P1

L5

$\overset{*}{\dot{O}}$ - univ. cover of $\dot{O}$

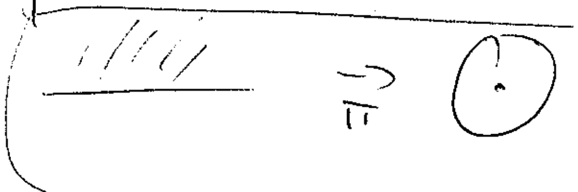
$\mathcal{Z} = \pi_1(\dot{O}, z_0)$ acts on $\overset{*}{O}$

~~Exr 5.1~~ π is princ. bundle with $G = \gamma$

$\ln \tilde{z} \in \mathbb{H}$

$z = \exp(2\pi i \ln \tilde{z})$ $\{0 < |z| < 1\}$

$G\tilde{z} = \tilde{z} + 1$



$$Y(\tilde{z}) = M(z)\,\tilde{z}^E$$

$\gamma^* f(\tilde{z}) = f(\gamma \tilde{z})$ – pullback of functions

Exr 5.2 $\quad G^* Y = Y \cdot G$ for $Y(\tilde{z})$ fund matrix.

Regular. $\iff \exists Q\ \forall$ sol. $y(\tilde{z}) \in X$ - space of sol.

$\forall$ sector $S < \dot{O}$

$$\frac{y(\tilde{z})}{|z|^Q} \to 0 \quad \text{when } z \to 0,\ z \in \dot{O},\ \tilde{z} \in \tilde{S}$$

Def $\quad y(\tilde{z})$ has polyn. growth at zero.

Def 5.1 $\quad \varphi(y) = \sup\{k \in \mathbb{Z} \mid \forall \lambda < K \ \ \frac{y(\tilde{z})}{|z|^\lambda} \to 0 \text{ if } z \to 0\}$

$\varphi(0) = \infty$

$\varphi(\text{matrix } M) = \inf(M_{ij})$

Ex. 5.1 $\quad \varphi\left(\frac{1}{z}\ln z\right) = -1 \qquad \varphi(\sqrt{z}) = 0$

$$\varphi\begin{pmatrix} 1 + z^2 \ln \tilde{z} \\ \frac{1}{z} + \ln \tilde{z} \end{pmatrix} = -1$$

Exr 5.3 $\quad 0 \le \operatorname{Re} \rho_i < 1 \implies \varphi(\tilde{z}^E) = 0$

Another def $\quad$ L4.1 & 4.2 $\implies y(\tilde{z}) = \sum_{j,l} f_{jl}(z)\,\tilde{z}^{\rho_j} (\ln \tilde{z})^{b_l}$

with ρ_j

$b_l \in \mathbb{Z}_+$ $\qquad \varphi(y(\tilde{z})) = \min \varphi(f_{il})$

Exr 5.4 2 def of φ are equiv.

~~Exr~~

$\varphi : \mathcal{M}^* \to \mathbb{Z}$

$\varphi(6^2) = 0$

$\varphi(z) = 1$

Prop 5.1 $\varphi: X \to \mathbb{Z} \cup \infty$

a) $\varphi(y_1+y_2) \geq \min(\varphi(y_1), \varphi(y_2))$

$= \text{ if } \varphi(y_1) \neq \varphi(y_2)$

b) $\varphi(cy) = \varphi(y) \quad \forall c \in \mathbb{C} \setminus 0$

c) $\varphi(6^* y) = \varphi(y)$

Proof c) $6^* \bar{z}^a = \exp(2\pi i a) \bar{z}^a$

$6^* \ln \bar{z} = \ln \bar{z} + 2\pi i$

$\Downarrow$

$\varphi(6^* y) \geq \varphi(y) \quad \forall y$

$\varphi((6^{-1})^* y) \geq \varphi(y) \implies \varphi(y) = \varphi((6^{-1})^* \circ 6^* y) \geq \varphi(6^* y)$

$\rotatebox{90}{$\geq$}\ \varphi(y)$

$=$ ok.

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a, b $\implies$ only finitely many values for $\varphi|_X$.

$\infty > \psi^1 > \dots > \psi^m$

filtration $0 \subset X^1 \subset \dots \subset X^m = X$ (Leveltis filtration)

$X^k = \{ y \in X \mid \varphi(y) \geq \psi^k \} \quad k = 1 \dots m$

6^* respects the filtration

Def $k_\ell = \dim X^\ell / X^{\ell-1} \qquad p = k_1 + \dots + k_m$

${}^\ell 6^* = 6^* |_{X^\ell / X^{\ell-1}}$

$e_1^1, \dots, e_{k_1}^1$ - base for X^1 s.t. ${}^1 6^*$ is $\begin{pmatrix} \ddots & * \\ 0 & \ddots \end{pmatrix}$

$e_1^1, \dots, e_{k_1}^1, e_1^2, \dots, e_{k_2}^2$ - base for X^2 s.t. ${}^2 6^*$ is $\begin{pmatrix} \ddots & * \\ 0 & \ddots \end{pmatrix}$

(choose $\bar{e}_1^\ell, \dots, \bar{e}_{k_\ell}^\ell$ for $X^\ell / X^{\ell-1}$, then take any lift s.t. $\begin{pmatrix} \ddots & * \\ 0 & \ddots \end{pmatrix}$)

$(e) = e_1 \dots e_p$

1) $\varphi(\{e_1, \dots, e_p\}) = k_1 \psi^1 + k_2 \psi^2 + \dots + k_m \psi^m$ (all ψ with mult. k_ℓ)

2) $\varphi(e_{\ell+1}) \leq \varphi(e_\ell)$

3) 6^* is $\begin{pmatrix} \ddots & * \\ 0 & \ddots \end{pmatrix}$

Def 5.2 is Leveltis bases, for solutions of $\frac{dy}{dz} = B(z) y$ with r.s.p. $z=0$.

Ex 5.2 If G^* is Jordan block, then Jordan base is Levelt's.

All Levelt's bases are equiv up to upper-triang. transformations.

Proof $Y^\ell = \langle e_1, \dots, e_\ell \rangle$

then $0 \subset Y^1 \subset \dots \subset Y^p = X$ is unique G^*-inv. filtr.

$G^*(e_\ell) = \lambda e_\ell + e_{\ell-1}$

$\Rightarrow X^\ell$ is G^*-inv $\dim X^\ell = d_\ell \Rightarrow X^\ell = Y^{d_\ell}$

(e) – Levelt's base $A = \begin{pmatrix} \varphi(e_1) & & 0 \\ & \ddots & \\ 0 & & \varphi(e_p) \end{pmatrix}$ $\varphi^\ell = \varphi(e_\ell)$

$G = G^*$, $E = \frac{1}{2\pi i} \ln G$

$0 \le \operatorname{Re} \rho_i < 1$

Lemma 5.1 $z^A G z^{-A}$ and $z^A E z^{-A}$ are holomorphic in $z=0$.

$$\varphi(z^A z^E z^{-A}) = 0$$

Proof G and E are $\begin{pmatrix} & \equiv \\ 0 & \end{pmatrix}$ $\quad \tilde{C} = z^A C z^{-A}$ $\quad C = \begin{pmatrix} & \equiv \\ 0 & \end{pmatrix}$

$$\tilde{C}_{km} = \begin{cases} c_{km} z^{\varphi^k - \varphi^m} & k \le m \\ 0 & k > m \end{cases}$$

$\Rightarrow \tilde{C}$ is holo $\Rightarrow \tilde{G}$ and $\tilde{E}$ are holo

$\varphi(z^E) \geq 0$ $\quad z^A z^E z^{-A} = \begin{pmatrix} z^{\rho_1} & & \equiv \\ 0 & & z^{\rho_p} \end{pmatrix}$

$\varphi(z^{\rho_i}) = 0$

Theorem 5.1 For Levelt's base (e) its fund. sol.

$$Y_e(z) = U(z) \cdot z^A \cdot z^E \qquad (5.3)$$

$U(z)$ is holom. and ~~uniquely~~ 1-valued.

Proof L4.1 $\Rightarrow U(z)$ is 1-valued.

let $r = \max \operatorname{Re} \rho_i$, $\quad 0 < \varepsilon < \frac{1-r}{2}$

let's show that $\lim\limits_{z \to 0} U(z) z^{r+2\varepsilon} = 0$ $\quad \left(\Rightarrow \begin{matrix} U \text{ is} \\ \text{holo} \end{matrix} \right)$

$$U(z) z^{r+2\varepsilon} = Y_e(z) z^{-E} z^{-A} z^{r+2\varepsilon} = N_1 \cdot N_2 \quad \left| \begin{array}{l} N_1 = Y_e \cdot z^{-A+\varepsilon Id} \\ N_2 = z^A z^{-E} z^{-A} z^{r+\varepsilon} \end{array} \right.$$

$N_1(z) = \{ e_j(z) \cdot z^{-\varphi(e_j)+\varepsilon} \}$

$\Rightarrow \lim_{z \to 0} N_1(z) = 0$ (by def of φ.)

L.4.2 $\Rightarrow z^{-E+r \cdot Id}$ $a_{ij} = \sum z^{\frac{r-\rho}{\geq 0}^{\ell}} P_{ij}^{\ell}(\ln z)$

so $\varphi(z^A z^{-E+rId} z^{-A}) = 0$

$\Rightarrow \lim N_2(z) = \lim z^A z^{-E+rId} z^{-A} z^{\varepsilon} = 0$

Def (Weak Leveltis base)

If Spec σ^* is 1 value, then weak Lev. = Lev.

$X = X_1 \oplus \ldots \oplus X_s$

$\lambda^1 \quad \ldots \quad \lambda^s$ - eigenvalues.

$\sigma^*|_{X_i} = \sigma_i^*$ — Construct Leveltis base for each X_i.

Weak Leveltis base(X) = $\bigsqcup$ Leveltis bases (X_i)

Exr 5.5 Show that weak Levelt (X) is ass. with „Leveltis" filtration of X, i.e. $\varphi(\text{weak Lev}) = k_1\varphi^1 + \ldots + k_m \varphi^m$

Lem 5.1 and T5.1 also true for weak Levelt.

Example 5.3 (4.3) $\frac{dy}{dz} = \begin{pmatrix} 0 & 1 \\ z^{-2} & -z^{-1} \end{pmatrix} y$

$\varphi\begin{pmatrix} z \\ 1 \end{pmatrix} = 0$ $\quad y(z) = \begin{pmatrix} z & z^{-1} \\ 1 & -z^{-2} \end{pmatrix}$

$\varphi\begin{pmatrix} z^{-1} \\ -z^{-2} \end{pmatrix} = -2$ so (5.3) (L.5.1) is $y(z) = \begin{pmatrix} z & z \\ 1 & -1 \end{pmatrix} z^{\begin{pmatrix} 0 & 0 \\ 0 & -2 \end{pmatrix}}$

Ex 5.4 (4.1 and 4.4) $\frac{dy}{dz} = \begin{pmatrix} z^{-1} & 1 \\ 0 & 0 \end{pmatrix} y$ $\quad \tilde{y}(z) = \begin{pmatrix} z & z \ln z \\ 0 & 1 \end{pmatrix}$

$\varphi\begin{pmatrix} z \\ 0 \end{pmatrix} = 1$, $\varphi\begin{pmatrix} z \ln z \\ 1 \end{pmatrix} = 0$ $\quad \tilde{y}(z) = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} z^{\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}} z^{\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}}$

Thm 5.2 (A.H.M. Levelt) $\frac{dy}{dz} = B(z) y$ with reg. sing. point $z = 0$ is Fuchsian $\Leftrightarrow$ $y(z) = U(z) z^A z^E$ (for (weak) Leveltis base (e)), $U(z)$ is in $\mathcal{O}^k$, $U(z) \in \mathcal{O}^*(\Delta \backslash 0)$

Exr 5.6 weak Lev $\to$ transpose Levelt

Exr 5.7 L5.1 & T5.1 for any base [illegible]