EXISTENCE AND NON-EXISTENCE FOR A MEAN CURVATURE EQUATION IN HYPERBOLIC SPACE: (VERSION WITH AN ERRATUM)

Elias M. Guio

Departamento de Matemática
Universidade Federal do Espirito Santo
Vitória 29060-900 ES, Brazil

Ricardo Sa Earp

Departamento de Matemática
Pontifícia Universidade Católica do Rio de Janeiro
Rio de Janeiro 22453-900 RJ, Brazil

Abstract. There exists a well-known criterion for the solvability of the Dirichlet Problem for the constant mean curvature equation in bounded smooth domains in Euclidean space. This classical result was established by Serrin in 1969. Focusing the Dirichlet Problem for radial vertical graphs P.-A. Nitsche has established an existence and non-existence results on account of a criterion based on the notion of a hyperbolic cylinder. In this work we carry out a similar but distinct result in hyperbolic space considering a different Dirichlet Problem based on another system of coordinates. We consider a non standard *cylinder generated by horocycles cutting ortogonally a geodesic plane* $\mathcal{P}$ *along the boundary of a domain* $\Omega \subset \mathcal{P}$. We prove that a non strict inequality between the mean curvature $\mathbf{H}'_{\mathcal{C}}(y)$ of this cylinder along $\partial\Omega$ and the prescribed mean curvature $\mathbf{H}(y)$, i.e $\mathbf{H}'_{\mathcal{C}}(y) \geqslant |\mathbf{H}(y)|$, $\forall\, y \in \partial\Omega$ and $|\mathbf{H}(x)| \leqslant 1$ or $|\mathbf{H}(x)| = a$ (constant) yields existence of our Dirichlet Problem. Thus we obtain existence of surfaces whose graphs have prescribed mean curvature $\mathbf{H}(x)$ in hyperbolic space taking a smooth prescribed boundary data φ. This result is sharp because if our condition fails at a point y a non-existence result can be inferred. The authors highly thank Friedrich Tomi, for pointing out to us gently an error in certain statements of the paper. The present version is the corrected version.

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1. Introduction

A classical result in Partial Differential Equations and Differential Geometry due to Serrin (1969) is the following: Given a constant a, there exists a condition on the boundary of the domain Ω such that the Dirichlet Problem for the mean equation $H = a$ is solvable. In fact, if the mean curvature of the Euclidean cylinder over $\partial\Omega$ is bigger than a, then some a priori gradient estimates ensures a posteriori existence for the Dirichlet Problem. This inequality is sharp in the following sense: if it does not hold at a point then it can be inferred non-existence for a certain smooth boundary data. In this article we carry out a similar program in hyperbolic space, taking into account a certain coordinates system and a certain geometric boundary condition in hyperbolic space based on a suitable but not standard notion of "cylinder". A novelty value in our paper is the deduction of a solution of the related Dirichlet Problem for prescribed (not necessarily constant) mean curvature $\mathbf{H}(x)$. Moreover, the sharpness of the result is assured by a non- existence result if our criterion in hyperbolic space "fails at a point". In fact, let $\mathcal{P}$ be a n-dimensional totally geodesic plane in hyperbolic space $\mathbb{H}^{n+1}$ and let Ω be a domain in $\mathcal{P}$ with C^2 boundary $\partial\Omega$.

We shall focus the upper halfspace model of hyperbolic space, i.e $\mathbb{H}^{n+1} := \{(x_0, x_1, \ldots, x_n);\ x_n > 0\}$ equipped with the metric $ds^2 = \dfrac{1}{x_n^2}(dx_0^2 + dx_1^2 + \cdots + dx_n^2)$. For sake of simplicity, we fix a hyperplane $\mathcal{P} := \{x_0 = 0\}$. Notice that the asymptotic boundary of $\mathbb{H}^{n+1}$ denoted $\partial_\infty\mathbb{H}^{n+1}$ is defined by $\partial_\infty\mathbb{H}^{n+1} := \{x_n = 0\} \cup \{\infty\}$. We represent by $\partial\Omega$ a $(n-1)$-dimensional closed connected embedded smooth submanifold of $\mathcal{P}$. We then define $\boldsymbol{\mathcal{C}}$ by the n-dimensional cylinder generated by horocycles with asymptotic boundary ∞ cutting orthogonally $\mathcal{P}$. That is,
$\boldsymbol{\mathcal{C}} := \{(t, x_1, \cdots, x_n); t \in \mathbb{R}, (0, x_1, \cdots, x_n) \in \partial\Omega\}$.

Notice that in this model $\mathcal{P}$ is a vertical hyperplane and our cylinder $\boldsymbol{\mathcal{C}}$ is an "Euclidean horizontal cylinder". We shall consider throughout this paper the following definition of *horizontal graph* of a function in hyperbolic space: Let $\Omega \subset \mathcal{P}$ be the domain whose boundary is $\partial\Omega$. Given a function $u : \overline{\Omega} \to \mathbb{R}$, the graph of u is defined as the set

$$G(u) = \{(u(x_1, \ldots, x_n), x_1, \ldots, x_n);\ (0, x_1, \ldots, x_n) \in \overline{\Omega}\}.$$

Indeed, this is the natural notion of graph in our system of coordinates given by horocycles cutting ortogonally $\mathcal{P}$.

We shall commence to describe the equation that we are going to focus:

Given a C^1 function $\mathbf{H} : \bar{\Omega} \subset \mathcal{P} \to \mathbb{R}$, and a C^2 function $\varphi : \overline{\Omega} \to \mathbb{R}$, we shall consider the following Dirichlet Problem for the prescribed *horizontal* mean curvature equation, say Problem (P):

$$\text{div}\left(\frac{Du}{W(u)}\right) = \frac{n}{x_n}\left(\mathbf{H} + \frac{D_n u}{W(u)}\right) \qquad \text{in } \Omega \tag{1}$$

$$u = \varphi \qquad \text{on} \quad \partial\Omega$$

where Du means the Euclidean gradient of u in $\mathcal{P}$, $W(u) = \sqrt{1+|Du|^2}$ $D_i u = \dfrac{\partial u}{\partial x_i}$, $i = 1, \ldots, n$ and the symbol "div" denotes the divergence in $\mathbb{R}^n = \{(x_1, \ldots, x_n)\}$. We shall explain with details some historical backgrounds and motivations. A main uniqueness result obtained by Lucas Barbosa and the second author (see [5], [6]) gives a strong motivation for the study of the above Dirichlet Problem. In fact, under some assumptions about the relation of the size of $\mathbf{H}$ with the geometry of $\partial\Omega$, uniqueness of the above Dirichlet Problem (*horizontal graphs*) can be inferred in the class of immersed compact hypersurfaces with same mean curvature and same boundary. Particularly, if $\mathbf{H}$ is constant with $\mathbf{H}^2 \leqslant 1$, and if the principal curvatures λ_i of the boundary $\partial\Omega$ satisfy an inequality $\lambda_i \geqslant h_0 > 1$, then any smooth compact connected immersed n-dimensional manifold with boundary $\partial\Omega$ is an horizontal graph. Lucas Barbosa and the second author have focus attention exclusively on the case of zero boundary conditions. Horizontal graphs satisfying the assumptions above were constructed to be used as barriers to obtain the cited main uniqueness theorem.

Before proceeding with the discussion of our problem in hyperbolic space, we pause to precise the criterion for the solvability of the Dirichlet Problem in Euclidean space, for the equation of constant mean curvature equation in a bounded C^2 domain Ω. Let $H^{'}$ be the mean curvature of $\partial\Omega$. Then the constant mean curvature equation $\text{div}\left(\dfrac{Du}{W(u)}\right) = nH$ in Ω , $u = \varphi$ on $\partial\Omega$ is solvable for constant H and arbitrary boundary continuous data φ if and only if $(n-1)H^{'} \geqslant n\,|H|$ everywhere on $\partial\Omega$. This theorem for smooth boundary data was established by J. Serrin in [26]. Notice that convexity of Ω is the sharp criterion of solvability for arbitrary continuous boundary data of the Dirichlet Problem on a bounded domain for the minimal surface equation in two variables; this was proved by R. Finn [10]. For more than two variables H. Jenkins and J. Serrin in [13] showed that the minimal hypersurface equation on a bounded $C^{2,\alpha}$ domain, say Ω, $0 < \alpha < 1$, is solvable for arbitrary $\varphi \in C^{2,\alpha}(\bar{\Omega})$ if and only if the mean curvature H' of $\partial\Omega$ is non-negative at every point of $\partial\Omega$.

Notice that the mean curvature equation in Euclidean space is now classical in the literature (see, for instance [3], [7], [9], [12]). Notice that there exist some recent interesting papers on minimal graphs over unbounded domains (see [29], [15]). We should mention the remarkable work of R. Schoen on the mean curvature equation in Euclidean space and its application to Geometry (see [27]).

L. Bers showed the classical minimal surface equation cannot have an isolated singularity (see [2]). The same result for the constant mean curvature equation was established by R. Finn (see [11]). The removable singularity theorem for prescribed mean curvature equations in $\mathbb{R}^{n+1}$ is due to the H. Rosenberg in a joint work with the second author (see [22]).

Returning to hyperbolic space, we observe that a removable singularity type theorem for the *horizontal* mean curvature equation in hyperbolic space was done by B. Nelli and the second author (see [20]).

* * *

Now we shall writedown the geometric condition (say, condition $(*)$) that plays a crucial role in the whole study of our Dirichlet Problem (P) for the prescribed mean curvature $\mathbf{H}(x)$:

$$\mathbf{H}'_{\mathcal{C}}(y) \geqslant |\mathbf{H}(y)|, \qquad \forall\, y \in \partial\Omega \tag{$*$}$$

where $\mathbf{H}'_{\mathcal{C}}(y)$ is the mean curvature of the cylinder generated by horocycles at a boundary point y.

We can express above inequality in the following equivalent analytic form

$$(n-1)\mathbf{H}'(y) + x_n(y)D_n d(y) \geqslant n|\mathbf{H}(y)|, \quad \forall\ y \in \partial\Omega \tag{$*$}$$

where $\mathbf{H}'(y)$, is the mean curvature of the boundary of Ω and $x_n(y)D_n d(y)$ is the n^{th} vertical component of the Euclidean unit interior normal to $\partial\Omega$ at a boundary point y ($d(x)$ is the hyperbolic distance from $x \in \overline{\Omega}$ to the boundary $\partial\Omega$). We feel that it is interesting to give now a geometric insight about the second term that appears on the left of the above inequality : We recall that Serrin's condition in Euclidean space ensures a priori boundary gradient estimates that are essential for the related Dirichlet Problem. On the other hand, looking to the simplest situation in hyperbolic space, it is not hard to see that there exist "spherical caps"of geodesic (minimal) planes cutting ortogonally the plan of the domain Ω at some point y, whose projections are convex curves (Euclidean ellipse) in the hyperbolic plane $\mathcal{P}$. Thus, Serrin inequality holds but we have no control of the gradient at y.

However, condition $(*)$ allows us to infer a priori boundary gradient estimates for the smooth solutions of Dirichlet Problem (P). We infer these estimates in Section 2. Then in Section 3 we prove existence for Dirichlet Problem (P) for the prescribed mean curvature $\mathbf{H}(x)$, given smooth boundary value data φ, if condition $(*)$ holds, and $|\mathbf{H}(x)| \leqslant 1$ or $|\mathbf{H}(x)| = a$ (constant). Finally, in Section 4, we obtain non-existence for the Dirichlet Problem (P), if condition $(*)$ fails at a point.

We would like to mention that in a preliminary version we have derived our existence result for $|\mathbf{H}(x)| \leqslant 1$. However Harold Rosenberg in a private conversation recalled to us the height estimates for compact embedded constant mean curvature hypersurfaces in hyperbolic space with constant mean curvature $\mathbf{H} > 1$ (see [14], [21]). In view of this, if condition $(*)$ holds, and $|\mathbf{H}(x)| \leqslant 1$ or $|\mathbf{H}(x)| = a$ (constant), our construction provide existence of the Dirichlet Problem (P) for prescribed $\mathbf{H}(x)$-hypersurfaces (taking prescribed smooth value data).

We remark that P.- A. Nitsche (see [19]) studied a different Dirichlet Problem on account of a different notion of graph, precisely radial vertical graphs. His condition is philosophically similar to our but geometrically distinct: Instead of cylinders generated by horocycles the author deals with usual hyperbolic cylinders. He proved that if the strict inequality between the mean curvature of the hyperbolic cylinder along the boundary of the domain (in a totally geodesic plane) and the prescribed mean curvature holds, then existence of his Dirichlet Problem for mean curvature less than 1 can be deduced. If the reverse strict inequality holds then he proved non existence.

At last, we would like to mention that there exists also a notion of *vertical graphs* in hyperbolic space. For instance, the reader is referred to [20] and [23]. We also point out that F. Lin (see [16]) studied regularity for vertical minimal equation in hyperbolic space. E. Toubiana and the second author (see [25]) established the *Perron process* for these vertical minimal equations in two independent variables.

The program accomplished in this paper was motivated by a question raised by the second author (see [23]), then investigated by the first author in his Doctoral Thesis under the supervision of the second author (see [8]).

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2. A priori boundary gradient estimates

Next, we begin our investigation of the Dirichlet Problem (P) by first deriving an estimate given by the following theorem

Theorem 2.1 (a priori boundary gradient estimate). *Let Ω be a bounded domain in $\mathbb{H}^n$ with C^3 boundary $\partial\Omega$, $\varphi \in C^2(\overline{\Omega})$ and $\mathbf{H}(x) \in C^1(\overline{\Omega})$. If $u \in C^2(\Omega) \cap C^1(\overline{\Omega})$ is a solution of Dirichlet Problem (P) and if $\mathbf{H}(x)$ satisfies condition $(*)$ then*

$$\sup_{\partial\Omega} |Du| \leqslant c$$

where

$$c = c(n,\, \Omega,\, \inf_{\Omega} x_n,\, \sup_{\Omega} x_n,\, \sup_{\Omega} |u|,\, |\varphi|_2,\, |\mathbf{H}|_1).$$

Proof. Before getting into details we are going to outline the proof. Recall that $d(x) = \operatorname{dist}(x, \partial\Omega)$ be the hyperbolic distance from $x \in \Omega$ to the boundary $\partial\Omega$. Notice that $d \in C^2(\Gamma)$ where $\Gamma = \{x \in \overline{\Omega}; d(x) < d_0\}$ for some $d_0 > 0$ small enough. To obtain the a priori estimate we shall make use of *barriers techniques*. We shall construct a *supersolution* ω^+ and a *subsolution* ω^- satisfying (Q is an elliptic operator) :

$$\begin{aligned} \pm Q(\omega^\pm) &\leqslant 0 \text{ in } \mathcal{N} \cap \Omega \\ \pm\omega^\pm &\geq \pm u \text{ in } \partial(\mathcal{N} \cap \Omega), \end{aligned} \tag{2}$$

where $\mathcal{N}$ is a neighborhood of $\partial\Omega$ lying in Γ. More precisely, we shall infer that suitable barriers can be chosen of the form

$$\omega^\pm = \pm\psi(d) + \varphi \tag{3}$$

where $\psi : [0, \infty) \to \mathbb{R}$, is a C^2 function such that $\psi(0) = 0$, $\psi' > 0$ and $\psi'' < 0$. Now notice that maximum principle yields

$$\begin{aligned} u(x) &\leqslant \psi(d(x)) + \varphi(x), \qquad \forall\, x \in \mathcal{N} \cap \Omega \\ -\psi(d(x)) + \varphi(x) &\leqslant u(x), \qquad \forall\, x \in \mathcal{N} \cap \Omega \end{aligned} \tag{4}$$

Let us recall how to proceed to obtain the a priori estimate by the method of barriers. Now let $x_0 \in \partial\Omega$ be an arbitrary point of the boundary of Ω and let $\vec{v}$ be an unit Euclidean . Let us assume that $\vec{v} \cdot \overrightarrow{\boldsymbol{n}} > 0$, where $\cdot$ is the inner product and $\overrightarrow{\boldsymbol{n}}$ is an Euclidean unit normal interior to Ω at x_0. Then from the above inequalities we get

$$\begin{aligned} \frac{-\psi \circ d(x_0 + \epsilon\vec{v})}{\epsilon} + \frac{\varphi(x_0 + \epsilon\vec{v}) - \varphi(x_0)}{\epsilon} \quad &\leqslant \quad \frac{u(x_0 + \epsilon\vec{v}) - u(x_0)}{\epsilon} \leqslant \\ &\leqslant \quad \frac{\psi \circ d(x_0 + \epsilon\vec{v})}{\epsilon} + \frac{\varphi(x_0 + \epsilon\vec{v}) - \varphi(x_0)}{\epsilon} \end{aligned}$$

for $\epsilon > 0$ small enough.

Now let $\epsilon \to 0^+$. We obtain

$$-\psi'(0)Dd(x_0)\cdot\vec{v}+D\varphi(x_0)\cdot\vec{v} \leqslant Du(x_0)\cdot\vec{v} \leqslant \psi'(0)Dd(x_0)\cdot\vec{v}+D\varphi(x_0)\cdot\vec{v}$$

Notice that $|Dd(x_0)| = \dfrac{1}{x_n(x_0)}$, where $|.|$ stands for the Euclidean norm. Now taking $\vec{v} = \dfrac{Du(x_0)}{|Du(x_0)|}$, by applying Schwarz inequality we deduce therefore

$$|Du(x_0)| \leqslant \frac{\psi'(0)}{\inf\limits_{\Omega} x_n} + \sup_{\Omega} |D\varphi|, \qquad \forall\, x_0 \in \partial\Omega.$$

Henceforth $|Du|_{\partial\Omega} \leqslant c$, moreover, for our future choice of ψ

$$c = c(n,\, \Omega,\, \inf_{\Omega} x_n,\, \sup_{\Omega} x_n,\, \sup_{\Omega} |u|,\, |\varphi|_2,\, |\mathbf{H}|_1),$$

where

$|\varphi|_2 = \max\{\sup\limits_{\Omega} |\varphi|,\, \sup\limits_{\Omega} |D\varphi|,\, \sup\limits_{\Omega} |D^2\varphi|\}$ and $|\mathbf{H}|_1 = \max\{\sup\limits_{\Omega} |\mathbf{H}|,\, \sup\limits_{\Omega} |D\mathbf{H}|\}$.

This will prove the desired a priori estimate. We are going now to carry out the details. First, notice that the mean equation (1) can be re-written in the form

$$Q(u) = \sum_{i,j=1}^{n} a^{ij}(x, Du)D_{ij}u + b(x, Du) = 0$$

where (using summation convention)

$$a^{ij}(x, Du) = (1 + |Du|^2)\delta_{ij} - D_i u D_j u$$
$$b(x, Du) = -\frac{nD_n u(1 + |Du|^2)}{x_n} - \frac{n\mathbf{H}(1 + |Du|^2)^{3/2}}{x_n}.$$

Let us assume momentarily that the barriers $\omega^{\pm}$ are chosen such that ψ satisfies

Property (i) $\quad \psi(a) \geqslant M = \sup\limits_{\Omega} |\varphi| + \sup\limits_{\Omega} |u|$

Property (ii) $\quad \psi'(d)\, d \leqslant 1$

Property (iii) $\quad \psi'|Dd| \geqslant \mu$

where $\mu \geqslant 3|D\varphi| + 8$ is a fixed constant. We shall first work with ω^+. For sake of simplicity we shall adopt the following notations and conventions: $\omega = \omega^+$ and $\xi = \sum\limits_{i,j=1}^{n} a^{ij}(x, D\omega)(D_i\omega - D_i\varphi)(D_j\omega - D_j\varphi)$. Recalling that the eigenvalues of the matrix $a^{ij}(x, D\omega)$ are 1 and $(1+|D\omega|^2)$, we observe that $|D\omega - D\varphi|^2 \leqslant \sum_{i,j=1}^{n} a^{ij}(x, D\omega)(D_i\omega - D_i\varphi)(D_j\omega - D_j\varphi)$. In view of our choice of barriers, see (3), we have $|D\omega - D\varphi| \geq \mu$. Employing the inequality above together with Cauchy' s inequality we infer that

$\xi \geqslant |D\omega - D\varphi|^2 \geqslant \dfrac{|D\omega|^2}{2} - |D\varphi|^2$. We also can deduce that $\mu\xi \geq 1 + |D\omega|^2$, with the aid of Schwarz's inequality. We begin now to estimate $Q(\omega)$ in Γ to seek a suitable upper bound of the form $Q(\omega) \leqslant (\mathrm{cst})\xi$:

$$\begin{aligned}
Q(\omega) &= \sum_{i,j=1}^{n} a^{ij}(x, D\omega) D_{ij}\omega + b(x, D\omega) \\
&= \sum_{i,j=1}^{n} a^{ij}(x, D\omega)(\psi'' D_i d D_j d + \psi' D_{ij} d + D_{ij}\varphi) + b(x, D\omega) \\
&= \psi' \sum_{i,j=1}^{n} a^{ij}(x, D\omega) D_{ij} d + \sum_{i,j=1}^{n} a^{ij}(x, D\omega) D_{ij}\varphi + b(x, D\omega) + \frac{\psi''}{\psi'^2}\,\xi \\
&:= A + B + C + D
\end{aligned}$$

(1) Estimate of A

$$\begin{aligned}
A = \psi' \sum_{i,j=1}^{n} a^{ij}(x, D\omega) D_{ij} d &= \psi' \sum_{i,j=1}^{n} ((1+|D\omega|^2)\delta_{ij} - D_i\omega D_j\omega) D_{ij} d \\
&= \psi'(1+|D\omega|^2) \sum_{i=1}^{n} D_{ii} d - \psi'^3 \sum_{i,j=1}^{n} D_i d D_j d D_{ij} d - \\
&- 2\psi'^2 \sum_{i,j=1}^{n} D_i d D_j \varphi D_{ij} d - \psi' \sum_{i,j=1}^{n} D_i\varphi D_j\varphi D_{ij} d
\end{aligned}$$

Now in view of the hyperbolic metric we record that

$$|Dd|^2 = \frac{1}{x_n^2} \tag{5}$$

Owing to (5) we obtain

$$
\begin{aligned}
-\psi'^3 \sum_{i,j=1}^{n} D_i d D_j d D_{ij} d =& \frac{\psi'^3 D_n d}{x_n^3} \\
=& \frac{\psi'(1+|D\omega|^2) D_n d}{x_n} - \\
& - \frac{\psi' D_n d}{x_n}(2\psi' Dd \cdot D\varphi + |D\varphi|^2 + 1)
\end{aligned} \tag{6}
$$

Now the last term above can be estimated as follows

(7)

$$
\begin{aligned}
&\left|\frac{\psi' D_n d}{x_n}(2\psi' Dd \cdot D\varphi + |D\varphi|^2 + 1)\right| \leqslant \frac{\psi'}{x_n^2}|2(D\omega - D\varphi)\cdot D\varphi + |D\varphi|^2 + 1| \\
\leqslant & \frac{|D\omega - D\varphi|}{x_n}(2|D\omega - D\varphi||D\varphi| + |D\varphi|^2 + 1) \\
\leqslant & \frac{|D\omega - D\varphi|^2}{x_n}(3|D\varphi| + 1)
\end{aligned}
$$

Notice that the expression $\psi'^2 \sum\limits_{i,j=1}^{n} D_i d D_j \varphi$ satisfies the inequality

$$
\begin{aligned}
|\psi'^2 \sum_{i,j=1}^{n} D_i d D_j \varphi| =& |\psi' \sum_{i,j=1}^{n} (D_i\omega - D_i\varphi) D_j \varphi| \\
\leqslant & n^2 \psi' |D\omega - D\varphi||D\varphi| \\
=& n^2 x_n |D\omega - D\varphi|^2 |D\varphi|
\end{aligned} \tag{8}
$$

Consequently

$$
\begin{aligned}
A =& \psi' \sum_{i,j=1}^{n} a^{ij}(x, D\omega) D_{ij} d \leqslant \psi'(1+|D\omega|^2)\left(\sum_{i=1}^{n} D_{ii} d + \frac{D_n d}{x_n}\right) + \\
& + |D\omega - D\varphi|^2 \left[\left(2n^2 x_n |D\varphi| + n^2 x_n |D\varphi|^2\right) \sup_{\Gamma} |D_{ij} d| + + \frac{1 + 3|D\varphi|}{x_n}\right]
\end{aligned}
$$

(2) Estimate of B

$$
B = \sum_{i,j=1}^{n} a^{ij}(x, D\omega) D_{ij}\varphi \leqslant (1+|D\omega|^2) \sum_{i,j=1}^{n} \sup_{\Omega} |D_{ij}\varphi|
$$

(3) Estimate of C

$$
C = b(x, D\omega) = \frac{-n}{x_n} D_n \omega (1+|D\omega|^2) - \frac{n}{x_n}(1+|D\omega|^2)^{3/2}\mathbf{H}
$$

Now write (Big-oh notation)

$$(1+|D\omega|^2)^{3/2} = (1+|D\omega|^2)\,(\psi'|Dd| + |D\varphi|)+(1+|D\omega|^2)\,O(1), \quad (as\ |Dw| \to \infty)$$

Then

$$\begin{aligned}
b(x, D\omega) \leqslant & \frac{-n\psi'}{x_n^2}(1+|D\omega|^2)\mathbf{H} + \frac{n}{x_n}(1+|D\omega|^2)\,(|\mathbf{H}| + |\mathbf{H}||D\varphi|)\\
& - \frac{n\psi' D_n d}{x_n}(1+|D\omega|^2) + + \frac{n}{x_n}|D\varphi|(1+|D\omega|^2)\\
= & \frac{\psi'(1+|D\omega|^2)}{x_n^2}(-n\mathbf{H} - n x_n D_n d) +\\
& + \frac{(1+|D\omega|^2)}{x_n}(n\,O(1)|\mathbf{H}| + n|\mathbf{H}||D\varphi| + n|D\varphi|)
\end{aligned}$$

where $O(1)$ will always represent a term which is bounded by a constant under control. From now on we will omit it. Combining these estimates we find

$$\begin{aligned}
Q(\omega) \leqslant & \frac{\psi'(1+|D\omega|^2)}{x_n^2}\left(x_n^2 \sum_{i=1}^n D_{ii}d - (n-1)x_n D_n d - n\mathbf{H}\right) +\\
& + |D\omega - D\varphi|^2 \left(\frac{1+3|D\varphi|}{x_n} + \left(2n^2 x_n|D\varphi| + n^2 x_n |D\varphi|^2\right)\sup_\Gamma |D_{ij}d|\right) +\\
& + (1+|D\omega|^2)\left(\frac{n\,|\mathbf{H}|}{x_n} + \sum_{i,j=1}^n \sup_\Omega |D_{ij}\varphi| + \frac{n|\mathbf{H}||D\varphi|}{x_n} + \frac{n}{x_n}\sup_\Omega |D\varphi|\right) + \frac{\psi''}{\psi'^2}\xi
\end{aligned}$$

Taking into account the two inequalities $\mu\xi \geq (1+|D\omega|^2)$ and $|D\omega - D\varphi|^2 \leqslant \xi$, we infer

$$\begin{aligned}
Q(\omega) \leqslant & \frac{\psi'(1+|D\omega|^2)}{x_n^2}\left(x_n^2 \sum_{i=1}^n D_{ii}d - (n-1)x_n D_n d - n\mathbf{H}\right) +\\
& + \Bigg[\left(2n^2 x_n|D\varphi| + n^2 x_n|D\varphi|^2\right)\sup_\Gamma |D_{ij}d| + \frac{1+3|D\varphi|}{x_n} +\\
& + \left(\frac{n\,|\mathbf{H}|}{x_n} + \sum_{i,j=1}^n \sup_\Omega |D_{ij}\varphi| + \frac{n|\mathbf{H}||D\varphi|}{x_n} + \frac{n}{x_n}\sup_\Omega |D\varphi|\right)\mu\Bigg]\,\xi + \frac{\psi''}{\psi'^2}\xi
\end{aligned}$$

We pause now to say a few words about basic hyperbolic geometry. We first note that fixing any two points $Q = (x_1, ..., x_n)$ and $P = (y_1, ..., y_n)$ in $\mathbb{H}^n$, a straigtforward computation shows that hyperbolic distance $\mathrm{dist}(P, Q)$ has an explicit formula given by

$$\operatorname{dist}(Q,P)=\ln\left(\frac{\sqrt{\sum_{i=1}^{n-1}(x_i-y_i)^2+(x_n+y_n)^2}+\sqrt{\sum_{i=1}^{n}(x_i-y_i)^2}}{\sqrt{\sum_{i=1}^{n-1}(x_i-y_i)^2+(x_n+y_n)^2}-\sqrt{\sum_{i=1}^{n}(x_i-y_i)^2}}\right)$$

We shall need now fundamental formulas in hyperbolic space. Let $t=d(x), 0\leqslant t\leqslant d_0$, where d_0 is chosen suficiently small. From a standard computation it follows that (see, for instance [8], [17], [28])

$$x_n^2(x)\sum_{i=1}^{n}D_{ii}d(x)-(n-2)x_n(x)D_nd(x)=\sum_{i=1}^{n-1}\frac{\tanh t-k_i}{1-k_i\tanh t} \tag{9}$$

where $k_1,k_2,\cdots,k_{n-1}$ the principal curvatures of $\partial\Omega$ at y. Here $y=y(x)$ is the closest point in $\partial\Omega$ to x.

On account of (9), we are now going to obtain a bound for the expression

$$x_n^2(x)\sum_{i=1}^{n}D_{ii}d(x)-(n-1)x_n(x)D_nd(x)-n\mathbf{H}(x)$$

As $\dfrac{-k_i}{1-k_i\tanh t}$, is a non-increasing function in the t variable we find

$$\begin{aligned}x_n^2(x)\sum_{i=1}^{n}D_{ii}d(x)&-(n-2)x_n(x)D_nd(x)\\&\leqslant-(n-1)\mathbf{H}'(y)+\sum_{i=1}^{n-1}\frac{\tanh t}{1-k_i\tanh t}\end{aligned} \tag{10}$$

where $\mathbf{H}'$ is the mean curvature of $\partial\Omega$ at y. As $\displaystyle\sum_{i=1}^{n-1}\frac{\tanh t}{1-k_i\tanh t}=t\,O(1)$, in view of our condition $(*)$ we derive

$$\begin{aligned}x_n^2(x)\sum_{i=1}^{n}D_{ii}d(x)&-(n-1)x_n(x)D_nd(x)-n\mathbf{H}(x)\leqslant\\&\leqslant n\mathbf{H}(y)+x_n(y)D_nd(y)+t\,O(1)-x_n(x)D_nd(x)-n\mathbf{H}(x)\\&=n(\mathbf{H}(y)-\mathbf{H}(x))+x_n(y)D_nd(y)-x_n(x)D_nd(x)+t\,O(1)\end{aligned}$$

Now utilizing the relation between Euclidean and hyperbolic metrics we get

$$x_n^2(x)\sum_{i=1}^{n} D_{ii}d(x)-(n-1)x_n(x)D_nd(x)-nH(x)\leqslant(1+K)t\,O(1)$$

where $K=\sup\limits_{x\in\Gamma}\dfrac{n|\mathbf{H}(y)-\mathbf{H}(x)|}{|x-y|}+\sup\limits_{x\in\Gamma}\dfrac{|x_n(y)D_nd(y)-x_n(x)D_nd(x)|}{|x-y|}$

Estimate of $Q(\omega)$

$$Q(\omega)\leqslant\Bigg[\frac{(1+K)t\,O(1)\psi'\mu}{x_n^2}+(2n^2x_n|D\varphi|+n^2x_n|D\varphi|^2)\sup_{\Gamma}|D_{ij}d|+\frac{1+3|D\varphi|}{x_n}+$$
$$+\left(\frac{n\,|\mathbf{H}|}{x_n}+\sum_{i,j=1}^{n}\sup_{\Omega}|D_{ij}\varphi|+\frac{n|\mathbf{H}||D\varphi|}{x_n}+\frac{n}{x_n}\sup_{\Omega}|D\varphi|\right)\mu+\frac{\psi''}{\psi'^2}\Bigg]\xi$$

Thus

$$Q(\omega)\leqslant\left(\nu_0+\frac{\psi''}{\psi'^2}\right)\xi$$

where

$$\begin{aligned}\nu_0=&\left(\frac{1+K}{\inf\limits_{\Omega}x_n^2}+\frac{n\sup\limits_{\Omega}|\mathbf{H}||D\varphi|+n\sup\limits_{\Omega}|D\varphi|+n|\mathbf{H}|}{\inf\limits_{\Omega}x_n}\right)\mu+\\&+\sum_{i,j=1}^{n}\sup_{\Omega}|D_{ij}\varphi|\mu+\\&+\left(2n^2\sup_{\Omega}|x_nD\varphi|+n^2\sup_{\Omega}x_n|D\varphi|^2\right)\sup_{\Gamma}|D_{ij}d|+\\&+\sup_{\Omega}\frac{1+3|D\varphi|}{x_n}\end{aligned}\tag{11}$$

We turn now to our primary task to build the barriers.

Choose then $\nu=\max\{\nu_0,1\}$. We have

$$Q(\omega)\leqslant(\nu+\frac{\psi''}{\psi'^2})\xi$$

We now choose ψ setting

$$\psi(d)=\frac{1}{\nu}\ln(1+kd),\ 0\leq d\leqslant a\ .$$

We must require that a and k satisfy
(12)
$$(i)\ \psi(a) \geqslant M = \sup_\Omega |\varphi| + \sup_\Omega |u|,\ (ii)\ \psi'(d)d \leqslant 1,\ (iii)\ \psi'|Dd| \geqslant \mu\ .$$

Notice that $\psi(0) = 0$, $\psi' > 0$ and $\psi'' < 0$. We must verify now the property $(i), (ii)$ and (iii).

Property (i) Let us take $a = \dfrac{e^{\nu M} - 1}{k}$, choosing k big enough to guarantee $a \leqslant d_0$. Hence $\dfrac{1}{\nu}\ln(1 + ka) = M$.

Property (ii) Since $\psi'(d)d$ in increasing, it follows that

$$\psi'(d)d \leqslant \psi'(a)a = \frac{ka}{\nu(1 + ka)} \leqslant 1$$

Property (iii) Notice that

$$\psi'|Dd| = \frac{\psi'}{x_n} \geqslant \frac{\psi'}{\sup\limits_\Omega x_n} \geqslant H\frac{\psi'(a)}{\sup\limits_\Omega x_n} = \frac{k}{\nu e^{\nu M} \sup\limits_\Omega x_n} \geq \mu,$$

if k is chosen big enough to ensure

$$k \geq \mu\nu e^{\nu M} \sup_\Omega x_n$$

In view of the above construction we get a function ψ satisfying the required conditions to obtain the supersolution $\omega = \omega^+$. Note also that $\dfrac{\psi''}{\psi'^2} + \nu = 0$. We have established therefore that the upper barrier ω satisfies (2) with the plus sign, namely

$$\begin{aligned} Q(\omega) \leqslant 0 &\qquad \text{in } \mathcal{N} \cap \Omega, \\ \omega \geq u &\qquad \text{on } \partial(\mathcal{N} \cap \Omega) \end{aligned}$$

Thus owing to maximum principle we infer the first inequality in (4), as desired.

We now must seek for the lower barrier ω^- satisfying the second inequality in (4). So we need to find ω^- satisfying (2) (with the minus sign). We will skip the details of this construction. We summarize it as follows. Computation of $Q(w)$ is similar as before. Notice that

$$\begin{aligned} Q(\omega) = &\sum_{i,j=1}^{n} a^{ij}(x, D\omega)(-\psi'' D_i d D_j d - \psi' D_{ij} d + D_{ij}\varphi) + b(x, D\omega) \\ = &- \psi' \sum_{i,j=1}^{n} a^{ij}(x, D\omega) D_{ij} d + \sum_{i,j=1}^{n} a^{ij}(x, D\omega) D_{ij}\varphi + b(x, D\omega) - \frac{\psi''}{\psi'^2}\xi \end{aligned}$$

Now working as in the estimate of the upper barrier we derive that

$$Q(\omega) \geq \frac{-\psi'(1+|D\omega|^2)}{x_n^2}\left(x_n^2\sum_{i=1}^{n} D_{ii}d-(n-1)x_nD_nd+nH\right)-$$
$$-\left[(2n^2x_n|D\varphi|+n^2x_n|D\varphi|^2)\sup_{\Gamma}|D_{ij}d|+\frac{1+3|D\varphi|}{x_n}+\right.$$
$$\left.+\left(\frac{n|\mathbf{H}|}{x_n}+\sum_{i,j=1}^{n}\sup_{\Omega}|D_{ij}\varphi|+\frac{n|\mathbf{H}||D\varphi|}{x_n}+\frac{n}{x_n}\sup_{\Omega}|D\varphi|\right)\mu\right]\xi-\frac{\psi''}{\psi'^2}\xi$$

Using once more condition $(*)$ and fundamental formula (9) we can calculate

$$x_n^2(x)\sum_{i=1}^{n} D_{ii}d(x)-(n-1)x_n(x)D_nd(x)+nH(x)\leqslant(1+K)tO(1)$$

We conclude therefore

$$Q(\omega)\geq-\left(\nu+\frac{\psi''}{\psi'^2}\right)\xi$$

since $\psi'(d)d\leq 1$ and $\psi'|Dd|\geq\mu$, where ν is defined as before, see (11). Consequently, choose now $\psi=\dfrac{1}{\nu}\ln(1+kd)$, as before to conclude that $\omega=-\psi\circ d+\varphi$ is a lower barrier satisfying (2) (with the minus sign), as desired. Thus the proof of Theorem 2.1 is now established.

□

We shall need a version of Theorem 2.1 for a family t, $t\in[0,1]$ to proceed further and to infer an existence theorem in Section 3, using degree theory. The proof is the same as before with $t\mathbf{H}$ in place of $\mathbf{H}$ and $t\varphi$ in place of φ. We next give the precise statement for future reference.

Theorem 2.2 (a priori boundary gradient estimate for a family t). *Let Ω be a bounded domain in $\mathbb{H}^n$ with C^3 boundary $\partial\Omega$, $\varphi\in C^2(\overline{\Omega})$ and $\mathbf{H}(x)\in C^1(\overline{\Omega})$. If $u=u^t\in C^2(\Omega)\cap C^1(\overline{\Omega})$ is a solution of the following Dirichlet Problem (P_t) (using summation convention)*

$$\begin{aligned}Q_t(u)&=a^{ij}(x,Du;t)D_{ij}u+b(x,Du;t)=0 \text{ in } \Omega\\ u&=t\varphi \text{ on } \partial\Omega\end{aligned}\tag{13}$$

where

$$a^{ij}(x,Du;t)=(1+|Du|^2)\delta_{ij}-D_iuD_ju$$
$$b(x,Du;t)=-\frac{nD_nu}{x_n}(1+|Du|^2)-\frac{tn\mathbf{H}(x)}{x_n}(1+|Du|^2)^{3/2}$$

If satisfies condition $(*)$ *holds then*

$$\sup_{\partial\Omega} |Du| \leqslant c$$

where

$$c = c(n,\, \Omega,\, \inf_{\Omega} x_n,\, \sup_{\Omega} x_n,\, \sup_{\Omega} |u^t|,\, |\varphi|_2,\, |\mathbf{H}|_1).$$

3. Existence results

In hyperbolic space horospheres play a role of geometric barriers and give rise to a geometric principle. This idea has been useful in many papers. We refer to [24] and to the references there. Let us briefly summarize it now. Consider M a smooth compact immersed surface into hyperbolic space with boundary a curve γ. Assume that the mean curvature of M is not bigger than 1. Now take any horosphere $\mathcal{H}$ that involves γ. Then $\mathcal{H}$ involves entirely M. In fact, the contrary we obtain a copy $\mathcal{H}_t$ by hyperbolic translating $\mathcal{H}$ involving M and touching M at an interior point. This will give a contradiction with the maximum principle. A similar result holds for hypersurfaces M with mean curvature less than some h, provide M is inside a n-sphere of mean curvature h. We will need the following theorem (see [5]).

Theorem 3.1 (height estimate). *Let* $\Omega \subset \mathbb{H}^n$ *be a bounded domain. Assume that* $\mathbf{H} \in C^1(\overline{\Omega})$, *and* $|\mathbf{H}(x)| \leqslant 1$ *or* $|\mathbf{H}(x)| = a$ *(constant). If* $u = u^t \in C^0(\overline{\Omega}) \cap C^2(\Omega)$ *is a solution of Dirichlet Problem* (P_t) *(see* (13)*)*

$$\begin{aligned} Q_t(u) = 0 \qquad & in \quad \Omega \\ u = t\varphi \ & on\ \partial\Omega \end{aligned} \tag{14}$$

then

$$\sup_{\Omega} |u^t| \leqslant c_1,$$

where c_1 *is a constant independent of* t.

Proof. On account of the previous discussion, we have a priori height estimates if the mean curvature is less or equal 1. We check the case $\mathbf{H}$ is constant with $|\mathbf{H}| > 1$. Notice that if we prove height estimates for constant mean curvature less or equal a, then a straightforward argument will provide height estimates for $|\mathbf{H}| \leqslant a$. We will proceed the proof as follows. Let S be a big n-sphere cutting orthogonally the geodesic hyperplane $\mathcal{P}$ containing Ω, such that S involves strictly both Ω and the graph of φ. Assume that S has mean curvature h (h is bigger than 1, but $h \approx 1$). This obviouslly can be done by compactness. Now notice that if M is an horizontal graph of mean curvature less than h taking boundary value data φ on $\partial\Omega$ then necessarily M lies inside S. Indeed, take a half part of S, say S^+ far away from M, doing horizontal Euclidean translations (hyperbolic isometries). Then move S^+ back towards M. By maximum principle S^+ (i.e a translates copy of S^+) cannot touch M during this movement, until it reaches the original position. Notice that since the graph of u_t has mean curvature $t\mathbf{H}$, where $\mathbf{H} = a$ (constant), we infer that for t varying in some interval $[t_0, 1]$, such that $t_0\mathbf{H} = h$ we have priori height estimates independent of t (see [14] or [21]). But we have also a priori height estimates independent of t for $0 \leqslant t \leqslant t_0$ according with the previous geometric argument. So then we have the desired estimates. $\square$

Combining together a priori boundary gradient estimates (see Theorem 2.1, Theorem 2.2) with height estimates (see Theorem 3.1) we can infer global gradient estimates by applying some elliptic theory, such as Lemma 3.1 in [20], Lemma 5.2 in [5], or Theorem 15.2 in [12] . Then using Ladyzhenskaya and Ural'tseva classical estimates we can infer a priori global Hölder estimates for the gradient. We have therefore the following result.

Theorem 3.2 (global gradient estimate). *Let Ω be a bounded domain in $\mathbb{H}^n$ with C^3 boundary $\partial\Omega$, $\varphi \in C^3(\overline{\Omega})$ and $\mathbf{H} \in C^2(\overline{\Omega})$, and $|\mathbf{H}(x)| \leqslant 1$ or $|\mathbf{H}(x)| = a$ (constant). If $u = u^t \in C^2(\overline{\Omega})$ is a solution of Dirichlet Problem (P_t) (see (13)) and if $\mathbf{H}(x)$ satisfies condition $(*)$ then*

$$\sup_{\Omega} |u^t|_{1,\beta} \leqslant c,$$

where c is a constant independent of t.

Remark 1. *By virtue of classical elliptic regularity, it suffices to assume that $\mathbf{H} \in C^{1,\alpha}(\overline{\Omega})$, $0 < \alpha < 1$. Indeed, with this assumption if u is a C^2 solution of Dirichlet Problem (P_t) then u is of classe C^3 and the argument developed in [20] or [5] can be applied. Moreover, a more refined derivation can be done to infer the same result for $\mathbf{H} \in C^1(\overline{\Omega})$, working with a weak form of (P_t). This remark can also be extended to the next existence theorem. But we will not go into such analysis in this article (see [12]).*

Of course we intent to solve Dirichlet Problem (P_t) for $t = 1$, i.e the horizontal mean equation in hyperbolic space. We will be able to carry out this plan by applying degree theory in the light of the a priori estimates. We now state our main existence result.

Theorem 3.3 (existence theorem). *Let Ω be a bounded domain in $\mathbb{H}^n$ with C^3 boundary $\partial\Omega$, $\varphi \in C^3(\overline{\Omega})$ and $\mathbf{H} \in C^2(\overline{\Omega})$, and $|\mathbf{H}(x)| \leqslant 1$ or*

$|\mathbf{H}(x)| = a$ *(constant). Assume condition* $(*)$ *holds. Then Dirichlet Problem* (P) *has a unique solution* $u \in C^2(\overline{\Omega})$.

Proof. In view of our a priori estimates we will be able to use degree theory methods to obtain existence of a fixed-point of certain map associated to the solutions of our Dirichlet Problem (P). We will proceed as follows.

Given $v \in C^{1,\beta}(\overline{\Omega})$ we then define a map $T_t : C^{1,\beta}(\overline{\Omega}) \to C^{1,\beta}(\overline{\Omega})$ by

$$T_t : v \mapsto u^t,$$

where u^t the unique solution of the second order elliptic linear problem

$$\sum_{i,j=1}^{n} \left((1+|Dv|^2)\delta_{ij} - D_i v D_j v\right) D_{ij}u^t - \frac{n(1+|Dv|^2)}{x_n} D_n u^t - \\ - \frac{tn(1+|Dv|^2)^{3/2}}{x_n}\mathbf{H} = 0 \quad \text{in } \Omega$$

$$u^t = t\varphi \text{ in } \partial\Omega$$

That is $u^t = T_t v$.

Existence and uniqueness of u^t is assured by linear elliptic theory. Moreover, $u^t \in C^{2,\alpha\beta}(\overline{\Omega})$, by regularity theory. With the aid of Arzela-Ascoli theorem and global Schauder estimates we deduce T_t is compact. A standard argument shows continuity of T_t (see [12]). From our estimates we can apply degree theory (see, [1], [4] or [18]) to conclude. We will proceed the final step of the proof as follows. We define the set

$$\mathcal{C} = \{v \in C^{1,\beta}(\overline{\Omega});\ |v|_{1,\beta} < c+1\}$$

where c is a constant given by Theorem 3.2. Notice that for our choice, the equation

$$u^t - T_t u^t = 0 \tag{15}$$

has no solutions in $\partial\mathcal{C}$.

Thus

$$\deg\left(I - T_t, \mathcal{C}, 0\right) = \text{cst}$$

is well defined and independent of $t \in [0,1]$, owing to homotopy invariance. But according to maximum principle, for $t = 0$ the trivial solution $u^0 \equiv 0$ is the unique solution for every $v \in C^{1,\beta}(\overline{\Omega})$. It follows that $T_0 \equiv 0$ and we deduce

$$\deg\left(I - T_0, \mathcal{C}, 0\right) = \deg\left(I, \mathcal{C}, 0\right) = 1$$

We have henceforth that Dirichlet Problem (P) has a solution, as desired.

$\square$

The following Corollaries are immediate consequences of Theorem 3.3.

Corollary 3.1. *Let Ω be a bounded domain in $\mathbb{H}^n$ with C^3 boundary $\partial\Omega$, $\varphi \in C^3(\overline{\Omega})$ and $\mathbf{H} \in C^2(\overline{\Omega})$. Assume $|\mathbf{H}(x)| \leqslant 1$, $\forall\, x \in \Omega$. Assume further $\mathbf{H}'(y) \geq \dfrac{n+1}{n-1}$, $\forall\, y \in \partial\Omega$. Then there exists a C^2 extension u of φ whose graph has prescribed mean curvature $\mathbf{H}(x)$.*

Corollary 3.2. *Let Ω be a ball entirely contained in n-dimensional hyperbolic space, i.e $\overline{\Omega} \subset \mathbb{H}^n$, and let φ be a smooth function defined on $\partial\Omega$. Then there exists a minimal solution u in Ω of the horizontal minimal equation taking the prescribed boundary value data φ on $\partial\Omega$.*

4. Non-existence results

Our construction to infer non-existence is based on classical maximum principle. We shall need the following issue stated here for future references.

Theorem 4.1 (see [12]). *Let Ω be a bounded domain in $\mathbb{R}^n$ and let Γ a C^1 open subset of $\partial\Omega$. Then if Q is an elliptic operator and if $u \in C^0(\overline{\Omega}) \cap C^2(\Omega \cup \Gamma)$,*
$v \in C^0(\overline{\Omega}) \cap C^2(\Omega)$ satisfies

$$Q(u) \geq Q(v) \text{ in } \Omega,$$
$$u \leqslant v \text{ in } \partial\Omega \setminus \Gamma,$$
$$\frac{\partial v}{\partial \nu} = -\infty \text{ in } \Gamma,$$

it follows that $u \leq v$ in Ω.

We will now established our non-existence result. Denote diam$\,(\Omega)$, the hyperbolic diameter of Ω.

Theorem 4.2. *Let Ω be a bounded domain in $\mathbb{H}^n$ with C^2 boundary $\partial\Omega$. Let us assume that condition $(*)$ fails at a point y , that is*

$$(n-1)\mathbf{H}'(y) + x_n(y) D_n d(y) < n|\mathbf{H}(y)|, \qquad \text{for some} \quad y \in \partial\Omega \qquad (\dagger)$$

where $\mathbf{H} \in C^0(\overline{\Omega})$ satisfies one of the following conditions:
(i) *It does not change sign in Ω and $n \geq 3$ or*
(ii) *It does not change sign in Ω, $n = 2$ and* diam$\,(\Omega) < \dfrac{\ln 3}{2}$ *or*

(iii) $|\mathbf{H}| \geq 1$ *in* Ω. *Under one of the above hypothesis it follows that there exists* $\varphi \in C^\infty(\overline{\Omega})$, *such that Dirichlet Problem* (P) *is not solvable. That is, there exists no solution* u *of the horizontal prescribed mean curvature equation in* Ω, *taking the prescribed boundary value data* φ *on* $\partial\Omega$.

Proof. We will proceed the proof in two steps. Firstly, we will work in a neighborhood of y to obtain an upper bound of a solution of Dirichlet Problem (P) in this neighborhood. Secondly, we will focus the region outside of this neighborhood to get a similar majoration. Then, by putting together these two estimates we will provide an upper bound of any solution of our Dirichlet Problem (P) satisfying condition (†). As we did to obtain a priori gradient estimates, we will again seek the desired majorations applying fairly estimates in the spirit of the calculations of the Section 2 by computing $Q(\omega)$.

First step of the proof: Let us fix $y \in \partial\Omega$, let $\delta = \operatorname{diam}(\Omega)$ and let a be a positive real number such that $a < \delta$. Define $\tilde{\Omega} := \{x \in \Omega;\, a < d(x) < \delta\}$. We then define ω setting $\omega(x) = \sup_{\partial\Omega \setminus B_a(y)} u + \psi(d)$, where $d(x) = \operatorname{dist}(x, y)$ and $\psi : (a, \delta) \to \mathbb{R}$ is a C^2 function to be chosen such that

$$\psi(\delta) = 0, \quad \psi' \leqslant 0, \quad \psi'(a) = -\infty \qquad \text{and} \quad \psi'' \geq 0 \tag{16}$$

We return now to our elliptic operator $Q(\omega)$. We intend to build $\psi = \psi(d)$ satisfying above properties such that ω is a supersolution for our Dirichlet Problem (P), i.e $Q(\omega) \leqslant 0$ in $\tilde{\Omega}$. Let us start our long and tedious computations to produce the barriers recalling

$$
\begin{aligned}
Q(\omega) = &\sum_{i,j=1}^{n} ((1+|D\omega|^2)\delta_{ij} - D_i\omega D_j\omega) D_{ij}\omega - \\
&- \frac{n}{x_n}\left(\frac{D_n\omega}{(1+|D\omega|^2)^{1/2}} + \mathbf{H}\right)(1+|D\omega|^2)^{3/2} \\
=&(1+|D\omega|^2)\sum_{i=1}^{n} D_{ii}\omega - \sum_{i,j=1}^{n} \psi'^2 D_i d D_j d D_{ij}\omega - \\
&- \frac{n}{x_n}\left(\frac{\psi' D_n d}{(1+|D\omega|^2)^{1/2}} + \mathbf{H}\right)(1+|D\omega|^2)^{3/2} \\
=&(1+|D\omega|^2)\left(\psi''\sum_{i=1}^{n}(D_i d)^2 + \psi'\sum_{i=1}^{n} D_{ii} d\right) - \\
&- \psi'^2 \sum_{i,j=1}^{n}\left(\psi''(D_i d)^2 (D_j d)^2 + \psi' D_i d D_j d D_{ij} d\right) \\
&- \frac{n}{x_n}\left(\frac{\psi' D_n d}{(1+|D\omega|^2)^{1/2}} + \mathbf{H}\right)(1+|D\omega|^2)^{3/2}
\end{aligned}
$$

We will need the following basic formulas in hyperbolic geometry.

$$
|Dd| = (\sum_{i=1}^{n}(D_i d)^2)^{1/2} = \frac{1}{x_n} \tag{17}
$$

$$
x_n^2 \sum_{i=1}^{n} D_{ii} d - (n-2) x_n D_n d = \frac{n-1}{\tanh d} \tag{18}
$$

Substituting (17) in the above expression we get

(19)

$$
\begin{aligned}
Q(\omega) =&(1+|D\omega|^2)\frac{\psi''}{x_n^2} + (1+|D\omega|^2)\psi'\sum_{i=1}^{n} D_{ii} d - \frac{\psi'^2\psi''}{x_n^4} + \frac{\psi'^3 D_n d}{x_n^3} \quad - \\
&- \frac{n(1+|D\omega|^2)}{x_n}\psi' D_n d - \frac{n}{x_n}\mathbf{H}(1+|D\omega|^2)^{3/2} \\
=&(1+|D\omega|^2)\psi'\sum_{i=1}^{n} D_{ii} d + \frac{\psi''}{x_n^2} - \frac{n(1+|D\omega|^2)}{x_n}\psi' D_n d \quad - \\
&- \frac{n}{x_n}\mathbf{H}(1+|D\omega|^2)^{3/2} + \psi'^3 \frac{D_n d}{x_n^3}
\end{aligned}
$$

Now substituting (18) we discover

$$\frac{(1+|D\omega|^2)}{x_n^2}\psi'\left(x_n^2\sum_{i=1}^{n}D_{ii}d-(n-2)x_nD_nd\right)-2\frac{(1+|D\omega|^2)}{x_n}\psi'D_nd+$$
$$+\psi'^3\frac{D_nd}{x_n^3}+\frac{\psi''}{x_n^2}-\frac{n}{x_n}\mathbf{H}(1+|D\omega|^2)^{3/2}$$
$$=\frac{(1+|D\omega|^2)}{x_n^2}\psi'\frac{n-1}{\tanh d}-2\frac{(1+|D\omega|^2)}{x_n}\psi'D_nd+$$
$$+\psi'^3\frac{D_nd}{x_n^3}+\frac{\psi''}{x_n^2}-\frac{n}{x_n}\mathbf{H}(1+|D\omega|^2)^{3/2}$$
$$=\frac{(1+|D\omega|^2)}{x_n^2}\psi'\frac{n-1}{\tanh d}-\frac{(1+|D\omega|^2)}{x_n}\psi'D_nd-$$
$$-\psi'\frac{D_nd}{x_n}+\frac{\psi''}{x_n^2}-\frac{n}{x_n}\mathbf{H}(1+|D\omega|^2)^{3/2}$$

We thereby obtain

$$Q(\omega)\leqslant\frac{(1+|D\omega|^2)}{x_n^2}\psi'(\frac{n-1}{\tanh d}-2)+\frac{\psi''}{x_n^2}-\frac{n}{x_n}\mathbf{H}(1+|D\omega|^2)^{3/2} \tag{20}$$

taking into account (17) and (16). Les us assume momentarily that $\mathbf{H}(x)$ is non-negative. The other case has a same analysis. Now owing to our choice we see that the quantity $\mu_1=\inf_{\Omega}(n-1-2\tanh d)$ is strictly positive, hence

$$\begin{aligned}Q(\omega)\leqslant&\frac{(1+|D\omega|^2)}{x_n^2}\left(\frac{n-1-2\tanh d}{\tanh d}\psi'+x_n^2\frac{\psi''}{\psi'^2}\right)\\ \leqslant&\frac{1+|D\omega|^2}{x_n^2}\left(\frac{\mu_1\psi'}{\tanh d}+\frac{\mu_2\psi''}{\psi'^2}\right)\end{aligned}$$

where $\mu_2=\sup_{\Omega}x_n^2$.

Of course, if $n\geq 3$ (case (i)) then μ_1 is always strictly positive. We then define

$$\psi(d)=\mu^{-1/2}\int_d^{\delta}\left(\ln\frac{\sinh t}{\sinh a}\right)^{-1/2}\mathrm{d}t,\ \text{where}\ \mu:=\frac{\mu_1}{\mu_2}=\mu(\mathrm{diam}(\Omega),\inf_{\Omega}x_n,\sup_{\Omega}x_n,n).$$

It is straigtforward to check that ω is the required barrier. Observe first that $Q(\omega)\leqslant 0$ in $\tilde{\Omega}$. Now in view of Theorem 4.1 we have therefore $u(x)\leqslant\omega(x)$, $x\in\tilde{\Omega}$. From this it follows that,

$$\sup_{\partial B_a(y)\cap\Omega}u\leqslant\omega(a)=\sup_{\partial\Omega\setminus B_a(y)}u+\psi(a). \tag{21}$$

By our assumptions if $n = 2$ (case (ii)) then μ_1 is positive hence we arrive analagously to the same conclusion. Let us turn to the estimates to treat the remaining case. We turn now to case(iii). Employing (20) using $\mathbf{H} \geq 1$ we infer

$$
\begin{aligned}
-n\frac{(1+|D\omega|^2)^{3/2}}{x_n}\mathbf{H} \leqslant & -n\frac{(1+|D\omega|^2)^{3/2}}{x_n} \\
& \leqslant -n\frac{(1+|D\omega|^2)}{x_n}|D\omega| = n\psi'\frac{(1+|D\omega|^2)}{x_n^2}
\end{aligned}
$$

We conclude henceforth

$$
\begin{aligned}
Q(\omega) \leqslant & \frac{(1+|D\omega|^2)}{x_n^2}\psi'(\frac{n-1}{\tanh d}+n-2)+\frac{\psi''}{x_n^2} \\
\leqslant & \frac{(1+|D\omega|^2)}{x_n^2}\psi'(\frac{n-1}{\tanh\, d}+n-2)+\frac{\psi''(1+|D\omega|^2)}{\psi'^2} \\
= & \frac{1+|D\omega|^2}{x_n^2}\left[\left(\frac{n-1}{\tanh d}+n-2\right)\psi'+x_n^2\frac{\psi''}{\psi'^2}\right] \\
\leqslant & \frac{1+|D\omega|^2}{x_n^2}\left(\frac{\mu_1}{\tanh d}\psi'+\mu_2\frac{\psi''}{\psi'^2}\right)
\end{aligned}
$$

where

$$\mu_1 = \inf_{\Omega}(n-1+(n-2)\tanh d) > 0,\ \mu_2 = \sup_{\Omega} x_n^2$$

We may therefore select

$$\psi(d) = \mu^{-1/2}\int_d^{\delta}\left(\ln\frac{\sinh t}{\sinh a}\right)^{-1/2} dt,\ \text{where } \frac{\mu_1}{\mu_2} = \mu = \mu(\mathrm{diam}(\Omega), \inf_{\Omega} x_n, \sup_{\Omega} x_n, n).$$

Clearly, ω satisfies $Q(\omega) \leqslant 0$ in $\tilde{\Omega}$ thus we arrive to the same conclusion as before, namely

$$\sup_{\partial B_a(y)\cap\Omega} u \leqslant \sup_{\partial\Omega\setminus B_a(y)} u + \psi(a)$$

First step of the proof is now completed.

Second step of the proof: We wish now examine the estimates in the complementary domain $\Omega_{\varepsilon,a} := \{x \in \Omega;\quad \epsilon < d(x) < a\}$. We observe that until now we have not use condition (†). We will use it now. Let us assume that there exists $\eta > 0$ such that

$$(n-1)\mathbf{H}'(y) + x_n(y)D_n d(y) \leqslant n\mathbf{H}(y) - 5\eta. \tag{22}$$

Let $S \subset \mathbb{R}^n$ be a smooth quadric surface satisfying
(i) $H_S(y) \leqslant \mathbf{H}'(y) + \eta$
(ii) $S \cap B_a(y) \subset \overline{\Omega}$ Let us consider now distance function $d(x) =$

dist(x, S), in $\Omega_{\varepsilon,a}$, choosing a suficiently small to ensures smoothness of $d(x)$. We want to find $\omega = \sup_{\partial B_a(y)\cap\Omega} u + \Psi(d)$ where $\Psi : (\epsilon, a) \to \mathbb{R},\ 0 < \epsilon < a$ such that $\Psi(a) = 0,\ \Psi' \leq 0,\ \Psi'(\epsilon) = -\infty$ and $\Psi'' \geq 0$. Furthermore, we also want $Q(\omega) \leqslant 0$ in $\Omega_{\varepsilon,a}$. As before, we will pass to perform the computation of $Q(\omega)$ using (17) once again. Hence

$$Q(\omega) = \Psi'(1 + |D\omega|^2)\sum_{i=1}^{n} D_{ii}d + \frac{\Psi'^3}{x_n^3}D_n d + \frac{\Psi''}{x_n^2} - \frac{n\Psi'}{x_n}(1 + |D\omega|^2)D_n d - \\ - \frac{n}{x_n}\mathbf{H}(1 + |D\omega|^2)^{3/2}$$

Using the identity

$$\frac{\Psi'^3}{x_n^3}D_n d = \frac{\Psi'}{x_n}(1 + |D\omega|^2)D_n d - \frac{\Psi' D_n d}{x_n}$$

We therefore obtain

$$Q(\omega) = \frac{\Psi'}{x_n^2}(1 + |D\omega|^2)\left(x_n^2\sum_{i=1}^{n} D_{ii}d - (n-1)x_n D_n d\right) - \frac{\Psi' D_n d}{x_n} + \frac{\Psi''}{x_n^2} \\ - \frac{n}{x_n}\mathbf{H}(1 + |D\omega|^2)^{3/2}$$

We observe that

$$(1+|D\omega|^2)^{3/2} = \frac{-\Psi'}{x_n}(1+|D\omega|^2)+\Psi'(1+|D\omega|^2)o(1) \quad (\text{as} \quad |D\omega| \to \infty)$$

Since

$$\frac{-\Psi'}{x_n}D_n d \leqslant \frac{-\Psi'}{x_n^2}$$

and

$$\frac{\Psi''}{x_n^2} \leqslant \frac{\Psi''}{x_n^2}\frac{(1 + |D\omega|^2)}{|Dw|^2} = (1 + |D\omega|^2)\frac{\Psi''}{\Psi'^2}$$

We infer

$$Q(\omega) \leqslant \frac{\Psi'}{x_n^2}(1 + |D\omega|^2)\left[x_n^2\sum_{i=1}^{n} D_{ii}d - (n-1)x_n D_n d + n\mathbf{H} + \right. \\ \left. +(n\sup_{\Omega}|\mathbf{H}| + 1)o(1) + \left(\sup_{\Omega} x_n^2\right)\frac{\Psi''}{\Psi'^3}\right].$$

Let us denote now by $S_t = \{x \in \Omega;\, d(x) = t\}$ the parallel hypersurface to S at a fixed distance t. In view of formula (9), we can write

$$x_n^2 \sum_{i=1}^{n} D_{ii} d - (n-2) x_n D_n d = -(n-1) H_{S_t}$$

where H_{S_t} is the mean curvature of S_t.

Now applying a standard continuity argument we have

(i) $|\mathbf{H}(x) - \mathbf{H}(y)| < \dfrac{\eta}{n}$

(ii) $|x_n^2(x) \sum_{i=1}^{n} D_{ii} d(x) - x_n^2(y) \sum_{i=1}^{n} D_{ii} d(y)| < \eta$

(iii) $|x_n(x) D_n d(x) - x_n(y) D_n d(y)| < \dfrac{\eta}{n-1}$

for $x \in \Omega_{\varepsilon,a}$ (recall that $y \in \partial\Omega$ is fixed). Hence

$$\begin{aligned} Q(\omega) \leqslant \frac{\Psi'(1+|D\omega|^2)}{x_n^2} & \left[x_n^2(y) \sum_{i=1}^{n} D_{ii} d(y) - (n-1) x_n(y) D_n d(y) + \right. \\ & \left. + n\mathbf{H}(y) - 3\eta + o(1) + \left(\sup_{\Omega} x_n^2 \right) \frac{\Psi''}{\Psi'^3} \right] \\ \leqslant \frac{\Psi'(1+|D\omega|^2)}{x_n^2} & \left[-(n-1) H_s(y) - x_n(y) D_n d(y) + n\mathbf{H}(y) - \right. \\ & \left. - 3\eta + o(1) + \left(\sup_{\Omega} x_n^2 \right) \frac{\Psi''}{\Psi'^3} \right] \\ \leqslant \frac{\Psi'(1+|D\omega|^2)}{x_n^2} & \left[-(n-1) \mathbf{H}'(y) - x_n(y) D_n d(y) + n\mathbf{H}(y) - \right. \\ & \left. - 4\eta + o(1) + \left(\sup_{\Omega} x_n^2 \right) \frac{\Psi''}{\Psi'^3} \right] \end{aligned}$$

We now using condition (†) in the form given by (22) we have

$$\begin{aligned} Q(\omega) \leqslant & \frac{\Psi'(1+|D\omega|^2)}{x_n^2} \left[5\eta - 4\eta + o(1) + \left(\sup_{\Omega} x_n^2 \right) \frac{\Psi''}{\Psi'^3} \right] \\ = & \frac{\Psi'(1+|D\omega|^2)}{x_n^2} \left[\eta + o(1) + \left(\sup_{\Omega} x_n^2 \right) \frac{\Psi''}{\Psi'^3} \right] \end{aligned}$$

We may define now $\Psi(d) = k((a-\epsilon)^{1/2} - (d-\epsilon)^{1/2})$, $\epsilon < d < a$, hence we obtain $Q(\omega) \leqslant 0$ in $\Omega_{\varepsilon,a}$, taking k suficiently big. According to Theorem 4.1 we have therefore

$$u(x) \leqslant \sup_{\partial B_a(y) \cap \Omega} u + \Psi(d(x)), \ \forall\, x \in \Omega_{\varepsilon,a}$$

Thus

$$\begin{aligned} u(x) &\leqslant \sup_{\partial B_a(y)\cap\Omega} u + \Psi(\epsilon),\ \forall\, x \in \Omega_{\varepsilon,a} \\ &= \sup_{\partial B_a(y)\cap\Omega} u + k(a-\epsilon)^{1/2},\ \forall\, x \in \Omega_{\varepsilon,a} \end{aligned} \tag{23}$$

We lastly combine estimates (21) and (23) letting $\epsilon \to 0^+$. We conclude therefore

$$u(y) \leqslant \sup_{\partial\Omega\setminus B_a(y)} u + \psi(a) + k(a)^{1/2}$$

This means u cannot be arbitrarily prescribed on $\partial\Omega$. That is, there exists a smooth function φ in $\overline{\Omega}$ such that our Dirichlet Problem (P) taking boundary value data φ on $\partial\Omega$ is not solvable.

□

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E-mail address: emguio@yahoo.com.br
E-mail address: earp@mat.puc-rio.br