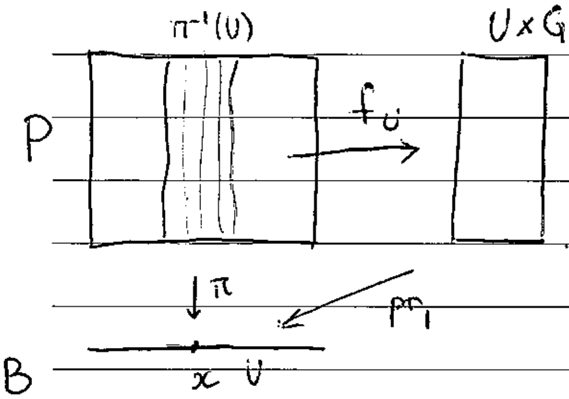
Lecture 1: Bundles and connections

1. Principal bundles.

$\begin{array}{c} P \\ \downarrow \pi \\ B \end{array}$ surjective map between manifolds

G : Lie group acting on P from the right

Def 1: (P, B, π, G) is called a principal G-bundle if :



$\forall x \in B$, $\exists$ open $\underset{\substack{\Downarrow \\ x}}{U} \subset B$ and an isomorphism

$f_U : \pi^{-1}(U) \longrightarrow U \times G$

such that ;

(1)
$$\begin{tikzcd} \pi^{-1}(U) \arrow[rr, "f_U"] \arrow[dr, "\pi"'] & & U \times G \arrow[dl, "pr_1"] \\ & B & \end{tikzcd}$$
commutes

and (2) $f_U(\bar{x} \cdot g) = f_U(\bar{x}) \cdot g$

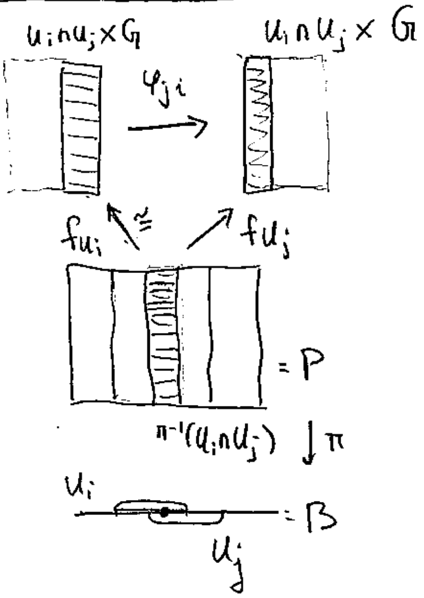
G-equivariant

Def 2: Two principal bundles (P, B, π, G) and (P', B, π', G) are equivalent if $\exists$ an isomorphism $h : P \to P'$, s.t.

G-equivariant

$$\begin{tikzcd} P \arrow[rr, "h"] \arrow[dr, "\pi"'] & & P' \arrow[dl, "\pi'"] \\ & B & \end{tikzcd}$$
commutes.

<u>transition functions:</u>



$f_{U_i} : \pi^{-1}(U_i) \xrightarrow{\cong} U_i \times G$

$f_{U_j} : \pi^{-1}(U_j) \xrightarrow{\cong} U_j \times G$

trivializations; then

$$\varphi_{ji} := f_{U_j}\circ f_{U_i}^{-1} : (U_i\cap U_j)\times G \xrightarrow{\cong} (U_i\cap U_j)\times G$$

is well <u>defined</u> → transition function

Note φ_{ji} is G-equivariant

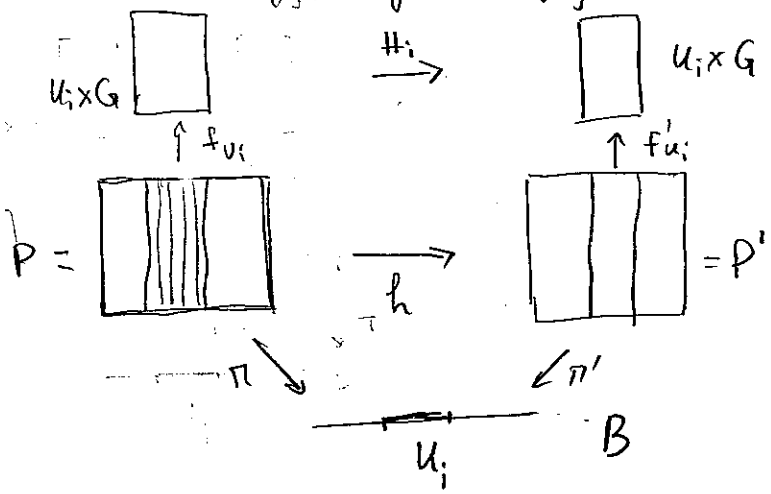
$$\Rightarrow \quad \varphi_{ji}(x,\underset{e\cdot g}{\overset{\shortparallel}{g}}) = \varphi_{ji}(x,e)\cdot g = (x, g_{ji}(x))\cdot g = (x, g_{ji}\cdot g)$$

where $g_{ji} : U_i \cap U_j \longrightarrow G$ is a collection of maps, called <u>gluing cocycle</u>

Note that

(1) $g_{ji}(x)\, g_{ij}(x) = e$, for $x \in U_i \cap U_j$

(2) $g_{ji}(x)\, g_{ik}(x)\, g_{kj}(x) = e$, for $x \in U_i \cap U_j \cap U_k$.



$H_i := f'_{U_i}\circ h\circ f_{U_i} : U_i\times G \xrightarrow{\cong} U_i\times G$

$\Rightarrow H_i(x,g) = (x, h_i(x)\, g)$

where $h_i : U_i \longrightarrow G$ is some map.

Assume $P \cong P'$.

Easy to check that

this defines equivalence between cocycles → $\boxed{g'_{ji}(x) = h_j(x)\, g_{ji}(x)\, h_i^{-1}(x)}$

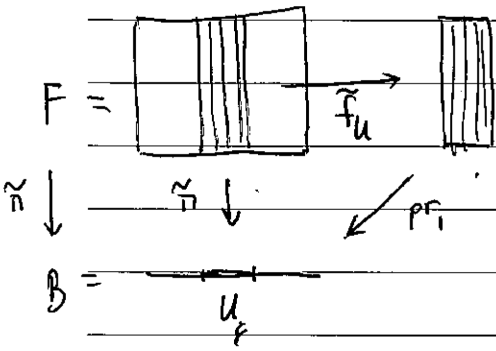
<u>Thm 1</u>. There is a one-to-one corresp. between

$\{$ isomorphism classes of principal G-bundles on B $\}$ $\xleftrightarrow{\text{and}}$ $\{$ equivalence classes of gluing cocycles $\}$

2. Vector bundles

$\begin{matrix} F \\ \tilde{\pi} \downarrow \\ B \end{matrix}$ surjective map between manifolds, s.t. $\forall x \in B$, $\tilde{\pi}^{-1}(x)$ is a vector space

<u>Def 3:</u> $(F, B, \tilde{\pi})$ is called a *vector bundle* if



$\forall x \in B$, $\exists$ open $\mathcal{U} \subset B$, $x \in \mathcal{U}$ and a homeomorphism

$$\tilde{f}_{\mathcal{U}} : \tilde{\pi}^{-1}(\mathcal{U}) \longrightarrow \mathcal{U} \times \mathbb{R}^p$$

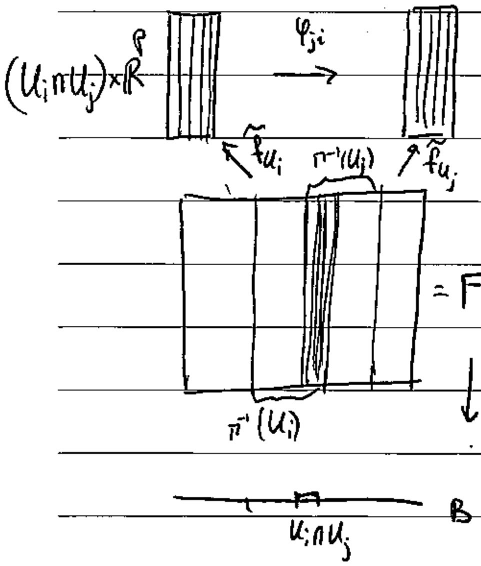
such that

(1)
$$\begin{tikzcd} \pi^{-1}(\tilde{\mathcal{U}}) \arrow[rr, "\tilde{f}_{\mathcal{U}}"] \arrow[dr, "\tilde{\pi}"'] & & \mathcal{U} \times \mathbb{R}^p \arrow[dl, "pr_1"] \\ & \tilde{\mathcal{U}} & \end{tikzcd}$$
commutes

and (2) The induced map

$$\tilde{f}^x_{\mathcal{U}} : \tilde{\pi}^{-1}(x) \longrightarrow \mathbb{R}^p$$

is a linear isomorphism.



Similarly $\varphi_{ji} := \tilde{f}_{\mathcal{U}_j} \circ \tilde{f}_{\mathcal{U}_i}^{-1}$ defines trans. functions

$$\Rightarrow \varphi_{ji}(x, v) = (x, g_{ji}(x) \cdot v)$$

where $g_{ji} : \mathcal{U}_i \cap \mathcal{U}_j \longrightarrow GL_p(\mathbb{R})$

is some collection of maps.

They satisfy the cocycle condition for $G := GL_p(\mathbb{R})$

$\Rightarrow$ we can construct a principal G-bundle P on B.

<u>Thm 2.</u> There is a one-to-one correspondence between

$$\left\{\begin{array}{l}\text{equivalence classes}\\ \text{of vector bundles}\\ \text{on } B \text{ (of rank } p)\end{array}\right\} \quad \text{and} \quad \left\{\begin{array}{l}\text{equivalence classes of}\\ \text{principal } GL_p\text{-bundles}\\ \text{on } B\end{array}\right\}.$$

Example. 1) Hopf fibration

$$P = S^3 = \{|z_1|^2 + |z_2^2| = 1\} \subset \mathbb{C}^2$$

$$\downarrow \pi \qquad\qquad \pi(z_1, z_2) = [z_1 : z_2]$$

$$B = \mathbb{P}^1 = (\mathbb{C}^2 \setminus \{0\}) / \mathbb{C}^* \qquad\qquad G = S^1 = \{|\lambda| = 1\} \subset \mathbb{C}$$

$$(z_1, z_2) \cdot \lambda = (z_1\lambda, z_2\lambda)$$

$$U_0 = \{z_1 \neq 0\} \cong \mathbb{C}, \quad z = \frac{z_2}{z_1} \qquad , \quad U_\infty = \{z_2 \neq 0\} \cong \mathbb{C}, \quad t = \frac{z_1}{z_2}$$

$$\pi^{-1}(U_0) \xleftarrow{\cong} U_0 \times S^1 \qquad\qquad \pi^{-1}(U_\infty) \xrightarrow{\cong} U_\infty \times S^1$$

$$(\lambda, z\lambda) \leftarrow (z, \lambda) : f_{U_0}^{-1} \qquad\qquad f_{U_\infty}(z_1, z_2) = \left(\frac{z_1}{z_2}, \frac{z_2}{|z_2|}\right)$$

$$f_{U_0}(z_1, z_2) = \left(\frac{z_2}{z_1}, \frac{z_1}{|z_1|}\right)$$

$$|z_1| \cdot z \quad \frac{1}{|z_2|} = g_{\infty 0}(z)$$

$$f_{U_0}((z_1, z_2)\cdot\lambda) = \left(\frac{z_2}{z_1}, \frac{z_1\lambda}{|z_1|}\right) = f_{U_0}(z_1, z_2)\cdot\lambda$$

$$h_\infty(z) = \frac{1}{|z|}, \quad h_0(t) = 1$$

S^1-equivariant

$$\varphi_{\infty 0}(z, \lambda) = f_{U_\infty} \circ f_{U_0}^{-1}(z, \lambda) = f_{U_\infty}(\lambda, z\lambda) = \qquad 1 \cdot t \cdot \frac{1}{|t|}$$

$$= \left(\frac{1}{z}, \frac{z\lambda}{|z|}\right)$$

$$\Rightarrow \boxed{g_{\infty 0}(z) = \frac{z}{|z|}}$$

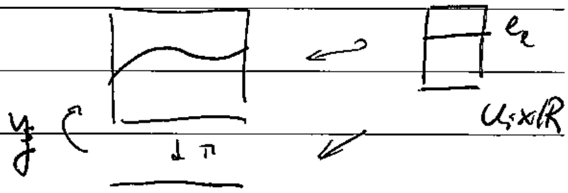
$$S^3 \times_{S^1} \mathbb{C} \cong \mathcal{O}(-1)$$

$$h_\infty(t) = |t|, \quad h_0(z) = 1$$

$$h_\infty g_{\infty,0} h_0^{-1} = |t| \frac{z}{|z|}.$$

Remark: If P is a principal G-bundle and $\rho : G \to GL_p(\mathbb{R})$ is a repr. of G in $\mathbb{R}^p$ then $F = P \times \mathbb{R}^p / (\bar{x}\cdot g, v) \sim (\bar{x}, \rho(g)\cdot v)$ is a vector bundle w/ transition functions gluing cocycle: $\rho \circ g_{ji} : U_i \cap U_j \to GL_p(\mathbb{R})$.

3. Connections



$F \xrightarrow{\tilde{\pi}} B$ vector bundle

Def: A connection on F is a linear map

$$\nabla : \Gamma(F) \longrightarrow \Gamma(T_B^* \otimes F)$$

Γ – sections of a vector bundle.

satisfying the Leibnitz rule:

$$\nabla(f \cdot y) = df \otimes y + f \, \nabla y.$$

Local expression: $\{U_i\}$ open cover of B, $f_{U_i} : \pi^{-1}(U_i) \to U_i \times \mathbb{R}^p$ trivialization

Note that $f_{U_i} \circ y(z) = (z, y_i(z))$

where $y_i(z) = \begin{pmatrix} y_i^1(z) \\ \vdots \\ y_i^p(z) \end{pmatrix} \in \mathbb{R}^p$, $y_i(z) = \sum_{\ell=1}^{p} y_i^\ell(z)\, e_\ell$

and $y_j(z) = g_{ji}(z) \cdot y_i(z)$, $z \in U_i \cap U_j$

A_i – matrix of 1-forms on U_i s.t.

$$\nabla e_\ell = \sum_{k=1}^{p} A_{i,k\ell} \otimes e_k \qquad \text{connection matrix}$$

then $\nabla y_i = dy_i + A_i \cdot y_i$

$\Rightarrow \{A_i\}_i$ must satisfy $A_i = dg_{ij} \cdot g_{ij}^{-1} + g_{ij} \cdot A_j \cdot g_{ij}^{-1}$.

Ass

Def: y is called a horizontal section if $\nabla y = 0$.

Locally: $dy_i + A_i \cdot y_i = 0$

$z = (z^1, \dots, z^m)$ local coords on U_i

$$\Updownarrow \qquad A_i = \sum_{a=1}^{m} A_{i,a}(z)\, dz^a$$

$$\frac{\partial y_i}{\partial z_a} = -A_{i,a}(z)\cdot y_i \quad , a = 1, 2, \dots, m$$

$$\frac{\partial^2 y_i}{\partial z_b \partial z_a} = -\frac{\partial A_a}{\partial z_b}\cdot y_i - A_a \cdot (-A_b \cdot y)$$

$$= \left(-\frac{\partial A_a}{\partial z_b} + A_a A_b\right)\cdot y = \left(-\frac{\partial A_b}{\partial z^a} + A_b A_a\right)\cdot y$$

i.e. integrability condition

$$\boxed{\frac{\partial A_a}{\partial z^b} - \frac{\partial A_b}{\partial z^a} = [A_a, A_b] \quad , \text{for all } 1 \le a, b \le m}$$

Def: ∇ is called flat if the above equation is satisfied.

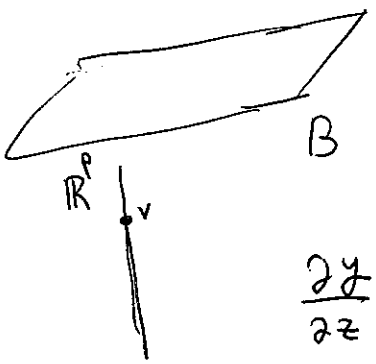
Assume the system is compatible, i.e., ∇ is flat, then

Thm. [Frobenius]: $dy = -A\cdot y$, $y(z_0) = V$ has a unique solution in a neighborhood of z_0

Assume now that $\dim_{\mathbb{C}} B = 1$.

A connection then is locally given by

$$\nabla = d + A(z)\, dz$$



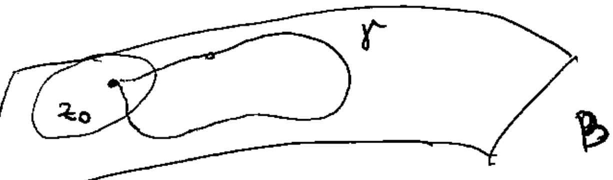
$\Rightarrow \nabla y = 0$ means

$$\frac{\partial y}{\partial z} = -A(z)\cdot y.$$

The Cauchy problem

$$\begin{cases} y'(z) = -A(z)\cdot y \\ y(z_0) = V \end{cases}$$

has a unique solution in a neighborhood of z_0



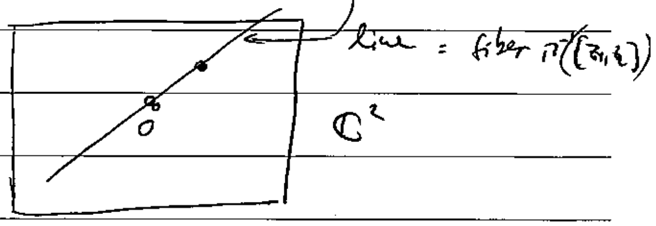
$\Rightarrow$ we get a monodromy repres. $\chi : \pi_1(B) \longrightarrow GL_p(\mathbb{R})$

Question: Given χ, can we construct a flat connection w/ monodromy repr. χ?

Example 2: $\mathcal{O}(-1)$

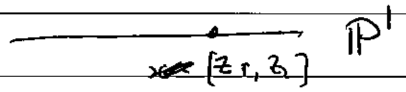
$F \subset \mathbb{P}^1 \times \mathbb{C}^2$, $\{([z_1, z_2], x_1, x_2) : z_1 x_2 = z_2 x_1\}$ $(= Bl_0 \mathbb{C}^2)$

$$\begin{array}{ccc} F & & \\ \tilde{\pi} \downarrow & \swarrow pr & \\ \mathbb{P}^1 & & \end{array}$$



$\mathbb{C} \cong U_0 = \{z_1 \neq 0\} \subset \mathbb{P}^1$

$z = \frac{z_2}{z_1}$



$f_{U_0} : \pi^{-1}(U_0) \longrightarrow U_0 \times \mathbb{C}$ $\qquad \mathbb{C} \cong U_\infty = \{z_2 \neq 0\} \subset \mathbb{P}^1$

$f_{U_0}([z_1, z_2], x_1, x_2) = \left(\frac{z_2}{z_1}, x_1\right)$ $\qquad t = \frac{z_1}{z_2}$

$f_{U_0}^{-1}(z, \lambda) = ([1, z], \lambda, \lambda z)$ $\qquad f_{U_\infty}([z_1, z_2], x_1, x_2) = ([z_1, z_2], x_2)$

$$\varphi_{\infty 0}([z_1, z_2], \lambda) = f_{U_\infty}\left([z_1, z_2], \lambda, \lambda \frac{z_2}{z_1}\right) = \left([z_1, z_2], \frac{z_2}{z_1}\lambda\right)$$

$$\Rightarrow g_{\infty 0}([z_1, z_2]) = \frac{z_2}{z_1} \in \mathbb{C}^*$$

or in terms of $z = \frac{z_2}{z_1}$ local coord. on $U_0 \cap U_\infty$:

$$\boxed{g_{\infty 0}(z) = z}$$

~~Remark~~: Rmk: $g_{\infty 0}(z) := z^{-k}$; then we get a bundle on $\mathbb{P}^1$ called $\mathcal{O}(k)$. $\qquad T_{\mathbb{P}^1} = \mathcal{O}(2)$.

$h_0(z)$, $h_\infty(t)$ s.t. $h_\infty(z) = z^k \, h_0(z)$ if $k>0$ $\lim_{z \to \infty} h_0(z) = 0 \Rightarrow h_0$ must be const. 0
$z \in \mathbb{C}$ $\quad t \in \mathbb{C}$ $\quad$ holom. at $t = \infty$ $\quad$ defined at $z = \infty \Rightarrow$
$t = \frac{1}{z}$

3 ~~Connection~~

$\frac{\partial y}{\partial z_2} = -A_2(z)\cdot y$

Lecture 2: Merom. connections w/ reg. singular pts.

1. Fuchsian and regular singular points.

F is $\mathbb{C}$-analytic v.b. $/B$ $\quad \dim_{\mathbb{C}} B = 1$

∇ - merom. connection on B, i.e.,

$$\nabla_{\partial/\partial z}\, y = \frac{\partial y}{\partial z} - B(z)\cdot y \quad , \quad B(z) \text{ is meromorphic on } B$$

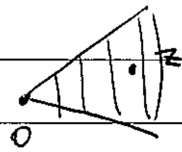
Assume $O \subset B$ is a neighborhood of $z_0 = 0$. Horizontal sections:

$$\frac{dy}{dz} = B(z)\cdot y \qquad (4.2)$$

Def: $z=0$ is a ~~regular~~ Fuchsian singular point if $B(z)$ has a pole of order ≤ 1. Moreover, if all singular points of $B(z)$ are Fuchsian, then ∇ is called Fuchsian connection.

Def 2: $z=0$ is a regular singular point of ∇ if every horizontal section y satisfies:

$\exists C, N$, s.t. $\quad |y(z)| \le C\cdot |z|^{-N}$



for every $z \in$ some sector w/ vertex $z=0$ and angle $< 2\pi$.

Example: $\frac{dy}{dz} = \begin{bmatrix} \frac{1}{z} & 1 \\ 0 & 0 \end{bmatrix}\cdot y \quad , \quad B = \mathbb{P}^1$

$z=0$ is Fuchsian

$y_1' = \frac{1}{z} y_1 + y_2 \qquad y_2 = c$

$y_2' = 0 \qquad$ [illegible]

$Y(z) = \begin{bmatrix} z & z\ln z \\ 0 & 1 \end{bmatrix}$

Ex2. $\frac{dy}{dz} = -\frac{y}{z^2}$ not Fuchsian and not regular

$y = e^{\frac{1}{z}}$

Thm 1. A Fuchsian singular point is always regular.

Rem: $\frac{dy}{dz} = \begin{bmatrix} 0 & 1 \\ \frac{1}{z^2} & -\frac{1}{z} \end{bmatrix} \cdot y$, $Y(z) = \begin{bmatrix} z & \frac{1}{z} \\ 1 & -1/z^2 \end{bmatrix}$ → regular

↑ not Fuchsian

2. Monodromy.

$\mathring{O} = O \setminus \{0\}$

if γ is a loop in O, analytic continuation of $Y(z)$ along γ gives a fundamental matrix $Y'(z) = Y(z) \cdot G_\gamma$ where $G_\gamma \in GL_p(\mathbb{C})$.

$$\Rightarrow \quad \chi_0 : \underset{\substack{\uparrow \\ \mathbb{Z}\cdot\gamma}}{\pi_1(\mathring{O}, z_0)} \longrightarrow GL_p(\mathbb{C})$$

γ

$\sigma := G_\gamma$ - monodromy matrix of Y.

Rem: If $\tilde{Y} = Y \cdot S$ is another fundam. matrix then $\tilde{\sigma} = S^{-1} \sigma S$.

If λ is an eigenvalue of a matrix H then we

G - monodromy matrix of $Y(z)$

Put $E = \frac{1}{2\pi i} \ln G$, where the eigen-values of E

$\rho^1, \ldots, \rho^p$ are s.t. $0 \leq \mathrm{Re}(\rho^i) < 1$

We define $z^E = \exp(E \ln z)$. Analytic continuation along γ transforms z^E into $z^E \cdot G$

Lemma 1. The fundamental matrix has a decomposition

$$Y(z) = M(z)\, z^E$$

where $M(z)$ is single valued in $\dot{O}$.

Example: $\frac{dy}{dz} = \begin{bmatrix} \frac{1}{z} & 1 \\ 0 & 0 \end{bmatrix} \cdot y$

$$Y(z) = \begin{bmatrix} z & z\ln z \\ 0 & 1 \end{bmatrix} \xrightarrow{\sigma} \begin{bmatrix} z & z(\ln z + 2\pi i) \\ 0 & 1 \end{bmatrix} = Y(z) \cdot \begin{bmatrix} 1 & (2\pi i) \\ 0 & 1 \end{bmatrix}$$

$$\Rightarrow G = \begin{bmatrix} 1 & 2\pi i \\ 0 & 1 \end{bmatrix} \Rightarrow E = \frac{1}{2\pi i} \log G = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$$

$$\Rightarrow Y(z) = \begin{bmatrix} z & 0 \\ 0 & 1 \end{bmatrix} \cdot z^{\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}}.$$

Lemma 2. The elements a_{ij} of the matrix z^E have the form

$$a_{ij} = \sum_{l=1}^{p} z^{\rho^l} P_{ij}^l (\ln z)$$

polynomial of degree $\leq$ size of the largest Jordan block of E.

Pf. Wlog $E = \rho I + N$ is a Jordan block, i.e.,
I - identity
N - uper triangular

$$z^E = z^\rho \cdot z^N = z^\rho \exp(N \log z) = $$
$$= \sum_{k=1}^{\infty} z^\rho \frac{(\ln z)^k}{k!} N^k = \sum_{k=1}^{p'-1} z^\rho (\ln z)^k \frac{N^k}{k!}$$

3. Scalar equations.

(3.1) $u^{(p)} + q_1(z) u^{(p-1)} + \dots + q_p(z) u = 0$

locally near $z=0$

Def: $z=0$ is regular singular at $z=0$ if every solution $u(z)$ satisfies: $|u(z)| < C|z|^{-N}$ for some C, N

$z \to 0$ in a sector $< 2\pi$

Def: $z=0$ is Fuchsian if $q_i(z) = \frac{r_i(z)}{z^i}$, where $r_i(z)$ is holomorphic at $z=0$.

Thm. Fuchsian $\Leftrightarrow$ regular.

pf: $\Rightarrow$) easy: put $y_1 = u$, $y_2 = zu'$, $\dots$, $y_p = z^{p-1} u^{(p-1)}$

$$y_1' = u' = \frac{1}{z} y_2$$
$$y_2' = u' + zu'' = \frac{1}{z}(y_2 + y_3)$$
$$\vdots$$
$\Rightarrow$ all poles are at most 1.

$$y_p' = \frac{p-1}{z} y_p + \frac{1}{z}(-r_1 \cdot y_1 - r_2 \cdot y_2 - \dots - r_{p-1} y_{p-1})$$

$$B(z) = \begin{vmatrix} 0 & z^{-1} & & & \\ & & & & \\ -r_1(z) & -r_2(z) & \dots & -r_{p-1}(z) & \frac{p-1}{z} \end{vmatrix} \quad \frac{1}{z}$$

$\Leftarrow$) $u_1, \dots, u_p$ a fundamental system of solutions of (3.1)

monodromy $Y(z) = [u_1, \dots, u_p] \longrightarrow [u_1, \dots, u_p] \cdot \underset{\uparrow}{G}$ monodromy matrix

note $Y(z) = M(z) \cdot z^E$

where $M(z) = [m_1(z), \dots, m_p(z)]$, $m_i(z)$ are single valued in $\dot{O}$.

The singularity is regular $\Rightarrow$ $M(z)$ is meromorphic.

May assume that E is uper triangular. Let $\rho = E_{11}$;

then $(z^E)_{1,1} = z^\rho \quad \Rightarrow \quad u_1(z) = m_1(z) \cdot z^\rho$

$\Rightarrow \quad u_1(z) = v(z) \cdot z^{\nu}$, v - holom. at $z=0$ and $v(0) \neq 0$

Induction on p: $\quad p = 1, \quad u' + q_1(z) \cdot u = 0$,

$u(z) = z^\rho v(z) \Rightarrow q_1(z) = -\partial_z(\ln u) = -\frac{\rho}{z} - \frac{v'(z)}{v(z)} \Rightarrow$ Fuchsian

Substitute $u(z) = x(z) \cdot u_1(z)$: set all integral coeff. to 1 for simplicity:

$$x^{(p)} + \left(q_1(z) + \frac{u_1'}{u_1}\right) x^{(p-1)} + \dots + \left(q_j(z) + \frac{u_1'}{u_1} + \dots + \frac{u_1^{(j)}}{u_1}\right) x^{(p-j)} + \dots +$$

$$\underbrace{\left(q_p(z) + \frac{u_1'}{u_1} + \dots + \frac{u_1^{(p)}}{u_1}\right)}_{=0} \cdot x = 0$$

Since $x = 1$ is a solution $\Rightarrow$

In particular,

$q_p(z)$ has a pole of order $\le p$.

Put $\tilde{u}(z) = X'(z)$ $\Rightarrow$ we get a diff. equation for $\tilde{u}$

of order $p-1$ w/ a regular singular point at $z=0$.

In particular, by induction

order of pole of $\Big(q_j + \underbrace{c_1 \cdot \frac{u'}{u} + c_2 \frac{u''}{u} + \dots + c_j \frac{u^{(j)}}{u}}_{\text{pole at most of order } j} \Big) \le j$

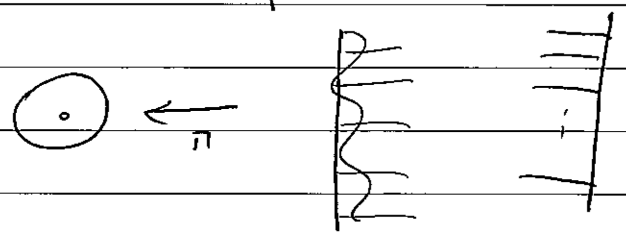
Lecture 3: Levelts theory

1. The universal cover of $\mathring{O}$.

$\mathring{O}$ - punctured disk $\{0<|z|<\delta\}$, $\pi_1(\mathring{O}, z_0) = \mathbb{Z}\sigma$

$\mathcal{O}^* \leftarrow$ right half-plane $= \{u \in \mathbb{C} \mid \operatorname{Re} u > \ln \delta\}$

$$\begin{array}{c} \mathcal{O}^* \\ \pi \downarrow \quad z = \exp u \\ \mathring{O} \end{array}$$



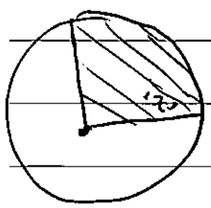
$\overline{z} = \pi^* z$; then $\ln \overline{z}$ is well defined

and $\begin{array}{c}\mathcal{O}^* \\ \downarrow \pi \\ \mathring{O}\end{array}$ is a principal $\mathbb{Z}$-bundle.

Note $\sigma^* u = u + 2\pi i$. For any function

$f(z)$ we define $(\sigma^* f)(z)$ ~~=~~

$z=0$ is a regular sing. point $\Longleftrightarrow$ $\exists\, Q \in \mathbb{Z}$ s.t. $\forall$ solutions $y(z)$ ~~$\forall$ sector~~ $S \subset \mathring{O}$



$$\frac{y(\overline{z})}{|\overline{z}|^Q} \to 0 \quad \text{as } \begin{array}{l} z \to 0 \\ z \in S \end{array}$$

Def: We say that $y(z)$ has a polynomial growth.

Def: Evaluation (Levelt's) $\varphi : X \to \mathbb{Z} \cup \{\infty\}$

$$\varphi(y) := \sup \left\{ l \;\middle|\; \lim_{\substack{z\to 0 \\ z \in S}} \frac{y(\overline{z})}{|z|^\lambda} = 0 \text{ for all } \lambda < l \right\}$$

$\varphi(0) := \infty$

Given a matrix $M = (f_{ij})_{1\le i,j\le p}$ we define

$$\varphi(M) = \min_{i,j} \varphi(f_{ij}).$$

Ex. $0 \le \mathrm{Re}\, \rho^i < 1$ where ρ^i are the eigenvalues of $E = \frac{1}{2\pi i} \ln G$

$\varphi(z^E) = 0.$

~~Exercise:~~

Proposition 1. The evaluation φ has the following properties:

a) $\varphi(y_1 + y_2) \ge \min\left(\varphi(y_1), \varphi(y_2)\right)$

with equality if $\varphi(y_1) \ne \varphi(y_2)$

b) $\varphi(cy) = \varphi(y)$ for $c \in \mathbb{C} \setminus \{0\}$

c) $\varphi^*(\sigma^* y) = \varphi(y)$ (monodromy invariance).

Pf: c) $\sigma^* z^a = \exp(2\pi\sqrt{-1}\, a)\, z^a$

$\sigma^* \ln z = \ln z + 2\pi i$

On the other hand $y(z) = \sum_{j,l} f_{j,l}(z)\, z^{\rho_j} (\ln z)^{b_l}$

$\Rightarrow \varphi(\sigma^* y) \le \varphi(y)$. Similarly $\varphi((\sigma^*)^{-1} y) \le \varphi(y)$. □

From a) and b) we get that $\{\varphi(X)\}$ is a finite

set: $\{\infty > \varphi^1 > \varphi^2 > \dots > \varphi^m\}$.

Def: [Levelt's filtration] $0 \subset X^1 \subset X^2 \subset \dots \subset X^m = X$

where $X^l = \{ y \in X \mid \varphi(y) \ge \varphi^l \}$.

The filtration is σ^*-invariant

Def: $k_\ell := \dim (X^\ell / X^{\ell-1})$. Note σ^* acts on $X^\ell / X^{\ell-1}$

$$^\ell\sigma^* = \sigma^* \big|_{X^\ell / X^{\ell-1}}$$

Let $e_1^1 \dots e_{k_1}^1$ base for X^1 s.t. σ^* is upper triangular

Take $\tilde{e}_1^2 \dots \tilde{e}_{k_2}^2$ base for X^2/X^1 s.t. $^2\sigma^*$ is upper triangular

and lift (arbitrary) to $\{e_1^2, \dots, e_{k_2}^2\} \in X^2$. Continuing this way we get a fundamental matrix:

$$Y(z) = [e_1^1 \dots e_{k_1}^1, e_1^2 \dots e_{k_2}^2, \dots, e_1^m \dots e_{k_m}^m] = [e_1, e_2, \dots, e_p]$$

the corresp. monodromy matrix G is upper-triangular.

The following properties hold:

1) φ takes all possible values $\varphi^1, \dots, \varphi^m$ w/ multipl. $k_1, \dots, k_m$

2) $\varphi(e_{\ell+1}) \le \varphi(e_\ell)$

3) σ^* is upper triangular

Def: Any basis $\{e_1, e_2, \dots, e_p\}$ of X satisfying 1), 2), and 3) then it is called Levelt's basis.

Exercise: If σ^* is a Jordan block, then a Jordan basis is a Levelt's basis. Any other Levelt's basis is obtained by conjugation by an upper triangular matrix.

Pf: $\{e_1, -, e_e\}$ Jordan basis

$\sigma^* e.$ ——

If $e = \{e_1, -, e_p\}$ is a Levelt's basis then

$$A := \begin{bmatrix} \varphi(e_1) & & 0 \\ & \ddots & \\ 0 & & \varphi(e_p) \end{bmatrix}, \quad G = \sigma^*, \quad E = \frac{1}{2\pi i} \ln G$$

$0 \leq \overset{Re}{\rho^i} < 1$

eigen-values of E

Lemma 5.1. Put $\tilde{C} = z^A C z^{-A}$, then $\tilde{G}$ and $\tilde{E}$ are holom. at $z=0$,

and $\varphi(z^A \bar{z}^E z^{-A}) = 0.$

Pf. if $C = (C_{ij})$ if $c_{ij} = 0$ for $i > j$ (upper triangular matrix)

$$\Rightarrow \tilde{C}_{ij} = \begin{cases} z^{\varphi_i - \varphi_j} \cdot c_{ij} & \text{for } i \leq j \\ 0 & \text{for } i > j \end{cases}$$

$\Rightarrow \tilde{G}$ and $\tilde{E}$ are holomorphic.

$$z^A \bar{z}^E z^{-A} = \begin{bmatrix} \bar{z}^{\rho_1} & & * \\ & \ddots & \\ 0 & & \bar{z}^{\rho_p} \end{bmatrix}$$

$$E = \begin{bmatrix} \rho_1 & & 0 \\ & \ddots & \\ 0 & & \rho_p \end{bmatrix} + \underset{\substack{\uparrow \\ \text{upper triangular}}}{N}$$

$$z^E = z^R \cdot \underset{\substack{\uparrow \\ \text{growth } 0}}{z^N}$$

<u>Thm 1.</u> If (e) is a Levelt's basis; then

$$Y(z) = U(z)\, z^{A}\, \bar{z}^{E} \quad \text{w/ } U(z) \text{ is holomorphic at } z=0.$$

<u>Pf.</u> We already saw that $U(z)\, z^{A}$ is single-valued $\Rightarrow$

$U(z)$ is single-valued.

Put $r = \max_i \rho_i$ and choose $\varepsilon > 0$ s.t. $2\varepsilon + r < 1$.

We want to show that $\lim_{z\to 0} U(z)\, \bar{z}^{r+2\varepsilon} = 0$.

$$U(z)\, \bar{z}^{r+2\varepsilon} = Y_e(z)\, z^{-A}\, \bar{z}^{-E}\, \bar{z}^{r+2\varepsilon} =$$

$$= \underbrace{\left(Y_e(z)\, z^{-A+\varepsilon}\right)}_{N_1} \cdot \underbrace{\left(z^{A}\, \bar{z}^{-E}\, z^{-A}\right)}_{N_2} \bar{z}^{r+\varepsilon}$$

By definition $e_i\, z^{-\varphi(e_i)+\varepsilon} \to 0$ as $z \to 0$ $\Rightarrow$ $\lim N_1(\bar{z}) = 0$ by definition

$\bar{z}^{-E+r}$ $\rightsquigarrow$ has entries $a_{ij} = \sum_{\ell=1}^{p} \bar{z}^{r-\rho_\ell}\, P_{ij}^{\ell}(\ln z)$

$\Rightarrow \varphi(a_{ij}) \geq 0$

$$\Rightarrow \lim_{z\to 0} \left(z^{A}\, \bar{z}^{-E+r}\, z^{-A}\right) \cdot z^{\varepsilon} = 0. \quad \square$$

<u>Def:</u> Weak Levelt's basis

- If only one eigenvalue then same as Levelt

$$X = X_1 \oplus \cdots \oplus X_s \quad \text{eigenspace decomposition w/ respect}$$
$$\quad \lambda_1 \qquad \lambda_s \qquad \text{to } \sigma^*$$

Put $\sigma_i^{*\prime} = \sigma^*|_{X_i}$.

Construct Levelt's basis for each X_i

$$\text{Weak Levelt } [X] = \bigsqcup_{i=1}^{s} \text{Levelt } [X_i]$$

Exercise 5.5. Show that $WL(X)$ is associated w/ Levelt of X as follows:

$$\varphi\{e_1, \ldots, e_p\} = \psi^{\ell} \text{ w/ mult. } k_\ell$$

Thm 1 holds for a weak Levelt basis.

Example: $\dfrac{dy}{dz} = \begin{pmatrix} 0 & 1 \\ z^{-2} & -z^{-1} \end{pmatrix} \cdot y$

$$Y(z) = \begin{vmatrix} z & z^{-1} \\ 1 & -z^{-2} \end{vmatrix}$$

$$\varphi\begin{pmatrix} z \\ 1 \end{pmatrix} = 0, \quad \varphi\begin{pmatrix} z^{-1} \\ -z^{-2} \end{pmatrix} = -2 \quad \Rightarrow \quad \text{Levelt's basis}$$

$$\Rightarrow Y(z) = \begin{bmatrix} z & z \\ 1 & -1 \end{bmatrix} \cdot z^{\begin{bmatrix} 0 & 0 \\ 0 & -2 \end{bmatrix}}.$$

Example: $\dfrac{dy}{dz} = \begin{bmatrix} z^{-1} & 1 \\ 0 & 0 \end{bmatrix} \cdot y, \quad Y(z) = \begin{bmatrix} z & z\ln z \\ 0 & 1 \end{bmatrix}$

$$\varphi\begin{pmatrix} z \\ 0 \end{pmatrix} = 1, \quad \varphi\begin{bmatrix} z\ln z \\ 1 \end{bmatrix} = 0$$

$$Y(z) = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \cdot z^{\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}} \cdot z^{\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}}$$

Thm 2. The matrix $U(0)$ is invertible if and only if the system is Fuchsian at $z=0$.