Equiaffine Darboux Frames for Codimension 2 Submanifolds contained in Hypersurfaces

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Abstract. Consider a codimension 1 submanifold $N^n \subset M^{n+1}$, where $M^{n+1} \subset \mathbb{R}^{n+2}$ is a hypersurface. The envelope of tangent spaces of M along N generalizes the concept of tangent developable surface of a surface along a curve. In this paper, we study the singularities of these envelopes.

    There are some important examples of submanifolds that admit a vector field tangent to M and transversal to N whose derivative in any direction of N is contained in N. When this is the case, one can construct transversal plane bundles and affine metrics on N with the desirable properties of being equiaffine and apolar. Moreover, this transversal bundle coincides with the classical notion of Transon plane. But we also give an explicit example of a submanifold that do not admit a vector field with the above property.

1. Introduction

Consider a surface $M \subset \mathbb{R}^3$ and let $\gamma : I \to \mathbb{R}$ be a smooth curve, where $I \subset \mathbb{R}$ is an interval. Under some general hypothesis, one can find a unique, up to sign, vector field ξ tangent to M along γ such that $\xi'(t)$ is tangent to γ, for any $t \in I$. The developable surface

$$OD_\gamma(t, u) = \gamma(t) + u\xi(t), \quad t \in I, \quad u \in \mathbb{R},$$

is called the *tangent developable* of M along γ and has been extensively studied. The natural Darboux frame to consider here is $\{\gamma'(t), \gamma''(t), \xi(t)\}$, where the parameterization $\gamma(t)$ satisfies $\gamma'''(t)$ tangent to M. Writing $\xi'(t) =$

$-\sigma(t)\gamma'(t)$, we have that, from the point of view of singularity theory, the behavior of the OD_γ in a neighborhood of $u = \sigma^{-1}(t)$ depends basically of the number of vanishing derivatives of σ ([4],[5]).

In this paper, we want to generalize this construction to arbitrary codimension 1 submanifolds $N^n \subset M^{n+1}$, where $M^{n+1} \subset \mathbb{R}^{n+2}$ is a hypersurface. As we shall see, under general hypothesis there exists a vector field ξ tangent to M along N such that $D_X\xi$ is tangent to M, for any X tangent to N. The direction determined by ξ is unique and we call it the *(osculating) Darboux direction*. The hypersurface

$$ET_N(p, u) = p + u\xi(p), \quad p \in N, \quad u \in \mathbb{R},$$

generalizes the tangent developable surface of a surface along a curve and we shall call *envelope of tangent spaces* of M along N. We shall verify that ET_N is smooth when u is not the inverse of a non-zero eigenvalue of the shape operator associated with ξ. The singularities of ET_N are also studied: We show by examples that all simple singularities are realizable.

For a fixed vector field ξ in the osculating Darboux direction, we can define a metric $g = g_\xi$ on N and a distinguished transversal plane bundle $\sigma = \sigma(\xi)$, the *affine normal plane bundle*. It is natural to consider a basis $\{\xi, \eta\}$ of σ such that $[X_1, ..., X_n, \eta, \xi] = 1$, where $\{X_1, ..., X_n\}$ is a g_ξ-orthonormal basis of TN. But this condition does not determine uniquely the vector η. In fact, any $\bar{\eta} = \eta + \lambda\xi$ does the same job.

The main difficulty here is to choose a "good" vector field ξ in the Darboux direction. In the case of curves, we can choose a vector field ξ in the Darboux direction such that $D_X\xi$ is tangent to N, for any X tangent to N. We shall refer to this latter property by saying that ξ is *parallel*. On the other hand, when M is non-degenerate, we can choose ξ such that g_ξ coincides with the restriction of the Blaschke metric of M to N and the affine Blaschke normal belongs to the affine normal plane bundle. But this choice of ξ is parallel only in very special cases.

The metric g_ξ and the affine transversal plane bundle $\sigma(\xi)$ have more desirable properties when ξ is parallel. Denote by $\omega = \omega(g_\xi)$ the volume form on N determined by the metric g_ξ and by $\nabla = \nabla(\xi)$ the connection determined by σ. The pair $(\nabla, g)(\xi)$ is equiaffine if $\nabla(\omega_g) = 0$. We can also define the cubic form C^2 and the metric h^2 relative to the vector field $\eta = \eta(\xi)$. We say that (C^2, h^2) is apolar if $tr_{h^2}C^2(X, \cdot, \cdot) = 0$, for any X tangent to N. We shall verify that the properties (∇, g) equiaffine and (C^2, h^2) apolar are both equivalent to the parallelism of ξ.

Consider now hyperplanes H containing the tangent space of N. The intersection of H with M determine a codimension 1 submanifold of the hyperplane H and thus we can consider its Blaschke normal vector $\eta(H)$. When we vary H, the vector $\eta(H)$ describes a 2-plane called the *Transon plane* of T_pN with respect to M. In the case of curves, this is a very classic result of A.Transon ([10]), see [6] for a modern reference. We shall verify that the Transon plane coincides with the affine normal plane if and only if ξ is parallel.

The latter two paragraphs show that the condition of ξ parallel is very significant. So it is natural to ask whether, for a given immersion $N \subset M$, a parallel vector field ξ exists or not. There are several examples of immersions $N \subset M$ that admit a parallel vector field: Curves in surfaces, submanifolds contained in hyperplanes, visual contour submanifolds, submanifolds contained in hyperquadrics. But there are also examples of immersions that do not admit parallel vector fields, and we give explicitly such an example. We prove also that the existence of a parallel vector field is equivalent to the flatness of the affine normal bundle of the immersion $N \subset M$.

The paper is organized as follows: In section 2 we discuss the generalization of the osculating Darboux direction and tangent developable surfaces to codimension 2 submanifolds contained in hypersurfaces. In section 3, we study the singularities of these hypersurfaces. In section 4, we recall the constructions of the affine metric and the affine normal plane bundle associated with a vector field ξ in the osculating Darboux direction. In section 5 we define the parallelism condition of ξ and show the equivalence of this property and the equiaffinity of (∇, g) and the apolarity of (C^2, h^2). We also give some important examples of immersions $N \subset M$ with parallel vector fields. In section 6 we recall the notion of Transon planes and prove that it coincides with the affine normal plane if and only if ξ is parallel. Finally in section 7 we show an example of an immersion $N \subset M$ that does not admit a parallel vector field and prove that the existence of a parallel vector field is equivalent to the flatness of the affine normal bundle connection.

2. Darboux directions and the envelope of tangent spaces

Consider a codimension 1 immersion $N^n \subset M^{n+1}$, where $M^{n+1} \subset \mathbb{R}^{n+2}$ is a hypersurface.

2.1. Basic equations

Fix vector fields η transversal to M and ξ tangent to M transversal to N. For X, Y vector fields tangent to N, we write

$$D_X Y = \nabla_X Y + h^1(X, Y)\xi + h^2(X, Y)\eta, \qquad (1)$$

where $\nabla_X Y$ is tangent to N. It is straightforward to verify that ∇ is a torsion-free connection on N and h^i, $i = 1, 2$ are bilinear symmetric forms. Write also

$$\nabla_{X_i} X_j = \sum_{k=1}^{n} \Gamma_{ij}^k X_k, \quad 1 \le i, j \le n, \qquad (2)$$

where Γ_{ij}^k are the Christoffel symbols of the connection.

The derivatives of η and ξ can be written as

$$\begin{aligned} D_X \xi &= -S_1 X + \tau_1^1(X)\xi + \tau_1^2(X)\eta, \\ D_X \eta &= -S_2 X + \tau_2^1(X)\xi + \tau_2^2(X)\eta, \end{aligned} \qquad (3)$$

where S_i, $i = 1, 2$, are $(1, 1)$-tensors of N called *shape operators* and τ_i^j are 1-forms on N.

2.2. Osculating Darboux direction

In this section, we generalize the notion of osculating Darboux direction from curves $\gamma \subset M$ to codimension 1 submanifolds $N \subset M$. Given a local frame $\{X_1, ..., X_n\}$ of TN, we say that the immersion $N \subset M \subset \mathbb{R}^{n+2}$ is *non-degenerate* if the matrix $h^2(X_i, X_j)$ is non-degenerate.

Lemma 2.1. *The non-degeneracy condition is independent of the choice of the local frame $\{X_1, ..., X_n\}$ of TN, of the vector field ξ tangent to M and of the transversal vector field η.*

Proof. Suppose we fix ξ and η and let $\{Y_1, ..., Y_n\}$ be a local frame of TN. Then we can write

$$Y_i = \sum_{i=1}^{n} a_{ij} X_j$$

for a certain invertible matrix $A = (a_{ij})$. It is not difficult to verify that $\left(h^2(Y_i, Y_j)\right) = A \left(h^2(X_i, X_j)\right) A^t$, thus proving that the non-degeneracy condition is invariant by a change of basis of TN. If we change ξ by $\bar{\xi}$ satisfying

$$\xi = \sum b_k X_k + \beta \bar{\xi},$$

then $\bar{h}^2(X_i, X_j) = h^2(X_i, X_j)$ and so the non-degeneracy condition is invariant by the choice of the vector ξ. Finally if we write

$$\eta = \sum b_k X_k + \beta \xi + \gamma \bar{\eta},$$

then $\bar{h}^2(X_i, X_j) = \gamma h^2(X_i, X_j)$, thus completing the proof of the lemma. $\square$

Proposition 2.2. *Assume that the immersion $N^n \subset M^{n+1} \subset \mathbb{R}^{n+2}$ is non-degenerate. There exist a unique direction ξ tangent to M along N and transversal to N such that $D_X \xi$ is tangent to M, for any $X \in T_p N$. We shall call this direction the osculating Darboux direction of $N \subset M$.*

Proof. We first remark that if $D_X \xi$ is tangent to M for any $X \in T_p M$, the same holds for $\lambda \xi$, for any $\lambda : N \to \mathbb{R}$. Take any ξ_1 tangent to M and write

$$\xi = \sum_{j=1}^{n} \alpha_j X_j + \xi_1.$$

Then the component of $D_{X_i} \xi$ in the direction η is $\sum_{j=1}^{n} \alpha_j h^2(X_i, X_j) + \tau_1^2(X_i)$. Thus we have to solve the system

$$\sum_{j=1}^{n} \alpha_j h^2(X_i, X_j) + \tau_1^2(X_i) = 0, \quad 1 \le i \le n,$$

which admits a unique solution by the non-degeneracy hypothesis. $\square$

Remark 2.3. In the case of curves, the non-degeneracy hypothesis is equivalent to $\gamma''(t) \notin T_{\gamma(t)} M$, i.e., the osculating plane of γ does not coincide with the tangent plane of M.

Next example shows that the non-degeneracy hypothesis is necessary:

Example 2.4. Consider M given by $\psi(u, v) = (u, v, uv)$ and N given by $\gamma(v) = (0, v, 0)$. Any tangent vector field along γ can be written as $B(v) = a(1, 0, v) + b(0, 1, 0)$. Thus $B'(v) = (a'(v), b'(v), a(v) + a'(v)v)$. We conclude that $B'(0) = (a'(0), b'(0), a(0))$ is tangent to M if and only if $a(0) = 0$. But then B is tangent to γ.

Remark 2.5. The hypersurface $M \subset \mathbb{R}^{n+2}$ is called *non-degenerate* if the $(n + 1) \times (n + 1)$ matrix $(h(X_i, X_j))$ is invertible, where

$$D_{X_i} X_j = \tilde{\nabla}_{X_i} X_j + h(X_i, X_j)\eta,$$

$\tilde{\nabla}_{X_i} X_j$ is tangent to M and $X_{n+1} = \xi$. In this case, the osculating Darboux direction ξ is h-orthogonal to the tangent space of N. In fact, since $h(X, \xi) = 0$, for any $X \in TN$, we have that $D_X \xi = \tilde{\nabla}_X X_{n+1}$ is tangent to M.

Remark 2.6. To define the osculating Darboux direction ξ we need only to know the tangent space to M at each point of N. Thus, instead starting with a codimension 2 submanifold N contained in a hypersurface M, we could also have started with a codimension 2 submanifold N together with a hyperspace bundle containing the tangent space of N, without an explicit mention to M.

2.3. Envelope of Tangent Spaces of M along N

Consider a curve $\gamma \subset$ where M is a surface in $\mathbb{R}^3$. The surface

$$OD_\gamma(t, u) = \gamma(t) + u\xi(t) \tag{4}$$

is called the *developable tangent surface* of M along γ and has been studied by many authors ([4],[5]).

   We can generalize the above definition to arbitrary dimensions. Let $\phi : U \subset \mathbb{R}^n \to \mathbb{R}^{n+2}$ be a parameterization of N and define the *envelope of tangent spaces* of M along N by

$$ET_N(t, u) = \phi(t) + u\xi(t) \tag{5}$$

for $t \in U$ and $\xi(t)$ in the osculating Darboux direction.

   Denote by $\{X_1, ..., X_n\}$ the local frame of N given by $X_i = D_{t_i}\phi$. The hypersurface ET_N can be studied by considering $F : U \times \mathbb{R}^{n+2} \to \mathbb{R}$ defined by

$$F(t, x) = [X_1(t), ..., X_n(t), \xi(t), x - \phi(t)]. \tag{6}$$

Observe that $F = 0$ is the equation of the tangent space of M at a point of N. The *discriminant set* or *envelope* of F is defined by

$$\mathcal{D}_F = \{x \in \mathbb{R}^{n+2} | F(t, x) = F_{t_1}(t, x) = ... = F_{t_n}(t, x) = 0, \text{for some}$$
$$t = (t_1, ..., t_n) \in U\}. \tag{7}$$

Next lemma justifies the name envelope of tangent spaces given to ET_N.

Lemma 2.7. *The envelope $\mathcal{D}_F$ of F coincides with the hypersurface defined by equation* (5).

Proof. Observe that

$$F_{t_i}(t,x) = a_i(t)F(t,x) + \sum_{l=1}^{n} h^2(X_i,X_l) \left[X_1,...,\eta,...,X_n,\xi,x-\phi(t)\right], \quad (8)$$

where $a_i = \tau_1^1(X_i) + \sum_{l=1}^{n} \Gamma_{li}^i$, and η is placed in the coordinate l in the second parcel of the second member. Since the matrix $(h^2(X_i,X_j))$ is non-degenerate, $F = F_{t_1} = ... = F_{t_n} = 0$ if and only if $F = G_1 = ... = G_n = 0$, where

$$G_l = \left[X_1,...,\eta,...,X_n,\xi,x-\phi(t)\right]. \quad (9)$$

This implies $x - \phi(t) = u\xi$, for some $u \in \mathbb{R}$, thus proving the lemma. $\qquad\square$

The regression points of F are points of its discriminant set where its hessian $D_{tt}F$ is degenerate.

Lemma 2.8. *The regression points of F correspond to points where u is the inverse of some non-zero eigenvalue of S_1.*

Proof. We may assume that $[X_1,...,X_n,\eta,\xi] = 1$. At a point of the discriminant set of F, we have $F = G_1 = ... = G_n = 0$. Using that the matrix $(h^2(X_i,X_j))$ is non-degenerate, it is not difficult to see that, at $\mathcal{D}(F)$, the matrix $(F_{t_it_j})$ is degenerate if and only if the matrix $((G_i)_{t_j})$ is degenerate. Differentiating equation (9) we obtain that, at $\mathcal{D}(F)$,

$$(G_i)_{t_j} = -\left[X_1,...,\eta,...,X_n,S_1(X_j),x-\phi(t)\right] + \left[X_1,...,\eta,...,X_n,\xi,-X_j\right]$$

where η is placed in coordinate i. Thus, at these points,

$$(G_i)_{t_j} = us_{ij} - \delta_{ij}$$

where s_{ij} is the (i,j)-entry of the matrix of S_1 in basis $\{X_1,...,X_n\}$. We conclude that the matrix $((G_i)_{t_j})$ at $\mathcal{D}(F)$ is exactly $uS_1 - Id$, thus proving the lemma. $\qquad\square$

Next corollary gives condition for the smoothness of ET_N:

Corollary 2.9. *If u is not the inverse of a non-zero eigenvalue of S_1, the hypersurface ET_N is smooth.*

Proof. Consider the map $G : U \times \mathbb{R}^{n+2} \to \mathbb{R}^{n+1}$ given by

$$G(t,x) = (F(t,x), F_{t_1}(t,x), ..., F_{t_n}(t,x)).$$

Then $ET_N = G^{-1}(0)$ and we shall verify that 0 is a regular value of G. But

$$DG = \begin{bmatrix} 0 & F_x \\ F_{tt} & F_{xt} \end{bmatrix}.$$

By the above lemma, F_{tt} is non-degenerate. On the other hand, by considering derivatives in the direction η one easily verifies that $F_x \neq 0$. This shows that 0 is a regular value of G, thus proving the corollary. $\qquad\square$

3. Singularities of the envelope of tangent spaces

In this section we study the singularities of ET_N. We begin with the case of curves, where a complete classification is given. For the general case, we show by examples that any simple singularity can occur.

3.1. Singularities of the tangent developable surface

Let $M \subset \mathbb{R}^3$ be a surface and $\gamma : I \to M$ a smooth curve. Denote $S_1\gamma'(t) = -\sigma(t)\gamma'(t)$ and $S_2\gamma'(t) = -\mu(t)\gamma'(t)$, where $\eta(t) = \gamma''(t)$. We may assume that $[\gamma'(t), \eta(t), \xi(t)] = 1$, for any $t \in I$, which implies that $\tau_1^1 + \tau_2^2 = 0$. The Frenet equations are then

$$\begin{cases} (\gamma')' = \eta \\ \eta' = -\mu\gamma' - \tau_1^1\eta + \tau_2^1\xi \\ \xi' = -\sigma\gamma' + \tau_1^1\xi. \end{cases}$$

Next proposition is proved in [4] using euclidean invariants. We give here a proof using affine invariants.

Proposition 3.1. *Let $\gamma : I \to M$ be a smooth curve and $t_0 \in I$ with $\sigma(t_0) \neq 0$. For $u_0 = \sigma^{-1}(t_0)$, we have that, at $OD_\gamma(t_0, u_0)$,*

1. *$OD(\gamma)$ is locally diffeomorphic to a cuspidal edge if $[\sigma_t - \sigma\tau_1^1](t_0) \neq 0$.*
2. *$OD(\gamma)$ is locally diffeomorphic to a swallowtail if $[\sigma_t - \sigma\tau_1^1](t_0) = 0$ and $[\sigma_t - \sigma\tau_1^1]_t(t_0) \neq 0$.*

Remark 3.2. We shall see in section 5.1 that it is possible to parameterize γ such that $\tau_1^1 = 0$. With such a parameterization, the formulas of the above proposition become much simpler.

We shall need a well-known result from singularity theory ([3],[4]).

Lemma 3.3. *Let $F : I \times \mathbb{R}^r \to \mathbb{R}$ denote a r parameter unfolding of $f(t) = F(t, x_0)$. Assume that $f(t)$ has an A_k-singularity at $t = t_0$. The unfolding $F(t, x)$ is $\mathcal{R}$-versal if the $k \times r$ matrix $j^{k-1}F_x$ has rank k, where $j^k g$ denotes the k-jet of g.*

Now we can prove proposition 3.1.

Proof. In the case of curves, $F : I \times \mathbb{R}^3 \to \mathbb{R}$ is given by

$$F(t, x) = [\gamma'(t), \xi(t), x - \gamma(t)]. \tag{10}$$

Then $F_t = G + \tau_1^1 F$, where

$$G(t, x) = [\eta(t), \xi(t), x - \gamma(t)].$$

Thus $F = F_t = 0$ at $t = t_0$ if and only if $x = \gamma(t_0) + \lambda(t_0)\xi(t_0)$. Moreover $G_t = H - 1 - \mu F$, where

$$H(t, x) = \sigma(t)\,[\gamma'(t), \eta(t), x - \gamma(t)].$$

Thus $F = F_t = F_{tt} = 0$ at $t = t_0$ if and only if $\sigma(t_0) \neq 0$ and $\lambda(t_0) = \sigma^{-1}(t_0)$. Differentiating again we obtain

$$H_t = \frac{\sigma_t}{\sigma}H - \tau_1^1 H + \tau_2^1\sigma F. \tag{11}$$

Thus $F = F_t = F_{tt} = 0$ and $F_{ttt} \neq 0$ at $t = t_0$ if and only if $x = \gamma(t_0) + \sigma^{-1}(t_0)\xi(t_0)$ and $\sigma_t - \tau_1^1\sigma \neq 0$. In this case, F has an A_2 singularity. Differentiating once more we obtain, at points where $F = F_t = F_{tt} = F_{ttt} = 0$,

$$H_{tt}(t_0, x) = (\sigma_t - \sigma\tau_1^1)_t\sigma^{-1}(t_0).$$

We conclude that $F = F_t = F_{tt} = F_{ttt} = 0$ and $F_{tttt} \neq 0$ at $t = t_0$ if and only if $x = \gamma(t_0) + \sigma^{-1}(t_0)\xi(t_0)$, $[\sigma_t - \sigma\tau_1^1](t_0) = 0$ and $[\sigma_t - \sigma\tau_1^1]_t(t_0) \neq 0$. In this case, F has an A_3 singularity.

To complete the proof, we must prove that F is a $\mathcal{R}$-versal unfolding of f. Observe that $F_x(t, x_0) = \gamma'(t) \times \xi(t)$, where $\times$ denotes vector product. For an A_2 point we write

$$j^1 F_x(t_0, x_0) = \begin{bmatrix} \gamma'(t_0) \times \xi(t_0) \\ \gamma''(t_0) \times \xi(t_0) \end{bmatrix},$$

which has rank 2. For an A_3 point we write

$$j^2 F_x(t_0, x_0) = \begin{bmatrix} \gamma'(t_0) \times \xi(t_0) \\ \gamma''(t_0) \times \xi(t_0) \\ \sigma(t_0)\gamma'(t_0) \times \gamma''(t_0) \end{bmatrix}.$$

Since $\sigma(t_0) \neq 0$, this matrix has rank 3. By lemma 3.3, F is a versal unfolding of a point A_k, $k = 2, 3$. $\square$

3.2. Realization of simple singularities of ET_N

In this section, we give several examples of singularities that occur in ET_N. Through these examples, we show that any simple singularity can appear in ET_N. We recall that any simple singularity is $\mathcal{R}$-equivalent to A_k, $k \geq 2$, D_k, $k \geq 4$, E_6, E_7 or E_8 (see [3], ch.11).

Consider $M \subset \mathbb{R}^{n+2}$ given by the graph of $f(t, y)$, $t = (t_1, ..., t_n)$. Then M is given by

$$\psi(t, y) = (t_1, ..., t_n, y, f(t, y)).$$

Thus

$$\psi_{t_i} = (e_i, 0, f_{t_i}); \quad \psi_y = (0, 1, f_y),$$

where $e_i = (0, .., 1, ...0)$ with 1 in the component i. We shall assume that $f = f_{t_i} = f_y = 0$ at the origin, for any $1 \leq i \leq n$. Let N be the submanifold $y = g(t)$ and assume that $g_{t_i} = 0$ at $t = 0$, i.e., the tangent plane of N is generated by $\{e_i\}$, $1 \leq i \leq n$.

Let $x = (x_1, ..., x_{n+2})$ and write the vector field ξ as

$$\xi(t) = \sum_{i=1}^{n} a_i(t)\psi_{t_i} + \psi_y.$$

Then $F(t, x) = \det(\psi_{t_1}(t), ..., \psi_{t_n}(t), \xi(t), x - \psi(t))$ can be written as

$$F(t, x) = \det(\psi_{t_1}(t), ..., \psi_{t_n}(t), \psi_y(t), x - \psi(t)).$$

We conclude that

$$F = f - x_{n+2} + \sum_{i=1}^{n} f_{t_i}(x_i - t_i) + f_y(x_{n+1} - g),$$

where f, f_{t_i}, f_y are taken at $(t, g(t))$.

Lemma 3.4. *Assume that $(f_{t_i t_j}(0))$ is the identity matrix and $f_{yt_i}(0) = 0$, for any i. Then ψ_y is the Darboux direction at 0. Moreover, the shape operator S_1 at the origin is given by $(f_{t_i t_j y}(0))$.*

Proof. First observe that $\psi_{yt_i}(0) = 0$, for any i. This implies that ψ_y is the Darboux direction at the origin. Moreover

$$\xi_{t_j} = \sum_{i=1}^{n} (a_i)_{t_j} \psi_{t_i} + \sum_{i=1}^{n} a_i \psi_{t_i t_j} + \psi_{yt_j}$$

Since these vectors are tangent to M and $\psi_{t_i t_j}, \psi_{yt_j}$ are co-linear and transversal to M, we obtain

$$\xi_{t_j} = \sum_{i=1}^{n} (a_i)_{t_j} \psi_{t_i}. \tag{12}$$

This implies that

$$\sum_{i=1}^{n} a_i f_{t_i t_j} + f_{yt_j} = 0 \tag{13}$$

Observe that $a_i = 0$ at 0. Differentiating equation (13) and taking $t = 0$ we obtain $(a_i)_{t_j}(0,0) = -f_{t_i t_j y}(0)$. Now equation (12) implies the second part of the lemma. □

Lemma 3.4 explicitly provides the Darboux direction and calculates the shape operator S_1 at the origin, thus indicating the way to find the realization of simple singularities of the envelope of tangent spaces of M along N. We shall now describe examples of functions $f(t, y)$ and $g(t)$ such that the corresponding families $F(t, x)$ given by equation (6) are versal unfoldings of functions $F(t, x_0)$, $x_0 = (0, ..., \sigma^{-1}, 0) \in ET_N$, with singular points of type A_k, $k \geq 2$, D_k, $k \geq 4$, E_6, E_7 and E_8 at $t = 0$. In each of the following examples, σ is eigenvalue of S_1, simple in case of A_k and double in cases of D_k and E_k. To simplify the formulas we have taken sometimes $\sigma = 1$.

Example 3.5. (1) Let

$$f(t, y) = \frac{t^2}{2} + \frac{1}{6} t^3 + \frac{\sigma}{2} t^2 y$$

and $g(t) = 0$. Then, close to $(0, \sigma^{-1}, 0)$,

$$F(t, x_1, x_2 + \sigma^{-1}, x_3) = -\frac{1}{3} t^3 + \frac{\sigma}{2} t^2 x_2 + (\frac{1}{2} t^2 + t) x_1 - x_3,$$

which is a versal unfolding of an A_2 point.

(2) Let

$$f(t, y) = \frac{t^2}{2} + \frac{1}{24} t^4 + \frac{\sigma}{2} t^2 y$$

and $g(t) = 0$. Close to $(0, \sigma^{-1}, 0)$,

$$F(t, x_1, x_2 + \sigma^{-1}, x_3) = -\frac{1}{8} t^4 + \frac{\sigma}{2} t^2 x_2 + (\frac{1}{6} t^3 + t) x_1 - x_3,$$

which is a versal unfolding of an A_3 point.

(3) Let

$$f(t_1, t_2, y) = \frac{1}{2}(t_1^2 + t_2^2) + \frac{\sigma}{2}t_1^2 y + t_1^3 t_2$$

For $\sigma = 1$, choose $g = -t_1^3 - 3t_1 t_2$. Then, close to $(0, 0, 1, 0)$,

$$F = t_1^5 - \frac{1}{2}t_2^2 + (t_1 - t_1^4)x_1 + (t_1^3 + t_2)x_2 + \frac{1}{2}t_1^2 x_3 - x_4,$$

which is a versal unfolding of an A_4 point.

(4) For general $k \geq 3$, let $\sigma = 1$, $t = (t_1, ..., t_{k-2})$, i.e., $n = k - 2$,

$$f(t, y) = \frac{1}{2}|t|^2 + \frac{1}{2}t_1^2 y + \sum_{j=2}^{k-2} t_1^{j+1} t_j,$$

and

$$g(t) = -t_1^{k-1} - \sum_{j=2}^{k-2}(j+1)t_1^{j-1}t_j$$

Then, close to $(0, ..., 1, 0)$,

$$F = t_1^{k+1} - \frac{1}{2}\sum_{j=2}^{k-2} t_j^2 + x_1(t_1 - t_1^k) + \sum_{j=2}^{k-2} x_j(t_j + t_1^{j+1}) + \frac{1}{2}t_1^2 x_{k-1} - x_k.$$

which is a versal unfolding of an A_k point.

Example 3.6. (1) Let

$$f(t_1, t_2, y) = \frac{1}{2}(t_1^2 + t_2^2) + \frac{\sigma}{2}(t_1^2 + t_2^2)y + t_1^3 + t_1 t_2^2$$

and $g = 0$. Then

$$F = -2(t_1^3 + t_1 t_2^2) - x_4 + x_1(t_1 + 3t_1^2 + t_2^2) + x_2(t_2 + 2t_1 t_2) + \frac{\sigma}{2}(t_1^2 + t_2^2)x_3$$

which is a versal unfolding of a D_4 point.

(2) For a general $k \geq 4$, take

$$f = \frac{1}{2}|t|^2 + \frac{y}{2}(t_1^2 + t_2^2) + t_1^{k-1} + t_1 t_2^2 + \sum_{j=3}^{k-2} t_1^j t_j + \sum_{j=3}^{k-2} t_1^{j-2} t_2^2 t_j$$

and $g = -\sum_{j=3}^{k-2} j t_j t_1^{j-2}$. Long but straightforward calculations show that, close to $(0, ..., 1, 0)$,

$$F = (2-k)t_1^{k-1} - 2t_1 t_2^2 - \frac{1}{2}\sum_{j=3}^{k-2} t_j^2 - x_k + \frac{1}{2}(t_1^2 + t_2^2)x_{k-1}$$

$$+ \sum_{j=3}^{k-2} x_j(t_1^j + t_1^{j-2}t_2^2 + t_j) + x_2 \left(t_2 + 2t_1 t_2 + \sum_{j=3}^{k-2}(2-j)t_1^{j-2}t_2 t_j \right)$$

$$+x_1\left(t_1+(k-1)t_1^{k-2}+t_2^2+\sum_{j=3}^{k-2}(j-2)t_1^{j-3}t_2^2t_j\right),$$

which is a versal unfolding of a D_k point.

Example 3.7. (1) Consider

$$f=\frac{1}{2}|t|^2+\frac{1}{2}(t_1^2+t_2^2)y+t_1^3+t_2^4+t_1t_2t_3+2t_1t_2t_3y+t_1t_2^2t_4+3t_1t_2^2t_4y$$

and $g=0$. Then

$$F=-2t_1^3-3t_2^4-\frac{1}{2}(t_3^2+t_4^2)-x_6+x_4(t_1t_2^2+t_4)+x_3(t_1t_2+t_3)$$

$$+x_1\left(t_1+3t_1^2+t_2^2t_4+t_2t_3\right)+x_2\left(t_2+4t_2^3+t_1t_3+2t_1t_2t_4\right)$$

$$+x_5\left(\frac{1}{2}(t_1^2+t_2^2)+2t_1t_2t_3+3t_1t_2^2t_4\right)$$

which is a versal unfolding of an E_6 point.

(2) Let

$$f=\frac{1}{2}|t|^2+\frac{1}{2}(t_1^2+t_2^2)u+t_1^3+t_1t_2^3+t_1t_2t_3+2t_1t_2t_3u+t_1^2t_2t_4+3t_1^2t_2t_4u+t_2^2t_5+2t_2^2t_5u$$

and $g=0$. Then

$$F=-2t_1^3-3t_1t_2^3-\frac{1}{2}(t_3^2+t_4^2+t_5^2)-x_7+x_5(t_2^2+t_5)+x_4(t_1^2t_2+t_4)+x_3(t_1t_2+t_3)$$

$$+x_1\left(t_1+2t_1t_2t_4+t_2^3+3t_1^2+t_2t_3\right)+x_2\left(t_2+3t_1t_2^2+2t_2t_5+t_1t_3+t_1^2t_4\right)$$

$$+x_6\left(\frac{1}{2}(t_1^2+t_2^2)+3t_1^2t_2t_4+2t_1t_2t_3+2t_2^2t_5\right),$$

which is a versal unfolding of an E_7 point.

(3) Let

$$f=\frac{1}{2}|t|^2+\frac{1}{2}(t_1^2+t_2^2)u+t_1^3+t_2^5+t_1t_2t_3+2t_1t_2t_3u$$

$$+t_1t_2^2t_4+3t_1t_2^2t_4u+t_2^3t_5+3t_2^3t_5u+t_1t_2^3t_6+4t_1t_2^3t_6u$$

and $g=0$. Then

$$F=-2t_1^3-4t_2^5-\frac{1}{2}(t_3^2+t_4^2+t_5^2+t_6^2)+x_4(t_1t_2^2+t_4)+x_3(t_1t_2+t_3)+x_5(t_2^3+t_5)+x_6(t_1t_2^3+t_6)$$

$$+x_1\left(t_1+3t_1^2+t_2^2t_4+t_2t_3+t_2^3t_6\right)+x_2\left(t_2+5t_2^4+t_1t_3+2t_1t_2t_4+3t_2^2t_5+3t_1t_2^2t_6\right)$$

$$+x_7\left(\frac{1}{2}(t_1^2+t_2^2)+2t_1t_2t_3+3t_1t_2^2t_4+4t_1t_2^3t_6+3t_2^3t_5\right)-x_8,$$

which is a versal unfolding of an E_8 point.

4. Affine metrics and normal plane bundles

4.1. Affine metric of a vector field

Fix a vector field ξ in the osculating Darboux direction. For a local frame $\{X_1, ..., X_n\}$ of TN and $X, Y \in TN$, define

$$G_\xi(X, Y) = [X_1, ..., X_n, D_X Y, \xi].$$

It is proved in [8] that

$$g_\xi(X, Y) = \frac{G_\xi(X, Y)}{\det G_\xi(X_i, X_j)^{\frac{1}{n+2}}} \tag{14}$$

is a metric in N (see also [9], ch.6). Assuming $[X_1, ..., X_n, \eta, \xi] = 1$, we get that $G_\xi(X_i, X_j) = h^2(X_i, X_j)$. Thus the non-degeneracy hypothesis of the matrix $(h^2(X_i, X_j))$ implies that the metric g_ξ is also non-degenerate.

4.2. Affine normal plane bundle

Assume for the moment that we have chosen a transversal bundle σ_1 generated by $\{\xi, \bar\eta_1\}$. Take a g_ξ-orthonormal frame $\{X_1, ..., X_n\}$ of the tangent space of N and change the basis of σ_1 by writing $\eta_1 = \mu\bar\eta_1 + \lambda\xi$. By choosing an adequate μ, we may assume that $[X_1, ..., X_n, \eta_1, \xi] = 1$. Note that λ remains arbitrary and $h^2(X_i, X_j) = \delta_{ij}$.

Now we shall make a particular choice for the transversal bundle. Write

$$\eta = \eta_1 - \sum_{k=1}^{n} \tau_2^2(X_k) X_k.$$

Direct computations show that $D_{X_i}\eta$ is tangent to M, for $1 \leq i \leq n$, and so $D_X\eta$ is tangent to M, for any X tangent to N. The transversal bundle σ generated by $\{\xi, \eta\}$ is called the *affine normal plane bundle*.

It is proved in ([8], props.3.5 and 3.6) that the affine normal plane bundle σ is independent of the choice of the initial bundle σ_1 and the g_ξ-orthonormal basis $\{X_1, ..., X_n\}$ of the tangent space of N. Thus σ depends only on the choice of the vector field ξ. We shall denote it by $\sigma = \sigma(\xi)$. The results of this section are summarized in next proposition:

Proposition 4.1. *Given a codimension 1 submanifold $N \subset M$ of a hypersurface $M \subset \mathbb{R}^{n+2}$ and a vector field ξ in the osculating Darboux direction, equation (14) defines a metric g_ξ in N. There exists a vector field η transversal to M such that*

$$[X_1, ...X_n, \eta, \xi] = 1, \tag{15}$$
$$h^2(X_i, X_j) = \delta_{ij}, \tag{16}$$

for any g_ξ-orthonormal frame $\{X_1, .., X_n\}$ of N, and

$$\tau_1^2 = \tau_2^2 = 0. \tag{17}$$

The transversal vector field η satisfying equations (15), (16) and (17) is not unique. In fact, any vector field $\bar\eta$ of the form

$$\bar\eta = \eta + \lambda\xi, \tag{18}$$

for some scalar function λ, also satisfies these equations. Conversely, any vector field $\bar{\eta}$ satisfying equations (15), (16) and (17) is given by equation (18), for some scalar function λ.

4.3. Blaschke metric and affine normal

In this section we assume that $M \subset \mathbb{R}^{n+2}$ is non-degenerate (see remark 2.5). Denote by ζ the affine Blaschke vector field of $M \subset \mathbb{R}^{n+2}$ and by h the Blaschke metric of M ([7]). We shall investigate the conditions under which ζ belongs to the affine normal plane (see [9], th.5.15).

Lemma 4.2. *Assume that ξ is a vector field in the osculating Darboux direction and let ζ be the affine Blaschke vector field of $M \subset \mathbb{R}^{n+2}$. Let $\{X_1, ..., X_n\}$ be an h-orthonormal local frame of TN. The following conditions are equivalent:*

1. *$h(\xi, \xi) = 1$.*
2. *$\{X_1, ..., X_n, \xi\}$ h-orthonormal local frame of TM.*
3. *$[X_1, ..., X_n, \zeta, \xi] = 1$.*
4. *$\{X_1, ..., X_n\}$ g_ξ-orthonormal local frame of TN.*
5. *The metric g_ξ is the restriction of the Blaschke metric of M to N.*
6. *ζ is contained in the affine normal plane bundle.*

Proof. It is easy to verify the equivalence between items 1,2 and 3. It is also easy to verify the equivalence of 3 and 4, while the equivalence between 4 and 5 is obvious. Finally, since equations (16) and (17) always hold, item 6 is equivalent to equation (15), thus to item 3. □

5. Parallel vector fields

Consider a vector field ξ in the osculating Darboux direction of $N \subset M$. We say that ξ is *parallel* if $D_X \xi$ is tangent to N, for any X tangent to N.

5.1. Darboux frames for curves

Consider a smooth curve $\gamma \subset M$, where $M \subset \mathbb{R}^3$ is a surface. We shall assume that the osculating plane of γ does not coincide with the tangent plane of M.

Let $\gamma(t)$, $t \in I$, be a curve contained in a surface M. We say that the parameterization is *adapted* to M if $\gamma'''(t) \in T_{\gamma(t)}M$, for each $t \in I$. Observe that when γ is contained in a hyperplane, the affine parameterization $\gamma(s)$ satisfies $\gamma'''(s) = -\mu(s)\gamma'(s)$, where $\mu(s)$ is the affine curvature of γ, and so this parameterization is adapted to M.

Lemma 5.1. *Assume that $\gamma''(s) \notin T_{\gamma(s)}M$, for each $s \in I$, i.e., the osculating plane of γ does not coincide with the tangent plane of M. Then γ admits an unique, up to linear changes, adapted re-parameterization.*

Proof. Let t be a new parameter with the change of variables given by $s = s(t)$. Then

$$\gamma_t = \gamma_s s_t,$$

$$\gamma_{tt} = \gamma_{ss} s_t^2 + \gamma_s s_{tt},$$

$$\gamma_{ttt} = \gamma_{sss} s_t^3 + 3\gamma_{ss} s_t s_{tt} + \gamma_s s_{ttt}.$$

Let ν be a co-normal vector field of M, i.e., $\nu(Z) = 0$, for any Z tangent to M. Then $\gamma_{ttt} \in T_{\gamma(t)}M$ if and only if $\nu(\gamma_{ttt}) = 0$, which is equivalent to solve the following differential equation

$$\nu(\gamma_{sss})s_t^3 + 3\nu(\gamma_{ss})s_t s_{tt} = 0.$$

Since $s_t > 0$, this equation is equivalent to

$$As_t^2 + 3Bs_{tt} = 0,$$

where $A = \nu(\gamma_{sss})$ and $B = \nu(\gamma_{ss})$. Since $B \neq 0$ by hypothesis, the lemma follows. $\qquad\square$

Assume now that $\gamma(t)$ is an adapted parameterization of γ and let ξ be a multiple of ξ_0 satisfying $[\gamma'(t), \gamma''(t), \xi(t)] = 1$. Differentiating this equation we obtain that $\xi'(t)$ is tangent to γ, and so ξ is parallel. We shall call the frame $\{\gamma'(t), \gamma''(t), \xi(t)\}$ the *affine Darboux frame* of $\gamma \subset M$. The structural equations of this frame are given by

$$\begin{cases} (\gamma'(t))' = \gamma''(t), \\ (\gamma''(t))' = -\mu(t)\gamma'(t) + \tau(t)\xi(t), \\ \xi'(t) = -\sigma(t)\gamma'(t). \end{cases}$$

5.2. Equiaffine transversal bundles

Fix a vector field ξ in the osculating Darboux direction and let $\sigma = \sigma(\xi)$ be the affine normal transversal bundle described in section 4. Then the corresponding connection ∇ given by equation (1) depends also on the choice of ξ. Thus we write $\nabla = \nabla(\xi)$.

Consider the volume form ω_g induced by the metric $g = g_\xi$. We say that the pair (∇, g) is *equiaffine* if $\nabla(\omega_g) = 0$. Let $\{X_1, ..., X_n\}$ be a g-orthonormal local frame of TN. Since

$$\nabla_{X_i}\omega_g = -\sum_{j=1}^n \omega_g(X_1, .., \nabla_{X_i}X_j, ..., X_n) = -\sum_{j=1}^n \Gamma_{ij}^j,$$

where Γ_{ij}^k are the Christoffel symbols of the connection, we conclude that (∇, g) is equiaffine if and only if

$$\Gamma_{i1}^1 + \Gamma_{i2}^2 + ... + \Gamma_{in}^n = 0, \quad 1 \leq i \leq n. \tag{19}$$

Proposition 5.2. *The pair $(\nabla(\xi), g_\xi)$ is equiaffine if and only if ξ is parallel.*

Proof. Differentiating $[X_1, ..., X_n, \eta, \xi] = 1$ in the direction X_i we obtain

$$\sum_{k=1}^n \Gamma_{ik}^k + \tau_1^1(X_i) = 0.$$

Since ξ is parallel if and only if $\tau_1^1(X_i) = 0$, for $1 \le i \le n$, the proposition is proved. □

5.3. The apolarity condition

The cubic forms are defined as

$$C^1(X,Y,Z) = (\nabla_X h^1)(Y,Z) + \tau_1^1(X)h^1(Y,Z) + \tau_2^1(X)h^2(Y,Z),$$
$$C^2(X,Y,Z) = (\nabla_X h^2)(Y,Z) + \tau_1^2(X)h^1(Y,Z) + \tau_2^2(X)h^2(Y,Z). \qquad (20)$$

One can verify that the cubic forms are symmetric in X, Y, Z.

Take any η in the affine normal plane described in section 4.2. In this case, h^2 coincides with g_ξ and the cubic form C^2 can be written as

$$C^2(X,Y,Z) = (\nabla_X h^2)(Y,Z). \qquad (21)$$

The cubic form C^2 is *apolar* with respect to h^2 if

$$tr_{h^2} C^2(X,\cdot,\cdot) = 0, \qquad (22)$$

for any vector field $X \in TN$.

Consider a g_ξ-orthonormal basis $\{X_1, ..., X_n\}$. Then the apolarity condition for (C^2, h^2) is equivalent to equation (19). Thus we conclude the following proposition:

Proposition 5.3. *The following statements are equivalent:*

1. *The cubic form C^2 is apolar with respect to h^2.*
2. *The pair (∇, g) is equiaffine.*
3. *The vector field ξ is parallel.*

5.4. Examples

We now give some examples of submanifolds $N \subset M$ that admit a parallel vector field ξ.

Example 5.4. When $n = 1$, the vector field ξ defined in section 5.1 is parallel. Assume that $\gamma(t)$ is an adapted parameterization. Then $\gamma'(t)$ is g_ξ-unitary and $\{\xi(t), \gamma''(t)\}$ is a basis for the affine normal plane $\sigma(\xi)$.

Example 5.5. *Hyperplanar submanifolds.* Assume $N = M \cap H$, where H is a hyperplane. Fix a point $p_0 \in N$ and let $\xi(p_0)$ be a vector in the osculating Darboux direction at p_0. We can extend ξ to a vector field along N in the osculating Darboux direction such that $\xi(p) = \xi(p_0) + \mathbf{e}(p)$, where $\mathbf{e}(p) \in H$, for any $p \in N$. Then $D_X\xi \in H \cap T_pM = T_pN$. The metric g_ξ defined by equation (14) coincides with the Blaschke metric of $N \subset H$ and the affine Blaschke normal ζ of $N \subset H$ belongs to the affine normal plane.

Example 5.6. *Visual contour submanifolds* ([2]). Suppose that all lines ξ_0 meet at a point O. Then we can choose $\xi = \lambda\xi_0$ such that $\xi(p) = O - p$. Differentiating we obtain $-S_1 X + \tau_1^1(X)\xi = -X$. We conclude that $S_1 = Id$ and $\tau_1^1 = 0$. Thus ξ is parallel.

Example 5.7. Suppose M is a hyperquadric and $N \subset M$ is arbitrary. Using the notation of remark 2.5, we have that $h(\xi, X) = 0$, for any $X \in T_p N$. Take ξ such that $h(\xi, \xi) = 1$. Differentiating and using that the cubic form C of $M \subset \mathbb{R}^{n+2}$ is zero we get $h(\tilde{\nabla}_X \xi, \xi) = 0$. Thus $\tilde{\nabla}_X \xi \in T_p N$ and so ξ is parallel. It is not difficult to see that in this case the affine normal Blaschke vector field is contained in the affine normal plane bundle and the metric g_ξ is the restriction of the Blaschke metric of M to N (see [7]).

5.5. Blaschke metric and the parallelism condition

In this section we shall assume that $M \subset \mathbb{R}^{n+2}$ is non-degenerate (see remark 2.5). Denote by ζ the affine Blaschke vector field of $M \subset \mathbb{R}^{n+2}$ and by h the Blaschke metric of M. We shall also assume that ξ is parallel and look for conditions under which ζ belongs to the affine normal plane.

Proposition 5.8. *Let ξ be a parallel vector field along N and assume that $h(\xi, \xi) = 1$ at a certain point $p_0 \in N$. The conditions of lemma 4.2 hold if and only if $C(X, \xi, \xi) = 0$, for any $X \in TN$.*

Proof. Assume that $h(\xi, \xi) = 1$. Differentiating this equation in the direction $X \in TN$ and using that ξ is parallel we obtain $C(X, \xi, \xi) = \tilde{\nabla}(X, \xi, \xi) = 0$, for any $X \in TN$. Conversely, if $C(X, \xi, \xi) = 0$ we obtain $X(h(\xi, \xi)) = 0$, for any $X \in TN$. Thus $h(\xi, \xi) = 1$ at N. $\qquad \square$

Example 5.9. Consider the surface $M \subset \mathbb{R}^3$ graph of

$$f(x, y) = \frac{1}{2}(x^2 + y^2) + \frac{c}{6}(x^3 - 3xy^2),$$

where $c \neq 0$. Let $\gamma = N$ be the intersection of M with the plane $y = 0$. Take ξ parallel as in section 5.1. Then the affine plane is generated by $\{\xi, \eta\}$, where η is the affine normal of γ (see example 5.4). Since $\eta(0, 0) = (-c/3, 0, 1)$, $\xi(0, 0) = (0, 1, 0)$ and $\zeta(0, 0) = (0, 0, 1)$, where ζ denotes the affine Blaschke normal of M, we conclude that, if $c \neq 0$, ζ does not belong to the affine normal plane.

6. Transon planes

6.1. Transon planes for curves

The following theorem is a very old result ([10],[6]):

Theorem 6.1. *Given a surface $M \subset \mathbb{R}^3$, a point $p_0 \in M$ and a vector $T \in T_{p_0} M$, consider sections of M by planes H containing T and passing through p_0. Then the affine normal vectors $\eta = \eta(H)$ of these sections at p_0 belong to a plane. This plane is called the Transon plane of $p_0 \in M$ in the direction T.*

The statement of this theorem needs an explanation: A parameterization $\gamma(s)$ of $\gamma = M \cap H$ by affine arc-length is defined by the condition $[\gamma'(s), \gamma''(s), \xi_0] = 1$, for some constant vector field ξ_0. Then the affine normal at $p_0 = \gamma(s_0)$ is just $\gamma''(s_0)$. We remark that, instead of a constant vector field ξ_0, we may also consider here a vector field ξ in the osculating Darboux

direction of $\gamma \subset M$ such that $\xi(p) = \xi(p_0) + \mathbf{e}(p)$, $\mathbf{e}(p) \in H$, for any $p \in \gamma$, where $\xi(p_0)$ is any vector in the osculating Darboux direction of $\gamma \subset M$ at p_0.

6.2. Transon planes in arbitrary dimensions

Let $M \subset \mathbb{R}^{n+2}$ be a hypersurface, $p_0 \in M$ and T a n-dimensional subspace contained in $T_{p_0}M$. For a hyperplane H containing T, consider the vector field ξ along $N = M \cap H$ given by $\xi(p) = \xi(p_0) + \mathbf{e}(p)$, $\mathbf{e}(p) \in H$, as in example 5.5. The metric g_ξ in N is, up to a constant, the Blaschke metric of N, and if we choose the unique $\eta \in H$ satisfying equations (15), (16), (17) and (18), then η is the affine Blaschke normal of $N \subset H$.

Next theorem says that, as in the case of curves, this affine normal plane at p_0 is independent of the hyperplane H, and we shall keep the name Transon plane for it.

Theorem 6.2. *Given a hypersurface $M \subset \mathbb{R}^{n+2}$, a point $p_0 \in M$ and a n-subspace $T \subset T_{p_0}M$, consider sections of M by hyperplanes H containing T and passing through p_0. Assume that one section $H_0 \cap M$ is non-degenerate in the sense of section 2.2. Then $H \cap M$ is non-degenerate for any H and the affine normal vectors $\eta = \eta(H)$ of these sections at p_0 belong to a plane.*

Proof. Assume that p_0 is the origin, the tangent to M is the plane $z = 0$ and that H_0 is the hyperplane $y = 0$. The non-degeneracy hypothesis implies that we can find a local coordinate system such that M is given by

$$z = \frac{1}{2}(x_1^2 + \ldots + x_n^2 + ay^2) + P_3(x) + yP_2(x) + y^2P_1(x) + P_0y^3 + O(4)(x, y). \quad (23)$$

where $a \in \mathbb{R}$, $P_k(x)$ is homogeneous of degree k in $x = (x_1, \ldots, x_n)$ and $O(4)(x, y)$ denote terms of degree ≥ 4 in (x, y). Consider the hyperplane H_λ of equation $y = \lambda z$. The projection of these sections in the xz-hyperplane is given by

$$z = \frac{1}{2}(x_1^2 + \ldots + x_n^2 + a\lambda^2 z^2) + P_3(x) + \lambda z P_2(x) + \lambda^2 z^2 P_1(x) + P_0\lambda^3 z^3 + O(4)(x, z),$$
$$\quad (24)$$

where $O(4)(x, z)$ denote terms of degree ≥ 4 in (x, z). This curve can be re-written as

$$z = \frac{1}{2}(x_1^2 + \ldots + x_n^2) + P_3(x) + O(4)(x). \quad (25)$$

where $O(4)(x)$ denotes terms of degree ≥ 4 in x. This implies that the projection of the affine normal vector does not depend on λ. $\qquad\square$

6.3. Transon planes for general submanifolds

Lemma 6.3. *Let $\bar{N}$ be the image of N by the projection $\pi : \mathbb{R}^{n+2} \to H$ in a hyperplane H along the constant direction $\xi(p_0)$. The Blaschke affine normal $\tilde{\eta}$ of $\bar{N}$ at p_0 belongs to the Transon plane.*

Proof. We may assume that N is defined by $y = P_2(x, z) + O(3)$. Then the same argument as above proves the proposition. $\qquad\square$

Denote by $\bar{\eta}$ the vector field along $\bar{N}$ such that $\bar{\eta}(\pi(x)) = \eta(x)$ and by $\bar{X} = \pi_*(X)$ the projection of X in H. Let η be the transversal vector field along N in the affine normal plane bundle such that η is parallel to H.

Lemma 6.4. *We have that*

1. $\bar{h}^2(X,Y)(p_0) = h^2(X,Y)(p_0)$, *for any* X,Y *tangent to* N.
2. $\bar{\nabla}(\bar{h}^2)(X,Y,Z)(p_0) = \nabla(h^2)(X,Y,Z)(p_0)$, *for any* X,Y,Z *tangent to* N.

Thus η *is apolar if and only if* $\bar{\eta}$ *is apolar.*

Proof. Write $\pi(p) = p + \lambda(p)\xi(p_0)$. Then $\pi_*X = X + X(\lambda)\xi(p_0)$, which implies that $X(\lambda)(p_0) = 0$. Differentiating again we obtain

$$D_Y\pi_*X = D_Y X + YX(\lambda)\xi + X(\lambda)D_Y\xi.$$

This implies that $\bar{h}^2(\bar{X},\bar{Y}) = h^2(X,Y)$ for any p and $\bar{\nabla}_{\bar{Y}}\bar{X} = \nabla_Y X$ at p_0. If $\{X_1,...,X_n\}$ is a h^2-orthonormal frame, then $\{\bar{X}_1,...,\bar{X}_n\}$ is a $\bar{h}^2$-orthonormal frame. Thus

$$\bar{\nabla}_{\bar{X}_k}\bar{h}^2(\bar{X}_i,\bar{X}_j) = -\bar{h}^2(\bar{\nabla}_{\bar{X}_k}\bar{X}_i,\bar{X}_j) - \bar{h}^2(\bar{\nabla}_{\bar{X}_k}\bar{X}_j,\bar{X}_i)$$

is equal, at p_0, to

$$\nabla_{X_k}h^2(X_i,X_j) = -h^2(\nabla_{X_k}X_i,X_j) - h^2(\nabla_{X_k}X_j,X_i),$$

thus proving the lemma. $\qquad\qquad\square$

Next lemma is a general result concerning codimension 1 immersions:

Lemma 6.5. *Let* $S \subset \mathbb{R}^{n+1}$ *be a hypersurface,* $p_0 \in S$, *and let* ζ *be a transversal vector field such that* $D_X\zeta$ *at* p_0 *is tangent to* S, *for any vector field* X *tangent to* S. *If* ζ *is apolar at* p_0, *then* $\zeta(p_0)$ *is a multiple of the affine normal vector. Conversely, if* $\zeta(p_0)$ *is a multiple of the affine normal vector, then* ζ *is apolar at* p_0.

Proof. Let $\tilde{\zeta}$ denote the affine Blaschke normal vector field and

$$\zeta = \sum_{l=1}^{n} a_l X_l + \delta\tilde{\zeta}.$$

Writing

$$D_{X_i}X_j = \nabla_{X_i}X_j + h(X_i,X_j)\zeta = \tilde{\nabla}_{X_i}X_j + \tilde{h}(X_i,X_j)\zeta,$$

we conclude that $\tilde{h}(X_i,X_j) = \delta h(X_i,X_j)$ and

$$\tilde{\Gamma}_{ij}^l = \Gamma_{ij}^l + h(X_i,X_j)a_l.$$

Moreover, since $D_{X_k}\zeta$ is tangent to S at p_0, we conclude that, at p_0,

$$X_k(\delta) = -\delta\sum_{l=1}^{n} a_l h(X_k,X_l).$$

Choose a basis $\{X_1,..,X_n\}$ h-orthonormal. Then

$$\tilde{C}(X_k,X_i,X_i) = X_k(\delta) - 2\delta\tilde{\Gamma}_{ik}^i, \quad C(X_k,X_i,X_i) = -2\Gamma_{ik}^i.$$

Since $\tilde{C}$ is apolar,

$$0 = \sum_{i=1}^{n} C(X_k, X_i, X_i) - (n+2)\delta a_k.$$

Thus C is apolar at p_0 if and only if $a_k(p_0) = 0$, for $1 \leq k \leq n$, which is equivalent to to ζ multiple of $\tilde{\zeta}$. □

Theorem 6.6. *The affine normal plane coincides with the Transon plane if and only if ξ is parallel.*

Proof. By proposition 5.3, ξ is parallel at p_0 if and only if η is apolar at p_0. By lemma 6.4, η is apolar at p_0 if and only if $\bar{\eta}$ is apolar at p_0. From lemma 6.5, $\bar{\eta}$ is a multiple of $\tilde{\eta}$ at p_0 if and only if $\bar{\eta}$ is apolar at p_0. Finally $\tilde{\eta}$ belongs to the Transon plane, by lemma 6.3. □

7. Existence of parallel vector fields

7.1. Submanifolds that admit parallel vector fields

In this section we characterize the submanifolds N that admit a parallel vector field ξ. We begin with the following lemma:

Lemma 7.1. *There exists a parallel vector field ξ if and only if τ_1^1 is exact.*

Proof. Fix a vector field ξ_0 in the osculating Darboux direction and look for λ such that $\xi = \lambda\xi_0$ is parallel. Differentiating this equation we obtain

$$D_X(\xi) = X(\lambda)\xi_0 + \lambda\left(-SX + \tau_1^1(X)\xi_0\right) = -\lambda SX + (X(\lambda) + \lambda\tau_1^1(X))\xi_0.$$

Then $X(\lambda) + \lambda\tau_1^1(X) = 0$, for any $X \in T_pN$, if and only if $\tau_1^1 = -d(\log(\lambda))$. □

Since $D_X\xi$ is tangent to M, we can write

$$D_X\xi = -S_1 X + \nabla_X^{\perp}\xi, \tag{26}$$

where $\nabla_X^{\perp}\xi = \tau_1^1(X)\xi$ is the affine normal connection.

Consider the normal bundle connection $\nabla^{\perp}$ defined by equation (26). The normal curvature $R^{\perp}$ is defined as

$$R^{\perp}(X,Y)\xi = \nabla_Y^{\perp}\nabla_X^{\perp}\xi - \nabla_X^{\perp}\nabla_Y^{\perp}\xi + \nabla_{[X,Y]}^{\perp}\xi.$$

We say that the normal bundle is *flat* if $R^{\perp} = 0$, for any $X, Y \in T_pN$, $\xi = \lambda\xi_0$ (see [1], ch.6).

Proposition 7.2. *There exists a parallel vector field ξ if and only the normal bundle is flat.*

Proof. Observe that $\nabla_X^{\perp}\xi_0 = \tau_1^1(X)\xi_0$. Thus

$$\nabla_Y^{\perp}\nabla_X^{\perp}\xi = Y\tau_1^1(X)\xi_0 + \tau_1^1(X)\tau_1^1(Y)\xi_0.$$

Now straightforward calculations shows that

$$R^{\perp}(X,Y)\xi_0 = \left(Y\tau_1^1(X) - X\tau_1^1(Y) + \tau_1^1([X,Y])\right)\xi_0 = d\tau_1^1(X,Y)\xi_0.$$

Using lemma 7.1 we prove the proposition. □

7.2. Example: A submanifold without a parallel vector field.

We now give an explicit example of an immersion $N \subset M$ that does not admit a parallel vector field.

Example 7.3. Take M to be the graph of

$$f(x_1, x_2, y) = \frac{1}{2}\left(x_1^2 + x_2^2 + y^2\right) + \frac{1}{2}\left(k_1 x_1^2 y + k_2 x_2^2 y\right)$$

and N given by the intersection of M with $y = x_1 x_2$. Thus N can be parameterized by

$$\phi(x_1, x_2) = (x_1, x_2, x_1 x_2, f(x_1, x_2, x_1 x_2))\,.$$

Let $X_1 = \phi_{x_1}$ and $X_2 = \phi_{x_2}$. Then the vector field $\xi(x_1, x_2) = (\xi_1, \xi_2, \xi_3, \xi_4)$,

$$\xi_1 = k_2^2 x_2^3 - k_2 x_1 x_2^2 - 2k_1 k_2 x_1^2 x_2 - k_1 x_1 - x_2$$

$$\xi_2 = k_1^2 x_1^3 - k_1 x_1^2 x_2 - 2k_1 k_2 x_1 x_2^2 - k_2 x_2 - x_1$$

$$\xi_3 = 1 + 2(k_1 + k_2)x_1 x_2 + 3k_1 k_2 x_1^2 x_2^2$$

$$\xi_4 = -\frac{1}{2}\left(k_1 x_1^2 + 2x_1 x_2 + k_2 x_2^2\right) + (k_1^2 - k_1 k_2)x_1^3 x_2 + (k_2^2 - k_1 k_2)x_1 x_2^3$$

$$+ \frac{1}{2}k_1 k_2 x_2^2 x_1^2 \left(k_1 x_1^2 + 2x_1 x_2 + k_2 x_2^2\right)$$

is tangent to M and $D_{X_i}\xi \in T_p M$, for $i = 1, 2$. Moreover,

$$\tau_1^1(X_1)(x_1, x_2) = x_1 + (3k_1 + 2k_2)x_2 + O(3)$$

$$\tau_1^1(X_2)(x_1, x_2) = (2k_1 + 3k_2)x_1 + x_2 + O(3)$$

We conclude that $d\tau_1^1(X_1, X_2)(0, 0) = k_1 - k_2 \neq 0$ and thus τ is not a closed form.

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