THE INFINITE DIMENSIONAL MANIFOLD OF HÖLDER EQUILIBRIUM PROBABILITIES HAS NON-NEGATIVE CURVATURE

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Abstract. Here we consider the discrete time dynamics described by a transformation $T : M \to M$, where T is either the action of shift $T = \sigma$ on the symbolic space $M = \{1, 2, ..., d\}^{\mathbb{N}}$, or, T describes the action of a d to 1 expanding transformation $T : S^1 \to S^1$ of class $C^{1+\alpha}$ (for example $x \to T(x) = d\,x$ (mod 1)), where $M = S^1$ is the unitary circle.

It is known that the infinite dimensional manifold $\mathcal{N}$ of Hölder equilibrium probabilities is an analytical manifold and carries a natural Riemannian metric. Given a certain normalized Hölder potential A denote by $\mu_A \in \mathcal{N}$ the associated equilibrium probability. The set of tangent vectors X (a function $X : M \to \mathbb{R}$) to the manifold $\mathcal{N}$ at the point μ_A coincides with the kernel of the Ruelle operator for the normalized potential A. The Riemannian norm $|X| = |X|_A$ of the vector X, which is tangent to $\mathcal{N}$ at the point μ_A, is described via the asymptotic variance, that is, satisfies

$|X|^2 \;=\; \langle X, X\rangle \;=\; \lim_{n\to\infty} \frac{1}{n} \int (\sum_{i=0}^{n-1} X \circ T^i)^2 \, d\mu_A.$

Given two unitary tangent vectors to the manifold $\mathcal{N}$ at μ_A, denoted by X and Y, we will show that the sectional curvature $K(X, Y)$ is always non-negative. In our proof for the above expression for the curvature it is necessary in some moment to show the existence of geodesics for such Riemannian metric.

1. Introduction

We denote by $T : M \to M$ a transformation acting on the metric space M, which is either the shift σ acting on $M = \{1, 2, ..., d\}^{\mathbb{N}}$, or, T is the action of a d to 1 expanding transformation $T : S^1 \to S^1$, of class $C^{1+\alpha}$, where $M = S^1$ is the unitary circle.

For a fixed $\alpha > 0$ we denote by Hol the set of α-Hölder functions on M.

For a Hölder potential $A : M \to \mathbb{R}$ we define the Ruelle operator (sometimes called transfer operator) - which acts on Hölder functions $f : M \to \mathbb{R}$ - by

$$f \to \mathscr{L}_A f(x) = \sum_{T(y)=x} e^{A(y)} f(y) \tag{1}$$

It is known (see for instance [15] or [1]) that $\mathscr{L}_A$ has a positive, simple leading eigenvalue λ_A with a positive Hölder eigenfunction h_A. Moreover, the dual operator acting on measures $\mathscr{L}_A^*$ has a unique eigenprobability ν_A which is associated to the same eigenvalue λ_A.

Given a Hölder potential A we say that the probability μ_A - defined on the Borel sigma-algebra of M - is the equilibrium probability for A, if μ_A maximizes the

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values

$$h(\mu) + \int A \;\, d\mu,$$

among Borel T-invariant probabilities μ and where $h(\mu)$ is the Kolmogorov-Sinai entropy of μ.

It is known that the probability μ_A is unique and is given by the expression $\mu_A = h_A\,\nu_A$.

In some particular cases the equilibrium probability (also called Gibbs probability) μ_A is the one observed on the thermodynamical equilibrium in the Statistical Mechanics of the one dimensional lattice $\mathbb{N}$ (under an interaction described by the potential A). As an example (where the spin in each site of the lattice $\mathbb{N}$ could be $+$ or $-$) one can take $M = \{+,-\}^{\mathbb{N}}$, $A : M \to \mathbb{R}$ and T is the shift.

We say that a Hölder potential A is normalized if $\mathscr{L}_A\, 1\; = 1$. In this case $\lambda_A = 1$ and $\mu_A = \nu_A$.

We say that two potentials A, B in Hol are cohomologous to each other (up to a constant), if there exists a continuous function $g : M \to \mathbb{R}$ and a constant c, such that,

$$A = B + g - g \circ T - c. \tag{2}$$

Note that the equilibrium probability for A, respectively B, is the same, if A and B are coboundary to each other. In each coboundary class (an equivalence relation) there exists a unique normalized potential A (see [15]). Therefore, the set of equilibrium probabilities for Hölder potentials $\mathcal{N}$ can be indexed by Hölder potentials A which are normalized. We will use this point of view here: $A \leftrightarrow \mu_A$.

The infinite dimensional manifold $\mathcal{N}$ of Hölder equilibrium probabilities μ_A is an analytical manifold (see [17], [12], [15], [6]) and it was shown in [13] that it carries a natural Riemannian structure. We will recall some definitions and properties described on [13].

The set of tangent vectors X (a function $X : M \to \mathbb{R}$) to $\mathcal{N}$ at the point μ_A coincides with the kernel of $\mathscr{L}_A$. The Riemannian norm $|X| = |X|_{\mu_A}$ of the vector X, which is tangent to $\mathcal{N}$ at the point μ_A, is described (see Theorem D in [13]) via the asymptotic variance, that is, satisfies

$$|X| \;=\; \sqrt{\langle X, X\rangle} \;=\; \sqrt{\lim_{n\to\infty} \frac{1}{n} \int (\sum_{j=0}^{n-1} X \circ T^j)^2 \, d\mu_A} \tag{3}$$

The associated bilinear form on the tangent space at the point μ_A can be described (see Theorem D in [13]) by

$$\langle\, X \,,\, Y \,\rangle \;=\; \int X\, Y \, d\mu_A. \tag{4}$$

This bilinear form is positive semi-definite and in order to make it definite one can consider equivalence classes (cohomologous up to a constant) as described by Definition 5.4 in [13]. In this way we finally get a Riemannian structure on $\mathcal{N}$ (as anticipated some paragraphs above). Elements X on the tangent space at μ_A have the property $\int X \, d\mu_A = 0$.

The tangent space to $\mathcal{N}$ at μ_A is denoted by $T_A \mathcal{N}$.

Our main result is Theorem 4.1 which claims:

Theorem 1.1. *Given two unitary orthogonal vectors X, Y tangent to $\mathcal{N}$ at the point μ_A we have that the sectional curvature $K(X, Y)$ is nonnegative.*

The explicit expression for the curvature is given by (19), Theorem 4.1.

We point out that section 8 in [13], which considers a simplified model for potentials that depend just on two coordinates on the symbolic space $\{1, 2\}^{\mathbb{N}}$, there was an indication that the curvature should be non negative. The curvature on this case can be obtained explicitly and one can check that there is no upper bound for the curvature.

We will show in section 5 the existence of geodesics for such Riemannian metric will substantially simplify the calculation of the sectional curvatures.

An important tool which will be used here is item (iv) on Theorem 5.1 in [13]: for all normalized $A \in \mathcal{N}$, $X \in T_A\mathcal{N}$ and φ a continuous function it holds:

$$\frac{\mathrm{d}}{\mathrm{d}t} \int \varphi \, d\mu_{A+tX}\bigg|_{t=0} = \int \varphi \, X \, d\,\mu_A. \tag{5}$$

In [14] , [3] and [16] the authors consider a similar kind of Riemannian structure. The bilinear form considered in [14] is the one we consider here divided by the entropy of μ_A. As mentioned in section 8 in [13] in this case the curvature can be positive and also negative in some parts.

The main motivation for the results obtained on [14] (and also [3]) is related to the study of a particular norm on the Teichmüller space.

A reference for general results in infinite dimensional Riemannian manifolds is [2].

In section 6 in [13] it is explained that the Riemannian metric considered here is not compatible with the 2-Wasserstein Riemannian structure on the space of probabilities.

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General references for analyticity (and moreover, inverse function theorems and implicit function theorems) in Banach spaces are [6] and [18].

2. Preliminaries of Riemannian geometry

Let us introduce some basic notions of Riemannian geometry. Given an infinite dimensional C^∞ manifold $(\tilde{M}, g)$ equipped with a smooth Riemannian metric g, let $T\,\tilde{M}$ be the tangent bundle and $T_1\tilde{M}$ be the set of unit norm tangent vectors of $(\tilde{M}, g)$, the unit tangent bundle. Let $\chi(\tilde{M})$ be the set of C^∞ vector fields of $\tilde{M}$.

In [2] several results for Riemannian metrics on infinite dimensional manifolds are presented. We will not use any of the results of that paper.

The only infinite dimensional manifold we will be interested here is $\mathcal{N}$ which is the set of Hölder equilibrium probabilities (which was initially defined in [13]). Tangent vectors, differentiability, analyticity, etc, should be always considered in the sense of the setting described in sections 2.3 and 5.1 in [13] (see also [5] and [12]). We will elaborate on this later.

In our case, where $\tilde{M} = \mathcal{N}$ and g is the L^2 metric $g_A(X, Y) = \int X\, Y d\mu_A$,

For practical purposes, we shall call *Energy* the function $E(v) = g(v, v)$, $v \in T\mathcal{N}$, although in mechanics the energy is rather defined by $\frac{1}{2} g(v, v)$.

Given a smooth function $f : \mathcal{N} \longrightarrow \mathbb{R}$, the derivative of f with respect to a vector field $X \in \chi(\mathcal{N})$ will be denoted by $X(f)$. The Lie bracket of two vector fields $X, Y \in \chi(\mathcal{N})$ is the vector field whose action on the set of functions $f : \mathcal{N} \longrightarrow \mathbb{R}$ is given by $[X,Y](f) = X(Y(f)) - Y(X(f))$.

The *Levi-Civita connection* of $(\mathcal{N}, g)$, $\nabla : \chi(\mathcal{N}) \times \chi(\mathcal{N}) \longrightarrow \chi(\mathcal{N})$, with notation $\nabla(X,Y) = \nabla_X Y$, is the affine operator characterized by the following properties:

(1) Compatibility with the metric g:
$$Xg(Y,Z) = g(\nabla_X Y, Z) + g(Y, \nabla_X Z)$$
for every triple of vector fields X, Y, Z.

(2) Absence of torsion:
$$\nabla_X Y - \nabla_Y X = [X,Y].$$

(3) For every smooth scalar function f and vector fields $X, Y \in \chi(\mathcal{N})$ we have
   - $\nabla_{fX} Y = f \nabla_X Y$,
   - Leibniz rule: $\nabla_X(fY) = X(f)Y + f\nabla_X Y$.

The expression of $\nabla_X Y$ can be obtained explicitly from the expression of the Riemannian metric, in dual form. Namely, given two vector fields $X, Y \in \chi(\mathcal{N})$, and $Z \in \chi(\mathcal{N})$ we have

$$g(\nabla_X Y, Z) = \frac{1}{2}(Xg(Y,Z) + Yg(Z,X) - Zg(X,Y) \tag{6}$$
$$- \quad g([X,Z],Y) - g([Y,Z],X) - g([X,Y],Z)), \tag{7}$$

A smooth curve $\gamma : (a,b) \longrightarrow \mathcal{N}$ is called a *geodesic* of $(\mathcal{N}, g)$ if $\nabla_{\gamma'(t)} \gamma'(t) = 0$ for every $t \in (a,b)$. If $\tilde{M}$ is finite dimensional, in any coordinate system the equation of geodesics gives rise to a second order, ordinary differential equation, so given any initial condition (p,v) in $T_1\tilde{M}$ there exists a unique solution $\gamma_{(p,v)}(t)$ such that $\gamma_{(p,v)}(0) = p$, $\gamma'_{(p,v)}(0) = v$. If $\tilde{M}$ is infinite dimensional, the existence of geodesics is a nontrivial issue that is usually tackled with the so-called Palais-Smale condition (see for instance [11]).

We shall show in the last section that:

Theorem 2.1. *Given $A \in \mathcal{N}$, $X \in T_A\mathcal{N}$, there exist $\rho > 0$ and a unique geodesic $\gamma : (-\rho, \rho) \longrightarrow \mathcal{N}$ such that $\gamma(0) = A$, $\gamma'(0) = X$.*

Although we won't show the Palais-Smale condition for $\mathcal{N}$, we shall show that the manifold $(\mathcal{N}, \langle , \rangle)$ has enough compactness to ensure the existence of geodesics provided that $T_A\mathcal{N}$ has a countable basis (as a Banach space). This is the case of normalized potentials of the expanding map $T(x) = 2x(mod.1)$ in S^1 (of course) and for the shift of two symbols (see for instance [10]).

Once we have geodesics we can solve the equation of parallel transport.

Theorem 2.2. *Under the assumptions of Theorem 2.1, given a unit vector $Y \in T_A\mathcal{N}$ there exists a unique smooth vector field $Y(t) \in T_{\gamma(t)}\mathcal{N}$, $t \in (-\epsilon, \epsilon)$, such that $Y(0) = Y$ and*
$$\nabla_{\gamma'(t)} Y(t) = 0, \tag{8}$$
for every $t \in (-\epsilon, \epsilon)$. This vector field is the parallel transport of Y along $\gamma(t)$.

The proof of this theorem is postponed to the last section, it is actually a consequence of the proof of the existence of geodesics.

2.1. **Fermi coordinates.** A parametrized local surface $S : (-\epsilon,\epsilon)\times(-\delta,\delta) \longrightarrow \mathcal{N}$, with parameters $S(t,s)$, is given in *Fermi* coordinates if

(1) $S(t,0)=\gamma(t)$ is a geodesic,
(2) The vector field $\frac{\partial S(t,0)}{\partial s}$ is parallel along $\gamma(t)$ and is perpendicular to $\gamma'(t)$,
(3) The curves $S_t(s) = S(t,s)$, $s \in (-\delta,\delta)$ are geodesics for each given $t \in (-\epsilon,\epsilon)$.

As a consequence of Theorems 2.1 and 2.2 we have

Proposition 2.3. *Given $A \in \mathcal{N}$, $X,Y \in T_A\mathcal{N}$ with unit norms, there exists a local surface $S : (-\epsilon,\epsilon)\times(-\delta,\delta) \longrightarrow \mathcal{N}$ parametrized in Fermi coordinates such that $S(t,0)=\gamma(t)$ is a geodesic with $\gamma(0)=A$, $\frac{\partial S(0,0)}{\partial s} = Y$, where $\gamma'(0)=X$.*

Proof. The proof goes as for Riemannian manifolds of finite dimensions. Let $X \in T_A\mathcal{N}$ with unit norm, let $\gamma(t)$ be the geodesic whose initial conditions are $\gamma(0)=A$, $\gamma'(0)=X$. Given $Y \in T_A\mathcal{N}$ with unit norm such that $\langle X,Y\rangle = 0$, let $Y(t)$ be the parallel transport of Y along $\gamma(t)$. It is clear that $\langle \gamma'(t), Y(t)\rangle = 0$ for every t because parallel transport is an isometry, so let us consider the local surface S defined by

$$S(t,s) = \beta_{(\gamma(t),Y(t))}(s), \tag{9}$$

for $s \in (-\delta,\delta)$ depending on Y, where $\beta_{(\gamma(t),Y(t))}(s)$ is the geodesic whose initial conditions are $\beta_{(\gamma(t),Y(t))}(0) = \gamma(t)$, $\beta'_{(\gamma(t),Y(t))}(0) = Y(t)$. Since $\mathcal{N}$ is analytic, the parallel transport is analytic and geodesics depend analytically on their initial conditions. So the local surface S is an analytic surface whose coordinates are Fermi coordinates according to the definition. □

2.2. **Curvature tensor and sectional curvatures.** We follow **??** for the definitions in the subsection. Let $\chi^\infty(\mathcal{N})$ be the set of C^∞ vector fields of $\mathcal{N}$. The curvature tensor

$$\mathcal{R} : \chi^\infty(\mathcal{N})\times\chi^\infty(\mathcal{N})\times\chi^\infty(\mathcal{N}) \longrightarrow \chi^\infty(\mathcal{N}) \tag{10}$$

is defined in terms of the Levi-Civita connection as follows

$$\mathcal{R}(X,Y)Z = \nabla_X\nabla_Y Z - \nabla_Y\nabla_X Z - \nabla_{[X,Y]}Z. \tag{11}$$

The sectional curvature of the plane generated by two vector fields X,Y at the point $A \in \mathcal{N}$, which are orthonormal at A, is given by

$$K(X,Y) = \langle \nabla_Y\nabla_X Y - \nabla_X\nabla_X Y + \nabla_{[X,Y]}X, X\rangle = \langle \mathcal{R}(X,Y)Y, X\rangle. \tag{12}$$

Let A be a normalized Hölder potential and $\gamma : (-\epsilon,\epsilon) \longrightarrow \mathcal{N}$ a geodesic of the Riemannian metric such that $\gamma(0)=A$, $\gamma'(t)=X(t)$, where $X(t)$ is a parallel unit vector field. Consider the local smooth surface $S(t,s)$ given in Fermi coordinates given by Lemma 2.3. Namely, let Y be a unit vector field in the tangent space of $\mathcal{N}$, that is perpendicular to $\gamma'(t)$ and is parallel in $\gamma(t)$, so $\nabla_X Y = 0$, let $\gamma_Y(t)(s)$ be the geodesic given by the initial conditions $\gamma_{Y(t)}(0) = \gamma(t)$, $\gamma'_{Y(t)}(0) = Y(t)$. Then,

$$S(t,s) = \gamma_{Y(t)}(s),$$

for every $\mid t \mid, \mid s \mid \leq \epsilon$. It is clear that $S(t,0)=\gamma(t)$, and that the image S of $S : (-\epsilon,\epsilon)\times(-\epsilon,\epsilon) \longrightarrow \mathrm{Hol}$ is a smooth embedded surface on Hol for ϵ suitably small. Let us calculate the sectional curvature $K(X,Y)$ at the point $A=\gamma(0)$. Through the section we shall use the notation for derivatives $\frac{d}{dt}Z = Z_t$ for any vector field or function.

Let $\bar{X}$ be the vector field tangent to the t-coordinate in S, it extends the vector field X and it is not necessarily geodesic in the whole surface. The vector fields $\bar{X}, Y$ commute, and moreover,

Lemma 2.4. *The vector fields $\bar{X}$ and Y are perpendicular in S.*

Proof. Since Y is parallel along γ and geodesic, we have

$$\begin{aligned} Y\langle \bar{X}, Y\rangle = \langle \nabla_Y \bar{X}, Y\rangle + \langle \bar{X}, \nabla_Y Y\rangle &= \langle \nabla_Y \bar{X}, Y\rangle = \langle \nabla_{\bar{X}} Y, Y\rangle \\ &= \frac{1}{2}\bar{X}\langle Y, Y\rangle = 0, \end{aligned}$$

where in the last equality we used the fact that $[\bar{X}, Y] = 0$. Therefore, the function $Y\langle \bar{X}, Y\rangle$ vanishes in S, and hence the function $\langle \bar{X}, Y\rangle$ is constant along the integral curves of Y. But at $\gamma(t)$ this function is $\langle X, Y\rangle$ which vanishes by hypothesis. So $\langle \bar{X}, Y\rangle$ vanishes everywhere in S thus proving our claim. □

Therefore, from the definition of sectional curvatures we deduce that,

Lemma 2.5. *Along the geodesic $\gamma(t)$ we have*

$$K(\bar{X}, Y) = -\frac{1}{2} Y(Y(\| \bar{X} \|)^2) \tag{13}$$

Proof. The symmetry properties of the curvature tensor imply that $\langle \mathcal{R}(X, Y)Y, X\rangle = -\langle \mathcal{R}(X, Y)X, Y\rangle$, so let us calculate $\langle \mathcal{R}(X, Y)X, Y\rangle$.

The fact that $\bar{X}$ and Y commute implies that

$$\langle \mathcal{R}(X, Y)X, Y\rangle = \langle \nabla_{\bar{X}}\nabla_Y \bar{X} - \nabla_Y \nabla_{\bar{X}} \bar{X}, Y\rangle.$$

The first term of the formula gives

$$\begin{aligned} \langle \nabla_{\bar{X}}\nabla_Y \bar{X}, Y\rangle &= \bar{X}\langle \nabla_Y \bar{X}, Y\rangle - \langle \nabla_Y \bar{X}, \nabla_{\bar{X}} Y\rangle \\ &= \bar{X}(Y\langle \bar{X}, Y\rangle - \langle \bar{X}, \nabla_Y Y\rangle) \\ &= 0 \end{aligned}$$

since $\nabla_Y Y = 0$, $\nabla_{\bar{X}} Y = 0$ along $\gamma(t)$ by assumption, and $\langle \bar{X}, Y\rangle = 0$ by Lemma 2.4.

The second term of the formula gives

$$\begin{aligned} -\langle \nabla_Y \nabla_{\bar{X}} \bar{X}, Y\rangle &= -Y\langle \nabla_{\bar{X}} \bar{X}, Y\rangle + \langle \nabla_{\bar{X}} \bar{X}, \nabla_Y Y\rangle) \\ &= -Y(\bar{X}\langle \bar{X}, Y\rangle - \langle \bar{X}, \nabla_{\bar{X}} Y\rangle) \\ &= Y\langle \bar{X}, \nabla_Y \bar{X}\rangle \\ &= \frac{1}{2} Y(Y\langle \bar{X}, \bar{X}\rangle) \end{aligned}$$

because $\langle \bar{X}, Y\rangle = 0$ by Lemma 2.4, $\bar{X}, Y$ commute so $\nabla_{\bar{X}} Y = \nabla_Y \bar{X}$, $\langle \bar{X}, Y\rangle = 0$ and $\nabla_Y Y = 0$. This proves the lemma

□

3. Curves of constant energy

In this section we show some technical results concerning curves of constant energy in $\mathcal{N}$ that will be important in the calculation of the sectional curvature $K(X, Y)$. The results involve some identities with a cohomological flavour satisfied by unitary vector fields.

We start with a technical result that is a consequence of formula 5. This lemma will be extensively used in the article.

Lemma 3.1. *Let $A \in \mathcal{N}$ and let $\gamma : (-\epsilon, \epsilon) \longrightarrow \mathcal{N}$ be a smooth curve such that $\gamma(0) = A$. Let $X(t) = \gamma'(t)$, and let Y be a smooth vector field tangent to $\mathcal{N}$ defined in an open neighborhood of A. Denote by $Y(t) = Y(\gamma(t))$. Then the derivative of $\int Y(t) d\mu_{\gamma(t)}$ with respect to the parameter t is*

$$\frac{d}{dt}\int Y(t) d\mu_{\gamma(t)} = \int \frac{dY(t)}{dt} d\mu_{\gamma(t)} + \int Y(t)X(t) d\mu_{\gamma(t)}$$

for every $t \in (-\epsilon, \epsilon)$.

Proof. The idea of the proof is very simple and based on the fact that the function $Q : \chi(\mathcal{N}) \times m_T \longrightarrow \mathbb{R}$ given by

$$Q(X, \mu) = \int X d\mu$$

is a bilinear form, where $\chi(\mathcal{N})$ is the set of C^1 vector fields tangent to $\mathcal{N}$ and m_T is the set of invariant measures of the map T. So the derivative of a function of the type $Q(X(t), \mu(t))$ satisfies a sort of Leibnitz rule. Let us check.

Let us calculate the derivative at $t = 0$, for every other $t \in (-\epsilon, \epsilon)$ the calculation is analogous. We have

$$\begin{aligned}
\frac{d}{dt}\int Y(t) d\mu_{\gamma(t)} \mid_{t=0} &= \lim_{t\to 0} \frac{1}{t}\Big(\int Y(t) d\mu_{\gamma(t)} - \int Y(0) d\mu_A\Big) \\
&= \int \lim_{t\to 0} \frac{1}{t}(Y(t) - Y(0)) d\mu_{\gamma(t)} \\
&+ \lim_{t\to 0} \frac{1}{t}\Big(\int Y(0) d\mu_{\gamma(t)} - \int Y(0) d\mu_A\Big) \\
&= \int \frac{dY(t)}{dt} d\mu_A + \lim_{t\to 0} \frac{1}{t}\Big(\int Y(0) d\mu_{A+tX(0)} - \int Y(0) d\mu_A\Big)
\end{aligned}$$

where in the last step we use the fact that the derivative with respect to t only depends on the vector $X(0)$ and not on the curve through A tangent to $X(0)$. By equation 5 the second term in the above equality is just $\frac{d}{dt}\int Y(0) d\mu_{A+tX(0)} \mid_{t=0}$, which equals $\int X(0)Y(0) d\mu_{A+tX(0)}$. This finishes the proof of the lemma. □

From now on, we shall adopt the notations $\frac{dY}{dt} = Y' = Y_t$, the first one applies when there is only one parameter involved in the calculations, the second one will be used otherwise.

Lemma 3.2. *Let $\gamma(t)$ be a smooth curve of normalized potentials such that $\gamma'(t) = X(t)$, for all t, has constant energy. Then the following formula holds in $\gamma(t)$:*

$$\int \frac{d}{dt}\Big(X' + \frac{1}{2}X^2\Big) d\mu_{\gamma(t)} = 0. \tag{14}$$

Proof. The constant energy assumption implies that $\int X^2(t) d\mu_{\gamma(t)} = c$ for every t in the domain of $\gamma(t)$. The constraint $\int X(t) d\mu_{\gamma(t)} = 0$ for the tangent vectors of curves in the manifold of normalized potentials gives, by taking derivatives and applying Lemma 3.1, the equality

$$0 = \frac{d}{dt} \int X(t) d\mu_{\gamma(t)} = \int (X' + X^2(t)) d\mu_{\gamma(t)}$$

So we get $\int X'(t) d\mu_{\gamma(t)} = -c$ and hence, taking again derivatives and applying Lemma 3.1

$$\frac{d}{dt} \int X'(t) d\mu_{\gamma(t)} = 0 = \int (X'' + X'X) d\mu_{\gamma(t)} = \int \frac{d}{dt}(X' + \frac{1}{2} X^2) d\mu_{\gamma(t)}.$$

This proves the Lemma.

□

Corollary 3.3. *Let $\gamma : (-a, a) \longrightarrow \mathcal{N}$ be an analytic curve of normalized potentials such that $\gamma'(t) = X(t)$, for all t, has energy equal to 1. Then there exist a Hölder function c, and an analytic curve $h(s)$, $s \in (-a, a)$ of Hölder functions with the following properties:*

1. $X'(t) + \frac{1}{2} X^2(t) = c + \int_0^t h(r) dr$
2. $\int h(t) d\mu_{\gamma(t)} = 0$ *for every* $t \in (-a, a)$,
3. $\int c d\mu_{\gamma(t)} = -\frac{1}{2}$ *for every* $t \in (-a, a)$.

Proof. By Lemma 3.2 we know that $\int \frac{d}{dt}(X' + \frac{1}{2} X^2) d\mu_{\gamma(t)} = 0$, so the function

$$h(t) = \frac{d}{dt}(X' + \frac{1}{2} X^2)$$

is an analytic function of t such that $\int h(t) d\mu_{\gamma(t)} = 0$ for every $t \in (-a, a)$. By the first fundamental Theorem of Calculus, there exists a function c such that

$$X'(t) + \frac{1}{2} X^2(t) = c + \int_0^t h(r) dr.$$

As we already noticed in the proof of Lemma 3.2, $\int X'(t) d\mu_{\gamma(t)} = -1$, if $\int X^2(t) d\mu_{\gamma(t)} = 1$, for every $t \in (-a, a)$. Therefore,

$$\int (X' + \frac{1}{2} X^2) d\mu_{\gamma(t)} = -\frac{1}{2} = \int c d\mu_{\gamma(t)},$$

for every $t \in (-a, a)$, since $\int (\int_0^t h(r) dr) d\mu_{\gamma(t)} = 0$. This finishes the proof of the Corollary.

□

Lemma 3.4. *Let A be a normalized potential, and let $S : (-\epsilon, \epsilon) \times (-\delta, \delta) \longrightarrow \mathcal{N}$ be an analytic local surface parametrized in Fermi coordinates with $S(0, 0) = A$, $\bar{X} = \frac{\partial}{\partial t}$, $Y = \frac{\partial}{\partial s}$, and $X = \bar{X}(A)$. Then we have:*

1. *There exist an analytic curve of Hölder functions $g(t)$, $t \in (-\epsilon, \epsilon)$, and analytic functions $q^t(s)$, for $(t, s) \in (-\epsilon, \epsilon) \times (-\delta, \delta)$, such that*

$$Y_s + \frac{1}{2} Y^2 = g(t) + \int_0^s q^t(r) dr.$$

The function $g(t)$ satisfies $\int g(t) d\mu_P = -\frac{1}{2}$ and the functions $q^t(s)$ have zero mean with respect to $d\mu_P$, $P = S(t, s)$.

(2) *There exist an analytic curve of Hölder functions $f(t)$ for $t \in (-\epsilon, \epsilon)$, and an analytic family of functions $p^t(s)$, both $f(t)$ and $p^t(s)$ with zero mean with respect to μ_P, for $P = S(t,s)$, such that*

$$Y_t + \frac{1}{2} Y \bar{X} = f(t) + \int_0^s p^t(r) dr. \tag{15}$$

Moreover, the vector field Y_t satisfies $\int Y_t d\mu_P = 0$ for every $P \in S(t,s)$.

(3) *There exist an analytic curve of Hölder functions $b(t)$ and a Hölder function C, $t \in (-\epsilon, \epsilon)$, such that along the geodesic $\gamma(t)$ we have*

$$\bar{X}_s + \frac{1}{2} Y \bar{X} = C + \int_0^t b(r) dr, \tag{16}$$

where $\int C d\mu_{\gamma(t)} = \int b(t) d\mu_{\gamma(t)} = 0$ for every $t \in (-\epsilon, \epsilon)$.

Proof. The vector field Y is geodesic and normalized, $\langle Y, Y \rangle = 1$ in the parameterized surface defined by the Fermi coordinates. So Corollary 3.3 implies item (1) for the geodesics tangent to the Fermi vector field Y. Since these geodesics cover the local surface $S(t,s)$, we have that $\int g(t) d\mu_P = -\frac{1}{2}$ and $\int q^t(s) d\mu_P = 0$ for every $P = S(t,s)$.

Moreover, by Lemma 2.4 we have that $\langle \bar{X}, Y \rangle = 0$ in the parametrized surface. So by Lemma 3.1 we have

$$\frac{\partial}{\partial t} \langle Y, Y \rangle = 0 = 2 \int Y \times (Y_t + \frac{1}{2} \bar{X} Y) d\mu_P$$

for every point P in the parametrized surface. By the previous equation we get

$$0 = \int Y \times (Y_t + \frac{1}{2} \bar{X} Y) d\mu_P = \frac{\partial}{\partial s} \int (Y_t + \frac{1}{2} \bar{X} Y) d\mu_P - \int \frac{\partial}{\partial s} (Y_t + \frac{1}{2} \bar{X} Y) d\mu_P. \tag{17}$$

Claim: The vector field Y_t has vanishing mean.

Indeed, we already have that $\int Y d\mu_P = 0$ for every P in the surface, so taking derivatives with respect to t and applying Lemma 3.1:

$$0 = \frac{\partial}{\partial t} \int Y d\mu_P = \int Y_t d\mu_P + \int \bar{X} Y d\mu_P = \int Y_t d\mu_P + \langle \bar{X}, Y \rangle = \int Y_t d\mu_P,$$

thus proving the Claim.

From equation 17, the Claim and the fact that $\int \bar{X} Y d\mu_P = \langle \bar{X}, Y \rangle_P = 0$, we deduce that

$$\int \frac{\partial}{\partial s} (Y_t + \frac{1}{2} \bar{X} Y) d\mu_P = 0, \tag{18}$$

therefore the function $(Y_t + \frac{1}{2} \bar{X} Y)$ and its derivative with respect to s must have zero means. The fundamental Theorem of Calculus implies that there exist $f(t)$ and $p^t(s)$ analytic in t, s such that

$$Y_t + \frac{1}{2} \bar{X} Y = f(t) + \int_0^s p^t(r) dr$$

where $p^t(r) = \frac{\partial}{\partial s} (Y_t + \frac{1}{2} \bar{X} Y)$. The functions $p^t(s)$ have zero mean for every t, s, and since $\int (Y_t + \frac{1}{2} \bar{X} Y) d\mu_P = 0$ for every $P = S(t,s)$ we get that $\int f(t) d\mu_P = 0$ for every $P = S(t,s)$.

To show item (3) we just interchange the roles of $\bar{X}$ and Y just along the geodesic $\gamma(t)$ (because $\bar{X}$ is not necessarily geodesic in the local surface S). Since $\bar{X} = X$ along $\gamma(t)$ we get an expression similar to the one in item (2) for X_s:

$$\bar{X}_s + \frac{1}{2}\bar{X}Y = C + \int_0^t b(r)dr$$

along $\gamma(t)$ for every $t \in (-\epsilon, \epsilon)$. This finishes the proof of the Lemma. $\square$

4. On the sectional curvatures of the Riemannian metric

The goal of the section is to calculate the sectional curvatures of the Riemannian metric in $\mathcal{N}$.

Theorem 4.1. *Let A be a normalized potential, and let $S : (-\epsilon, \epsilon) \times (-\delta, \delta) \longrightarrow \mathcal{N}$ be a local surface parametrized in Fermi coordinates with $S(0,0) = A$, $\bar{X} = \frac{\partial}{\partial t}$, $Y = \frac{\partial}{\partial s}$, and $X = \bar{X}(A)$. Then the sectional curvature $K(X,Y)$ at A is*

$$K(X,Y) = \int f^2 d\mu_A, \tag{19}$$

where f is the function defined equation 15 **at time** $t = 0$**.** *So it is always nonnegative.*

We assume the existence of geodesics that will be proved on section 5.

According to Lemma 2.5, we have that the sectional curvature of a plane generated by two unit vectors X, Y tangent to $\mathcal{N}$ at a normalized potential A is

$$K(X,Y) = -\frac{1}{2}Y(Y(\| X \|)^2). \tag{20}$$

To estimate this function we shall need some preparatory lemmas. Let us first define some notations. Let $\bar{X}_t$ be the derivative of the vector field $\bar{X}$ with respect to the parameter t and $\bar{X}_s$ be the derivative of the vector field $\bar{X}$ with respect to the parameter s. The same convention applies to Y_t, Y_s. The notations $\bar{X}(Y) = \frac{\partial}{\partial t}Y = Y_t$ will always represent derivatives with respect to the vector field X, while $\bar{X}Y$ or $\bar{X} \times (Y)$ will represent the product of the functions X and Y. Through the section this double character of the vectors tangent to the manifold $\mathcal{N}$ which are also functions will show up in all statements and proofs.

Lemma 4.2. *We have that $\bar{X}_s = Y_t$ in the local surface S.*

Proof. This is due to the fact that the vector fields $\bar{X}, Y$ commute in S, so

$$0 = [\bar{X}, Y] = \bar{X}(Y) - Y(\bar{X}) = Y_t - \bar{X}_s.$$

$\square$

Lemma 4.3. *Let $Y_t + \frac{1}{2}Y\bar{X} = G = \bar{X}_s + \frac{1}{2}\bar{X}Y$ be the equation defined in Lemma 3.4. The following identity holds in the surface $S(t,s)$:*

$$Y \int \bar{X}^2 d\mu = 2\int \bar{X}\bar{X}_s d\mu + \int Y\bar{X}^2 d\mu = 2\int \bar{X} G d\mu \tag{21}$$

Proof. By Lemma 4.2 we know that $\bar{X}_s = Y_t$ so the functions $G = Y_t + \frac{1}{2}Y\bar{X}$ and $H = \bar{X}_s + \frac{1}{2}Y\bar{X}$ coincide. By the Leibnitz rule (Lemma 3.1) we have

$$Y\int \bar{X}^2 d\mu = 2\int \bar{X}\bar{X}_s d\mu + \int Y\bar{X}^2 d\mu = 2\int \bar{X}\times(\bar{X}_s + \frac{1}{2}\bar{X}Y)d\mu = 2\int \bar{X}G d\mu$$

thus proving the Lemma.

□

Lemma 4.4. *Let* $Y_t + \frac{1}{2}\bar{X}Y = G(t,s) = f(t) + \int_0^s p^t(r)dr, X_s + \frac{1}{2}\bar{X}Y = H = C + \int_0^t b(r)dr$ *be the functions in* $S(t,s)$ *defined in Lemma 3.4 item (2). Then* $\int f d\mu = 0$, $\int G d\mu = 0$ *in* $S(t,s)$ *and moreover,*

1. $\int G_s d\mu = 0$, $\int f_t d\mu = 0$,
2. $\int YG d\mu = 0$, $\int X f d\mu = 0$,
3. $\int \bar{X}YG d\mu = -2\int G^2 d\mu = -2\int f^2 d\mu$
4. $\int YG_t d\mu = 0$ *and* $\int \bar{X}_s G d\mu = 0$

at the points of $\gamma(t) = S(t,0)$.

Proof. Indeed, by Lemma 3.4 item (2) we already knew that $\int f d\mu = 0$, $\int G d\mu = 0$ in the local surface $S(t,s)$. Moreover, since $G_s = p^t$ we also have that $\int G_s d\mu = 0$ which is item (1). Since by Lemma 4.2 $X_s = Y_t$ we have that $H = G$ and therefore, $f(t) = C + \int_0^t b(r)dr$ along the geodesic $\gamma(t)$. This implies that $f_t = b(t)$ and hence $\int f_t d\mu 0$ at the points of $\gamma(t)$.

Now, to show item (2), observe that

$$0 = Y\int G d\mu = \int YG d\mu + \int G_s d\mu$$

so by item (1) we get $\int YG d\mu = 0$ at the points of $\gamma(t)$. Analogously, $\int X f d\mu = 0$ along $\gamma(t)$ and item (2) proceeds.

To show item (3) notice first that $Z = G\bar{X}$ is tangent to $\mathcal{N}$ at $\gamma(t)$ because $\int XG d\mu = 0$. Moreover, since $G = f$ depends only on t the vector field $Z = G\bar{X}$ is a reparametrization of the vector field $\bar{X}$ along $\gamma(t)$: $Z(t) = \bar{X}(\int_0^t g(r)dr, 0)$. This follows from the chain rule: given a smooth function $\sigma : \mathcal{N} \longrightarrow \mathbb{R}$ we have that the derivative of σ with respect to Z is

$$Z(\sigma) = \frac{d\sigma}{dt} = g(t)\bar{X}(\sigma).$$

So we can apply Lemma 3.1 to get

$$\int \bar{X}YG d\mu = (G\bar{X})\int Y d\mu - \int (Y)_{G\bar{X}} d\mu = -\int GY_t d\mu \tag{22}$$

along $\gamma(t)$, where Y_{GX} is the derivative of Y with respect to GX (recall that $\int Y d\mu = 0$). Moreover, by Lemma 3.4 item (2), equation (15), we have at the points of $\gamma(t)$

$$\begin{aligned}\int GY_t d\mu &= \int f\times(-\frac{\bar{X}Y}{2} + G)d\mu \\ &= -\frac{1}{2}\int \bar{X}YG d\mu + \int f^2 d\mu\end{aligned}$$

so replacing this equality in equation 22 we get item (3).

Item (4) is a byproduct of item (3), since by the Leibnitz rule and equation 22 we have

$$\int \bar{X} Y G d\mu = \bar{X} \int Y G d\mu - \int Y_t G d\mu - \int Y G_t d\mu = - \int Y_t G d\mu$$

and we know by item (2) that $\int Y G d\mu = 0$ along $\gamma(t)$, so its derivative with respect to $\bar{X}$ vanishes. This yields item (4).

□

Corollary 4.5. *Following the notations of Lemma 4.4, we have that*

$$\frac{1}{2} Y(Y \int \bar{X}^2 d\mu) = Y \int \bar{X} G d\mu = \int \bar{X} G_s d\mu + \int \bar{X} Y G d\mu$$

at the points of the geodesic $\gamma(t) = S(t, 0)$.

Proof. We just apply the Leibnitz rule and Lemma 4.4 to get

$$\begin{aligned} \frac{1}{2} Y(Y \int \bar{X}^2 d\mu) &= Y \int \bar{X} G d\mu \\ &= \int \bar{X}_s G d\mu + \int \bar{X} G_s d\mu + \int \bar{X} Y G d\mu \\ &= \int \bar{X} G_s d\mu + \int \bar{X} Y G d\mu \end{aligned}$$

at the points of $\gamma(t)$. □

Lemma 4.6. *Let $Q = Y_s + \frac{1}{2} Y^2$ be the function defined in Lemma 3.4 item (1). The following assertions hold at the point $A = \gamma(0)$:*

(1) $\int Q d\mu = \int Q_s d\mu = \int Q_t d\mu = 0$
(2) $\int \bar{X} Q d\mu = \int Y Q d\mu = 0$
(3) $\int \bar{X}^2 Q d\mu = 0$.
(4) $\int \bar{X} Q_t d\mu = 0$.

Proof. The proof is quite analogous to the proof of Lemma 4.4 since the functions Q and G have similar properties. We already know that $\int Q d\mu = 0$ in the local surface $S(t, s)$ by Lemma 3.4. Moreover, since $Q = g(t) + \int_0^s q^t(r) dr$ and therefore, $Q_s = q^t(s)$, we have that $\int Q_s d\mu = \int q^t(s) d\mu = 0$ in $S(t, s)$ by Lemma 3.4.

To see that $\int Q_t d\mu = 0$, notice that $Q_t = Y_{st} + Y Y_t$, so

$$\int Y_{st} d\mu = \int Y_{ts} d\mu = Y \int Y_t d\mu - \int Y Y_t d\mu = - \int Y Y_t d\mu$$

since we know that $\int Y_t d\mu = 0$ in the surface $S(t, s)$. Hence, $\int Q_t d\mu = 0$ as we claimed.

Item (2) is straightforward from item (1):

$$\int \bar{X} Q d\mu = \bar{X} \int Q d\mu - \int Q_t d\mu = 0$$

and

$$\int Y Q d\mu = Y \int Q d\mu - \int Q_s d\mu = 0$$

To prove item (3) we proceed as in the proof of Lemma 4.4. The function $Q\bar{X}$ is tangent to $S(t, s)$ and restricted to the geodesic $\gamma(t)$ it is a reparametrization of the vector field X. So at the points of $\gamma(t)$ we have

$$\int X^2 Q d\mu = \int (QX) X d\mu = (QX) \int X d\mu - \int X_{QX} d\mu = - \int Q X_t d\mu$$

since we know that $\int \bar{X} d\mu = 0$ in the surface $S(t,s)$. This implies that $\int \bar{X} Q_t d\mu = 0$ along $\gamma(t)$ since

$$\begin{aligned}\int X^2 Q d\mu &= \bar{X}\int \bar{X} Q d\mu - \int \bar{X}_t Q d\mu - \int \bar{X} Q_t d\mu \\ &= -\int \bar{X}_t Q d\mu - \int \bar{X} Q_t d\mu\end{aligned}$$

where in the last equality we applied item (2). This is actually the proof of item (4). Now, let $\bar{X}_t^T$ be the projection of $\bar{X}_t$ in the tangent space of $S(t,s)$. Since $\int \bar{X}_t Q d\mu$ is the inner product of the functions $\bar{X}_t$ and Q we have that $\int \bar{X}_t Q d\mu = \int \bar{X}_t^T Q d\mu$. Let $\bar{X}_t^T = aX + bY$ at the point $A = \gamma(0)$. Then

$$\int \bar{X}_t Q d\mu = a\int \bar{X} Q d\mu + b\int Y Q d\mu = 0$$

by item (2), thus proving item (3) and the lemma. □

Proof of Theorem 4.1

By Corollary 4.5 we have that

$$\begin{aligned}\frac{1}{2}Y(Y\int \bar{X}^2 d\mu) &= \int \bar{X} G_s d\mu + \int \bar{X} Y G d\mu \\ &= \int \bar{X} \times (\bar{X}_s + \frac{1}{2}\bar{X} Y) d\mu + \int \bar{X} Y G d\mu \\ &= \int \bar{X}\bar{X}_{ss} d\mu + \frac{1}{2}\int \bar{X}\bar{X}_s Y d\mu + \frac{1}{2}\int Y_s \bar{X}^2 d\mu + \int \bar{X} Y G d\mu\end{aligned}$$

By Lemma 4.2 we have that

$$\int \bar{X}\bar{X}_{ss} d\mu = \int \bar{X} Y_{ts} d\mu = \int \bar{X} Y_{st} d\mu$$

which gives

$$\int X Y_{st} d\mu = \int X \times (-\frac{1}{2}Y^2 + Q)_t d\mu \tag{23}$$

$$= -\int X Y Y_t d\mu + \int X Q_t d\mu \tag{24}$$

$$= -\int X Y X_s d\mu \tag{25}$$

where in the last equality we applied Lemma 4.6 item (4).

Moreover, we have

$$\int Y_s X^2 d\mu = \int X^2 \times (-\frac{1}{2}Y^2 + Q) d\mu \tag{26}$$

$$= -\frac{1}{2}\int X^2 Y^2 d\mu + \int X^2 Q \tag{27}$$

$$= -\frac{1}{2}\int X^2 Y^2 d\mu \tag{28}$$

where we applied item (3) of Lemma 4.6 to get the last equation. Replacing equations (25) and (28) in the expression of $\frac{1}{2}Y(Y\int X^2 d\mu)$ we obtain

$$\frac{1}{2}Y(Y\int X^2 d\mu) = -\frac{1}{2}\int XYX_s d\mu - \frac{1}{4}\int X^2Y^2 d\mu + \int XYG d\mu$$

Since we have that

$$\int XYX_s d\mu = \int XY \times (-\frac{1}{2}XY + G) d\mu = -\frac{1}{2}\int X^2Y^2 d\mu + \int XYG d\mu$$

we conclude

$$\begin{aligned} \frac{1}{2}Y(Y\int X^2 d\mu) &= \frac{1}{2}\int XYG d\mu \\ &= -\int G^2 d\mu \end{aligned}$$

at the points of $\gamma(t)$ by Lemma 4.4 item (3). Since the sectional curvature $K(X,Y)$ at $A = \gamma(0)$ is $-\frac{1}{2}Y(Y\int X^2 d\mu$ and $G = f$ at the points of $\gamma(t)$ we get Theorem 4.1.

5. The geodesics of the space of normalized potentials

The goal of the section is to show Theorems 2.1 and 2.2. Namely, given an element $A \in \mathcal{N}$, and a vector $X \in T_A\mathcal{N}$, we shall show that there exists a geodesic $\gamma(t)$ in the space such that $\gamma(0) = A$, $\gamma'(0) = X(0) = X$, and that the parallel transport of vectors along $\gamma(t)$ is well defined. Since the manifold of normalized potentials is an infinite dimensional manifold, the usual way of proving the existence of geodesics via solutions of an ordinary differential equations with coefficients in the set of Cristoffel symbols may not proceed.

One of the most common approaches to the problem of existence of geodesics in Hilbert manifolds is to show the Palais-Smale condition (see [11]) for the Riemannian metric. This is an issue in infinite dimensional Lagrangian calculus of variations: the Palais-Smale condition depends very much on each particular Riemannian metric and in our case it is not clear that such a condition is satisfied (anyway, we will not use it in the classical form). However, what we shall show is in some sense a weak Palais-Smale condition for our Riemannian manifold: roughly speaking, we shall construct a sequence of approximated solutions of the Euler-Lagrange equation having as a limit a true solution of the equation.

We would like to point out that we will not use any of the classical results on Hilbert manifolds.

We shall develop a strategy to prove the existence of geodesics under the following assumption: there exists a countable basis $\{v_n\}$, $n \in \mathbb{N}$, of tangent vectors in each tangent space $T_A\mathcal{N}$. We know that in every Banach space, the existence of a countable, dense subset gives a countable basis, so the above assumption holds for instance if our dynamics acts on a smooth manifold (the space of polynomial functions is dense for instance). This will do the job in the case $M = S^1$.

Remark 1: When $M = \{1, 2..., d\}^{\mathbb{N}}$ and μ the equilibrium probability for a Holder potential A it was shown in Theorem 3.5 in [10] that there exist a (countable) complete orthogonal set φ_n, $n \in \mathbb{N}$, on $\mathcal{L}^2(\mu_A)$.

Definition 5.1. Let $(X, |.|)$ and $(Y, |.|)$ Banach spaces and V an open subset of X. Given $k \in \mathbb{N}$, a function $F : V \to Y$ is called k-differentiable in x, if for each $j = 1, ..., k$, there exists a j-linear bounded transformation

$$D^j F(x) : \underbrace{X \times X \times ... \times X}_{j} \to Y,$$

such that,

$$D^{j-1}F(x+v_j)(v_1, ..., v_{j-1}) - D^{j-1}F(x)(v_1, ..., v_{j-1}) = D^j F(x)(v_1, ..., v_j) + o_j(v_j),$$

where

$$o_j : X \to Y, \;\; \text{satisfies,} \; \lim_{v \to 0} \frac{|o_j(v)|_Y}{|v|_X} = 0$$

By definition F has derivatives of all orders in V, if for any $x \in V$ and any $k \in \mathbb{N}$, the function F is k-differentiable in x.

Definition 5.2. Let X, Y be Banach spaces and V an open subset of X. A function $F : V \to X$ is called analytic on V, when F has derivatives of all orders in V, and for each $x \in V$ there exists an open neighborhood V_x of x in V, such that, for all $v \in V_x$, we have that

$$F(x + v) \; - \; F(x) = \sum_{j=1}^{\infty} \frac{1}{n!} \; D^j F(x) v^j,$$

where $D^j F(x) v^j = D^j F(x)(v, ..., v)$ and $D_j F(x)$ is the j-th derivative of F in x.

Above we use the notation of section 3.2 in [12].

$\mathcal{N}$ can be expressed locally in coordinates via analytic charts (see [13]).

5.1. **Some more estimates from Thermodynamic Formalism.** Given a potential $B \in \text{Hol}$ we consider the associated Ruelle operator $\mathcal{L}_B$ and the corresponding main eigenvalue λ_B and eigenfunction h_B.

The function

$$\Pi(B) = B + \log(h_B) - \log(h_B(T)) - \log(\lambda_B) \tag{29}$$

describes the projection of the space of potentials B on Hol onto the analytic manifold of normalized potentials $\mathcal{N}$.

We identify below $T_A\mathcal{N}$ with the affine subspace $\{A + X \,:\, X \in T_A\mathcal{N}\}$.

The function Π is analytic (see [13]) and therefore has first and second derivatives. Given the potential B, then $D_B\Pi$ should be considered as linear map from Hol to itself (with the Holder norm on Hol). Moreover, the second derivative $D_B^2\Pi$ should be interpreted as a bilinear form from Hol $\times$ Hol to Hol.

When B is normalized the eigenvalue is 1 and the eigenfunction is equal to 1. We would like to study the geometry of the projection Π restricted to the tangent space $T_A\mathcal{N}$ into the manifold $\mathcal{N}$ (namely, to get bounds for its first and second derivatives with respect to the potential viewed as a variable) for a given normalized potential A.

The space $T_A\mathcal{N}$ is a linear subspace of functions and the derivative map $D\,\Pi$ is analytic when restricted to it. The goal of the subsection is to estimate the first and second derivatives of Π restricted to $T_A\mathcal{N}$ in a small neighborhood of A in the sup norm. This is of course linked to the geometry of the transfer operator in a small neighborhood of a normalized potential A. The geometry of $\Pi\,|_{T_A\mathcal{N}}$ will be important to show the existence of geodesics as we shall see in the forthcoming subsections.

To get such estimates we recall some well known results of the analytic theory of the Ruelle operator.

Proposition 5.3. *Given a normalized potential $A \in \mathcal{N}$ and $\delta > 0$ there exists $r > 0$, such that, for every Hölder continuous function B in the ball $B_r(A)$ of radius r around A, the norms of $D_B\Pi$ and $D^2_B\Pi$ restricted to the functions in $T_A\mathcal{N}$ satisfy*

$$\| \,(D_B\Pi)\,|_{T_A\mathcal{N}} - I\,\| \leq \delta$$

$$\| \,(D^2_B\Pi)\,|_{T_A\mathcal{N}}\| \leq 1 + \delta.$$

In the above for linear operators we use the operator norm (in Hol we consider the sup norm) and for bilinear forms we use also the sup norm (see section 2.3 in [13]).

Proposition 5.3 is perhaps well known, we sketch its proof for the sake of completeness. Let us recall some well known results of the theory of the transfer operator.

The following results are taken from Theorem 3.5 in [12] and Theorems A, B, C in [5].

Lemma 5.4. *Let $\Lambda : Hol \longrightarrow \mathbb{R}$, $H : Hol \longrightarrow Hol$ be given, respectively, by $\Lambda(B) = \lambda_B$, $H(B) = h_B$. Then we have*

1. *The maps Λ, H, and $A \longrightarrow \mu_A$ are differentiable.*
2. $D_B \log(\Lambda)(\psi) = \int \psi d\mu_B$,
3. $D^2_B \log(\Lambda)(\eta, \psi) = \int \eta\psi d\mu_B$, *where ψ, η are L^2 functions.*
4. $D_A H(X) = h_A \int [\,(I - \mathcal{L}_{T,A})^{-1}\,(1 - h_A)\,].\,X)\,d\mu_A$.
5. *If A is a normalized potential, then for every function $X \in T_A\mathcal{N}$ we have $\int X d\mu_A = 0$.*

Remark 2: The expression of item (4) appears in an old ArXiv version of [5] (see Proposition 4.6. in the 2012 version arXiv:1205.5361v1). Note that the derivative linear operator $X \to D_A H(X)$ is zero when A is normalized.

Remark 3: Note that item (2) implies by item (5) that $D_B \log(\Lambda)(\psi) = \int \psi d\mu_B = 0$, when B is normalized and $\psi \in T_{\mu_B}(\mathcal{N})$.

Remark 4: Item (1) above means that for a fixed Holder function f the map $A \to \int f d\mu_A$ is differentiable on A (see theorem B in [5])

Questions related to second derivatives on Thermodynamic Formalism are considered in [8] and [16].

From the above lemma we deduce the following:

Lemma 5.5. *Given a normalized potential A and $\delta \in (0,1)$, there exists $r > 0$, such that, for every Hölder continuous B in the C^0 ball $B_r(A)$ of radius r centered at A, we have that the L^2 norms of $D_B\Lambda$, $D_B H$ and $D^2_B\Lambda$ satisfy*

1. $\| \,D_B\Lambda\,|_{T_A\mathcal{N}}\| \leq \delta$,
2. $\| \,D_B H\,|_{T_A\mathcal{N}}\| \leq \delta$,
3. $\| \,D_B\Pi\,|_{T_A\mathcal{N}} - I\,\| \leq \delta$,
4. $\| \,D^2_B\Lambda\,|_{T_A\mathcal{N}}\| \leq 1 + \delta$, *for every $B \in B_r(A) \cap T_A\mathcal{N}$.*

Proof. Since the map $A \longrightarrow \mu_A$ is analytic given $\epsilon > 0$ there exists $r > 0$ such that for every Hölder function $X : S^1 \longrightarrow \mathbb{R}$ with unit norm with respect to μ_A we have

$$\mid \int X d\mu_A - \int X d\mu_B \mid < \epsilon$$

for every Hölder function B in the ball $B_r(A)$ of radius r around A in the C^0 topology. Let $X \in T_A\mathcal{N}$, items (2) and (5) in Lemma 5.4 imply that $D_A \log(\Lambda)(X) = 0$, and moreover,

$$\mid D_A \log(\Lambda)(X) - D_B \log(\Lambda)(X) \mid = \mid \int X d\mu_B \mid < \epsilon,$$

so the L^2 norm of $D_B \log(\Lambda)$ restricted to $B(1, A, L^2)$ - the L^2 ball of radius 1 **in Hol** with respect to the measure μ_A - is bounded above by $\epsilon \sup_{X \in B(1,A,L^2)} \int X d\mu_B$. From this assertion follows the estimate for $D_B\Lambda$.

The estimate for the second derivative of Λ follows from items (2) and (3) in Lemma 5.4, since the second derivative of $\log(\Lambda)$ at B is just the L^2 inner product with respect to the measure $d\mu_B$.

To show item (2), observe that according to item (4) in Lemma 5.4, **for all** x

$$\| D_B H(X) \| \leq h_B(x) \ \| (I - \mathcal{L}_{T,A})^{-1} \|_\infty \| (1 - h_B) \|_\infty \| X \|_\infty .$$

Since A is a normalized potential, we have $h_A = 1 = \lambda_A$ and we can suppose that in the ball $B_r(A)$ we also have $\mid 1 - h_B \mid < \epsilon$ by the analyticity of the function H. The operator $(I - \mathcal{L}_{T,A})^{-1}$ is uniformly bounded as well because of the spectral gap of the operator $\mathcal{L}_{T,A}$. This yields that the norms $\| \ \ \|_\infty$, $\| \ \ \|_{L^1}$, $\| \ \ \|_{L^2}$ are small for $D_B H$, $B \in B_r(A)$.

The proof of item (3) is a consequence of the definition of Π and the already proved items in the lemma.

□

Notice that item (3) in the previous lemma is the first inequality of Proposition 5.3. So it remains to show the second inequality.

In a future section we will need to control the second order derivative of the function Π acting on Hölder potentials B close to a normalized potential A. On that moment we will have to use the next lemma. We point out that the continuous dependence (follows from analyticity) on all parameters which are involved on the computations.

Lemma 5.6. *Let $A \in \mathcal{N}$, $r > 0$, $B_r(A)$ be given in Lemma 5.5. Then there exists $\delta(r) > 0$ small enough, such that, the second order derivative bilinear form of the function*

$$B \to \Pi(B) \ = \ B + \log(h_B) - \log(h_B(T)) - \log\lambda(B) \tag{30}$$

restricted to $T_A\mathcal{N}$ is $\delta(r)$-close to the zero bilinear form in L^2 for every $B \in B_r(A) \cap T_A\mathcal{N}$.

Proof. Remember that when A is normalized $\Pi(A) \ = \ A + \log(h_A) - \log(h_A(T)) - \log\lambda(A) = A$. Moreover, the first and second derivatives of Π on B are close to the corresponding ones of A.

It is known that for a normalized potential A we have

$$D_A\Pi = I,$$

where I is the identity.

Let us analyze the first derivative of Π at a point $B \in B_r(A)$ not necessarily normalized and a variable increment ψ.

By the analyticity of H, and the fact that $\log(H(A)) - \log(H(A)(T)) = 0$ if A is normalized, there exists $\delta_1 > 0$ small such that $\| \log(h_B) - \log(h_B(T)) \|_\infty < \delta_1$ for every $B \in B_r(A)$.

We get from item (3) of Lemma 5.5

$$D_B \Pi \ \sim I,$$

in L^2 norm (the error of this approximation is bounded above by δ in Lemma 5.5).

We denote by $\frac{\partial}{\partial \psi} \Pi(B + \psi) = D_B \Pi(\psi)$ the derivative on the direction of the tangent vector ψ.

Moreover, for the single increment $\psi \in T_A \mathcal{X}$ we get (by the rule of the derivative of the product)

$$\frac{\partial}{\partial \psi} \Pi(B + \psi) = \psi + \int \psi d\mu_B + \int \frac{\partial}{\partial \psi} (\log h_{B+\psi} - \log h_{B+\psi} \circ T) \, d\mu_B +$$
$$\int (\log h_{B+\psi} - \log h_{B+\psi} \circ T) \ \psi \ d\mu_B \, .$$

As we mentioned before $(\log h_{B+\psi} - \log h_{B+\psi} \circ T)$ (and its first derivative) is small when ψ is small by Lemma 5.5.

Now, we analyze the second derivative. For the pair of tangent vectors ψ, φ we get (using the rule of the derivative of the product) the bilinear form

$$(\psi, \varphi) \ \to \int \frac{\partial}{\partial \psi} \frac{\partial}{\partial \varphi} (\log h_B - \log h_B \circ T) \, d\mu_B +$$
$$\int \frac{\partial}{\partial \varphi} (\log h_B - \log h_B \circ T) \ \psi \ d\mu_B +$$
$$\int \frac{\partial}{\partial \psi} (\log h_B - \log h_B \circ T) \ \varphi \ d\mu_B +$$
$$\int (\log h_B - \log h_B \circ T) \ \psi \ \varphi \ d\mu_B \, .$$

The claim of the lemma follows from the following facts:

1) the first term of the sum above is zero by the coboundary property,

2) the linear derivative of $B \to (\log h_B - \log h_B \circ T)$ is δ_r small (second and third terms by Lemma 5.5),

3) $(\log h_B - \log h_B \circ T)$ is small when ψ and φ are small and close to a normalized potential (fourth term).

□

5.2. **The system of differential equations of geodesic vector fields.** Let us begin with the same ideas of the finite dimensional case. Suppose that $\gamma(t)$ exists, we are going to characterize γ in terms of a differential equation in the space $\mathcal{N}$ that has a unique solution. Let $X(t) = \gamma'(t)$, since it is geodesic, $\nabla_X X = 0$, where ∇ is the Levi-Civita connection of the Riemannian metric in $\mathcal{N}$. Actually, we have to show that this equation has a solution, we shall reduce this problem to solve another differential equation. So we have that

$$\langle \nabla_X X, Y \rangle = 0, \tag{31}$$

for every $Y \in T_{\gamma(t)}\mathcal{N}$. By the compatibility properties of the Riemannian metric and the covariant derivative (see (2) and (3) on properties of the Levi-Civita connection in the beginning of section 2)

$$\langle \nabla_X X, Y\rangle = X\langle X, Y\rangle - \frac{1}{2}Y\langle X, X\rangle - \langle X, [X, Y]\rangle, \tag{32}$$

where $X(f)$ means the derivative of a scalar function f with respect to X.

In particular, the energy of geodesics is constant,

$$\langle \nabla_X X, X\rangle = 0 = \frac{1}{2}X\langle X, X\rangle = \frac{1}{2}\int (2XX' + X^3)d\mu_{\gamma(t)}. \tag{33}$$

So let us restrict ourselves to the energy level of vector fields X with constant norm equal to 1. In this case, the equation of geodesics and (32) gives

$$0 = \langle \nabla_X X, Y\rangle = X\langle X, Y\rangle - \langle X, [X, Y]\rangle,$$

or equivalently,

$$X\langle X, Y\rangle = \langle X, [X, Y]\rangle, \tag{34}$$

for every vector field Y. In particular, if the elements of the basis v_n generate vector fields we have

$$X\langle X, v_n\rangle = \langle X, [X, v_n]\rangle.$$

In the case where the vector fields v_n correspond to a finite number of coordinate vector fields this set of equations might be used to show the existence of the geodesic vector field. Indeed, say that $n \leq m$, then the above system of equations is equivalent to a system of first order partial differential equations whose solution always exists by the theory of characteristics. Let us write down the system explicitly.

Let $\Phi : U_m \longrightarrow V_m$, $\Phi(t_1, t_2, .., t_m)$, be a coordinate system defined in an open neighborhood of $0 \in \mathbb{R}^m$ whose image is a smooth m-dimensional manifold in $\mathcal{N}$ containing A. Let e_n be vector fields in $\mathbb{R}^m$ tangent to the coordinates t_n, and let $v_n = D\Phi(e_n)$ define the coordinate vector fields in $\mathcal{N}$.

Let $X = \sum_{i=1}^m x_i v_i$, $\bar{x}_i = \langle X, v_i\rangle$. The differential equation of the geodesic vector field X is equivalent to

$$X\langle X, v_n\rangle = \langle X, [X, v_n]\rangle = \langle X, [\sum_{i=1}^m x_i v_i, v_n]\rangle$$

and observe that

$$[\sum_{i=1}^m x_i v_i, v_n] = \sum_{i=1}^m [x_i v_i, v_n] = \sum_{i=1}^m (x_i[v_i, v_n] - v_n(x_i)v_i)$$

and since the vector fields v_n commute we get

$$[\sum_{i=1}^m x_i v_i, v_n] = \sum_{i=1}^m -v_n(x_i)v_i = -(\sum_{i=1}^m v_n(x_i v_i)) + x_n\bar{v}_n = -v_n(X) + x_n\bar{v}_n$$

because the derivatives $v_n(v_i) = \frac{d}{dt_n}v_i$ are equal to 0 if $n \neq i$ and **where** $\bar{v}_n = D^2(\Phi)(v_n)$ if $n = i$. Hence we can write the differential equation for X as

$$X(\bar{x}_n) = X\langle X, v_n\rangle \quad = \quad -\langle X, v_n(X)\rangle + \langle X, x_n\bar{v}_n\rangle.$$

In terms of $\frac{d}{dt}$, $\frac{d}{dt_n}$ we obtain a system S_m of first order partial differential equations

$$S_m := \frac{d}{dt}(\bar{x}_n) \quad = \quad -\langle X, \frac{d}{dt_n}(X)\rangle + \langle X, x_n \bar{v}_n\rangle, \; n = 1, 2, .., m. \tag{35}$$

The above system of differential equations gives rise to a system of partial differential equations for the functions $\bar{x}_i$. Indeed, let $\bar{X} = (\bar{x}_1, \bar{x}_2, .., \bar{x}_m)$, and let M_m be the matrix of the first fundamental form in the basis v_i, namely,

$$(M_m)_{ij} = \langle v_i, v_j\rangle.$$

We have that $\bar{X} = M_m X$, so $X = M_m^{-1}\bar{X}$. Replacing this identity in the initial system (35) we get a system of first order, quasi-linear partial differential equations (see chapter 7 in [4] for definition and properties) for the functions $\bar{x}_n$ whose coefficients depend on the matrices M_m^{-1} and $\frac{d}{dt_n}(M_m^{-1})$.

5.3. **Uniform bounds for the PDE geodesic systems.** A natural way to obtain geodesics from the family of systems S_m would be to solve each of the systems with a given initial condition and take the limit $m \to \infty$. A limit function would be the desired geodesic. However, the limit process might not give any limit function, this depends on uniform bounds for the coefficients of the matrices M_m. This is the subject of the next lemma which considers the case $M = S^1$ where it is well known the existence of a countable basis (independent of the equilibrium probability).

For the case when M is the symbolic space we shall use the Remark 1 on section 5 and the next lemma will work in a similar way.

Lemma 5.7. *Let $A : S^1 \to \mathbb{R}$ be normalized potential and the function $B \in B_r(A)$ (where $B_r(A)$ is the open neighborhood of A given in Proposition 5.3). Let f_n be any countable basis of analytic functions of the circle (interval) in the L^2 norm, and let*

$$\bar{f}_n = f_n - \int f_n d\mu_A.$$

Then,

1. *The set of functions $\bar{f}_n$ is a basis of $T_A\mathcal{N}$.*
2. *Let e_n be an orthonormal basis of $T_A\mathcal{N}$ obtained from $\bar{f}_n$ (via Gram-Schmidt). Then, the functions*
$$v_n(B) = D_B\Pi(e_n)$$
form a basis for $T_B\mathcal{N}$ and
$$\mid \langle v_n(B), v_m(B)\rangle - \delta_{nm} \mid \leq \delta$$
where δ_{nm} is the Kronecker function : $\delta_{nm} = 1$ if $n = m$, and 0 otherwise.
3. *There exists $b > 0$ such that map Π restricted to the sets*
$$U_m = \{\sum_{i=1}^{m} t_i e_i, \;\; \mid t_i \mid < b\}$$
is an embedding into a m-dimensional submanifold $V_m \subset \mathcal{N}$.

Proof. The map $f \to f - \int f d\mu_A$ is a linear map from the set of functions to $T_A\mathcal{N}$. Therefore, if f_n is a basis of the set of functions the image of the set $\{f_n\}$ by this linear map is a basis in the image of the map, that is precisely $T_A\mathcal{N}$. From the basis $\bar{f}_n$ we can of course obtain an orthonormal basis e_n by Gram-Schmidt.

From Proposition 5.3, we know that $D_A\Pi \mid_{T_A\mathcal{A}}= I$ and that $D_B\Pi \mid_{T_A\mathcal{A}}$ is close to the identity if $B \in B_r(A)$. Hence, if we chose $B = A + \sum_{i=1}^m t_i w_i$ in a way that $\| B - A \| < r$ then the vectors $v_n(B) = D_B\Pi(e_n)$ will be almost perpendicular at $T_B\mathcal{N}$. This yields that the vectors $v_n(B)$ are linearly independent in $T_B\mathcal{N}$ and therefore, the map Π has constant rank m in U_m. By the local form of immersions, the image $V_m = \Pi(U_m)$ is an analytic submanifold of $\mathcal{N}$ of dimension m. □

By virtue of Lemma 5.7, we shall consider the collection of m-dimensional coordinate systems given by the restrictions of Π to the sets U_m. Let us estimate the norms of the associated matrices M_m, M_m^{-1} and its derivatives.

Lemma 5.8. *There exists $C > 0$ such that the norms of the matrices M_m^{-1}, $\frac{d}{dt_n}(M^{-1})$ are uniformly bounded by C in the neighborhood $B_r(A)$.*

Proof. The coefficients of the first fundamental form M_m at a point $B \in B_r(A)$ are

$$\langle v_i(B), v_j(B)\rangle = \int v_i(B)v_j(B)d\mu_B.$$

By Lemma 5.7 and Lemma 5.6, the matrix M_m is a perturbation of the identity at every point $B \in B_r(A)$. This yields that the norm of M_m^{-1} is uniformly bounded above in $B_r(A)$.

The derivative of M_m^{-1} with respect to t_n is $-M_m^{-1}\frac{d}{dt_n}(M_m)M_m^{-1}$. The derivatives at $B \in B_r(A)$ of the coefficients of M_m are determined by the derivatives of the terms $\langle v_i, v_j\rangle$. We have

$$\frac{d}{dt_n}\langle v_i, v_j\rangle = \int (\frac{d}{dt_n}(v_i)v_j + v_i\frac{d}{dt_n}(v_j) + v_iv_jv_n)d\mu_B$$

by the definition of the Riemannian metric. The norms of the terms in the equation are bounded by the products of the norms of $D^2(\Pi)$, v_i, v_j, v_n, which have uniform bounds according to Proposition 5.3.

□

5.4. **First order systems of ordinary differential equations equivalent to first order PDE's.** Let us start this subsection with some standard basic results of the theory of first order partial differential equations. We follow the book by L. C. Evans [7] Chapter n. 3 but the subject is quite well known and there are many classical references.

Let $F : \mathbb{R}^n \times \mathbb{R} \times \bar{U} :\longrightarrow \mathbb{R}$ be a C^2 function where U is an open subset of $\mathbb{R}^n$ and $\bar{U}$ is its closure. The system of first order, partial differential equations defined by F is given by

$$F(Du, u, x) = 0$$

where $u : \bar{U} \longrightarrow \mathbb{R}$ is the unknown. Let us write

$$F(p, z, x) = F(p_1, p_2, .., p_n, z, x_1, x_2, .., x_n)$$

and denote by

$$D_pF = (F_{p_1}, F_{p_2}, .., F_{p_n}),\ D_zF = F_z,\ D_xF = (F_{x_1}, F_{x_2}, .., D_{x_n})$$

the differentials of F with respect to the variables p, z, x. The theory of the characteristics associates a system of first order differential equations to the system $F(Du, u, x) = 0$ in the following way. We look for smooth curves $x(s) =$

$(x^1(s),..,x^n(s))$ for $s \in \mathcal{I}$ defined in some open interval, and consider the function $z(s) = u(x(s))$. Let $p(s) = Du(x(s))$, where $p(s) = (p^1(s),..,p^n(s))$ is given by $p^i(s) = u_{x_i}(x(s))$. Differentiating with respect to s we obtain the characteristic equations

$$\begin{aligned} p'(s) &= -D_x F(p(s), z(s), x(s)) - D_z F(P(s), z(s), x(s)) p(s) \\ z'(s) &= D_p F(p(s), z(s), x(s)) p(s) \\ x'(s) &= D_p F(p(s), z(s), x(s)) \end{aligned}$$

This setting extends of course to smooth finite dimensional manifolds, by taking local coordinate systems.

Euler-Lagrange equations in a Riemannian manifold, a system of second order differential equations, is equivalent to a first order system of partial differential equations in the tangent bundle of the manifold. The above procedure applied to this system gives rise to the Hamilton equations in the cotangent bundle, a system of ordinary first order differential equations.

Euler-Lagrange equations in the case of Riemannian metrics are expressed in terms of the Levi-Civita connection by the system

$$\langle \nabla_X X, e_i \rangle = 0$$

where X is the vector field tangent to a geodesic and e_i, $i = 1, 2, .., n$ is a coordinate basis of the tangent space of the n-dimensional manifold. This is exactly what we did in the previous subsection. The tangent space $T\mathcal{N}$ and the cotangent space $T^*\mathcal{N}$ of $\mathcal{N}$ are analytic manifolds as well, and we are looking for solutions of Euler-Lagrange equations in finite dimensional submanifolds of $T\mathcal{N}$.

Therefore, as a consequence of Lemma 5.8 and Theorem 6.4 in the last section we get an existence of solutions result for the partial differential equation of geodesics.

Lemma 5.9. *Under the assumptions of Lemma 5.8, there exist $\rho > 0$, $D > 0$, such that given a unit vector $X(0) \in T_A\mathcal{N}$ there exists a unique analytic curve $\gamma : (-\rho, \rho) \longrightarrow \mathcal{N}$ such that $\gamma(0) = A$, and $\gamma'(t) = X(t)$ is the unique solution of the equation (35) whose initial condition is $X(0)$. The solution $X(t)$ is defined in an interval $\mid t \mid \leq \rho$, and the norms of $X(t)$, $X'(t)$ are bounded by D for every $\mid t \mid \leq \rho$.*

Proof. By the theory of first order partial differential equations, the system (35) that is a second order, partial differential system in the curve $\gamma(t)$ is equivalent to a system of first order ordinary differential equations $\frac{d}{dt}Y = F_m(Y)$ where the function F_m depends on the first fundamental form A and its derivatives with respect to the coordinates t_n. These functions have uniformly bounded norm in the neighborhood $B(r)$ and are analytic. Then, Theorem 6.4 implies the existence and uniqueness of solutions of the ordinary differential equations, namely, there exists $\rho > 0$ such that the solution $\gamma_m(t)$ of (1) with initial condition $\gamma_m(0) = A$, $\gamma_m'(0) = X(0)$, is unique and defined in $(-\rho, \rho)$.

$$\frac{d}{dt} \parallel Y \parallel \leq \parallel F \parallel \parallel Y \parallel$$

which yields that

The uniform bound for the sup norm of F_m in $B(r)$ implies that there exists $\rho > 0$ such that the analyitic solutions $\gamma_m(t)$ are defined in $(-\rho, \rho)$ and are uniformly bounded in this interval.

Then Theorem 6.3 implies that there exists a convergent subsequence with limit $\gamma(t)$ analytic in the interval $(-\rho,\rho)$. The function $\gamma(t)$ is tangent to the curve of vectors $X(t)$ that s the limit of the convergent subsequence of the curves $\gamma_m'(t) = X_m(t)$ in $(-\rho,\rho)$.

Claim: The curve $\gamma(t)$ is a geodesic.

Since $X_m(t)$ converges uniformly to $X(t)$ in the interval $(-\rho,\rho)$ we have that given $\epsilon > 0$ there exists m_ϵ such that for every $m \geq m_\epsilon$ we have

$$\| F_m(X_m'(t)) - F_m(X(t)) \|_\infty \leq k \| X_m(t)) - X(t) \|_\infty \leq \epsilon$$

where k is a constant depending on the (unform) bounds of the first derivatives of the funcitons F_m. So we get that $X(t)$ is an approximate solution of the systems defined by the functions F_m:

$$\begin{aligned} \| X' - F_m(X) \|_\infty &\leq \| X' - X_m' \|_\infty + \| X_m' - F_m(X_m) \|_\infty + \| F_m(X_m - F_m(X) \|_\infty \\ &\leq 2\epsilon \end{aligned}$$

if we choose m_ϵ such that $\| X_m' - X' \|_\infty < \epsilon$ for every $m \geq m_\epsilon$ as well. Now, notice that the equation $\frac{d}{dt}Y = F_m(Y)$ is equivalent to the system $\langle \nabla_Y Y, v_k\rangle = 0$, for every $0 < k \leq m$, which means that

$$| \langle \nabla_X X, v_k \rangle | \leq \epsilon$$

for every $0 < k \leq m$. Since ϵ may be chosen arbitrarily, we conclude that $\langle \nabla_X X, v_m\rangle = 0$ for every m, which implies that the vector field $\nabla_X X$ is identically zero, because the collection of the vectors v_m is a base for the L^2 inner product in $T\mathcal{N}$. This yields that the curve $\gamma(t)$ is a geodesic as we claimed.

□

5.5. **Parallel transport and Fermi coordinates for local surfaces.** The proof of Theorem 2.2 is similar to the proof of Theorem 2.1, we shall sketch the proof in some steps to avoid repetition of arguments. Let $A \in \mathcal{N}$, $X \in T_A\mathcal{N}$, $\gamma : (-\epsilon,\epsilon) \longrightarrow \mathcal{N}$ the geodesic such that $\gamma(0) = A$, $\gamma'(0) = X$. Let $\gamma'(t) = X(t)$, and consider a countable basis e_n of $T_A\mathcal{N}$ such that $e_1 = X$.

Let us define a family of local n-dimensional submanifolds S_n of $\mathcal{N}$ in the following way. Let $v : (-\epsilon,\epsilon) \longrightarrow T_A\mathcal{N}$ be the curve $v(t) = (\Pi_A)^{-1}(\gamma(t))$, where $\Pi_A : T_A\mathcal{N} \longrightarrow \mathcal{N}$ is the restriction of Π to $T_A\mathcal{N}$. Since Π_A is a local diffeomorphism in a small ball around $0 \in T_A\mathcal{N}$ the curve $v(t)$ is analytic and tangent to X at $t = 0$. Let us consider the subsets W_n of functions in $T_A\mathcal{N}$

$$W_n = \cup_{|t_i|<\epsilon}\{X(t_1) + \sum_{i=2}^{n} t_i e_i\}.$$

It is a n-dimensional submanifold of functions whose tangent space at A contains the vectors $X, e_2, .., e_n$. Since $D\Pi$ is close to the identity in an open neighborhood of $T_A\mathcal{N}$ we have that

$$S_n = \Pi(W_n)$$

is a family of parametrized smooth n-dimensional submanifolds in $\mathcal{N}$. Notice that this family is slightly different from the family V_n considered in the previous subsection. The point is that the geodesic $\gamma(t)$ now is a coordinate axis of S_n, $t = t_1$ is the first coordinate of the parametrization. We can suppose that the coordinate tangent vector fields $\sigma_n = D_{v(t)}\Pi(e_n)$ are perpendicular to $X(t) = \sigma_1(\gamma(t))$ for every $t \in (-\epsilon, \epsilon)$ (otherwise we just orthogonalize them along $\gamma(t)$).

To find a local surface S parametrized in Fermi coordinates we start by choosing a vector $Y \in T_A\mathcal{N}$, and we would like to solve the equation

$$\nabla_{X(t)} Y(t) = 0, \tag{36}$$

where $Y(t)$ is a vector field defined in $\gamma(t)$ such that $Y(0) = Y$, which amounts to solve the system of equations

$$\langle \nabla_X Y, \sigma_n \rangle = 0$$

in each S_n for every n. In the finite dimensional case, we can parametrize open neighborhoods of the Riemannian manifold with Fermi coordinates. We are not going to show that in our case (we do not need for the proof of Theorem 2.2). However, we shall make the following assumption on $Y(t)$ that is satisfied in the finite dimensional case: Y is a vector field defined in an open neighborhood of A which commutes with the coordinate vector fields σ_n at $\gamma(t)$. If we show that the above system has a solution under this hypothesis we find the parallel transport of Y along $\gamma(t)$ and Theorem 2.3 proceeds.

Let us orthogonalize the vector fields σ_n to get vector fields $\bar{\sigma}_n$ that might not be coordinate vector fields, although they are along $\gamma(t)$. The vector fields $\bar{\sigma}_n$ continue to form a basis of $T_B\mathcal{N}$ for B in an open neighborhood of A. The expression of the parallel transport system in this base is, according to the equation of the Levi-Civita connection,

$$\langle \nabla_X Y, \bar{\sigma}_n \rangle = 0 = \frac{1}{2}(X\langle Y, \bar{\sigma}_n\rangle - \bar{\sigma}_n\langle X, Y\rangle). \tag{37}$$

Let us consider the orthogonal projection Y_n of Y in the subspace generated by the vectors $\bar{\sigma}_i$, $i = 1, 2, .., n$. We have $Y_n = \sum_{i=1}^n y_i\bar{\sigma}_i$, for $y_i = \langle Y, \bar{\sigma}_i\rangle$. Replacing in the system we get

$$X(y_n) = \sum_{i=1}^n \bar{\sigma}_n(y_i\langle X, \bar{\sigma}_i\rangle).$$

The functions $\langle X, \bar{\sigma}_i\rangle$ are known, and we can get from this system another system in terms of the coordinate vector fields σ_i that is close to it (let us remind that $\sigma_i = \bar{\sigma}_i$ along γ). Both systems are first order, partial differential equations systems with uniformly bounded coefficients by Proposition 5.3. As in the proof of Lemma 5.9, we get a family $Y_n(B)$ of solutions defined in an open neighborhood of A because of Theorem 6.4, and letting n tend to ∞ we get a solution $Y(t)$ for the parallel transport of $Y = Y(0)$ along $\gamma(t)$ by Theorem 6.3.

6. On the existence and uniqueness of solutions of differential equations in $\mathcal{N}$

Let us now proceed to the proof of Picard's Theorem in our infinite dimensional setting. We start with the Arzela-Ascoli theorem. We shall state the main resuts for the shift, for the expanding map $T(x) = 2x(mod.1)$ in S^1 the results are analogous.

Theorem 6.1. *Let (X, d) be a second countable compact metric space (namely, there exists a countable dense subset). Let $\mathcal{F}$ be a family of functions $f : X \longrightarrow \mathbb{R}$ that is uniformly bounded and equicontinuous. Then every sequence in $\mathcal{F}$ has a convergent subsequence in the set of continuous functions.*

Proof. The proof follows from the same steps of the usual version of the theorem for compact subsets of $\mathbb{R}^n$. □

This implies

Lemma 6.2. *Let $\Sigma = \{0,1\}^{\mathbb{N}}$, endowed with the metric*

$$d(\{a_n\},\{b_n\}) = \frac{1}{2}\sum_{i=0}^{\infty} \frac{\mid a_i - b_i \mid}{2^i}.$$

Let $Hol_{C,\alpha}(\Sigma)$ be the set of Hölder continuous functions $f : \Sigma \longrightarrow \mathbb{R}$ with constant C and exponent α endowed with the sup norm. Then every subset of $Hol_{C,\alpha}$ of uniformly bounded functions is precompact.

Proof. First of all, observe that (Σ, d) is a compact metric space with a countable dense subset, the set of periodic sequences of 0's and 1's. Then Theorem 6.1 holds, and since the set of functions in $Hol_{C,\alpha}$ is equicontinuous, every uniformly bounded subset has a convergent subsequence. □

Next, let us study the precompactness of the set of analytic curves of normalized potentials $\gamma : (a,b) \longrightarrow Hol_{C,\alpha}(X)$ endowed with the sup norm. Analytic means that $\gamma(t)$ depends analytically on the parameter $t \in (a,b)$.

Proposition 6.3. *Let $\Gamma_{C,\alpha}([a,b],\Sigma)$ be the set of curves $\gamma : [a,b] \longrightarrow Hol_{C,\alpha}(\Sigma)$ of normalized potentials which are analytic in (a,b) and continuous in $[a,b]$, endowed with the sup norm. Then every family of functions in $\Gamma_{C.\alpha}([a,b],\Sigma)$ that is uniformly bounded and equicontinuous has a convergent subsequence. Namely, there exists a continuous function $\gamma_\infty : [a,b] \longrightarrow Hol_{C,\alpha}(\Sigma)$ that is analytic on (a,b) and a sequence of functions in $\Gamma_{C.\alpha}([a,b],\Sigma)$ converging uniformly to γ_∞.*

Proof. Let $\gamma_n \in \Gamma_{C,\alpha}([a,b],\Sigma)$ be a sequence of uniformly bounded curves. For simplicity, let us suppose that $a = -r, b = r$ for some $0 < r \leq 1$, and let us center the series expansion at $t_0 = 0$ (for different center of expansion the argument is just analogous). This implies that we get an expression in power series for each $\gamma_n(t)$ of the form

$$\gamma_n(t) = \sum_{m=0}^{\infty} a_m^n(p) t^m$$

where $a_m^n : \Sigma \longrightarrow \mathbb{R}$ are functions in $Hol_{C,\alpha}(\Sigma)$. Since the functions γ_n are uniformly bounded by a constant $L > 0$ in $(-r,r)$, we have that $\| a_0^n \|_\infty \leq L$ for every n and by Lemma 6.2 there exists a convergent subsequence $a_o^{n_i^0}$ whose limit is a function A_0. Since the radius of convergence of all the series is r, we have that $\limsup_n (\mid a_m^n(p) \mid)^{\frac{1}{n}} = \frac{1}{r}$ and therefore

$$\| a_m^n \|_\infty \leq \frac{1}{r^m}$$

for every n, m. So the family of functions $\mathcal{F}_m = \{a_m^n\}$ is uniformly bounded and we can apply again Lemma 6.2. So there exists a subsequence $n^0_{n^1_j}$ of the indices n^0_j such that the functions $a_m^{n_j^0}$ converge to a function $A_1 \in Hol_{C,\alpha}(\Sigma)$. By induction, we get a subsequence γ_{N_k} of the functions γ_n such that the first $k+1$ coefficients of their series expansions converge to functions $A_0, A_1, .., A_k$ in $Hol_{C,\alpha}(\Sigma)$.

Consider the function

$$\gamma_\infty(t) = \sum_{m=0}^{\infty} A_m(t).$$

By the choice of the A_m's, the above series converges with the same convergence radius of the functions γ_n. Moreover, it is easy to check that $\gamma_\infty(t)$ is a curve of functions in $Hol_{C,\alpha}(\Sigma)$, and we have that the sequence γ_{N_k} converges uniformly on compact sets to γ_∞ in the sup norm. Indeed, let $[a,b] \subset (-r,r)$, since the functions γ_n are uniformly bounded given $\epsilon > 0$ there exists $m_\epsilon > 0$ such that for every $n \in \mathbb{N}$, $k \geq m_\epsilon$ we have

$$\mid \sum_{k}^{\infty} a_k^n(p) t^k \mid \leq \epsilon$$

for every $p \in \Sigma$. The same holds for the series γ_∞. This yields

$$\begin{aligned} \parallel \gamma_\infty(t) - \gamma_n(t) \parallel_\infty &\leq \sum_{m=0}^{m_\epsilon} \parallel A_m - a_m^{N_k} \parallel_\infty t^m + \parallel \sum_{m_\epsilon+1}^{\infty} (A_m - a_m^{N_k}) \parallel_\infty t^m \\ &\leq \sum_{m=0}^{m_\epsilon} \parallel A_m - a_m^n \parallel_\infty t^m + 2\epsilon. \end{aligned}$$

Since the functions $a_m^{N_k}$ converge uniformly to the function A_m, we can chose k large enough such that $\parallel (A_m - a_m^{N_k}) \parallel_\infty \leq \frac{\epsilon}{m}$ and therefore

$$\parallel \gamma_\infty(t) - \gamma_n(t) \parallel_\infty \leq 3\epsilon$$

for every $t \in [-r,r]$ and since ϵ can be chosen arbitrarily we get the lemma. $\square$

Now, we can state Picard's Theorem for differential equations in $\mathcal{N}$.

Theorem 6.4. *Let $F : [x,y] \times U \longrightarrow Hol_{C,\alpha}(\Sigma)$ be an analytic function in $t \in (x,y)$ and in $Hol_{C,\alpha}(\Sigma)$, where U is an open subset of $(Hol_{C,\alpha}(\Sigma))^n$. Then, given $(t_0, f_1, f_2, .., f_n) \in (x,y) \times U$ there exists a unique solution of the differential equation $\frac{d}{dt}X(t) = F(t, X(t))$ defined in a certain interval $X : (t_0 - \epsilon, t_0 + \epsilon) \longrightarrow U$ that is analytic and satisfies $X(t_0) = (f_1, f_2, .., f_n)$.*

Proof. The proof mimics the usual proof of Picard's theorem applying the idea of contracion operators. The operator

$$L(g)(t) = (f_1, f_2, .., f_n) + \int_{t_0}^{t} F(s, g(s)) ds$$

is defined in the set of continuous curves $g : [x,y] \longrightarrow (Hol_{C,\alpha}(\Sigma))^n$ that are analytic on (x,y). According to Lemma 6.3, this set of curves endowed with the sup norm is co-compact. Now, as in the proof of the usual version of Picard's theorem, there exists a small interval $(t_0 - \epsilon, t_0 + \epsilon)$, where $\epsilon > 0$ depends on the sup norm of the first derivatives of the function F, where the above operator restricted to curves defined in $(t_0 - \epsilon, t_0 + \epsilon)$ is a contraction. Therefore, by Lemma 6.3, there exists a unique fixed point $X(t)$ that must be the solution of the equation claimed in the statement. The solution is analytic in since the function F is analytic. $\square$

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