STABILIZATION OF HETERODIMENSIONAL CYCLES

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ABSTRACT. We consider diffeomorphisms f with heteroclinic cycles associated to saddles P and Q of different indices. We say that a cycle of this type can be stabilized if there are diffeomorphisms close to f with a robust cycle associated to hyperbolic sets containing the continuations of P and Q. We focus on the case where the indices of these two saddles differ by one. We prove that, excluding one particular case (so-called twisted cycles that additionally satisfy some geometrical restrictions), all such cycles can be stabilized.

1. INTRODUCTION

In [17] Palis proposed a program whose main goal is a geometrical description for the behavior of most dynamical systems. This program pays special attention to the generation of non-hyperbolic dynamics and to robust dynamical properties (i.e., properties that hold for open sets of dynamical systems). An important part of this program is the *Density Conjecture olicity versus cycles)*: the two main sources of non-hyperbolic dynamics are heterodimensional cycles and homoclinic tangencies (shortly, *cycles*), see [17, Conjecture 1][1]. The goal of this paper is to study the *generation of robust heterodimensional cycles* (see Definition 1.1).

Besides Palis' program, we have the following two motivations for this paper:

Motivation I ([16, 21, 22]). *Every C^2-diffeomorphism having a homoclinic tangency associated with a saddle P is in the C^2-closure of the set of diffeomorphisms having C^2-robust homoclinic tangencies. Moreover, these robust homoclinic tangencies can be taken associated to hyperbolic sets containing the continuations of the saddle P.*

Using the terminology that will be introduced in this paper this means that homoclinic tangencies of C^2-diffeomorphisms can be *stabilized,* see Definition 1.1. On the other hand, for C^1-diffeomorphisms of surfaces homoclinic tangencies cannot be stabilized, see in [14]. This leads to the following motivation.

Motivation II ([8]). *Every diffeomorphism with a heterodimensional cycle associated with a pair of hyperbolic saddles P and Q with $\dim E^s(P) = \dim E^s(Q) \pm 1$ belongs to the C^1-closure of the set of diffeomorphisms having C^1-robust heterodimensional cycles. Here E^s denotes the stable bundle of a saddle.*

[1]This conjecture was proved by Pujals-Sambarino for C^1-surface diffeomorphisms in [18] (since heterodimensional cycles only can occur in manifolds of dimension $n \geq 3$, for surface diffeomorphisms it is enough to consider homoclinic tangencies).

One may think of the result in Motivation II as a version of the results in Motivation I for heterodimensional cycles in the C^1-setting. However, the results in [8] does not provide information about the relation between the hyperbolic sets involved in the robust cycles and the saddles in the initial one. Thus, one aims for an extension of [8] giving some information about the hyperbolic sets displaying the robust cycles, see [8, Question 1.9].

In this paper we prove that, with the exception of a special type of heterodimensional cycles (so-called *twisted cycles*, see Definition 4.6), the hyperbolic sets exhibiting the robust cycles can be taken containing the continuations of the saddles in the initial cycle. In fact, by [10] our results cannot be improved: there are twisted cycles that cannot be stabilized, that is, the hyperbolic sets with robust cycles cannot be taken containing the continuations of the saddles in the initial cycle.

To state precisely our results we need to introduce some definitions. Recall that if Λ is a hyperbolic basic set of a diffeomorphism $f \colon M \to M$ then there are a neighborhood $\mathcal{U}_f$ of f in the space of C^1-diffeomorphisms and a continuous map $\mathcal{U}_f \to M \colon g \mapsto \Lambda_g$, such that $\Lambda_f = \Lambda$, Λ_g is a hyperbolic basic set, and the dynamics of $f|_\Lambda$ and $g|_{\Lambda_g}$ are conjugate. The set Λ_g is called the *continuation* of Λ for g. Note that these continuations are uniquely defined.

Definition 1.1 (Heterodimensional cycles).

- The s-*index* (u-*index*) of a transitive hyperbolic set is the dimension of its stable (unstable) bundle.
- A diffeomorphism f has a *heterodimensional cycle* associated to transitive hyperbolic basic sets Λ and Σ of f if these sets have different s-indices and their invariant manifolds meet cyclically, that is, if $W^{\mathrm{s}}(\Lambda, f) \cap W^{\mathrm{u}}(\Sigma, f) \neq \emptyset$ and $W^{\mathrm{u}}(\Lambda, f) \cap W^{\mathrm{s}}(\Sigma, f) \neq \emptyset$.
- The heterodimensional cycle has *coindex* k if s-index$(\Lambda) = $ s-index$(\Sigma) \pm k$. In such a case we just write *coindex k cycle*.
- A diffeomorphism f has a C^1-*robust heterodimensional cycle* associated to its hyperbolic basic sets Λ and Σ if there is a C^1-neighborhood $\mathcal{U}$ of f such that every diffeomorphism $g \in \mathcal{U}$ has a a heterodimensional cycle associated to the continuations Λ_g and Σ_g of Λ and Σ, respectively.
- Consider a diffeomorphism f with a heterodimensional cycle associated to a pair of saddles P and Q. This cycle *can be C^1-stabilized* if every C^1-neighborhood $\mathcal{U}$ of f contains a diffeomorphism g with hyperbolic basic sets $\Lambda_g \ni P_g$ and $\Sigma_g \ni Q_g$ having a robust heterodimensional cycle. Otherwise the cycle is said to be C^1-*fragile*.

Remark that, by the Kupka-Smale genericity theorem (invariant manifolds of hyperbolic periodic points of generic diffeomorphisms are in general position), at least one of the hyperbolic sets involved in a robust cycle is necessarily *non-trivial,* that is, not a periodic orbit.

Definition 1.2 (Homoclinic class). The *homoclinic class* of a saddle P is the closure of the transverse intersections of the stable and unstable manifolds $W^{\mathrm{s}}(P, f)$ and $W^{\mathrm{u}}(P, f)$ of the orbit of P. We denote this class by $H(P, f)$. A homoclinic class is *non-trivial* if it contains at least two different orbits.

A homoclinic class can be also defined as the closure of the set of saddles that are homoclinically related with P. Here we say that a saddle Q is *homoclinically*

related with P if the invariant manifolds of the orbits of P and Q meet cyclically and transversely, that is, $W^{\mathrm{s}}(P, f) \pitchfork W^{\mathrm{u}}(Q, f) \neq \emptyset$ and $W^{\mathrm{s}}(Q, f) \pitchfork W^{\mathrm{u}}(P, f) \neq \emptyset$.

The following is a consequence of our results (see Theorems 2 and 3 below).

Theorem 1. *Let f be a C^1-diffeomorphism with a coindex one cycle associated to saddles P and Q. Suppose that at least one of the homoclinic classes of these saddles is non-trivial. Then the heterodimensional cycle of f associated to P and Q can be C^1-stabilized.*

A simple consequence of this result is the following:

Corollary 1. *Let f be a C^1-diffeomorphism with a heterodimensional cycle associated to saddles P and Q such that $\mathrm{s}\text{-}\mathrm{index}(P) = \mathrm{s}\text{-}\mathrm{index}(Q) + 1$. Suppose that the intersection $W^{\mathrm{u}}(P, f) \cap W^{\mathrm{s}}(Q, f)$ contains at least two different orbits. Then the cycle can be C^1-stabilized.*

The question of the stabilization of cycles is relevant for describing the global dynamics of diffeomorphisms (indeed this is another motivation for this paper). Let us explain this point succinctly. Following [12, 15, 1], this global dynamics is structured by means of homoclinic or/and chain recurrence classes. The goal is to describe the dynamics of these classes and their relating cycles. In general, homoclinic classes are (properly) contained in chain recurrence classes. For C^1-generic diffeomorphisms and for hyperbolic periodic points, these two kinds of classes coincide, [4]. However, there are non-generic situations where two different homoclinic classes are "joined" by a cycle. In this case these classes are contained in one common chain recurrence class which hence is strictly larger. We would like to know under which conditions after small perturbations these two homoclinic classes explode and fall into the very same homoclinic class C^1-robustly. Indeed this occurs if the cycle can be C^1-stabilized. Examples where this stabilization is used for describing global dynamics can be found in [5, 23, 24]. See [11, Chapter 10.3-4] and [3] for a broader discussion of these questions.

To prove our results we analyze the dynamics associated to different types of coindex one cycles. This analysis essentially depends on two factors: the *central multipliers of the cycle* and its *unfolding map*. Let us now discuss this point briefly, for further details we refer to Section 4.

1.1. **Multipliers and unfolding map of a cycle.** Let f be a diffeomorphism with a coindex one cycle associated to saddles P and Q. In what follows we will assume that $\mathrm{s}\text{-}\mathrm{index}(P) = \mathrm{s}\text{-}\mathrm{index}(Q) + 1$. Denote by $\pi(R)$ the period of a periodic point R.

We say that the cycle is *partially hyperbolic* if there are heteroclinic points $X \in W^{\mathrm{s}}(P, f) \cap W^{\mathrm{u}}(Q, f)$ and $Y \in W^{\mathrm{u}}(P, f) \cap W^{\mathrm{s}}(Q, f)$ such that the closed set formed by the orbits of P, Q, X, and Y has a partially hyperbolic splitting of the form $E^{\mathrm{ss}} \oplus E^c \oplus E^{\mathrm{uu}}$, where E^c is one-dimensional, E^{ss} is uniformly contracting, and E^{uu} is uniformly expanding. We call E^c the *central bundle*. Note that, in particular, this implies that X is a transverse intersection and Y is a quasi-tranverse intersection of the invariant manifolds. Also observe that the bundle E^c is necessarily non-hyperbolic. Bearing in mind this property we introduce the following definition.

Definition 1.3 (Central multipliers). The *cycle has real central multipliers* if there are a contracting real eigenvalue λ of $Df^{\pi(P)}(P)$ and an expanding real eigenvalue β

of $Df^{\pi(Q)}(Q)$ such that: **(i)** λ and β have multiplicity one, **(ii)** $|\lambda| > |\sigma|$ for every contracting eigenvalue σ of $Df^{\pi(P)}(P)$, and **(iii)** $|\beta| < |\eta|$ for every expanding eigenvalue η of $Df^{\pi(Q)}(Q)$. In this case, we say that λ and β are the *real central multipliers of the cycle.*

Similarly, the *cycle has non-real central multipliers* if either **(i)** there are a pair of non-real (conjugate) contracting eigenvalues λ and $\bar\lambda$ of $Df^{\pi(P)}(P)$ such that $|\lambda| = |\bar\lambda| \geq |\sigma|$ for every contracting eigenvalue σ of $Df^{\pi(P)}(P)$, or **(ii)** there are a pair of non-real (conjugate) expanding eigenvalues β and $\bar\beta$ of $Df^{\pi(Q)}(Q)$ such that $|\beta| = |\bar\beta| \leq |\eta|$ for every expanding eigenvalue η of $Df^{\pi(Q)}(Q)$.

Let us note that cycles with central real multipliers can be perturbed to get partially hyperbolic ones (associated to the continuations of the saddles in the initial one).

In the case of cycles with real central multipliers we will distinguish so-called *twisted* and *non-twisted* cycles, see Definition 4.6. An intuitive explanation of these two sorts of cycles goes as follows, see Figure 1.

In order to study the dynamics of the cycle we select heteroclinic points $X \in W^{\mathrm{s}}(P, f) \cap W^{\mathrm{u}}(Q, f)$ and $Y \in W^{\mathrm{u}}(P, f) \cap W^{\mathrm{s}}(Q, f)$. Typically, X is a transverse intersection point and Y is a quasi-transverse intersection point (due to dimension deficiency). The next step is to consider a *neighborhood of the cycle,* that is, an open set V containing the orbits of P, Q, X, and Y, and study the dynamics of perturbations of f in such a neighborhood. If the neighborhood V is small enough, possibly after a perturbation of f, the dynamics of f in V is partially hyperbolic with a splitting of the form $E^{\mathrm{ss}} \oplus E^{\mathrm{c}} \oplus E^{\mathrm{uu}}$ (recall the definition above).

Replacing Y by some backward iterate, we can assume that the heteroclinic point Y is close to P. We pick some large number k such that $f^k(Y)$ is nearby Q and consider the map $\mathfrak{T}_1 = f^k$ defined in a small neighborhood of Y. This map is called the *unfolding map.* If it is possible to pick k in such a way that Df^k preserves the orientation of the central bundle then we say that the cycle is *non-twisted.* Otherwise, the cycle is *twisted.* Note that in the previous discussion the choice of the heteroclinic point X does not play any relevant role.
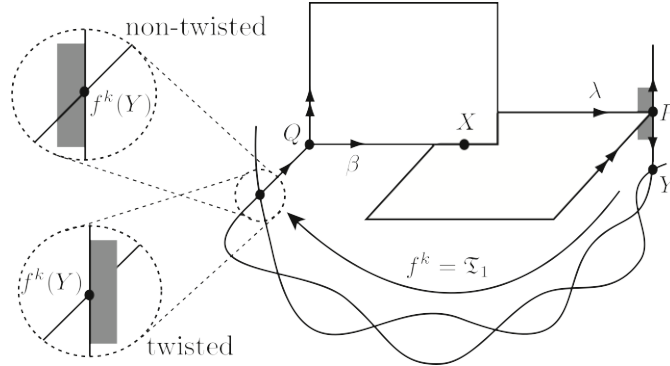

FIGURE 1. Twisted and non-twisted cycles

More precisely, the dynamics of the unfolding of the cycle mostly depends on the signs of the central eigenvalues λ (associated to P) and β (associated to Q) and

on the restriction of $\mathfrak{T}_1$ to the central bundle. We associate to the cycle the *signs*, $\mathrm{sign}(Q)$, $\mathrm{sign}(P)$, and $\mathrm{sign}(\mathfrak{T}_1)$ in $\{+,-\}$ determined by the following rules:

- $\mathrm{sign}(Q) = +$ if $\beta > 0$ and $\mathrm{sign}(Q) = -$ if $\beta < 0$;
- $\mathrm{sign}(P) = +$ if $\lambda > 0$ and $\mathrm{sign}(P) = -$ if $\lambda < 0$; and
- $\mathrm{sign}(\mathfrak{T}_1) = +$ if $\mathfrak{T}_1$ preserves the orientation in the central direction and $\mathrm{sign}(\mathfrak{T}_1) = -$ if the orientation is reversed.

A cycle is twisted if $\mathrm{sign}(Q) = +$, $\mathrm{sign}(P) = +$, and $\mathrm{sign}(\mathfrak{T}_1) = -$. Otherwise the cycle is non-twisted. For details see Definition 4.6.

Let us observe that the discussion above is reminiscent of the one in [19, Section 2] about bifurcations of homoclinic tangencies of surface diffeomorphisms. It involves similar ingredients to the ones above: the signs of the eigenvalues of the derivatives, the sides of the tangencies, and the connections (homoclinic and heteroclinic intersections).

We are now ready to state our main results.

Theorem 2. *Consider a diffeomorphism f having a coindex one cycle associated to saddles P and Q. Suppose that*

(A) *either the cycle has a non-real central multiplier,*

(B) *or the cycle has real multipliers and is non-twisted.*

Then the cycle of f associated to P and Q can be C^1-stabilized.

Let us observe that Theorem 2 cannot be improved. Indeed, there are examples of diffeomorphisms with twisted cycles that cannot be stabilized, see [10]. On the other hand, we prove that cycles with the *bi-accumulation property* can be C^1-stabilized. Let us state this result more precisely.

Given a periodic point R of f, consider the eigenvalues $\lambda_1(R), \ldots, \lambda_n(R)$ of $Df^{\pi(R)}(R)$ ordered in increasing modulus and counted with multiplicity. If R is hyperbolic, has s-index k, and $|\lambda_{k-1}(R)| < |\lambda_k(R)|$ then there is a unique invariant manifold $W^{\mathrm{ss}}(R, f)$ (the strong stable manifold of R) tangent to the eigenspace associated to $\lambda_1(R), \ldots, \lambda_{k-1}(R)$ (the strong stable bundle). The manifold $W^{\mathrm{ss}}_{\mathrm{loc}}(R, f)$ has codimension one in $W^{\mathrm{s}}_{\mathrm{loc}}(R, f)$ and $W^{\mathrm{ss}}_{\mathrm{loc}}(R, f)$ splits each component of $W^{\mathrm{s}}_{\mathrm{loc}}(R, f)$ into two parts.

Definition 1.4 (Bi-accumulation property). A saddle R of s-index k such that $|\lambda_{k-1}(R)| < |\lambda_k(R)|$ is s-*bi-accumulated* (by homoclinic points) if every component of $W^{\mathrm{s}}_{\mathrm{loc}}(R, f) \setminus W^{\mathrm{ss}}_{\mathrm{loc}}(R, f)$ contains transverse homoclinic points of R.

A heterodimensional cycle associated to saddles P and Q with s-index$(P) =$ s-index$(Q) + 1$ is *bi-accumulated* is either P is s-bi-accumulated for f or Q is s-bi-accumulated for f^{-1}.

In the next result we consider cycles with real central multipliers.

Theorem 3.

(A) *Every non-twisted cycle can be C^1-stabilized.*

(B) *Every twisted cycle with the bi-accumulation property can be C^1-stabilized.*

Indeed, Theorems 1 and 2 are consequence of Theorem 3.

Finally, our results can be summarized as follows:

Corollary 2. *Consider a diffeomorphism f with a fragile cycle associated to saddles P and Q with s-index$(P) =$ s-index$(Q) + 1$. Then*

- *the cycle has positive central real multipliers,*
- *the cycle is persistently twisted (i.e., the cycle cannot be perturbed to get a non-twisted cycle associated to P and Q),*
- *the intersection $W^u(P, f) \cap W^u(Q, f)$ consists of exactly one orbit, and*
- *the homoclinic classes of P and Q are both trivial.*

Examples of fragile cycles satisfying the four properties in the corollary can be found in [10].

2. Ingredients of the proofs

In this section we review some tools of our constructions.

2.1. **Reduction to the case of cycles with real multipliers.** A first step is to see that to prove our results it is enough to consider cycles with real central multipliers. For that let us recall a result from [8].

Theorem 2.1 (Theorem 2.1 in [8])**.** *Let f be a diffeomorphism with a coindex one cycle associated to saddles P and Q. Then there are diffeomorphisms g arbitrarily C^1 close to f with a coindex one cycle with real central multipliers associated to saddles P'_g and Q'_g which are homoclinically related to the continuations P_g and Q_g of P and Q. In this result one may have $P = P_g$ and/or $Q = Q_g$.*

Note that the previous theorem means the following.

Remark 2.2. Assume that the saddle P in Theorem 2.1 has non-real central multipliers. Then the homoclinic class of P'_g is non-trivial and contains P.

There is also the following simple fact:

Lemma 2.3. *Consider a diffeomorphism f with a heterodimensional cycle associated to P and Q. Suppose that there are saddles P'_g and Q'_g homoclinically related to P_g and Q_g, respectively, with a heterodimensional cycle that can be C^1-stabilized. Then the initial cycle can also be C^1-stabilized.*

Proof. The stabilization of the cycle associated to P'_g and Q'_g means that there is h arbitrarily close to g having a pair of basic hyperbolic sets $\Lambda'_h \ni P'_h$ and $\Sigma'_h \ni Q'_h$ with a robust cycle. Since the saddles P_h and P'_h are homoclinically related there is a basic set Λ_h containing Λ'_h and P_h. Similarly, there is a basic set Σ_h containing Σ'_h and Q_h. Since $W^i(\Lambda_h, h) \supset W^i(\Lambda'_h, h)$ and $W^i(\Sigma_h, h) \supset W^i(\Sigma'_h, h)$, $i = s, u$, it is immediate that there is a robust cycle associated to $\Lambda_h \ni P_h$ and $\Sigma_h \ni Q_h$. $\square$

Remark 2.4. Theorem 2.1 and Lemma 2.3 mean that to prove Theorems 1 and 2 it is enough to stabilize cycles with real central multipliers (indeed this is the sort of cycles considered in Theorem 3). Thus in what follows we will focus on this type cycles.

2.2. **Strong homoclinic intersections and blenders.** A key ingredient for obtaining robust heterodimensional cycles in [8] is the notion of a *blender*. A blender is a hyperbolic set with some additional geometrical intersection properties that guarantee some robust intersections, see Section 3.1 and Definition 3.1. The key step in [8] to obtain robust cycles is that coindex one cycles yield periodic points of saddle-node/flip type with *strong homoclinic intersections*: the strong stable

manifold of the saddle-node/flip intersects its strong unstable manifold, see Definition 3.3. These strong homoclinic intersections generate blenders yielding robust cycles, see Proposition 3.4.

In [8] the generation of blenders is not controlled and in general the saddle-node/flip has "nothing to do" with the saddles in the initial cycle. This is why in [8] the hyperbolic sets with robust cycles are not related (in general) to the saddles in the initial cycle. Here we control the "generation" of the saddle-node/flip with strong homoclinic intersections, obtaining blenders that contains the continuation of a saddle in the initial cycle and intersecting the invariant manifolds of the other saddle in the cycle. This configuration provides robust cycles associated to hyperbolic sets containing the continuation of both initial saddles, see Theorem 3.5.

We next explain the "generation" of saddle-node/flip poits with strong homoclinic intersections.

2.3. **Simple cycles and iterated function systems (IFSs).** To analyze the dynamics of cycles with real multipliers we borrow some constructions and the notion of a *simple cycle* from [8], see Section 4.

In very rough terms, if a diffeomorphism has a simple cycle then its dynamics in a neighborhood of the cycle is affine and preserves a partially hyperbolic splitting $E^{\mathrm{ss}} \oplus E^{\mathrm{c}} \oplus E^{\mathrm{uu}}$, where E^{ss} is uniformly contracting, E^{uu} is uniformly expanding, and E^{c} is one-dimensional and non-hyperbolic, see Proposition 4.1. Following [8], to prove our results it is enough to consider simple cycles and their (suitable) unfoldings.

We consider one-parameter families of diffeomorphisms $(f_t)_t$ unfolding a simple cycle at $t = 0$ and preserving the affine structure associated to the splitting $E^{\mathrm{ss}} \oplus E^{\mathrm{c}} \oplus E^{\mathrm{uu}}$. In particular, the foliation of hyperplanes parallel to $E^{\mathrm{ss}} \oplus E^{\mathrm{uu}}$ is preserved. Considering the *central dynamics* given by the quotient of the dynamics of the diffeomorphism f_t by these hyperplanes one gets a one-parameter family of iterated function systems (IFSs). Some properties of these IFSs are translated to properties of the diffeomorphisms f_t, see Proposition 4.9. This IFS provides relevant information about the dynamics of the the diffeomorphisms f_t such as, for example, the existence of saddle-nodes with strong homoclinic intersections. Such IFSs play a role similar to the one of the quadratic family in the setting of homoclinic bifurcations, compare [20, Chapter 6.3].

2.4. **Organization of the paper.** The discussion above corresponds to the contents in Sections 3 and 4. The key step is to analyze the dynamics of the IFSs associated to simple cycles. Using these IFSs, in Section 5 we analyze non-twisted cycles (which is the principal case) and explain how they yield saddle-nodes/flips with strong homoclinic intersections as well as further intersection properties, see Proposition 5.3. We study (twisted and non-twisted) cycles with the bi-accumulation property in Section 5.3. In Section 6 we prove Theorem 3, which is the main technical step in the paper. Finally, in Section 7 we see how Theorems 1 and 2 can be easily derived from Theorem 3.

3. Robust cycles and blenders

In this section, we recall the definition and main properties of blenders. We also state the tools to get the stabilization of heterodimensional cycles, see Proposition 3.4 and Theorem 3.5.

3.1. **Blenders.** Let us recall the definition of a cu-blender in [9]. See also the examples in [6] and the discussion in [11, Chapter 6]:

Definition 3.1 (*cu*-blender, Definition 3.1 in [9]). Let $f\colon M \to M$ be a diffeomorphism. A transitive hyperbolic compact set Γ of f with u-index$(\Gamma) = k$, $k \geq 2$, is a cu-*blender* if there are a C^1-neighborhood $\mathcal{U}$ of f and a C^1-open set $\mathcal{D}$ of embeddings of $(k-1)$-dimensional disks D into M such that for every $g \in \mathcal{U}$ and every disk $D \in \mathcal{D}$ the local stable manifold $W^s_{\mathrm{loc}}(\Gamma_g)$ of Γ_g intersects $\mathcal{D}$. The set $\mathcal{D}$ is called the *superposition* region of the blender.

Remark 3.2. Let Γ be a blender of f. Then for every g close enough to f the continuation Γ_g of Γ is a blender of g.

In fact, the cu-blenders considered in [8] to obtain robust cycles are a special class of blenders, called *blender-horseshoes,* see [9, Definition 3.8]. In this definition, the blender-horseshoe Γ is the maximal invariant set in a "cube" C and has a hyperbolic splitting with three non-trivial bundles $T_\Gamma M = E^s \oplus E^{cu} \oplus E^{uu}$, such that the unstable bundle of Γ is $E^u = E^{cu} \oplus E^{uu}$ and E^{cu} is one-dimensional. Moreover, the set Γ is conjugate to the complete shift of two symbols. Thus it has exactly two fixed points, say A and B, called *distinguished points of the blender,* and that play a special role in the definition of a blender-horseshoe.

The definition of a blender-horseshoe involves a Df-invariant strong unstable cone-field $\mathcal{C}^{uu}$ corresponding to the strong unstable direction E^{uu}, the local stable manifolds $W^s_{\mathrm{loc}}(A, f)$ and $W^s_{\mathrm{loc}}(B, f)$ of the distinguished saddles A and B (defined as the connected component of $W^s(R, f) \cap C$ containing R, $R = A$, B), and the local strong unstable manifolds $W^{uu}_{\mathrm{loc}}(A, f)$ and $W^{uu}_{\mathrm{loc}}(B, f)$ of A (the component of $W^{uu}(R, f) \cap C$ containing R). Recall that the strong unstable manifold of R is the only invariant manifold of dimension $\dim(E^{uu})$ that is tangent to E^{uu} at R.

Let $\dim(E^{uu}) = u$. One considers *vertical disks* through the blender, that is, disks Δ of dimension u tangent to the cone-field $\mathcal{C}^{uu}$ joining the "top" and the "bottom" of the cube C. Then there are two isotopy classes of vertical disks that do not intersect $W^s_{\mathrm{loc}}(A, f)$ (resp. $W^s_{\mathrm{loc}}(B, f)$), called disks at the right and at the left of $W^s_{\mathrm{loc}}(A, f)$ (resp. $W^s_{\mathrm{loc}}(B, f)$). For instance, $W^{uu}_{\mathrm{loc}}(B, f)$ (that is a vertical disk) is at the right of $W^s_{\mathrm{loc}}(A, f)$. Similarly, $W^{uu}_{\mathrm{loc}}(A, f)$ is at the left of $W^s_{\mathrm{loc}}(B, f)$. The superposition region $\mathcal{D}$ of the blender-horseshoe consists of the vertical disks in between $W^s_{\mathrm{loc}}(A, f)$ and $W^s_{\mathrm{loc}}(B, f)$ (i.e., at the right of $W^s_{\mathrm{loc}}(A, f)$ and at the left of $W^s_{\mathrm{loc}}(B, f)$). See Figure 2.
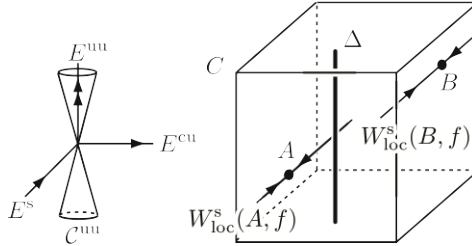

FIGURE 2. Vertical disks in a blender.

3.2. **Generation of blenders and robust cycles.** To state a criterion for the existence of robust cycles we need some definitions.

Definition 3.3. Let S be a periodic point of a diffeomorphism f.

- We say that S is a *partially hyperbolic saddle-node (resp. flip)* of f if the derivative of $Df^{\pi(S)}(S)$ has exactly one eigenvalue σ of modulus 1, the eigenvalue σ is equal to 1 (resp., -1), and there are eigenvalues λ and β of $Df^{\pi(S)}(S)$ with $|\lambda| < 1 < |\beta|$.
- Consider the strong unstable (resp. stable) invariant direction E^{uu} (resp. E^{ss}) corresponding to the eigenvalues κ of $Df^{\pi(S)}(S)$ with $|\kappa| > 1$ (resp. $|\kappa| < 1$). The *strong unstable manifold* $W^{\mathrm{uu}}(S, f)$ of S is the unique f-invariant manifold tangent to E^{uu} of the same dimension as E^{uu}. The *strong stable manifold* $W^{\mathrm{ss}}(S, f)$ of S is defined similarly considering E^{ss}.
- We say that S has a *strong homoclinic intersection* if $W^{\mathrm{ss}}(S, f) \cap W^{\mathrm{uu}}(S, f)$ contains points which do not belong to the orbit of S.

Proposition 3.4 (Criterion for robust cycles. Theorem 2.4 in [8])**.** *Let f be a diffeomorphism having a partially hyperbolic saddle-node/flip S with a strong homoclinic intersection. Then there is a diffeomorphism h arbitrarily C^1-close to f with a robust heterodimensional cycle.*

Note that this result does not provide information about the sets involved in the robust cycle. We state in Theorem 3.5 a version of this proposition providing some information about these sets. Before proving this theorem let us explain the main steps of the proof of Proposition 3.4, for further details see [8].

Sketch of the proof of Proposition 3.4. For simplicity, let us assume that S is a saddle-node of f of period one. After a perturbation, we can suppose that the saddle-node S splits into two hyperbolic fixed points S_g^- (contracting in the central direction) and S_g^+ (expanding in the central direction), here g is a diffeomorphism obtained by a small the perturbation of f. The saddles S_g^+ and S_g^- have different indices and the manifolds $W^{\mathrm{s}}(S_g^-)$ and $W^{\mathrm{u}}(S_g^+)$ have a transverse intersection that contains the interior of a "central" curve joining S_g^- and S_g^+. Note that this intersection property is C^1-robust. The proof has three steps (see Figure 3):

(I) There is a blender-horseshoe Γ_g having S_g^+ as a distinguished fixed point.

(II) The unstable manifold of S_g^- contains a vertical disk Δ in the superposition region $\mathcal{D}$ of the blender-horseshoe Γ_g. Thus, by the definition of blender-horseshoe, $W^{\mathrm{s}}(\Gamma_g, g)$ intersects $W^{\mathrm{u}}(S_g^-, g)$. Hence, as $S_g^+ \in \Gamma_g$ and $W^{\mathrm{u}}(S_g^-, g) \pitchfork W^{\mathrm{s}}(S_g^+, g) \neq \emptyset$, there is a heterodimensional cycle associated to Γ_g and S_g^-.

(III) The following properties are open ones: **i)** the continuation of the hyperbolic set Γ_g to be a blender (the elements in the definition of a blender depend continuously on g, see Remark 3.2), **ii)** $W^{\mathrm{u}}(S_g^-, g)$ to contain a vertical disk in the superposition region $\mathcal{D}$ of the blender, and **iii)** $W^{\mathrm{s}}(S_g^-, g) \pitchfork W^{\mathrm{u}}(S_g^+, g) \neq \emptyset$.

Therefore, every diffeomorphism h that is C^1-close to g has a heterodimensional cycle associated to S_h^- and Γ_h. Since g can be taken arbitrarily close to f this concludes the proof. $\square$
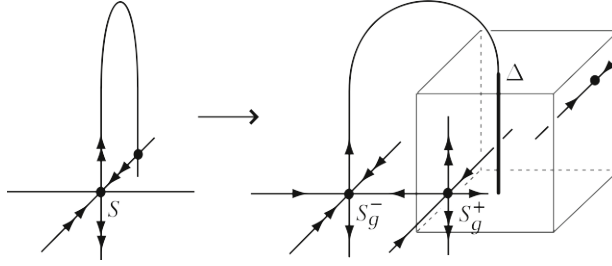
FIGURE 3. Proof of Proposition 3.4.

Next result is just a reformulation of the construction above that allows us to get robust cycles associated to sets that contain the continuations of a given saddle. This theorem will be the main tool for stabilizing cycles.

Theorem 3.5. *Let f be a diffeomorphism, P a saddle of f, and S a partially hyperbolic saddle-node/flip of f such that:*

(1) s*-index$(P) = \dim(W^{\mathrm{ss}}(S)) + 1 = s + 1$,*
(2) S *has a strong homoclinic intersection,*
(3) $W^{\mathrm{u}}(P, f) \cap W^{\mathrm{ss}}(S, f) \neq \emptyset$, *and*
(4) $W^{\mathrm{s}}(P, f) \pitchfork W^{\mathrm{uu}}(S, f) \neq \emptyset$.

Then there is a diffeomorphism h arbitrarily C^1-close to f with a robust heterodimensional cycle associated to the continuation P_h of P and a transitive hyperbolic set Γ_h containing a hyperbolic continuation S_h^+ of S of s-*index s.*

Proof. One proceeds as in the proof of Proposition 3.4, considering a perturbation h of g with saddles $S_h^{\pm}$ satisfying conditions (I) and (II) above and such that

$$W^{\mathrm{u}}(P_h, h) \pitchfork W^{\mathrm{s}}(S_h^-, h) \neq \emptyset.$$

Since $W^{\mathrm{u}}(S_h^-, h) \pitchfork W^{\mathrm{s}}(S_h^+, h) \neq \emptyset$, the inclination lemma now implies that

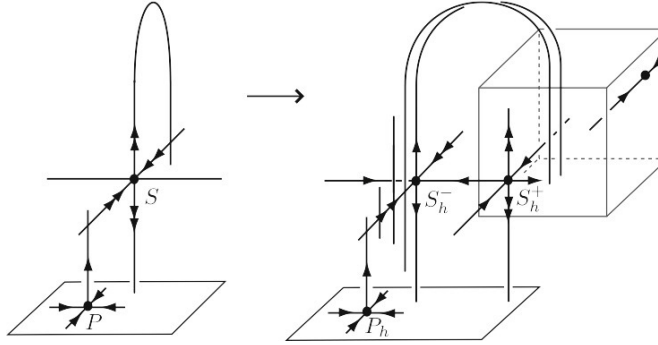$$W^{\mathrm{s}}(P_h, h) \pitchfork W^{\mathrm{u}}(S_h^+, h) \neq \emptyset,$$

see Figure 4.

FIGURE 4. Proof of Theorem 3.5.

Recall that $W^{\mathrm{u}}(S_h^-, h)$ contains a vertical disk in the superposition region of the blender Γ_h. Since $W^{\mathrm{u}}(P_h, h) \pitchfork W^{\mathrm{s}}(S_h^-, h) \neq \emptyset$, the inclination lemma implies that the same holds for $W^{\mathrm{u}}(P_h, h)$. Thus we can repeat the construction in Proposition 3.4 replacing S_h^- by P_h. Hence $W^{\mathrm{u}}(P_\varphi, \varphi)$ intersects $W^{\mathrm{s}}(\Gamma_\varphi, \varphi)$ for any diffeomorphism φ close to h. Since $W^{\mathrm{s}}(P_\varphi, h) \cap W^{\mathrm{u}}(S_\varphi^+, \varphi) \neq \emptyset$ and $S_\varphi^+ \in \Gamma_\varphi$ for every φ close to h, there is a robust heterodimensional cycles associated to P_φ and Γ_φ, ending the proof of the theorem. $\qquad\square$

4. Simple cycles and systems of iterated funtions

In this section, following [8], we introduce simple cycles (Section 4.1) and their associated one-dimensional dynamics (Section 4.3). We see that given any diffeomorphism f with a co-index one cycle with real central multipliers (associated to saddles P and Q) there is a diffeomorphism g arbitrarily C^1-close to f with a cycle associated to P and Q whose dynamics in a neighborhood of the cycle is affine, see Proposition 4.1. In such a case we say that this cycle of g is simple.

In fact, for a diffeomorphism g with a simple cycle there is a one-parameter family of diffeomorphisms $(g_t)_t$, $g_0 = g$, preserving a (semi-local) partially hyperbolic splitting $E^{\mathrm{ss}} \oplus E^c \oplus E^{\mathrm{uu}}$ such that the bundles E^{ss} and E^{uu} are non-trivial and hyperbolic (uniformly contracting and uniformly expanding, respectively) and the bundle E^c is not hyperbolic and one-dimensional. We consider the quotient dynamics by the hyperplanes $E^{\mathrm{ss}} \oplus E^{\mathrm{uu}}$, obtaining a one-parameter family of one-dimensional iteration function systems (IFSs) which describe the central dynamics of the maps g_t. Properties of these IFSs are translated to properties of the diffeomorphisms g_t, see Proposition 4.9.

In Section 5 we will write intersection properties implying the existence of robust cycles (similar to the ones in Theorem 3.5) in terms of properties of the IFSs associated to simple cycles. We now discuss simple cycles and their IFSs.

4.1. **Simple cycles.** Next proposition summarizes the results in [8] about *simple cycles* and their *unfoldings*. This proposition means that if (f_t) is a "model arc" unfolding a simple cycle then the dynamics of the maps f_t in a neighborhood of the cycle is given by suitable compositions of two linear maps (the dynamics nearby the saddles in the cycle) and two affine maps (iterations corresponding to the "transition" and the "unfolding maps").

Proposition 4.1 (Proposition 3.5 and Section 3.2 in [8]). *Let f be a diffeomorphism having a co-index one cycle with real central multipliers associated to saddles P and Q such that*

$$\mathrm{s\text{-}index}(Q) + 1 = \mathrm{s\text{-}index}(P).$$

Then there is a one-parameter family of diffeomorphisms $(f_t)_{t \in [-\epsilon, \epsilon]}$, $\epsilon > 0$, such that it satisfies properties (C1)–(C3) below and f_0 is arbitrarily close to f.

Let s and u be the dimensions of $W^{\mathrm{s}}(Q, f)$ and of $W^{\mathrm{u}}(P, f)$, respectively. There are linear maps

- $\phi_\lambda, \psi_\beta \colon \mathbb{R} \to \mathbb{R}$, $\phi_\lambda(x) = \lambda\, x$ *and* $\psi_\beta(x) = \beta\, (x)$,
- $A^{\mathrm{s}}, B^{\mathrm{s}}, T_1^{\mathrm{s}}, T_2^{\mathrm{s}} \colon \mathbb{R}^s \to \mathbb{R}^s$, *which are contractions (i.e., their norms are strictly less than one),*
- $A^{\mathrm{u}}, B^{\mathrm{u}}, T_1^{\mathrm{u}}, T_2^{\mathrm{u}} \colon \mathbb{R}^u \to \mathbb{R}^u$, *which are expansions (i.e., their inverse maps are contractions),*

such that:

(C1) *There are local charts U_P and U_Q centered at P and Q such that in these coordinates we have, for all t,*

$$
\begin{aligned}
f_t^{\pi(P)}(x^s, x^c, x^u) &= (A^s(x^s), \phi_\lambda(x^c), A^u(x^u)), \\
f_t^{\pi(Q)}(x^s, x^c, x^u) &= (B^s(x^s), \psi_\beta(x^c), B^u(x^u)),
\end{aligned}
$$

where $|\lambda| \in (0,1)$ and $|\beta| > 1$, $x^s \in \mathbb{R}^s$, $x^c \in \mathbb{R}$, and $x^u \in \mathbb{R}^u$, and $\pi(P)$ and $\pi(Q)$ are the periods of P and Q, respectively.

(C2) *There is a quasi-transverse heteroclinic point $Y_P \in W^s(Q, f_0) \cap W^u(P, f_0)$ in U_P such that, in the coordinates in the chart U_P, it holds:*

(1) For every t, $Y_P = (0^s, 0, a^u) \in W^u_{loc}(P, f_t)$, $a^u \in \mathbb{R}^u$.

(2) There is a neighborhood $C^s(Y_P)$ of Y_P in $W^s(Q, f_0) \cap U_P$ of the form $(-1,1)^s \times \{(0, a^u)\}$.

(3) There is $\tau_{p,q} \in \mathbb{N}$ such that for all t

$$
Y_{Q,t} = (a^s, t, 0^u) = f_t^{\tau_{p,q}}(Y_P) \in U_Q \cap W^u(P, f_t), \quad a^s \in \mathbb{R}^s,
$$

and

$$
Y_{Q,t} \in C^u(Y_{Q,t}) = \{(a^s, t)\} \times (-1,1)^u \subset W^u(P, f_t) \cap U_Q.
$$

(4) There is a neighborhood U_{Y_P} of Y_P, $U_{Y_P} \subset U_P$, such that

$$
\mathfrak{T}_{1,t} = f_t^{\tau_{p,q}} : U_{Y_P} \to f^{\tau_{p,q}}(U_{Y_P}) \subset U_Q
$$

is an affine map of the form

$$
\begin{aligned}
\mathfrak{T}_{1,t}(x^s, x^c, x^u) &= \mathfrak{T}_1(x^s, x^c, x^u) + (0, t, 0) \\
&= \left(T_1^s(x^s), \pm x^c, T_1^u(x^u)\right) + \left(a^s, t, -T_1^u(a^u)\right) \\
&= \left(T_1^s(x^s) + a_t^s, \theta_{1,t}(x^c), T_1^u(x^u) - T_1^u(a^u)\right).
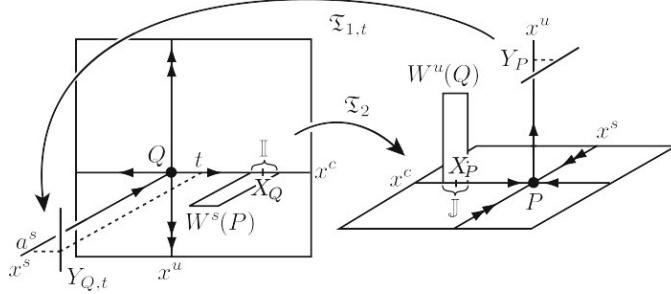\end{aligned}
$$

FIGURE 5

(C3) *For every t, there is a point $X_Q \in U_Q$ in $W^u(Q, f_t) \pitchfork W^s(P, f_t)$ (independent of t) such that, in the coordinates in the chart U_Q, it holds:*

(1) $X_Q = (0^s, 1, 0^u)$ and there is $\delta > 0$ such that

$$
X_Q \subset \mathbb{I} = \{0^s\} \times [1 - \delta, 1 + \delta] \times \{0^u\} \subset W^u(Q, f_t) \pitchfork W^s(P, f_t).
$$

(2) There is $\tau_{q,p} \in \mathbb{N}$ such that $X_P = f_t^{\tau_{q,p}}(X_Q) = (0, -1, 0) \in U_P$ and

$$
X_P \in \mathbb{J} = f_t^{\tau_{q,p}}(\mathbb{I}) = \{0^s\} \times [-1 - \delta, -1 + \delta] \times \{0^u\} \subset U_P.
$$

(3) *There is a neighborhood U_{X_Q} of X_Q, $U_{X_Q} \subset U_Q$, such that*

$$\mathfrak{T}_{2,t} = \mathfrak{T}_2 = f_t^{\tau_{q,p}} : U_{X_Q} \to f^{\tau_{q,p}}(U_{X_Q}) \subset U_P$$

is an affine map of the form

$$\mathfrak{T}_2(x^s, x^c, x^u) = \big(T_2^s(x^s), \pm(x^c - 1), T_2^u(x^u)\big) + (0^s, -1, 0^u)$$
$$= \big(T_2^s(x^s), \theta_2(x^c), T_2^u(x^u)\big).$$

According to [8, Sections 3.1-2], we give the following definition.

Definition 4.2 (Simple cycles)**.** The map f_0 in Proposition 4.1 has a *simple cycle* and $(f_t)_{t \in [-\epsilon,\epsilon]}$ is a *model unfolding family* of f_0.

- $\mathfrak{T}_{1,t}$ and $\mathfrak{T}_2$ are the *unfolding* and the *transition maps*,
- $\theta_{1,t}$ and θ_2 are the *central unfolding* and the *central transition maps*,
- $\tau_{p,q}$ and $\tau_{q,p}$ are the *unfolding* and the *transition times*,
- λ and β are the *central multipliers,* and
- $\phi_\lambda(x) = \lambda x$ and $\psi_\beta(x) = \beta x$ are the *linear central maps* of the cycle.

Remark 4.3. Since we are only interested in the dynamics in the central direction of the simple cycle, we denote the simple cycle and its unfolding model by $\mathbf{sc}(f, Q, P, \beta, \lambda, \pm_1, \pm_2)$, where the symbols $\pm_1$ and $\pm_2$ refer to the orientation preservation or reversion of the maps $\mathfrak{T}_1$ and $\mathfrak{T}_2$, respectively. These symbols coincide with the choices of $\pm$ in (C2)(4) and (C3)(3). To emphasize the unfolding and the transition times $\tau_{p,q}$ and $\tau_{q,p}$ we will write $\mathbf{sc}(f, Q, P, \beta, \lambda, \pm_1, \pm_2, \tau_{p,q}, \tau_{q,p})$.

We now state some generalizations of the simple cycles above.

4.1.1. *Simple cycles with homoclinic intersections and semi-simple cycles.* In our constructions we will consider cycles associated to saddles with non-trivial homoclinic classes. We want that some of these homoclinic intersections associated to this saddle were "detected" by the cycle and "well posed" in relation to it. This leads to the next definition.

Definition 4.4 (Simple cycles with adapted homoclinic intersections)**.** Consider a simple cycle $\mathbf{sc}(f, Q, P, \beta, \lambda, \pm_1, \pm_2)$. Write $f = f_0$ and let $(f_t)_{t \in [-\varepsilon,\varepsilon]}$ be a model unfolding family of f_0. The family $(f_t)_{t \in [-\varepsilon,\varepsilon]}$ has *adapted homoclinic intersections* (associated to P) if it satisfies conditions (C1)–(C3) in Proposition 4.1 and

(C4) In the local coordinates in U_Q, there is $\bar{a}^s \in (-1,1)^s$ such that

$$\Delta_0 = \{(\bar{a}^s, 1)\} \times [-1,1]^u \subset W^u(P, f_t), \quad \text{for every } t \text{ close to } 0.$$

This implies that $(\bar{a}^s, 1, 0)$ is a transverse homoclinic point of P of f_t for all t close to 0.

The family $(f_t)_{t \in [-\varepsilon,\varepsilon]}$ has a *sequence of adapted homoclinic intersections* (associated to P) if it satisfies conditions (C1)–(C4) and

(C5) In the local coordinates in U_Q, for every t close to 0 there are sequences

$$\bar{a}_i^s \to \bar{a}^s \quad \text{and} \quad x_i \to 1, \qquad \bar{a}_i^s \in (-1,1)^s \quad \text{and} \quad x_i \in (1-\delta, 1+\delta),$$

such that

$$\Delta_i = \{(\bar{a}_i^s, x_i)\} \times [-1,1]^u \subset W^u(P, f_t) \quad \text{for every } t \text{ close to } 0.$$

Moreover, the orbits by f_t of the disks Δ_i, $i \geq 0$, are pairwise disjoint.

As above, this implies that $(\bar{a}_i^s, x_i, 0)$ is a transverse homoclinic point of P of f_t.

In these cases, we say that f_0 has a *simple cycle with an adapted (sequence of) homoclinic intersection(s).*

Since we will consider perturbations of simple cycles, in some cases we will need to consider diffeomorphisms with "simple cycles" such that the maps ψ_β and ϕ_λ in Proposition 4.1 are not linear.

Definition 4.5 (Semi-simple cycles)**.** A diffeomorphism f has a *semi-simple cycle* associated to saddles P and Q if it satisfies Proposition 4.1 where the linear central maps ϕ_λ and ψ_β in (C1) are replaced by maps $\tilde{\phi}_\lambda, \tilde{\psi}_\beta \colon \mathbb{R} \to \mathbb{R}$ with

$$\tilde{\phi}_\lambda(0) = \tilde{\psi}_\beta(0) = 0, \quad \tilde{\phi}'_\lambda(0) = \lambda, \quad \tilde{\psi}'_\beta(0) = \beta.$$

For such a semi-simple cycle we use the notation $\mathbf{ssc}(f, Q, P, \tilde{\psi}_\beta, \tilde{\phi}_\lambda, \pm_1, \pm_2)$.

4.2. **Twisted and non-twisted cycles.** To a simple cycle $\mathbf{sc}(f, Q, P, \beta, \lambda, \pm_1, \pm_2)$ we associate signs $\mathrm{sign}(Q)$, $\mathrm{sign}(P)$, and $\mathrm{sign}(\mathfrak{T}_1)$ in $\{+, -\}$ by the following rules:

- $\mathrm{sign}(Q) = +$ if $\beta > 0$ and $\mathrm{sign}(Q) = -$ if $\beta < 0$,
- $\mathrm{sign}(P) = +$ if $\lambda > 0$ and $\mathrm{sign}(P) = -$ if $\lambda < 0$, and
- $\mathrm{sign}(\mathfrak{T}_1) = +$ if $\pm_1 = +$ (i.e., $\theta_{1,0}(x^c) = x^c$) and $\mathrm{sign}(\mathfrak{T}_1) = -$ if $\pm_1 = -$ (i.e., $\theta_{1,0}(x^c) = -x^c$).

Definition 4.6 (Twisted and non-twisted cycles)**.** We say that a simple cycle $\mathbf{sc}(f, Q, P, \beta, \lambda, \pm_1, \pm_2)$ is *twisted* if $(\mathrm{sign}(Q), \mathrm{sign}(P), \mathrm{sign}(\mathfrak{T}_1)) = (+, +, -)$. Otherwise the cycle is *non-twisted*.

A diffeomorphism f with a co-index one cycle with real central multipliers (associated to P and Q) is *twisted* (resp. *non-twisted*) if there is a diffeomorphism h arbitrarily C^1-close to f with a twisted (resp. non-twisted) simple cycle associated to P and Q.

Next lemma means that after a perturbation non-twisted cycles can be chosen satisfying $(\mathrm{sign}(Q), \mathrm{sign}(P), \mathrm{sign}(\mathfrak{T}_1)) = (\pm, \pm, +)$ (i.e., the case $(-, -, -)$ can be discarded).

Lemma 4.7. *Consider a non-twisted simple cycle* $\mathbf{sc}(f, Q, P, \beta, \lambda, \pm_1, \pm_2)$. *Then there is a diffeomorphism g arbitrarily close to f with a simple cycle associated to P and Q of such that*

$$(\mathrm{sign}(Q), \mathrm{sign}(P), \mathrm{sign}(\mathfrak{T}_1^g)) = (\pm, \pm, +).$$

This notation emphasizes that $\mathfrak{T}_1^g$ is the unfolding map of the cycle associated to g.

Proof. If $\mathrm{sign}(\mathfrak{T}_1) = +$ we are done. If $\mathrm{sign}(\mathfrak{T}_1) = -$ then the definition of non-twisted cycle implies that at least one of the central multipliers λ and β of the cycle is negative. To prove the lemma we fix a constant $K > 0$ (with $K > |\beta|^2$ and $K^{-1} < |\lambda|^2$) and replace the unfolding map $\mathfrak{T}_{1,0}$ by a composition of the form

$$(Df^{\pi(Q)})^m \circ \mathfrak{T}_{1,0} \circ (Df^{\pi(P)})^n,$$

where n and m are arbitrarily large and

$$\lambda^n \beta^m < 0 \quad \text{and} \quad K^{-1} < |\lambda^n \beta^m| < K.$$

In this way, we get a new "unfolding map" $\bar{\mathfrak{T}}_{1,0} = f^m \circ \mathfrak{T}_{1,0} \circ f^n$, defined on a small neighborhood of $f^{-n}(Y_P)$, where $Y_P \in W^s(Q, f) \cap W^u(P, f)$ is the heteroclinic

point in (C2) in Proposition 4.1. By construction, the central component $\bar{\theta}_{1,0}$ of $\bar{\mathfrak{T}}_{1,0}$ satisfies

$$\bar{\theta}_{1,0}(x^c) = -\lambda^n \beta^m x^c = |\lambda^n \beta^m| x^c.$$

Consider now the segment of orbit

$$\{f^{-n}(Y_P), \ldots, Y_P, \ldots, f^{\tau_{p,q}}(Y_P), \ldots, f^{\tau_{p,q}+m}(Y_P)\}.$$

Since n and m are arbitrarily big and $K^{-1} < |\lambda^n \beta^m| < K$, we can modify the map f along this segment of orbit to get $\bar{\theta}_{1,0}(x^c) = x^c$. This perturbation can be taken arbitrarily small if n and m are arbitrarily large. Therefore the new simple cycle is of type $(\pm, \pm, +)$. This completes the sketch of the proof of the lemma. For further details see [8, Proposition 3.5]. $\qquad\square$

4.3. Quotient dynamics. Families of iterated function systems.
In what follows, $(f_t)_{t \in [-\epsilon, \epsilon]}$ is a model unfolding family associated to a diffeomorphism $f = f_0$ with a semi-simple cycle. We use the notation in Proposition 4.1. Next remark allows us to consider (in a neighborhood of a semi-simple cycle) the quotient dynamics by the strong stable/unstable hyperplanes.

Remark 4.8. Consider a semi-simple cycle $\mathbf{ssc}(f, Q, P, \tilde{\psi}_\beta, \tilde{\phi}_\lambda, \pm_1, \pm_2)$ and its model unfolding map $(f_t)_{t \in [-\varepsilon, \varepsilon]}$, where $f_0 = f$. Consider the partially hyperbolic splitting $E^{ss} \oplus E^c \oplus E^{uu}$, defined over the orbits of P and Q, that in the local charts U_P and U_Q is of the form

$$E^{ss} = \mathbb{R}^s \times \{(0, 0^u)\}, \quad E^c = \{0^s\} \times \mathbb{R} \times \{0^u\}, \quad E^{uu} = \{(0^s, 0)\} \times \mathbb{R}^u.$$

This splitting is extended to $U_P \cup U_Q$ as constant bundles. Proposition 4.1 implies that the maps $\mathfrak{T}_{1,t}$ and $\mathfrak{T}_2$ are affine maps preserving $E^{ss} \oplus E^c \oplus E^{uu}$.

The open set V defined by

$$(4.1) \qquad V = U_P \cup U_Q \cup \left(\bigcup_{i=0}^{\tau_{q,p}} f_0^i(U_{X_Q}) \right) \cup \left(\bigcup_{i=0}^{\tau_{p,q}} f_0^i(U_{Y_P}) \right)$$

is the *neighborhood associated to the cycle*. For small t, we consider the maximal invariant set $\Lambda_t(V)$ of f_t in V,

$$\Lambda_t(V) = \bigcap_{i \in \mathbb{Z}} f_t^i(V).$$

By construction, for f_t there is a partially hyperbolic extension of the splitting $E^{ss} \oplus E^c \oplus E^{uu}$ over the set $\Lambda_t(V)$. With a slight abuse of notation, we also denote this extension by $E^{ss} \oplus E^c \oplus E^{uu}$.

This remark implies that the returns of points $X \in U_{X_Q} \cap \Lambda_t(V)$ to U_{X_Q},

$$X \in U_{X_Q} \cap \Lambda_t(V) \mapsto f_t^i(X) \in U_{X_Q},$$

preserve the codimension one foliation $\mathbb{R}^s \times \{x^c\} \times \mathbb{R}^u$ tangent to $E^{ss} \oplus E^{uu}$. We consider the "quotient dynamics" by these hyperplanes, obtaining a one parameter family of iterated function systems (IFS) defined on the interval $\mathbb{I} = [1 - \delta, 1 + \delta]$ (see item (1) in (C3) in Proposition 4.1). This family describes the "central" dynamics of these returns. We will provide in Proposition 4.9 a "dictionary" translating properties of this IFS to properties of the diffeomorphisms f_t. These properties are about the existence of periodic orbits, homoclinic and heteroclinic intersections, and cycles.

4.3.1. *Families of IFSs induced by the quotient dynamics.* Consider a semi-simple cycle $\mathbf{ssc}(f, Q, P, \psi_\beta, \phi_\lambda, \pm_1, \pm_2)$ and its model unfolding family $(f_t)_{t \in [-\epsilon, \epsilon]}$, here $f = f_0$. Consider the segment $\mathbb{I}$ in condition (C3)(1) in Proposition 4.1. For each pair (k, n) of large natural numbers and small t, define the map

(4.2) $$\Gamma_t^{k,n} \colon \mathbb{I}_t^{k,n} \to \mathbb{I}, \quad \Gamma_t^{k,n}(x) = (\psi_\beta^k \circ \theta_{1,t} \circ \phi_\lambda^n \circ \theta_2)(x),$$

where $\mathbb{I}_t^{k,n}$ is the maximal subinterval of $\mathbb{I}$ where the map $\Gamma_t^{k,n}$ is defined. Note that there are choices of k, n, t such that the set $\mathbb{I}_t^{k,n}$ is empty.

The one-parameter family $(\Gamma_t^{k,n})_{t \in [-\epsilon, \epsilon]}$ is *the IFS associated to* $(f_t)_{t \in [-\epsilon, \epsilon]}$.

4.3.2. *Dictionary IFS – Global dynamics.* Using the invariance of the spitting $E^{\mathrm{ss}} \oplus E^c \oplus E^{\mathrm{uu}}$ above one gets the following extension of [8, Proposition 3.8]:

Proposition 4.9 (Quotient dynamics – Global dynamics). *Consider a semi-simple cycle* $\mathbf{ssc}(f, Q, P, \psi_\beta, \phi_\lambda, \pm_1, \pm_2, \tau_{p,q}, \tau_{q,p})$, *its model unfolding family* $(f_t)_{t \in [-\varepsilon, \varepsilon]}$, *here* $f = f_0$, *and its associated IFS* $(\Gamma_t^{n,m})_{t \in [-\epsilon, \epsilon]}$. *Suppose that the saddles P and Q have s-indices $(s + 1)$ and s, respectively.*

(A) Periodic points: *Suppose that there is* $r \in \mathbb{I}_t^{k,n}$ *such that*

$$\Gamma_t^{k,n}(r) = r.$$

Then there are $r^s \in \mathbb{R}^s$ *and* $r^u \in \mathbb{R}^u$ *such that*

$$R = (r^s, r, r^u) \in U_Q \cap \Lambda_t(V)$$

is a periodic point of f_t of period

$$\pi(R) = k \, \pi(Q) + n \, \pi(P) + \tau_{p,q} + \tau_{q,p}.$$

The eigenvalue of $Df_t^{\pi(R)}(R)$ *corresponding to central direction* $\{0^s\} \times \mathbb{R} \times \{0^u\}$ *is*

$$\left(\Gamma_t^{k,n}\right)'(r) = \left(\psi_\beta^k\right)'\left(\theta_{1,t}(\phi_\lambda^n(\theta_2(r)))\right) \left(\phi_\lambda^n\right)'\left(\theta_2(r)\right).$$

In particular, if $\left|\left(\Gamma_t^{k,n}\right)'(r)\right| > 1$ *(resp.* < 1*) the periodic point R has s-index s (resp. s-index $s + 1$).*

Moreover, the periodic point R also satisfies

(4.3) $$W^{\mathrm{ss}}(R, f_t) \pitchfork W^{\mathrm{u}}(Q, f_t) \neq \emptyset \quad and \quad W^{\mathrm{uu}}(R, f_t) \pitchfork W^{\mathrm{s}}(P, f_t) \neq \emptyset.$$

In what follows, let r, R, and (k, n) be as in item (A).

(B) Strong homoclinic intersections: *Suppose that there is a pair* $(\bar{k}, \bar{n}) \neq (k, n)$ *such that*

$$\Gamma_t^{\bar{k}, \bar{n}}(r) = r.$$

Then $W^{\mathrm{ss}}(R, f_t) \cap W^{\mathrm{uu}}(R, f_t)$ *contains points that do not belong to the orbit of R.*

(C) Heterodimensional cycles: *Suppose that there are* $d \in \mathbb{I}$ *and* $d^s \in \mathbb{R}^s$ *such that (in the coordinates in U_Q)*

$$\Upsilon = \Upsilon(d^s, d) = \{(d^s, d)\} \times [-1, 1]^u \subset W^{\mathrm{u}}(P, f_t).$$

If there is $i \in \mathbb{N}$ *such that*

$$\theta_{1,t} \circ \phi_\lambda^i \circ \theta_2(d) = 0$$

then

$$W^{\mathrm{u}}(P, f_t) \cap W^{\mathrm{s}}(Q, f_t) \neq \emptyset.$$

Thus, as $W^{\mathrm{s}}(P, f_t) \cap W^{\mathrm{u}}(Q, f_t) \neq \emptyset$, the diffeomorphism f_t has a heterodimensional cycle associated to P and Q.

In particular, if there are $i, h \in \mathbb{N}$ such that

$$\theta_{1,t} \circ \phi_\lambda^i \circ \theta_2 \circ \psi_\beta^h(t) = \theta_{1,t} \circ \phi_\lambda^i \circ \theta_2 \circ \psi_\beta^h \circ \theta_{1,t}(0) = 0$$

then f_t has a heterodimensional cycle associated to P and Q^2.

(D) Heteroclinic intersections (I): *Suppose that there are* $i, \tilde{k}, \tilde{n} \in \mathbb{N}$ *such that*

$$\theta_{1,t} \circ \phi_\lambda^i \circ \theta_2 \circ \Gamma_t^{\tilde{k}, \tilde{n}}(r) = 0.$$

Then

$$W^{\mathrm{uu}}(R, f_t) \cap W^{\mathrm{s}}(Q, f_t) \neq \emptyset.$$

If $(\tilde{k}, \tilde{n}) = (0, 0)$ *the previous identity just means* $\theta_{1,t} \circ \phi_\lambda^i \circ \theta_2(r) = 0$.

(E) Heteroclinic intersections (II): *Let* (d^s, d) *be as in item (C) (i.e.,* $\Upsilon(d^s, d) \subset W^{\mathrm{u}}(P, f_t)$*). If there are* $i, j \in \mathbb{N}$ *such that*

$$\Gamma_t^{i,j}(d) = r$$

then

$$W^{\mathrm{u}}(P, f_t) \cap W^{\mathrm{ss}}(R, f_t) \neq 0.$$

In particular, if

    (1) either $r = d$ *and* $(i, j) = (0, 0)$,
    (2) or there is i *such that* $\psi_\beta^i \circ \theta_{1,t}(0) = \psi_\beta^i(t) = r$

then

$$W^{\mathrm{u}}(P, f_t) \cap W^{\mathrm{ss}}(R, f_t) \neq \emptyset.$$

(F) Homoclinic points: *Suppose that there is* i *such that*

$$\psi_\beta^i \circ \theta_{1,t}(0) = \psi_\beta^i(t) = \hat{h} \in [1 - \delta, 1 + \delta].$$

Then there is $\hat{h}^s \in (-1, 1)^s$ *such that* $\hat{H} = (\hat{h}^s, \hat{h}, 0^u) \in U_Q$ *is a transverse homoclinic point of* P *for* f_t *and*

$$\{(\hat{h}^s, \hat{h})\} \times [-1, 1]^u \subset W^{\mathrm{u}}(P, f_t).$$

Proof. For notational simplicity, let us assume that P and Q are fixed points.

Items (A) and (B) are stated in [8, Proposition 3.8]. To prove item (A) it is enough to observe that the definition of the pair (k, n) and the product structure provide a pair of cubes $\Delta^u \subset [-1, 1]^u$ and $\Delta^s \subset [-1, 1]^s$ such that

$$f_t^\ell \big([-1, 1]^s \times \{r\} \times \Delta^u \big) = \Delta^s \times \{r\} \times [-1, 1]^u, \quad \ell = k + n + \tau_{p,q} + \tau_{q,p},$$

if k and n are large enough (note that $k, n \to \infty$ as $t \to 0$). Note that Df_t^ℓ uniformly contracts vectors parallel to $\mathbb{R}^s \times \{(0, 0^u)\}$ and uniformly expands vectors parallel to $\{(0^s, 0)\} \times \mathbb{R}^u$. This gives the periodic point $R = (r^s, r, r^u)$ of period ℓ. Note that our arguments also imply that

(4.4) $W^{\mathrm{uu}}(R, f_t) \supset \{(r^s, r)\} \times [-1, 1]^u, \quad W^{\mathrm{ss}}(R, f_t) \supset [-1, 1]^s \times \{(r, r^u)\}.$

---

[2]In the previous expression one implicitly assumes that $\psi_\beta^h(t) \in [1 - \delta, 1 + \delta]$, otherwise one cannot apply θ_2.

Note also that from (C3)(1) in Proposition 4.1, in the coordinates in U_Q, one has that

(4.5)
$$\{0^s\} \times [1 - \delta, 1 + \delta] \times [-1, 1]^u \subset W^{\mathrm{u}}(Q, f_t), \quad \text{and}$$
$$[-1, 1]^s \times [1 - \delta, 1 + \delta] \times \{0^u\} \subset W^{\mathrm{s}}(P, f_t).$$

The intersection properties between the invariant manifolds of R, P, and Q in item (A) follow immediately from equations (4.4) and (4.5) and $r \in [1 - \delta, 1 + \delta]$.

To prove item (B) one argues exactly as in item (A). Note that the choice of $(\bar{k}, \bar{n})$ (large $\bar{k}, \bar{n}$) provides a cube $\tilde{\Delta}^u \subset [-1, 1]^u$ and a point $\tilde{r}^s \in [-1, 1]^s$ such that

$$f_t^m\big(\{(r^s, r)\} \times \tilde{\Delta}^u\big) = \{(\tilde{r}^s, r)\} \times [-1, 1]^u, \quad m = \bar{k} + \bar{n} + \tau_{p,q} + \tau_{q,p}.$$

Since $\{(r^s, r)\} \times \tilde{\Delta}^u \subset W^{\mathrm{uu}}(R, f_t)$ and $(\tilde{r}^s, r, r^u) \in W^{\mathrm{ss}}(R, f_t)$ there is a strong homoclinic intersection associated to R.

To prove the first part of item (C) note that if t is small then i is large and thus

$$\begin{aligned}
f_t^{\tau_{p,q}+i+\tau_{q,p}}(\Upsilon) \cap U_Q &= f_t^{\tau_{p,q}+i+\tau_{q,p}}\big(\{(d^s, d)\} \times [-1, 1]^u\big) \cap U_Q \\
&\supset \{(\bar{d}^s, \theta_{1,t} \circ \phi_\lambda^i \circ \theta_2(d))\} \times [-1, 1]^u \\
&= \{(\bar{d}^s, 0)\} \times [-1, 1]^u,
\end{aligned}$$

for some $\bar{d}^s \in (-1, 1)^s$. Since $[-1, 1]^s \times \{(0, 0^u)\} \subset W^{\mathrm{s}}(Q, f_t)$ and $\Upsilon \subset W^{\mathrm{u}}(P, f_t)$ we get $W^{\mathrm{u}}(P, f_t) \cap W^{\mathrm{s}}(Q, f_t) \neq \emptyset$.

To prove the second part of item (C) consider $a^s \in \mathbb{R}^s$ and the linear map B^s as in (C2)(3) and (C1) in Proposition 4.1, respectively. Note that

$$\big((B^s)^h(a^s), \psi_\beta^h(t), 0^u\big) = (\tilde{d}^s, \tilde{d}, 0^u), \quad \tilde{d} = \psi_\beta^h(t) = \psi_\beta^h \circ \theta_{1,t}(0) \in [1 - \delta, 1 + \delta]$$

is a transverse homoclinic point of P such that

$$\tilde{\Upsilon} = \{(\tilde{d}^s, \tilde{d})\} \times [-1, 1]^u \subset W^{\mathrm{u}}(P, f_t) \cap U_Q.$$

The intersection between $W^{\mathrm{u}}(P, f_t)$ and $W^{\mathrm{s}}(Q, f_t)$ now follows applying the first part of item (C) to the disk $\tilde{\Upsilon}$: just note that by hypothesis and the definition of $\tilde{d} = \psi_\beta^h \circ \theta_{1,t}(0)$ one has $\theta_{1,t} \circ \phi_\lambda^i \circ \theta_2(\tilde{d}) = 0$.

Item (D) follows similarly. Let (in the coordinates in U_Q)

$$\Delta = \{(r^s, r)\} \times [-1, 1]^u \subset W^{\mathrm{uu}}(R, f_t).$$

In the coordinates in U_Q, we have

$$\begin{aligned}
f_t^{\tau_{p,q}+i+\tau_{q,p}+\tilde{n}+\tau_{p,q}+\tilde{k}+\tau_{q,p}}(\Delta) &\supset \{(\tilde{r}^s, \theta_{1,t} \circ \phi_\lambda^i \circ \theta_2 \circ \Gamma_t^{\tilde{n},\tilde{k}}(r))\} \times [-1, 1]^u \\
&= \{(\tilde{r}^s, 0)\} \times [-1, 1]^u,
\end{aligned}$$

for some $\tilde{r}^s$. As $[-1, 1]^s \times \{(0, 0^u)\} \subset W^{\mathrm{s}}(Q, f_t)$ we get $W^{\mathrm{uu}}(R, f_t) \cap W^{\mathrm{s}}(Q, f_t) \neq \emptyset$.

The remainder assertions (E) and (F) in the proposition follow analogously, so we omit their proofs. $\qquad \square$

5. Simple non-twisted cycles

In this section we first consider non-twisted cycles and explain how these cycles yield partially hyperbolic saddle-node/flip points with strong homoclinic intersections as well as further intersection properties, see Proposition 5.3. Using Proposition 4.9 we will write these properties in terms of the IFSs associated to the cycle. We also see how these intersections are realized by perturbations (model families) of the initial cycle. These intersection properties are the main ingredient for the stabilization of cycles. Finally, in Section 5.3 we consider cycles involving a saddle with a non-trivial homoclinic class and introduce the bi-accumulation property.

5.1. **Non-twisted simple cycles with adapted homoclinic intersections.** The first step is to see that non-twisted simple cycles yield simple cycles with adapted homoclinic intersections.

Lemma 5.1. *Consider a non-twisted cycle* $\mathbf{sc}(f, Q, P, \beta, \lambda, \pm_1, \pm_2)$. *There is* g *arbitrarily* C^1-*close to* f *having a non-twisted simple cycle (associated to* Q *and* P) *with a sequence of adapted homoclinic intersections (associated to* P).

Proof. Note that by Lemma 4.7 we can assume that $\theta_{1,t}(x) = x + t$. The proof has two steps. We first perturb the cycle to get a cycle with one adapted homoclinic intersection. In the second step we perturb this new cycle with an adapted homoclinic intersection to get a cycle with a sequence of adapted homoclinic intersections.

A cycle with one adapted homoclinic intersection. Observe that, after an arbitrarily small perturbation, we can assume that the central multipliers of the cycle satisfy $\lambda^k = \beta^{-m} > 0$ for some arbitrarily large k and m. We fix small $t_k > 0$ such that

$$(5.1) \qquad\qquad t_k = \lambda^k = \beta^{-m}.$$

This choice gives

$$\psi_\beta^m\big(\theta_{1,t_k}(0)\big) = \psi_\beta^m(t_k) = 1.$$

Therefore, by (F) in Proposition 4.9, the point $H = (h^s, 1, 0) \in U_Q$ is a transverse homoclinic point of P such that

$$\{(h^s, 1)\} \times [-1, 1]^u \subset W^u(P, f_{t_k}).$$

The point H will provide the adapted homoclinic point in Definition 4.4.

To see that f_{t_k} has a cycle associated to P and Q just note that

$$(5.2) \qquad \theta_{1,t_k} \circ \phi_\lambda^k \circ \theta_2 \circ \psi_\beta^m \circ \theta_{1,t_k}(0) = \theta_{1,t_k} \circ \phi_\lambda^k \circ \theta_2(1) = -\lambda^k + t_k = 0.$$

Item (C) in Proposition 4.9 implies that $W^u(P, f_{t_k}) \cap W^s(Q, f_{t_k}) \neq \emptyset$.

Let $\tilde{Y}_P$ be the heteroclinic point in $W^u(P, f_{t_k}) \cap W^s(Q, f_{t_k})$ corresponding to the condition in (5.2). This implies that f_{t_k} has a cycle associated to P and Q and that the points $X_Q \in W^s(P, f_{t_k}) \cap W^u(Q, f_{t_k})$ (in condition (C3)(1)) and $\tilde{Y}_P \in W^u(P, f_{t_k}) \cap W^s(Q, f_{t_k})$ are heteroclinic points associated to this cycle. Using the transverse homoclinic point H of P and arguing as in Lemma 4.7, we will get a cycle with an adapted homoclinic intersection.

Indeed, repeating the previous argument we can assume that the cycle has two "adapted homoclinic points". The additional one is of the form $V = (v^s, 1 + v, v^u)$, where $1 + v \in [1 - \delta, 1 + \delta]$ (in principle $v \neq 0$) and $\Delta_V = \{(v^s, 1 + v)\} \times [-1, 1]^u \subset W^u(P, f_t)$. We also have that the disks Δ_V and $\Delta_H = \{(h^s, 1)\} \times [-1, 1]^u \subset W^u(P, f_t)$ have disjoint orbits. We use the disk Δ_V to get the sequence of adapted homoclinic intersections.

A cycle with a sequence of adapted homoclinic intersections. To get a cycle with a sequence of adapted homoclinic intersections we argue as above, but now starting with a cycle with "two adapted homoclinic intersections", say H and V as above. Let us assume that $\theta_2(1+x) = (-1+x)$. The case $\theta_2(1+x) = (-1-x)$ is analogous. As above we can assume that equation (5.1) holds for infinitely many m and k.

To get a sequence of homoclinic points H_i accumulating to H write

$$\delta_i = \beta^m \, \lambda^{2\,i} \, (1-v) > 0$$

and consider the sequence

$$\psi_\beta^m \circ \theta_{1,t_k} \circ \phi_\lambda^{2\,i} \circ \theta_2(1+v) = \psi_\beta^m \big(t_k - \lambda^{2\,i} \, (1-v) \big)$$
$$= 1 - \beta^m \, \lambda^{2\,i} \, (1-v) = 1 - \delta_i.$$

Item (F) in Proposition 4.9 implies that for each i there is h_i^s such that

(5.3) $$H_i = (h_i^s, 1 - \delta_i, 0) \in U_Q, \quad \delta_i > 0,$$

is a transverse homoclinic point of P and

$$\Delta_i = \{(h_i^s, 1 - \delta_i)\} \times [-1,1]^u \subset W^u(P, f_{t_k}).$$

This sequence accumulates to Δ_H and the disks Δ_i and Δ_H have disjoint orbits by construction.

Finally, arguing exactly as above we have that f_{t_k} has a heterodimensional cycle associated to P and Q.

Write $\tilde{f} = f_{t_k}$. We perturb $\tilde{f}$ to get a simple cycle with a sequence of adapted homoclinic intersections. Note that $\tilde{f}$ preserves the partially hyperbolic splitting $E^{ss} \oplus E^c \oplus E^{uu}$ in the neighborhood V of the initial simple cycle (recall (4.1)). For this new cycle we have "transition maps" say $\tilde{\mathfrak{T}}_{1,t_k}$ and $\tilde{\mathfrak{T}}_2$ (in principle, these maps do not satisfy all the properties of "true" transitions). These new "transitions" $\tilde{\mathfrak{T}}_{1,t_k}$ and $\tilde{\mathfrak{T}}_2$ are obtained considering compositions of the maps $\mathfrak{T}_{1,t_k}$, $\mathfrak{T}_2$, $Df^{\pi(P)}(P)$, and $Df^{\pi(Q)}(Q)$ defined for the initial cycle and replacing the heteroclinic points X_Q and $\tilde{Y}_P$ by some backward iterates of them. Note that the central maps $\tilde{\theta}_{1,0}$ and $\tilde{\theta}_2$ associated to the "new transitions" may fail to be isometries.

Now, exactly as in the proof of Lemma 4.7, we consider an arbitrarily small perturbation of $\tilde{f}$ obtained taking multiplications (in the central direction) by numbers close to one throughout long segments of the orbits of X_Q and $\tilde{Y}_P$. This is possible since t_k can be taken arbitrarily small and k and m arbitrarily big. The resulting diffeomorphism has a simple cycle with a sequence of adapted homoclinic intersections associated to P (obtained considering appropriate iterations of the points H_i and H). This completes the proof of the lemma.   □

Remark 5.2. Using equation (5.3), we can assume that in the coordinates in U_Q, the adapted transverse homoclinic points of P are such that

$$\begin{aligned} H &= (h^s, 1, 0^u) \quad \text{and} \quad \{(h^s, 1)\} \times [-1,1]^u \subset W^u(P, f), \\ H_i &= (h_i^s, \zeta_i, 0^u) \quad \text{and} \quad \{(h_i^s, \zeta_i)\} \times [-1,1]^u \subset W^u(P, f), \end{aligned}$$

where (ζ_i) is an increasing sequence converging to 1.

5.2. **Dynamics generated by non-twisted cycles.** Consider a diffeomorphism f with a simple cycle and its associated neighborhood V in (4.1). For g close to f let $\Lambda_g(V) = \cap_{i\in\mathbb{Z}} g^i(V)$ be the maximal invariant set of g in V. Note that the set $\Lambda_g(V)$ has a partially hyperbolic splitting of the form $E_g^{\mathrm{ss}} \oplus E_g^{\mathrm{c}} \oplus E_g^{\mathrm{uu}}$, where E_g^{c} is one-dimensional and E_g^{ss} and E^{uu} uniformly contracting and expanding, respectively.

Proposition 5.3. *Consider a non-twisted cycle* $\mathbf{sc}(f,Q,P,\beta,\lambda,+,\pm_2)$ *with a sequence of adapted homoclinic intersections (associated to P). Then there is a diffeomorphism g arbitrarily C^1-close to f with a partially hyperbolic saddle-node/flip $S_g \in \Lambda_g(V)$ of arbitrarily large period satisfying the following properties:*

(1) $W^{\mathrm{ss}}(S_g,g) \pitchfork W^{\mathrm{u}}(Q,g) \neq \emptyset$,
(2) $W^{\mathrm{uu}}(S_g,g) \pitchfork W^{\mathrm{s}}(P,g) \neq \emptyset$,
(3) $W^{\mathrm{u}}(P,g) \cap W^{\mathrm{ss}}(S_g,g) \neq \emptyset$,
(4) $W^{\mathrm{u}}(P,g) \cap W^{\mathrm{s}}(Q,g) \neq \emptyset$ and this intersection is quasi-transverse, and
(5) the homoclinic class of P for g is non-trivial.

Remark 5.4. Indeed, the proof of this proposition will imply that the strong unstable manifold of S_g transversely intersects the disk $[-1,1]^s \times \mathbb{I} \times \{0^u\}$ contained in $W^{\mathrm{s}}(P,g)$ in (C3)-(1) in Proposition 4.1. Now item (3) in Proposition 5.3 implies that $W^{\mathrm{u}}(P,g)$ accumulates to $W^{\mathrm{uu}}(S_g,g)$ (may be after a perturbation). Thus after a perturbation we can assume that $W^{\mathrm{u}}(P,g) \pitchfork \left([-1,1]^s \times \mathbb{I} \times \{0^u\}\right) \neq \emptyset$.

5.2.1. *Proof of Proposition 5.3.* The main step in the proof of the proposition is the next lemma about the IFS associated to a simple cycle.

Lemma 5.5. *Consider a non-twisted cycle* $\mathbf{sc}(f,Q,P,\beta,\lambda,+,\pm_2)$ *with an increasing sequence of adapted homoclinic intersections $(h_i^s,\zeta_i,0^u)$ as in Remark 5.2.*

Then there are sequences of parameters $(t_i)_i$, $t_i \to 0$, and of perturbations $\psi_{\beta,i}$ of $\psi_\beta(x) = \beta\,x$, $\psi_{\beta,i} \to \psi_\beta$, such that the IFS $\tilde{\Gamma}_{t_i}^{n,k}$ associated to ϕ_λ, $\psi_{\beta,i}$, θ_{1,t_i}, and θ_2 in equation (4.2) satisfies the following properties:

(1) There is a sequence of pairs (v_i,w_i), $v_i, w_i \to \infty$, such that

$$\tilde{\Gamma}_{t_i}^{v_i,w_i}(1) = 1,$$
$$\frac{\lambda^2}{2\,(1-|\lambda|^2)} < |(\tilde{\Gamma}_{t_i}^{v_i,w_i})'(1))| < \frac{2\,|\lambda|}{1-|\lambda|}.$$

(2) There are large j and $\ell \in \mathbb{N}$ such that

$$\theta_{1,t_i} \circ \phi_\lambda^\ell \circ \theta_2(\zeta_j) = 0.$$

(3) There are $j_0 \in \{j-1,j+1\}$ (j as in item (2)) and $\bar{n},\bar{\ell} \in \mathbb{N}$ such that

$$\Gamma_{t_i}^{\bar{n},\bar{\ell}}(\zeta_{j_0}) = 1.$$

We postpone the proof of this lemma to the next subsection.

Proof of Proposition 5.3. Note that for each t_i there is a perturbation f_i of f, $f_i \to f$ as $i \to \infty$, having a semi-simple cycle $\mathbf{ssc}(f_i,Q,P,\psi_{\beta,i},\lambda,+,\pm_2)$ "close" to the initial cycle $\mathbf{sc}(f,Q,P,\beta,\lambda,+,\pm_2)$ (i.e., we replace the linear map ψ_β by its perturbation $\psi_{\beta,i}$, while preserving the cycle configuration).

For large i, write $g = f_i$ and select the pair (v_i,w_i) in item (1) of Lemma 5.5. Let $S_g = (s^s,1,s^u)$ be the saddle associated to this pair and the central coordinate

"1" given by (A) in Proposition 4.9. By construction, the eigenvalue $\lambda_c(S_g)$ of $Dg^{\pi(S_g)}(S_g)$ corresponding to the central direction E_g^c satisfies

$$\frac{|\lambda|^2}{2\,(1-|\lambda|^2)} < |\lambda_c(S_g)| < \frac{2\,|\lambda|}{1-|\lambda|}.$$

We claim that S_g also satisfies the intersection properties in the proposition (note that in principle S_g is not yet a saddle-node/flip).

- Itens (1) and (2) in the proposition follow from equation (4.3) in item (A) of Proposition 4.9.
- Item (3) in the proposition follows from (3) in Lemma 5.5 and (E) in Proposition 4.9, where $d = \zeta_{j\pm1}$ corresponds to adapted homoclinic points (recall also Remark 5.2). Note that using these points we also get that $W^u(P,g)$ transversely intersects $[-1,1]^s \times \mathbb{I} \times \{0^u\}$, proving Remark 5.4.
- Item (4) in the proposition follows from (2) in Lemma 5.5 and (C) in Proposition 4.9, where $d = \zeta_j$ corresponds to an adapted homoclinic point.
- Since transverse homoclinic intersections persist and the saddle P has transverse homoclinic points for the diffeomorphism f, we get (5) in the proposition.

It remains to see that we can take S_g with $\lambda_c(S_g) = \pm1$. Observe that the period $\pi(S_g)$ of S_g can be taken arbitrarily large and $|\lambda_c(S_g)|$ is uniformly bounded (independent of the period). Arguing as in Lemma 4.7, we perturb g along the orbit of S_g in order to transform this point into a saddle-node (if $\lambda_c(S_g) > 0$) or a flip (if $\lambda_c(S_g) < 0$). In this way one gets a partially hyperbolic saddle-node/flip. This perturbation can be done preserving the intersection properties in the proposition. This concludes the proof of the proposition. □

5.2.2. *Proof of Lemma 5.5.* Let us first consider the case $\beta > 0$ and $\lambda > 0$.

Positive central multipliers: As above, after an arbitrarily small perturbation of the central multipliers of cycle, we can assume that there are arbitrarily large m and k with

(5.4) $$\beta^{-m} = \lambda^k\,(1-\lambda).$$

Consider the parameter $t_k = \lambda^k$. This choice gives

$$\begin{aligned}\Gamma_{t_k}^{m,k+1}(1) &= \psi_\beta^m \circ \theta_{1,t_k} \circ \phi_\lambda^{k+1} \circ \theta_2(1) = \psi_\beta^m \circ \theta_{1,t_k}(-\lambda^{k+1})\\ &= \beta^m\,(\lambda^k - \lambda^{k+1}) = \beta^m\,\lambda^k\,(1-\lambda) = 1.\end{aligned}$$

Take $(v_k, w_k) = (m, k+1)$ and note that

$$(\Gamma_{t_k}^{v_k,w_k})'(1) = \pm\beta^m\,\lambda^{k+1} = \pm\frac{\lambda}{1-\lambda}.$$

This gives (1) in the lemma. To obtain the other conditions we consider perturbations $\tilde\psi_\beta$ of ψ_β preserving the condition $\Gamma_{t_k}^{v_k,w_k}(1) = 1$. From now on we fix the parameter t_k. We first consider the case where θ_2 has derivative $+1$.

Case $\theta_2(1+x) = -1+x$: For every small enough μ, define $\beta(\mu)$ by

$$\beta(\mu)^m\,(\lambda^k(1-\lambda)+\mu) = 1$$

and consider its associated linear map $\psi_{\beta(\mu)}(x) = \beta(\mu)\,x$. Write $\phi_\lambda(x) = \lambda\,x$. Note that the IFS $\tilde{\Gamma}^{i,j}_{t_k+\mu}$ associated to ϕ_λ, $\psi_{\beta(\mu)}$, $\theta_{t_k+\mu}$, and θ_2 satisfies

$$(5.5) \qquad \tilde{\Gamma}^{m,k+1}_{t_k+\mu}(1) = \psi^m_{\beta(\mu)} \circ \theta_{1,t_k+\mu} \circ \phi^{k+1}_\lambda \circ \theta_2(1) = 1, \quad \text{for all small } \mu.$$

Thus, for $(v_k, w_k) = (m, k+1)$,

$$(5.6) \qquad (\tilde{\Gamma}^{v_k,w_k}_{t_k+\mu})'(1) = \beta(\mu)^m\, \lambda^{k+1} = \frac{\lambda}{1 - \lambda + \mu}.$$

Thus, for small μ, these derivatives also satisfy (1).

Consider ζ_i as in Remark 5.2, that is $\zeta_i = 1 - \delta_i$, $\delta_i \to 0^+$ and $\delta_i > \delta_{i+1}$. For large i define

$$(5.7) \qquad \omega_i(\mu) = \theta_{1,t_k+\mu} \circ \phi^k_\lambda \circ \theta_2(\zeta_i) = \theta_{1,t_k+\mu}(-\lambda^k - \lambda^k\,\delta_i) = \mu - \lambda^k\,\delta_i.$$

Note that

$$(5.8) \qquad \omega_{i+1}(\mu) - \omega_i(\mu) = \lambda^k\,(\delta_i - \delta_{i+1}).$$

Define small $\mu_j > 0$ by the condition

$$\omega_j(\mu_j) = 0, \qquad \mu_j = \lambda^k\,\delta_j, \qquad \lim_{j\to\infty} \mu_j \to 0.$$

By the choice of μ_j and (5.8) one has

$$\omega_{j+1}(\mu_j) = \lambda^k\,(\delta_j - \delta_{j+1}), \qquad \lim_{j\to\infty} \omega_{j+1}(\mu_j) \to 0^+.$$

In particular, $\omega_{j+1}(\mu_j)$ can be taken arbitrarily small in comparison with $\beta(\mu_j)^{-m} = \lambda^k\,(1 - \lambda) + \mu_j$. This immediately implies the following:

Fact 5.6. *Given any $N > 0$ there is large j such that $[\omega_{j+1}(\mu_j), \beta(\mu_j)^{-m}]$ contains at least N consecutive fundamental domains of $\psi_{\beta(\mu_j)}$.*

Using this fact, we get that for every large j there is a small perturbation $\tilde{\psi}_{\beta(\mu_j)}$ of the linear map $\psi_{\beta(\mu_j)}$ such that:

- $\tilde{\psi}_{\beta(\mu_j)}(x) = \psi_{\beta(\mu_j)}(x)$ if $x \in [\beta(\mu_j)^{-m-1}, 1]$.
- There is large n_j such that $\tilde{\psi}^{n_j}_{\beta(\mu_j)}(\omega_{j+1}(\mu_j)) = \beta(\mu_j)^{-m}$.
- The maps $\tilde{\psi}_{\beta(\mu_j)}$ and $\psi_{\beta(\mu_j)}$ coincide in a small neighborhood of 0.
- The size of the perturbation goes to 0 as $j \to \infty$.

Remark 5.7. Note that the first two conditions above imply that

$$(5.9) \qquad \tilde{\psi}^{n_j+m}_{\beta(\mu_j)}(\omega_{j+1}(\mu_j)) = 1.$$

Also important, note that this perturbation can be done (and we do) in such a way previous conditions (5.5), (5.6), and (5.7) are preserved.

The previous construction can be summarized as follows. Fix large k and the sequence of parameters $t_{k,j} = t_k + \mu_j$. For each large j, consider the perturbation $\tilde{\psi}_{\beta(\mu_j)}$ of ψ_β and the IFS $\tilde{\Gamma}^{\ell,n}_{t_{k,j}}$ corresponding to $\tilde{\psi}_{\beta(\mu_j)}$, ϕ_λ, $\theta_{1,t_{k,j}}$, and θ_2. Then

(i) $\tilde{\Gamma}^{v_k,w_k}_{t_{k,j}}(1) = 1$, (recall (5.5)),

(ii) $(\tilde{\Gamma}^{v_k,w_k}_{t_{k,j}})'(1) = \dfrac{\lambda}{1 - \lambda + \mu_j}$, (recall (5.6)),

(iii) $\theta_{1,t_{k,j}} \circ \phi^k_\lambda \circ \theta_2(\zeta_j) = \omega_j(\mu_j) = 0$, (recall the choice of μ_j and (5.7)), and

(iv) $\tilde{\Gamma}_{t_{k,j}}^{n_j+m,k}(\zeta_{j+1}) = \tilde{\psi}_{\beta(\mu_j)}^{n_j+m} \circ \theta_{1,t_{k,j}} \circ \phi_\lambda^k \circ \theta_2(\zeta_{j+1}) = \tilde{\psi}_{\beta(\mu_j)}^{n_j+m}(\omega_{j+1}(\mu_j)) = 1$, (recall (5.9)).

To conclude the proof the lemma in this first case (positive multipliers and $\theta_2(1+x) = -1+x$) just note that (i)–(ii) correspond to (1) in the lemma, (iii) to (2) in the lemma, and (iv) to (3) in the lemma.

Case $\theta_2(1+x) = -1-x$: We proceed as in the previous case and define the sequence $\omega_i(\mu)$ similarly. In this case, instead equation (5.7) we get

$$\omega_i(\mu) = \theta_{1,t_k+\mu} \circ \phi_\lambda^k \circ \theta_2(\zeta_i) = \theta_{1,t_k+\mu}(-\lambda^k + \lambda^k \delta_i) = \mu + \lambda^k \delta_i.$$

We define μ_j as above, $\omega_j(\mu_j) = 0$, and consider $\omega_{j-1}(\mu_j) > 0$ instead of $\omega_{j+1}(\mu_j)$. The proof now follows as above.

Non-positive central multipliers: In this case, after an arbitrarily small perturbation of the central multipliers of cycle, we can assume that there are arbitrarily large m and k with

$$(5.10) \qquad\qquad \beta^{-2m} = \lambda^{2k}(1 - \lambda^2).$$

We consider the parameter $t_k = \lambda^{2k}$. The proof now follows exactly as in the case where the multipliers are both positive considering the sequences

$$\omega_i(\mu) = \theta_{1,t_k+\mu} \circ \phi_\lambda^{2k} \circ \theta_2(\zeta_i) = \theta_{1,t_k+\mu}(-\lambda^{2k} \pm \lambda^{2k}\delta_i) = \mu \pm \lambda^{2k}\delta_i.$$

This completes the proof of Lemma 5.5. □

5.3. Cycles associated to a bi-accumulated saddles.
Given a periodic point R of f, consider the eigenvalues $\lambda_1(R), \dots, \lambda_n(R)$ of $Df^{\pi(R)}(R)$ ordered in increasing modulus and counted with multiplicity. Denote by $\mathrm{Per}^k(f)$ the set of (hyperbolic) saddles R of f of s-index k satisfying $|\lambda_{k-1}(R)| < |\lambda_k(R)| < 1$. Given such a saddle $R \in \mathrm{Per}^k(f)$, its local strong stable manifold $W^{\mathrm{ss}}_{\mathrm{loc}}(R, f)$ is well defined (recall that $W^{\mathrm{ss}}(R, f)$ is the unique invariant manifold tangent to the eigenspace associated to $\lambda_1(R), \dots, \lambda_{k-1}(R)$). Moreover, $W^{\mathrm{ss}}_{\mathrm{loc}}(R, f)$ has codimension one in $W^{\mathrm{s}}_{\mathrm{loc}}(R, f)$ and $W^{\mathrm{s}}_{\mathrm{loc}}(R, f) \setminus W^{\mathrm{ss}}_{\mathrm{loc}}(R, f)$ has $2\pi(R)$ connected components (indeed $W^{\mathrm{ss}}_{\mathrm{loc}}(R, f)$ splits each component of $W^{\mathrm{s}}_{\mathrm{loc}}(R, f)$ into two parts).

Given a saddle P of s-index $s + 1$, we consider the following subsets of $H(P, f)$:

- $\mathrm{Per}_h(H(P, f))$ is the subset of $H(P, f)$ of hyperbolic periodic points R which are homoclinically related to P (thus R also has index $(s + 1)$),
- $\mathrm{Per}_h^{s+1}(H(P, f)) = \mathrm{Per}_h(H(P, f)) \cap \mathrm{Per}^{s+1}(f)$.

Definition 5.8 (Bi-accumulation property). A saddle $R \in \mathrm{Per}^{s+1}(f)$ is s-*bi-accumulated* (by homoclinic points) if every component of $(W^{\mathrm{s}}_{\mathrm{loc}}(R, f) \setminus W^{\mathrm{ss}}_{\mathrm{loc}}(R, f))$ contains transverse homoclinic points of R.

We have the following result.

Lemma 5.9. *Let f be a diffeomorphism with a coindex one cycle associated to P and Q such that $H(P, f)$ is non-trivial. Let $\mathrm{s}\text{-index}(P) = s + 1$. Then there is g arbitrarily C^1-close to f such that*

- *there is a saddle $\bar{P}_g \in \mathrm{Per}_h^{s+1}(H(P_g, g))$ that is s-bi-accumulated and*
- *the diffeomorphism g has a cycle associated to $\bar{P}_g$ and Q_g.*

Proof. The lemma follows from [2, 13]. From [2, Proposition 2.3], if $H(P, f)$ is non-trivial then there is g arbitrarily C^1-close to f with a cycle associated to P_g and Q_g and such that $\mathrm{Per}_h^{s+1}(H(P_g, g))$ is infinite.

By [13, Lemma 3.4], if the set $\mathrm{Per}_h^{s+1}(H(P, f))$ is infinite then there is a diffeomorphism g arbitrarily C^1-close to f with a cycle associated to P_g and Q_g and such that $\mathrm{Per}_h^{s+1}(H(P_g, g))$ contains infinitely many s-bi-accumulated saddles. Pick one of these saddles $\bar{P}_g$ and note that to be bi-accumulated is a property that persists under perturbations. We can now perturb g to get h with a cycle associated to $\bar{P}_h$ and Q_h, ending the proof of the lemma. □

6. STABILIZATION OF CYCLES. PROOF OF THEOREM 3

6.1. **Stabilization of non-twisted cycles.** Next proposition is the main step to prove the stabilization of non-twisted cycles.

Proposition 6.1. *Let f be a diffeomorphism with a non-twisted cycle associated to saddles P and Q such that $\mathrm{s}\text{-}\mathrm{index}(P) = \mathrm{s}\text{-}\mathrm{index}(Q) + 1$. Then there is a diffeomorphism g arbitrarily C^1-close to f with a partially hyperbolic saddle-node/flip S_g such that:*

(1) $W^{\mathrm{ss}}(S_g, g) \pitchfork W^{\mathrm{u}}(Q_g, g) \neq \emptyset$,
(2) $W^{\mathrm{uu}}(S_g, g) \pitchfork W^{\mathrm{s}}(P_g, g) \neq \emptyset$,
(3) $W^{\mathrm{uu}}(S_g, g) \cap W^{\mathrm{ss}}(S_g, g)$ *contains a point that is not in the orbit of S_g (strong homoclinic intersection),*
(4) $W^{\mathrm{ss}}(S_g, g) \cap W^{\mathrm{u}}(P_g, g) \neq \emptyset$, *and*
(5) $W^{\mathrm{uu}}(S_g, g) \cap W^{\mathrm{s}}(Q_g, g) \neq \emptyset$.

The dynamical configuration in the proposition is depicted in Figure 6.

We postpone the proof of this proposition to Section 6.1.1. We now prove (A) in Theorem 3.
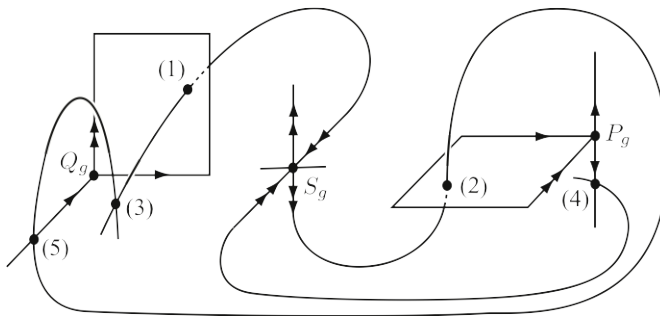

FIGURE 6. Dynamical configuration in Proposition 6.1.

6.1.1. *Proposition 6.1 implies (A) in Theorem 3.* Note that the transverse intersection conditions immediately imply that $\mathrm{s}\text{-}\mathrm{index}(P) = \dim(W^{\mathrm{ss}}(S)) + 1 = s + 1$ (condition (1) in Theorem 3.5). Moreover, conditions (2)–(4) in Proposition 6.1 imply that S and P satisfy (2)–(4) in Theorem 3.5. Thus the diffeomorphism g satisfies all conditions in Theorem 3.5 and hence there is h arbitrarily C^1-close to g having a robust heterodimensional cycle associated to P_h and a (transitive) hyperbolic set Γ_h containing a continuation S_h^+ of s-index s of S_g.

Observe that items (1) and (5) in Proposition 6.1 imply that the saddle S_h^+ of h can be chosen such that

$$W^s(S_h^+, h) \pitchfork W^u(Q_h, h) \neq \emptyset \quad \text{and} \quad W^u(S_h^+, h) \cap W^s(Q_h, h) \neq \emptyset.$$

Thus the saddles S_h^+ and Q_h are homoclinically related and then there is a transitive hyperbolic set Σ_h containing Q_h and Γ_h. In particular, for every diffeomorphism φ close to h it holds $W^{s,u}(\Gamma_\varphi, \varphi) \subset W^{s,u}(\Sigma_\varphi, \varphi)$. Thus, by the first step of the proof, the diffeomorphism h has a robust cycle associated to Σ_h and P_h, ending the proof of (A) in Theorem 3.   □

6.1.2. *Proof of Proposition 6.1.* This proposition follows from Proposition 5.3. First note that by Lemma 5.1, after a small perturbation, we can assume that the cycle (associated to P and Q) has a sequence of adapted homoclinic intersections associated to the saddle P. Thus applying Proposition 5.3 we obtain g close to f with a partially hyperbolic saddle-node/flip satisfying conditions (1), (2), and (4) in Proposition 6.1. It remains to obtain conditions (3) ($W^{uu}(S_g, g) \cap W^{ss}(S_g, g)$ contains a point that is not in the orbit of S_g) and (5) ($W^{uu}(S_g, g) \cap W^s(P_g, g) \neq \emptyset$) in Proposition 6.1. To get these two properties we use arguments analogous to the ones in Lemmas 5.1 and 5.5.

Since in what follows we do not modify the orbits of P_g, Q_g, and S_g let us omit the dependence on g. Note that since $W^{uu}(S, g) \pitchfork W^s(P, g)$ (condition (2) in Proposition 5.3) we have that $W^{uu}(S, g)$ accumulate to $W^u(P, g)$. Since by condition (4) in Proposition 5.3 we have that $W^u(P, g) \cap W^s(Q, g) \neq \emptyset$, thus $W^{uu}(S, g)$ also accumulates to $W^s(Q, g)$. In particular there are segments of $W^{uu}(S, g)$ (with disjoint orbits) arbitrarily close to $W_{loc}^s(Q, g)$. We use one of these segments to get $W^{uu}(S, h) \cap W^s(Q, h) \neq \emptyset$ for some h close to g (condition (5) in Proposition 6.1).

Moreover, the previous perturbation can be done in such a way there are segments of $W^{uu}(S, h)$ close to $W^s(Q, h)$ in the "same side" of $W^s(Q, h)$ as $W^{ss}(S, h)$. See Figures 7 and 8. Thus modifying the derivative of Q in the central direction we get that $W^{uu}(S, h)$ intersects $W^{ss}(S, h)$ (condition (3) in Proposition 6.1). Note that these perturbations can be done preserving the saddle-node/flip S and the intersections properties (1), (2), and (4) in Proposition 6.1.   □
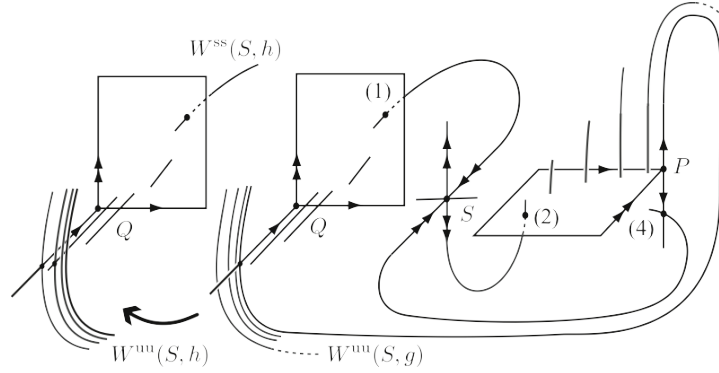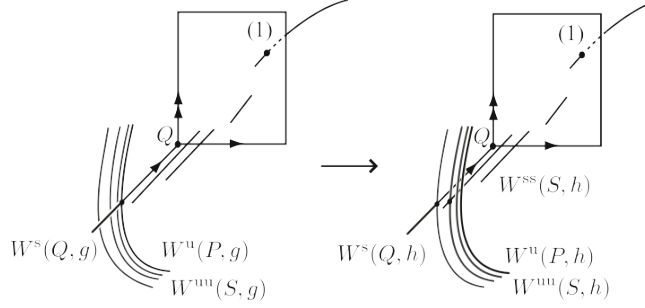

FIGURE 7. Accumulation of $W^{uu}(S)$ to $W_{loc}^s(Q)$.

FIGURE 8. Accumulation of $W^{\mathrm{uu}}(S)$ to $W^s_{\mathrm{loc}}(Q)$.

6.2. **Stabilization of bi-accumulated twisted cycles.** In this section we prove item (B) in Theorem 3.

Proposition 6.2 (Generation of non-twisted cycles). *Let f be a diffeomorphism with a twisted cycle associated to saddles P and Q with $\mathrm{s}\text{-}\mathrm{index}(P) = \mathrm{s}\text{-}\mathrm{index}(Q) + 1$. Assume that P is s-bi-accumulated. Then there is g arbitrarily C^1-close to f with a non-twisted cycle associated to Q_g and a saddle R_g that is homoclinically related to P_g.*

Item (A) in Theorem 3 implies that the cycle associated to R_g and Q_g can be stabilized. Since R_g is homoclinically related to P_g, Lemma 2.3 implies that the cycle associated to P_f and Q_f can also be stabilized. Thus Proposition 6.2 implies (B) in Theorem 3.

6.2.1. *Proof of Proposition 6.2.* The proposition is an immediate consequence of the following two lemmas:

Lemma 6.3. *Under the hypotheses of Proposition 6.2, there is g arbitrarily C^1-close to f with a twisted simple cycle associated to P and Q and with an adapted homoclinic point of P.*

Lemma 6.4. *Consider a twisted cycle $\mathbf{sc}(f, Q, P, \beta, \lambda, -, \pm_2)$, $\lambda, \beta > 0$, with an adapted homoclinic intersection (associated to P). Then there is g arbitrarily C^1-close to f with a saddle R_g such that*

- *R_g is homoclinically related to P_g and*
- *g has a non-twisted cycle associated to R_g and Q_g.*

6.2.2. *Proof of Lemma 6.3.* We claim that (in the coordinates in U_Q in Proposition 4.1) there are sequences of points $(x_i)_i$ and $(a^s_i)_i$, $x_i \in \mathbb{R}$ and $a^s_i \in \mathbb{R}^s$, and of disks Δ_i of dimension u such that

- $(a^s_i, x_i, 0) \in \Delta_i$ where $x_i \to 0^+$ and $a^s_i \to a^s$, and
- $\Delta_i \to \{(a^s, 0)\} \times [-1, 1]^u$ and $\Delta_i \subset W^{\mathrm{u}}(P, f)$,

here $(a^s, 0, 0^u)$ is the heteroclinic intersection between $W^{\mathrm{u}}(P, f)$ and $W^{\mathrm{s}}(Q, f)$ in (C2) in Proposition 4.1.

To see why this assertion is so just note that, by the bi-accumulation property, there is as sequence of unstable disks $\tilde{\Delta}_i \subset W^{\mathrm{u}}(P, f)$ of dimension u approaching to $W^{\mathrm{u}}_{\mathrm{loc}}(P, f)$ from the "negative side", see Figure 9. Since the cycle is twisted the map $\mathfrak{T}_{1,0}$ reverses the ordering in the central direction. Thus these disks are
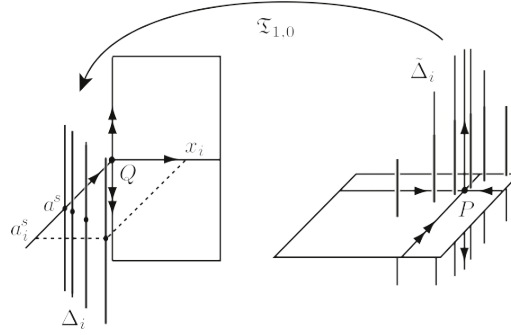
FIGURE 9. The disks Δ_i.

mapped by $\mathfrak{T}_{1,0}$ into a disks Δ_i that approaches $(a^s, 0, 0^u)$ from the "positive side". See Figure 9. We need to perform a perturbation in order to put these disks in "vertical" position.

Arguing exactly as in the proof of Lemma 5.5, after an arbitrarily small perturbation we can assume that β is such that $\psi_\beta^{k_i}(x_i) = 1$ for some arbitrarily large i and k_i. This provides a transverse homoclinic point of P of the form $(h^s, 1, 0)$. This follows from (F) in Proposition 4.9. Note that this perturbation can be done preserving the cycle between P and Q.

Finally, using this transverse homoclinic point and after an arbitrarily small perturbation, we get the simple cycle with an adapted homoclinic intersection associated to P and Q (the argument is similar to the one in Lemma 4.7.) □

6.3. **Proof of Lemma 6.4.** The lemma follows arguing as in [8, Lemma 3.13] and using Proposition 4.9. Note that we can assume (after a small modification of β and λ) that $\beta^{-m} = \lambda^k$. Noting that the cycle is twisted (i.e., $\theta_{1,t}(x) = t - x$) we have that this equality implies that

$$\Gamma_0^{m,k}(1) = \psi_\beta^m \circ \theta_{1,0} \circ \phi_\lambda^k \circ \theta_2(1) = \psi_\beta^m\big(-\phi_\lambda^k(-1)\big) = 1.$$

In this case we also have,

$$(\Gamma_0^{m,k})'(1) = (\psi_\beta^m)'\big(-\phi_\lambda^k(-1)\big)(\phi_\lambda^k)'(-1) = \pm\beta^m\,\lambda^k = \pm 1.$$

Thus modifying the central derivatives at P and Q, we can assume that the cycle is semi-simple with central maps $\tilde{\psi}_\beta$ and $\tilde{\phi}_\lambda$ such that there are large m, k, and ℓ, with $\ell >> k$, satisfying

$$(6.1) \qquad\qquad \tilde{\psi}_\beta^m\big(-\tilde{\phi}_\lambda^k(-1) + \tilde{\phi}_\lambda^\ell(-1)\big) = 1$$

and

$$(6.2) \qquad\qquad |(\tilde{\psi}_\beta^m)'\big(-\tilde{\phi}_\lambda^k(-1) + \tilde{\phi}_\lambda^\ell(-1)\big)(\tilde{\phi}_\lambda^k)'(-1)| < 1.$$

For that note that $\tilde{\phi}_\lambda^\ell(-1)$ is arbitrarily small in comparison with $\tilde{\phi}_\lambda^k(-1)$.

We now select the parameter $t = \tilde{\phi}^\ell(-1) < 0$. By equation (6.1) one has

$$\tilde{\Gamma}_t^{m,k}(1) = \tilde{\psi}_\beta^m \circ \theta_{1,t} \circ \tilde{\phi}_\lambda^k \circ \theta_2(1) = \tilde{\psi}_\beta^m \circ \theta_{1,t}\big(\tilde{\phi}_\lambda^k(-1)\big)$$
$$= \tilde{\psi}_\beta^m\big(-\tilde{\phi}_\lambda^k(-1) + \tilde{\phi}_\lambda^\ell(-1)\big) = 1.$$

Let $R = (r^s, 1, r^u) \in U_Q$ be the saddle of f_t associated to 1 and the itinerary (m, k) given by (A) in Proposition 4.9. Note that

$$|(\tilde{\Gamma}_t^{m,k})'(1)| = |(\tilde{\psi}_\beta^m)'(\theta_{1,t}(\tilde{\phi}_\lambda^k(\theta_2(1)))) (\tilde{\phi}_\lambda^k)'(\theta_2(1))|$$
$$= |(\tilde{\psi}_\beta^m)'(\tilde{\phi}_\lambda^k(-1) + \tilde{\phi}_\lambda^\ell(-1)) (\tilde{\phi}_\lambda^k)'(-1)| < 1,$$

where the inequality follows from (6.2). By (A) in Proposition 4.9 the saddle R has index $s + 1$. Indeed, since $\theta_{1,t}(x) = -x + t$, the central multiplier of R is positive if θ_2 reverses the orientation and negative otherwise.

We claim that the saddle R is homoclinically related to P and has a cycle associated to Q. Note that $W^{uu}(R, f_t) = W^u(R, f_t)$.

By equation (4.3) in Proposition 4.9 we have that

(6.3) $W^s(R, f_t) \pitchfork W^u(Q, f_t) \neq \emptyset$ and $W^u(R, f_t) \pitchfork W^s(P, f_t) \neq \emptyset$.

From the existence of an adapted homoclinic intersection and item (E)(1) in Proposition 4.9:

- $H = (h^s, 1, 0)$ is a transverse homoclinic point of P,
- $\{(h^s, 1)\} \times [-1, 1]^u \subset W^u(P, f_t) \cap U_Q$, and
- $\{[-1, 1]^s \times \{(1, r^u)\} \subset W^{ss}(R, f_t)$.

This implies that $W^u(P, f_t) \pitchfork W^s(R, f_t)$. Thus, by the second part of (6.3), the saddles P and R are homoclinically related for f_t.

To get cycle associated to R and Q note that the choice of t implies that

$$\theta_{1,t} \circ \tilde{\phi}_\lambda^\ell \circ \theta_2(1) = -\tilde{\phi}_\lambda^\ell(-1) + t = 0.$$

Since $R = (r^s, 1, r^u)$, condition (D) in Proposition 4.9 implies that $W^u(R, f_t) \cap W^s(Q, f_t) \neq \emptyset$. Thus by the first part of (6.3) the diffeomorphism f_t has a cycle associated to R and Q.

We claim that this cycle is non-twisted. If θ_2 reverses the orientation then the central multiplier of R is negative and the cycle is non-twisted. Otherwise, we have a cycle whose central "unfolding map" is obtained considering the composition $\theta_{1,t} \circ \tilde{\phi}_\lambda \circ \theta_2$. This map preserves the central orientation: just note that $\theta_{1,t}$ and θ_2 both reverse the orientation and $\tilde{\phi}_\lambda$ preserves this orientation (recall that $\lambda > 0$). This completes the proof of the lemma. $\square$

7. Proof of Theorems 1 and 2

7.1. **Proof of Theorem 2.** Note that (B) in Theorem 2 is an immediate consequence from (A) in Theorem 3.

To prove item (A) let us assume that, for instance, the saddle P has non-real central multipliers. By Theorem 2.1 (see also Remark 2.2) there is g close to f having saddles P_g' and Q_g' such that

- there is a cycle with real central multipliers associated to P_g' and Q_g',
- P_g' and Q_g' are homoclinically related to P_g and Q_g,
- the homoclinic class of P_g' is non-trivial (note that we may have $Q_g' = Q_g$ and a trivial homoclinic class $H(Q_g, g)$).

By Lemma 2.3 it is enough to prove that this new cycle can be stabilized.

If the cycle associated to P_g' and Q_g' is non-twisted the stabilization follows from (A) in Theorem 3. Otherwise, if the cycle is twisted, by Lemma 5.9 there is a diffeomorphism h close to g having a saddle $\bar{P}_h$ such that

- $\bar{P}_h$ is homoclinically related to P'_h and has the bi-accumulation property,
- there is a cycle associated to Q'_h and $\bar{P}_h$. Note that this cycle has real central multipliers.

As above, it is enough to prove that this cycle can be stabilized. The stabilization of this cycle follows from Theorem 3. This ends the proof of the theorem. $\qquad\square$

7.2. Proof of Theorem 1.

By Theorem 2.1 and Lemma 2.3 we can assume that the cycle associated to the saddles P and Q has real central multipliers and that, for instance, the homoclinic class of P is non-trivial. If the cycle is non-twisted the result follows from (A) in Theorem 3.

Otherwise, if the cycle is twisted, arguing as in the proof of Theorem 2, there is a diffeomorphism g close to f having a cycle associated to Q_g and to a saddle $\bar{P}_g$ that is homoclinically related to P_g and satisfies the s-bi-accumulation property. By (B) in Theorem 3 this cycle can be stabilized. Since $\bar{P}_g$ is homoclinically related to P_g the initial cycle also can be stabilized, ending the proof of the theorem. $\qquad\square$

7.3. Proof of Corollary 1.

This result follows immediately from Theorem 1 considering the following perturbation of the initial cycle. First, we preserve one of the heterocinic orbits in $W^{\mathrm{u}}(P, f) \cap W^{\mathrm{s}}(Q, f)$. We can also assume that $W^{\mathrm{s}}(P, f)$ transversely intersects $W^{\mathrm{u}}(Q, f)$ and thus accumulates to $W^{\mathrm{s}}(Q, f)$. We can now use the second heteroclinic orbit in $W^{\mathrm{u}}(P, f) \cap W^{\mathrm{s}}(Q, f)$ to get a transverse homoclinic point of P. In this way we obtain a cycle satisfying Theorem 1.

ACKNOWLEDGMENTS

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