) If the system is Fuchsian

$$\begin{tikzcd} E, \nabla \arrow[d] \\ B \end{tikzcd}$$ trivial v.b. w/ Fuchsian connection.

χ' subrepr. of χ, dim $\chi' = l$

$X_l \subset X$: monodromy invariant subspace of dim l.

$\uparrow$ solutions

near a_i we have fund. matrix (from Levelt (=Jordan) filtr. for E_i)

$$Y_i(z) = U_i(z)\, (z-a_i)^{\Lambda_i} (z-a_i)^{E_i} \cdot S_i$$

$$\Lambda_i = \begin{vmatrix} \lambda_i^1 \geq & & 0 \\ & \ddots & \\ 0 & & \geq \lambda_i^p \end{vmatrix} \Rightarrow \exists \text{ subbundle } F' \subset E$$

$\operatorname{rk} F' = l$

$$\kappa(E) = \qquad \frac{\lambda_i^1 + \cdots + \lambda_i^l}{l} \geq \frac{\lambda_i^1 + \cdots + \lambda_i^p}{p}$$

$$\Rightarrow \kappa(F') = \sum_{i=1}^{n} \left(\frac{\lambda_i^1 + \cdots + \lambda_i^l}{l} + f_i \right) \geq \kappa(E) = 0$$

but we know $\kappa(F') \leq 0$ $\Longrightarrow$ $c_1(F') = 0$

$\Rightarrow$ $\Lambda_i = c_i \cdot I$ are scalar matrices $\quad c = \sum_{i=2}^{m} c_i$, $Y_1(t) = z-$

$$Y' = \frac{(z-a_1)^{c}}{\quad} Y(z),$$

Birkhoff normal form

(1) $z\frac{dy}{dz} = C(z)\cdot y$ near $z=\infty$

$C(z) = z^r \sum_{n=0}^{\infty} C_n z^{-n}$, $C_0 \neq 0$, $r \geq 0$ Poincaré rank

$\uparrow$ converges in $\mathcal{O}_\infty = \{z \in \mathbb{P}^1 \mid |z| > R\}$

$x = \Gamma(z) y$ $\leftarrow$ anal. invertible in $\mathcal{O}_\infty$ or merom.

$z\frac{dx}{dz} = \tilde{C}(z)\, x$

$\tilde{C}(z) = d\Gamma(z)\ \Gamma(z)^{-1} + \Gamma(z)\, C(z)\, \Gamma(z)^{-1}$

Thm [Birkhoff] There is some anal. transf. $\Gamma(z)$ s.t.

$\tilde{C} = (\tilde{C}_0 + \tilde{C}_1\cdot z + \dots + \tilde{C}_r\cdot z^r)$

provided the system (1) satisfies some conditions. e.g. irreducible monodromy

Remark: such $\tilde{C}$ is called Birkhoff standard form. (BSF)

Example 12.1 [Gantmacher 50's]

$z\frac{dy}{dz} = \left(\begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} + z^{-1}\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}\right)\cdot y$, $r = 0$

BSF should be $\tilde{C} = \tilde{C}_0$ – does not exist!

Define a vector bundle $F \to \mathbb{P}^1$ from (1) :

$\mathcal{O}_\infty \subset \mathbb{P}^1 \setminus \{0\}$

$Y(z) = T(z)\, z^E$ $(\mathbb{C},$ E[illegible]$,\ g_{\infty}(z) = T(z))$

∇ given by $\frac{C(z)}{z}dz = \omega_\infty$, $\omega_0 = \frac{E}{z}\,dz$

$\Rightarrow$ we can reduce the connection (by choosing a holom. trivializ. over $\mathbb{P}^1 \setminus \{0, \infty\}$) to

$$\tilde{C} = \sum_{i=-k}^{\textcircled{r}} c_i z^i .$$

We can similarly construct bundles F^{Λ}, $\Lambda = \begin{bmatrix} \lambda_1 & & 0 \\ & \ddots & \\ 0 & & \lambda_p \end{bmatrix}$

and get connection ∇^{Λ}. We get a class of v.b. $\mathcal{E}$

Thm 12.2. The system (1) admits a (BSF) iff $\mathcal{E}$ contains a trivial vector bundle.

Thm 12.3. Assume the system is irreducible (i.e. gauge $\cdot\ C \neq \begin{bmatrix} C_1 & * \\ 0 & C_2 \end{bmatrix}$) and $E \in \mathcal{E}$ then

$$k_i - k_{i+1} \leq r , \qquad i = 1, \dots, p-1 .$$

Thm 12.4. If C is irreducible; then BSF exists.