Lecture 11, part 2 $\uparrow$ P1

Def 11.3 B-representation is monodromy rep. with reducible total space s.t. all G_i are Jordan 1-blocks $\begin{pmatrix} \lambda & & 0 \\ & \ddots & \\ 0 & & \lambda \end{pmatrix}$

Thm 11.2 B-repr. is monodromy of Fuchsian system $\Leftrightarrow$

F^0 is semi-stable $(= \mathcal{O}(k) \otimes \mathbb{C}^p)$.

Proof $\Leftarrow$ Thm 10.2 (Plemelj) and (10.3) $Y_1(z) = (z-a_i)^{-K} V(z) (z-a_i)^{\Lambda_i} (z-a_i)$

implies $\exists$ system Fuchsian in $a_2, \ldots, a_n$ with $\varphi_i = 0$

and $\Lambda_1 = 0$, $K = k \cdot Id$:

$$(11.3) \quad Y_1(z) = (z-a_1)^{-k Id} V(z) \, (z-a_1)^{E_1}$$

commute since k-scalar.

system is Fuchsian in a_1 also.

$\Rightarrow$ Thm 8.1 $\Rightarrow$ E is holom. fam.

Let $\dim X' = l$, X_l – subspace of sol. monodromy-inv.

$$G_i = \begin{pmatrix} G_i' & * \\ 0 & G_i'' \end{pmatrix} \quad i = 1 \ldots n$$

Exam 5.2 and Thm 5.2 $\Rightarrow$ $Y(z) = U_i(z) (z-a_i)^{\Lambda_i} (z-a_i)^{E_i} S_i$

$$\Lambda_i = \begin{pmatrix} \Lambda_i' & 0 \\ 0 & \Lambda_i'' \end{pmatrix}$$

Def slope $\quad \mathfrak{X}(F) = \dfrac{c_1(F)}{\operatorname{rk} F}$

$\Rightarrow$ $\exists$ subbundle $F' \subset E$ $\quad \operatorname{rk} E' = l$

(G_i) $\quad X', \Lambda_i'$

Statement: $\deg(F') \geq 0$ and $\deg(F') = 0 \Leftrightarrow \Lambda_i = c_i \cdot Id \quad i = 1 \ldots n$

$$\lambda_i^1 \geq \ldots \geq \lambda_i^p \Rightarrow \frac{\lambda_i^1 + \ldots + \lambda_i^l}{l} \geq \frac{\lambda_i^1 + \ldots + \lambda_i^p}{p}$$

$\Lambda_i = \begin{pmatrix} \lambda_i^1 & & 0 \\ & \ddots & \\ 0 & & \lambda_i^p \end{pmatrix}$ So $\mathfrak{X}(F') = \dfrac{1}{l} \sum_{i=1}^{n} \operatorname{tr}(\Lambda_i' + E) = \sum_{i=1}^{n} \left(\rho_i + \dfrac{\lambda_i^1 + \ldots + \lambda_i^l}{l} \right) \geq$

$$\geq \sum_{i=1}^{n} \left(\rho_i + \frac{\lambda_i^1 + \ldots + \lambda_i^p}{p} \right) = \mathfrak{X}(E) = c_1(E) = 0$$

Prop 11.1 $\Rightarrow c_1(F') \leq 0 \quad \mathfrak{X}(F') \leq 0$

$\Rightarrow \mathfrak{X}(F') = c_1(F') = 0, \quad \Lambda_i = c_i \cdot Id$

$$y'(z) = \prod_{i=2}^{n} (z-a_i)^{-c_i} (z-a_1)^{c}\Big) y(z), \quad c = \sum_{i=2}^{n} c_i$$

Fuchsian in all points.

$\varphi_2, \dots \varphi_n = 0$

In $a_1 \Rightarrow$ (11.3) with $k = \xi - (c + c_1)$

$$F = \mathcal{O}(k) \oplus \dots \oplus \mathcal{O}(k) \quad \square$$

<u>Cor 11.2</u> B-rep. is Fuchsian monodromy, then

$$c_1(F^0) = 0 \mod (\operatorname{rk} F^0)$$

<u>Example</u> 11.1 (R-H counterexample)

$$G_1 = \begin{pmatrix} 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \end{pmatrix} \quad G_2 = \begin{pmatrix} 3 & 1 & 1 & -1 \\ -4 & -1 & 1 & 2 \\ 0 & 0 & 3 & 1 \\ 0 & 0 & -4 & -1 \end{pmatrix} \quad G_3 = \begin{pmatrix} -1 & 0 & 2 & -1 \\ 4 & -1 & 0 & 1 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 4 & -1 \end{pmatrix}$$

$$G_1 \cdot G_2 \cdot G_3 = Id, \quad S_2^{-1} G_2 S_2 = G_1, \quad S_3^{-1} G_3 S_3 = \begin{pmatrix} -1 & 1 & & 0 \\ & -1 & 1 & \\ & & -1 & 1 \\ 0 & & & -1 \end{pmatrix}$$

$$S_2 = \frac{1}{3} \begin{pmatrix} 3 & 0 & 0 & 0 \\ -6 & 3 & -3 & 4 \\ 0 & 0 & 1 & -1 \\ 0 & 0 & -\cancel{2} & 3 \end{pmatrix} \quad S_3 = \frac{1}{64} \begin{pmatrix} 0 & 16 & 4 & 3 \\ 64 & 0 & 0 & 0 \\ 0 & 0 & 0 & -4 \\ 0 & 0 & -16 & -12 \end{pmatrix}$$

<u>Def 7.1</u> $c_1(F^0) = \operatorname{tr} E_1 + \operatorname{tr} E_2 + \operatorname{tr} E_3 = 0 + 0 + 4 \cdot \frac{1}{2} = 2 \not\vdots 4$

so Cor 11.2 $\Rightarrow$ $\not\exists$. $\square$

<u>Rem 11.1</u> $\prod_{i=1}^{n} \mu_i = 1$ ← eigenvalues of monodromies.

<u>Rem 11.2</u> Fuchsian + B-repr after $y' = Sy$ (S - constant) becomes $\omega' = \begin{pmatrix} \omega^1 & * \\ 0 & \omega^2 \end{pmatrix}$ (l, l)

<u>Proof</u> exercise.