A CRITERION FOR ZERO AVERAGES AND FULL SUPPORT OF ERGODIC MEASURES

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Abstract. Consider a homeomorphism f defined on a compact metric space X and a continuous map $\phi\colon X \to \mathbb{R}$. We provide an abstract criterion, called *control at any scale with a long sparse tail* for a point $x \in X$ and the map ϕ, that guarantees that any weak$*$ limit measure μ of the Birkhoff average of Dirac measures $\frac{1}{n}\sum_0^{n-1}\delta(f^i(x))$ is such that μ-almost every point y has a dense orbit in X and the Birkhoff average of ϕ along the orbit of y is zero.

As an illustration of the strength of this criterion, we prove that the diffeomorphisms with nonhyperbolic ergodic measures form a C^1-open and dense subset of the set of robustly transitive partially hyperbolic diffeomorphisms with one dimensional nonhyperbolic central direction. We also obtain applications for nonhyperbolic homoclinic classes.

1. Introduction

1.1. **Motivation and general setting.** This work is a part of a long-term project to attack the following general problem which rephrases the opening question in [GIKN] from a different perspective: *To what extent does ergodic theory detect the nonhyperbolicity of a dynamical system?*

More precisely, we say that a diffeomorphism f is *nonhyperbolic* if its non-wandering set is nonhyperbolic. We aim to know if such f possesses *nonhyperbolic ergodic measures* (i.e. with some zero *Lyapunov exponent*) and if some of them fully reflect the nonhyperbolic behaviour of f. For instance, we would like to know

- what is their support,
- what is their entropy, and
- how many Lyapunov exponents of the measures are zero.

In this generality, the answer to this question is negative. There are simple examples of (even analytic) nonhyperbolic dynamical systems whose invariant measures are all hyperbolic and even with Lyapunov exponents uniformly far from zero, see for instance the logistic map $t \mapsto 4t(1-t)$ or the surgery examples in [BBS] where a saddle of a uniformly hyperbolic set is replaced by non-uniformly hyperbolic sets, among others (more examples of different nature can be found in [CLR, LOR]).

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Nevertheless, these examples are very specific and fragile. Thus, one hopes that the "great majority" of nonhyperbolic systems have nonhyperbolic ergodic measures which detect and truly reflect the nonhyperbolic behaviour of the dynamics.

Concerning this sort of questions, a first wave of results, initiated with [GIKN], continued in [DG, BDG], and culminated in [CCGWY], show that the existence of nonhyperbolic ergodic measures for nonhyperbolic dynamical systems is quite general in the C^1-setting: for C^1-generic diffeomorphisms, every nonhyperbolic homoclinic class supports a nonhyperbolic ergodic measure, furthermore under quite natural hypotheses the support of the measure is the whole homoclinic class[1].

Given a periodic point p of a diffeomorphism f denote by $\mu_{\mathcal{O}(p)}$ the unique f-invariant measure supported on the orbit of p. We say that such a measure is *periodic*. The previous works follow the strategy of periodic approximations in [GIKN] for constructing a nonhyperbolic ergodic measure as weak$*$ limits of periodic measures $\mu_{\mathcal{O}(p_n)}$ supported on orbits $\mathcal{O}(p_n)$ of hyperbolic periodic points p_n with decreasing "amount of hyperbolicity". The main difficulty is to obtain the ergodicity of the limit measure. [GIKN] provides a criterion for ergodicity summarised in rough terms as follows. Each periodic orbit $\mathcal{O}(p_n)$ consists of two parts: a "shadowing part" where $\mathcal{O}(p_n)$ closely shadows the previous orbit $\mathcal{O}(p_{n-1})$ and a "tail" where the orbit is far from the previous one. To get an ergodic limit measure one needs some balance between the "shadowing" and the "tail" parts of the orbits. The "tail part" is used to decrease the amount of hyperbolicity of a given Lyapunov exponent (see [GIKN]) and also to spread the support of the limit measure, (see [BDG]).

Nonhyperbolic measures seem very fragile as small perturbations may turn the zero Lyapunov exponent into a nonzero one. However, in [KN] there are obtained (using the method in [GIKN]) certain C^1-open sets of diffeomorphisms having nonhyperbolic ergodic measures. Bearing this result in mind, it is natural to ask if the existence of nonhyperbolic measures is a C^1-open and dense property in the space of nonhyperbolic diffeomorphisms. In this direction, [BBD2, Theorem 4] formulates an abstract criterion called *control at any scale*[2] that leads to the following result (see [BBD2, Theorems 1 and 3]): *The C^1-interior of the set of diffeomorphisms having a nonhyperbolic ergodic measure contains an open and dense subset of the set of C^1-diffeomorphisms having a pair of hyperbolic periodic points of different indices robustly in the same chain recurrence class.*

The method in [BBD2] provides a partially hyperbolic invariant set with positive topological entropy whose central Lyapunov exponent vanishes uniformly. This set only supports nonhyperbolic measures and the existence of a measure with positive entropy is a consequence of the variational principle for entropy [W]. A con of this method is that the "completely" nonhyperbolic nature of the (obtained) set where a Lyapunov exponent vanishes uniformly prevents the measures to have full support in nonhyperbolic chain recurrence classes. This shows that, in some sense, the criterion in [BBD2] may be "too demanding" and "rigid".

[1]See [CCGWY, Main Theorem] and also [CCGWY, Theorem B and Proposition 1.1]. This last result states that the support of the nonhyperbolic measure in a nonhyperbolic homoclinic class of a saddle is the whole homoclinic class. This result requires neither that the stable/unstable splitting of the saddle extends to a dominated splitting on the class (compare with [BDG]) nor that the homoclinic class contains saddles of different type of hyperbolicity (compare with [DG]).

[2]This construction also involves the so-called *flip-flop families,* we will review these notions below as they play an important role in our constructions.

The aim of this paper is to introduce a new criterion that relaxes the "control at any scale criterion" and allows to get nonhyperbolic measures with "full support" (in the appropriate ambient space: homoclinic class, chain recurrence class, the whole manifold, according to the case). To be a bit more precise, given a point x and a diffeomorphism f consider the *empirical measures* $\mu_n(x)$, $n \in \mathbb{N}$, associated to x defined as the averages of the Dirac measures $\delta(f^i(x))$ in the orbit segment $\{x, \dots, f^{n-1}(x)\}$,

$$\mu_n(x) \stackrel{\text{def}}{=} \frac{1}{n} \sum_{i=0}^{n-1} \delta(f^i(x)). \tag{1.1}$$

The criterion in this paper, called *control at any scale with a long sparse tail* with respect to a continuous map φ of a point x, allows to construct ergodic measures with full support (in the appropriate ambient space) and a prescribed average with respect to φ, see Theorem 1. This construction involves two main aspects of different nature: density of the orbits of μ-generic points and control of averages. The existence of ergodic measures satisfying both properties is a consequence of the construction.

A specially interesting case occurs when the map φ is the derivative of a diffeomorphism with respect to a continuous one-dimensional center direction (taking positive and negative values). In such a case we get that every measure μ that is a weak$*$ limit of a sequence of empirical measures of x is such that μ-almost every point has a zero Lyapunov exponent and a dense orbit (in the corresponding ambient space), see Theorems 7 and 8.

To state more precisely the dynamical consequences of the criterion let us introduce some notation (the precise definitions can be found below). In what follows we consider a boundaryless Riemannian compact manifold M and the following two C^1-open subsets of diffeomorphisms:

- The set $\mathcal{RT}(M)$ of all *robustly transitive diffeomorphisms*[3] with a partially hyperbolic splitting with one-dimensional (nonhyperbolic) center,
- The set $\mathcal{Z}(M)$ defined as the C^1-interior of the set of C^1-diffeomorphisms having a nonhyperbolic ergodic measure with full support in M.

As an application of our criterion we get that set $\mathcal{Z}(M) \cap \mathcal{RT}(M)$ is C^1-open and C^1-dense in $\mathcal{RT}(M)$, see Theorem 9. We also get semi-local versions of this result formulated in terms of nonhyperbolic homoclinic classes or/and chain recurrence classes, see Theorems 7 and 8. These results turn the C^1-generic statements in [BDG] into C^1-open and C^1-dense ones. We observe that a similar result involving different methods was announced in [BZ][4]. Applications of the criterion in hyperbolic-like contexts, as for instance full shifts and horseshoes, are discussed in Section 1.4.

In this paper we restrict ourselves to the control of the support and the averages of the measures, omitting questions related to the entropy of these measures. Nevertheless it seems that our method is well suited to construct nonhyperbolic ergodic measures with positive entropy and full support. This is the next step of

[3]A diffeomorphism is called transitive if it has a dense orbit. The diffeomorphism is C^1-robustly transitive if C^1-nearby diffeomorphisms are also transitive.

[4]The construction in [BZ] combines the criteria of periodic approximations in [GIKN] and of the control at any scale in [BBD2] and a shadowing lemma by Gan-Liao, [G].

an ongoing project whose ingredients involve tools of a very different nature beyond the scope of this paper.

In the dynamical applications we focus on partially hyperbolic diffeomorphisms with a one-dimensional center bundle and therefore the measures may have at most one zero Lyapunov exponent. Here we do not consider the case of higher dimensional central bundles and the possible occurrence of multiple zero exponents. Up to now, there are quite few results on multiple zero Lyapunov exponents. The simultaneous control of several exponents is much more difficult, essentially due to the non-commutativity of $\mathrm{GL}(n,\mathbb{R})$ for $n > 1$. We refer to [BBD] for examples of (C^1 and C^2) robust existence of ergodic measures with multiple zero exponents in the context of iterated function systems. Recently, [WZ] announces the locally C^1-generic vanishing of several Lyapunov exponents in homoclinic classes of diffeomorphisms.

We now describe our methods and results in a more detailed way.

1.2. **A criterion for controlling averages of continuous maps.** Consider a compact metric space (X,d), a homeomorphism f defined on X, and a continuous map $\varphi\colon X \to \mathbb{R}$. Given a point $x \in X$ consider the set of *empirical measures* $\mu_n(x)$ associated to x defined as in (1.1). Consider the following notation for finite Birkhoff averages of φ,

$$\varphi_n(x) \stackrel{\text{def}}{=} \frac{1}{n}\sum_{i=0}^{n-1}\varphi(f^i(x)), \tag{1.2}$$

and limit averages of φ

$$\varphi_\infty(x) \stackrel{\text{def}}{=} \lim_{n\to+\infty} \varphi_n(x) = \lim_{n\to+\infty}\frac{1}{n}\sum_{i=0}^{n-1}\varphi(f^i(x)), \tag{1.3}$$

if such a limit exists.

Consider a measure μ that is a weak$*$ limit of empirical measures of x and a subsequence $\mu_{n_k}(x)$ with $\mu_{n_k}(x) \to \mu$ in the weak$*$ topology. The convergence of the sequence of Birkhoff averages $\int \varphi\, d\mu_{n_k}(x)$ to some limit α implies that $\int \varphi\, d\mu = \alpha$. But since μ may be non-ergodic this does not provide any information about the Birkhoff averages $\varphi_n(y)$ of μ-generic points y. We aim for a criterion guaranteeing that μ-generic points have the same limit average as x. Naively, in [BBD2] the way to get this property is to require that "all large orbit intervals of the forward orbit of x have average close to the limit average (say) α". This was formalised in the criterion *control of Birkhoff averages at any scale* of a point x with respect to a map φ in [BBD2]. This criterion implies that there are sequences of times $t_n \to \infty$ and of "errors" $\varepsilon_n \to 0$ such that every orbit interval with length $t \geq t_n$ of the forward orbit of x has φ-Birkhoff average in $[\alpha - \varepsilon_n, \alpha + \varepsilon_n]$. When φ-Birkhoff averages are controlled at any scale then the φ-Birkhoff averages of any ω-limit point of x converge uniformly to α (see [BBD2, Lemma 2.2]).

To get a limit measure whose support is the whole ambient space the requirement "all long orbit intervals satisfy the limit average property" is extremely restrictive. Roughly, in the criterion in this paper we only require that "most of large orbit intervals of the forward orbit of x have average close to the limit average α". Let us explain a little more precisely this rough idea.

If the limit measure has full support then the orbit of the point x must necessarily visit "all regions" of the ambient space and these visits require an arbitrary large time. Moreover, to get limit measures whose generic points have dense orbits in the ambient space these "long visits" must occur with some frequency. During these long visits the control of the averages can be lost.

To control simultaneously Birkhoff averages and support of the limit measure, one needs some "balance" between the part of the orbit where there is a "good control of the averages" and the part of the orbit used for spreading the support of the measure to get its density (roughly, these parts play the roles of the "shadowing" and "tail parts" of the method in [GIKN]). The criterion in this paper formalizes an abstract notion for this balance that we call *control at any scale with a long sparse tail with respect to* φ *and* X (see Definitions 2.10 and 2.11). Our main technical result is that this criterion provides ergodic measures having simultaneously a prescribed average and a prescribed support.

Theorem 1. *Let* (X,d) *be a compact metric space,* $f\colon X\to X$ *a homeomorphism, and* $\varphi\colon X\to\mathbb{R}$ *a continuous map. Consider*

- *a point* $x_0\in X$ *that is controlled at any scale with a long sparse tail with respect* φ *and* X *and*
- *a measure* μ *that is a weak$*$ limit of the sequence of empirical measures* $(\mu_n(x_0))_n$ *of* x_0*.*

Then for μ*-almost every point* x *the following holds:*

a) the forward orbit of x *for* f *is dense in* X *and*
b) $\lim_{n\to\infty}\frac{1}{n}\sum_{i=0}^{n-1}\varphi(f^i(x))=\int\varphi\,d\mu$.

In particular, these two assertions hold for almost every ergodic measure of the ergodic decomposition of μ*.*

We now exhibit some dynamical configurations where the criterion holds. Indeed, we see that such configurations are quite "frequent".

1.3. **Flip-flop families with sojourns: control at any scale with a long sparse tail.** To present a mechanism providing orbits controlled at any scale we borrow the following definition from [BBD2]:

Definition 1.1 (Flip-flop family)**.** *Let* (X,d) *be a compact metric space,* $f\colon X\to X$ *a homeomorphism, and* $\varphi\colon X\to\mathbb{R}$ *a continuous function.*

A flip-flop family *associated to* φ *and* f *is a family* $\mathfrak{F}=\mathfrak{F}^+\bigsqcup\mathfrak{F}^-$ *of compact subsets of* X *such that there are* $\alpha>0$ *and a sequence of numbers* $(\zeta_n)_n$*,* $\zeta_n>0$ *and* $\zeta_n\to 0$ *as* $n\to\infty$*, such that:*

a) for every $D\in\mathfrak{F}^+$ *(resp.* $D\in\mathfrak{F}^-$*) and every* $x\in D$ *it holds* $\varphi(x)\geq\alpha$ *(resp.* $\varphi(x)\leq-\alpha$*);*
b) for every $D\in\mathfrak{F}$*, there are sets* $D^+\in\mathfrak{F}^+$ *and* $D^-\in\mathfrak{F}^-$ *contained in* $f(D)$*;*
c) for every $n>0$ *and every family of sets* $D_i\in\mathfrak{F}$*,* $i\in\{0,\dots,n\}$ *with* $D_{i+1}\subset f(D_i)$ *it holds*

$$d(f^{n-i}(x),f^{n-i}(y))\leq\zeta_i\cdot d(f^n(x)f^n(y))$$

for every $i\in\{0,\dots,n\}$ *and every pair of points* $x,y\in f^{-n}(D_n)$*.*

We call plaques[5] *the sets of the flip-flop family* $\mathcal{F}$*.*

[5] We pay special attention to the case when the sets of the flip-flop family are discs tangent to a strong unstable cone field. This justifies this name.

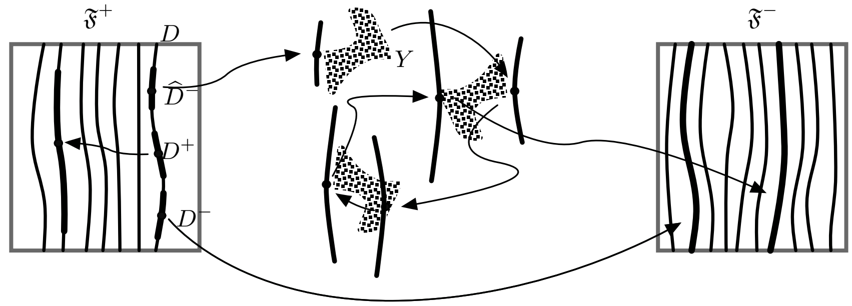

FIGURE 1. A flip-flop family with sojourns: the plaques D^+, D^-, and $\widehat{D}^-$ (the plaque $\widehat{D}^+$ is omitted for visual simplicity).

With the notation in Definition 1.1, [BBD2, Theorem 2.1] claims that for every number $t \in (-\alpha, \alpha)$ and every set $D \in \mathfrak{F}$ there is a point $x_t \in D$ whose orbit is controlled at any scale for the function $\varphi_t = \varphi - t$. Hence the Birkhoff average of φ along the orbit of any point $y \in \omega(x_t)$ is t. Furthermore, the ω-limit set of x_t has positive topological entropy.

Since we aim to obtain measures with full support we need to relax the control of the averages. For that we introduce a "sojourn condition" for the returns of the sets of the flip-flop family (item (a) in the definition below). These"sojourns" will be used to get dense orbits and to spread the support of the measures and play a role similar to the "tails" in [GIKN].

Definition 1.2 (Flip-flop family with sojourns). *Let (X, d) be a compact metric space, Y a compact subset of X, $f\colon X \to X$ a homeomorphism, and $\varphi\colon X \to \mathbb{R}$ a continuous function.*

Consider a flip-flop family $\mathfrak{F} = \mathfrak{F}^+ \bigsqcup \mathfrak{F}^-$ associated to φ and f. We say that the flip-flop family $\mathfrak{F}$ has sojourns along Y *(or that $\mathfrak{F}$* sojourns along Y*) if for every $\delta > 0$ there is an integer $N = N_\delta$ such that every plaque $D \in \mathfrak{F}$ contains subsets $\widehat{D}^+, \widehat{D}^-$ such that:*

a) for every $x \in \widehat{D}^+ \cup \widehat{D}^-$ the orbit segment $\{x, \dots, f^N(x)\}$ is δ-dense in Y (i.e., the δ-neighbourhood of the orbit segment contains Y);

b) $f^N(\widehat{D}^+) \in \mathfrak{F}^+$ and $f^N(\widehat{D}^-) \in \mathfrak{F}^-$;

c) for every $i \in \{0, \dots, N\}$ and every pair of points $x, y \in \widehat{D}^+$ or $x, y \in \widehat{D}^-$ it holds

$$d(f^{N-i}(x), f^{N-i}(y)) \le \zeta_i \cdot d(f^N(x) f^N(y)),$$

where $(\zeta_i)_i$ is a sequence as in Definition 1.1.

The conditions in Definition 1.2 are depicted in Figure 1.

Next theorem corresponds to [BBD2, Theorem 2.1] in our setting:

Theorem 2. *Let (X, d) be a compact metric space, Y a compact subset of X, $f\colon X \to X$ a homeomorphism, and $\varphi\colon X \to \mathbb{R}$ a continuous function. Consider a flip-flop family $\mathfrak{F}$ associated to φ and f having sojourns along Y.*

Then every plaque $D \in \mathfrak{F}$ contains a point $x_D \in D$ that is controlled at any scale with a long sparse tail with respect to φ and Y.

As a corollary of Theorems 1 and 2 we get (recall the notation for Birkhoff limits in (1.3)):

Corollary 3. *Under the hypotheses of Theorem 2 and with the same notation, any measure μ that is a weak$*$ limit of the empirical measures $(\mu_n(x_D))_n$ satisfies the following properties:*

- *the orbit of μ-almost every point is dense in Y and*
- *for μ-almost every point x it holds $\varphi_\infty(x) = 0$.*

As a consequence, almost every measure ν in the ergodic decomposition of μ has full support in Y and satisfies $\int \varphi \, d\nu = 0$.

We now explore some consequences of the results above.

1.4. **Birkhoff averages in homoclinic classes.** An important property of our methods is that they can be used in nonhyperbolic and non-Markovian settings. We now present two applications of our criteria in the "hyperbolic" setting of a mixing sub-shift of finite type that are, as far as we are aware, unknown. The key point of Proposition 4 is that it only requires continuity of the potential φ. When the potential is Hölder continuous this sort of result is well-known[6].

Proposition 4. *Let $\sigma\colon \Sigma \to \Sigma$ be a mixing sub-shift of finite type and $\varphi\colon \Sigma \to \mathbb{R}$ a continuous function. Let α and β be the infimum and maximum, respectively, of $\int \varphi \, d\mu$ over the set of σ-invariant probability measures μ (or equivalently of the Birkhoff averages along periodic orbits). Then for every $t \in (\alpha, \beta)$ the following holds:*

a) (Application of the criterion in [BBD2]) *There is a σ-invariant compact set K_t with positive topological entropy such that the Birkhoff average of φ along the orbit of any point in K_t is t.*

b) (Application of the new criterion) *There is an ergodic measure μ_t with full support in Σ such that $\int \varphi \, d\mu_t = t$.*

This proposition deals with systems satisfying *specification properties*. An important property of our two criteria is that they do not involve and do not depend on specification-like properties. Indeed, they are introduced to control averages of functions in partially hyperbolic settings where specification fails. We now present an application of our criterion in settings without specification properties.

In what follows let M be a boundaryless compact Riemannian manifold and $\mathrm{Diff}^1(M)$ the space of C^1-diffeomorphisms endowed with the standard uniform topology. The *homoclinic class* of a hyperbolic periodic point q of a diffeomorphism $f \in \mathrm{Diff}^1(M)$, denoted by $H(q, f)$, is the closure of the set of transverse intersection points of the stable and unstable manifolds of the orbit of q. Two hyperbolic periodic points p and q of f are *homoclinically related* if the stable and unstable manifolds of their orbits intersect cyclically and transversely. The homoclinic class of q can also be defined as the closure of the periodic points of f that are homoclinically related to q. A homoclinic class is a *transitive set* (existence of a dense orbit) whose periodic points form a dense subset of it. Homoclinic classes are in many cases the "elementary pieces of the dynamics" of a diffeomorphism and are used to structure its dynamics, playing a similar role of the basic sets of the

[6]For instance, techniques from multifractal analysis provide the following: Given a Hölder continuous function φ, there is a parametrised family of Gibbs states μ_t, $t \in (\alpha, \beta)$, where α, β are as above, such that $\int \varphi \, d\mu_t = t$. Each μ_t has full support and positive entropy. The conclusion in this statement is stronger than the than the one in b) as it guarantees also positive entropy. For a survey of this topic see for instance [PW].

hyperbolic theory (indeed each basic set is a homoclinic class), for a discussion see the survey in [B].

The u-*index* of a hyperbolic periodic point is the dimension of its unstable bundle. We analogously define s-*index*. Two saddles which are homoclinically related have necessarily the same u- and s-indices. However two saddles with *different indices* (it is not necessary to specify the index type) may be in the same homoclinic class. In such a case the class is necessarily nonhyperbolic. Indeed, the property of a homoclinic class containing saddles of different indices is a typical feature in the nonhyperbolic dynamics studied in this paper (see also [S, M1, BD1]).

The next result is a generalisation of the second part of Proposition 4 to a non-necessarily hyperbolic context, observe that we do not require hyperbolicity of the homoclinic class. Recall that if p is a periodic point of f we denote by $\mu_{\mathcal{O}(p)}$ the f-invariant probability supported on the orbit of p.

Theorem 5. *Let $f\colon M \to M$ be a C^1-diffeomorphism defined on a boundaryless compact manifold and $\varphi\colon M \to \mathbb{R}$ a continuous function. Consider a pair of hyperbolic periodic points p and q of f that are homoclinically related and satisfy*

$$a_p \stackrel{\text{def}}{=} \int \varphi \, d\mu_{\mathcal{O}(p)} < \int \varphi \, d\mu_{\mathcal{O}(q)} \stackrel{\text{def}}{=} a_q.$$

Then for every $t \in (a_p, a_q)$ there is an ergodic measure μ_t whose support is the whole homoclinic class $H(p,f) = H(q,f)$ and satisfies $\int \varphi \, d\mu_t = t$.

Note that the hypotheses in the theorem are C^1-open. Observe that the difficulty in the theorem is to get simultaneously the three properties *ergodicity*, *prescribed average*, and *full support*. It is easier (and also known) to build measures satisfying simultaneously only two of these properties.

We also aim to apply the criterion in Theorem 1 to saddles p and q that have different indices and are in the same homoclinic class (or, more generally, chain recurrence class) and thus the saddles are not homoclinically related.

Before stating the next corollary let us recall the definition of a chain recurrence class. Given $\epsilon > 0$, a finite sequence of points $(x_i)_{i=0}^n$ is an *ϵ-pseudo-orbit* of a diffeomorphism f if $d(f(x_i), x_{i+1}) < \epsilon$ for every $i = 0, \dots, n-1$ (here d denotes the distance in M). A point x is *chain recurrent* for f if for every $\epsilon > 0$ there is an ϵ-pseudo-orbit $(x_i)_{i=0}^n$ with $x_0 = x = x_n$. The *chain recurrent set* of f, denoted by $\mathcal{R}(f)$, is the union of the chain recurrent points of f. The *chain recurrence class* $C(x,f)$ of a point $x \in \mathcal{R}(f)$ is the set of points y such that for every $\epsilon > 0$ there are ϵ-pseudo-orbits joining x to y and y to x. Two chain recurrence classes are either disjoint or equal. Thus the set $\mathcal{R}(f)$ is the union of pairwise disjoint chain recurrence classes. Let us observe that two points in the same homoclinic class are also in the same chain recurrence class (the converse is false in general, although C^1-generically homoclinic classes and chain recurrence classes of periodic points coincide, see [BC]). Thus if p is a hyperbolic periodic point then $H(p,f) \subseteq C(p,f)$.

Corollary 6. *Let M be a boundaryless compact manifold and $\mathcal{U}$ be a C^1-open set in $\mathrm{Diff}^1(M)$ such that every $f \in \mathcal{U}$ has a pair of hyperbolic periodic orbits p_f and q_f of different indices depending continuously on f whose chain recurrence classes are equal. Let $\varphi\colon M \to \mathbb{R}$ be a continuous function such that*

$$\int \varphi \, d\mu_{\mathcal{O}(p_f)} < 0 < \int \varphi \, d\mu_{\mathcal{O}(q_f)}, \quad \textit{for every } f \in \mathcal{U}.$$

Then there are two C^1-open sets $\mathcal{V}_p$ and $\mathcal{V}_q$ whose union is C^1-dense in $\mathcal{U}$ such that every $f \in \mathcal{V}_p$ (resp. $f \in \mathcal{V}_q$) has an ergodic measure μ_f whose support is the homoclinic class $H(p_f, f)$ (resp. $H(q_f, f)$) and satisfies $\int \varphi \, d\mu_f = 0$.

Note that the saddles in the corollary cannot be homoclinically related and hence Theorem 5 cannot be applied. We bypass this difficulty by transferring the desired averages to pairs of homoclinically related periodic points (then the proof follows from Theorem 5), see Section 5.3 for the proof of the corollary.

Remark 1.3. By [BDPR, Theorem E], if in Corollary 6 we assume that the chain recurrence class is partially hyperbolic with one-dimensional center (see definition below) then there is a C^1-open and dense subset $\mathcal{V}$ of $\mathcal{U}$ such that $H(p_g, g) = H(q_g, g)$ for all $g \in \mathcal{V}$. Without this extra hypothesis the equality of the homoclinic classes is only guaranteed for a residual subset of $\mathcal{U}$, see [BC].

1.5. **Nonhyperbolic ergodic measures with full support.** In what follows we focus on *partially hyperbolic diffeomorphisms with one-dimensional center*. Our aim is to get results as above when φ is the "logarithm of the center derivative". This will allow us to obtain nonhyperbolic ergodic measures with large support in quite general nonhyperbolic settings. Before going to the details we need some definitions.

Given a diffeomorphism f we say that a compact f-invariant set Λ is *partially hyperbolic with one-dimensional center* if there is a Df-invariant dominated[7] splitting with three non-trivial bundles

$$T_\Lambda M = E^{\mathrm{uu}} \oplus E^{\mathrm{c}} \oplus E^{\mathrm{ss}} \tag{1.4}$$

such that E^{uu} is uniformly expanding, E^{c} has dimension 1, and E^{ss} is uniformly contracting. We say that E^{uu} and E^{ss} are the *strong unstable* and *strong stable bundles*, respectively, and that E^{c} is the *central bundle*. We denote by d^{uu} and d^{ss} the dimensions of E^{uu} and E^{ss}, respectively.

Given an ergodic measure μ of a diffeomorphism f the Oseledets' Theorem gives numbers $\chi_1(\mu) \ge \chi_2(\mu) \ge \cdots \ge \chi_d(\mu)$, the *Lyapunov exponents*, and a Df-invariant splitting $E_1 \oplus E_2 \oplus \cdots \oplus E_d$, the *Oseledets' splitting,* where $d = \dim(M)$, with the following property: for μ-almost every point

$$\lim_{n\to\pm\infty} \frac{\log \|Df_x^n(v_i)\|}{n} = \chi_i(\mu), \quad \text{for every } i \text{ and } v \in E_i \setminus \{\bar{0}\}.$$

If the measure is supported on a partially hyperbolic set with one-dimensional center as above then

$$E^{\mathrm{uu}} = E_1 \oplus \cdots \oplus E_{d^{\mathrm{uu}}}, \quad E^{\mathrm{c}} = E_{d^{\mathrm{uu}}+1}, \quad E^{\mathrm{ss}} = E_{d^{\mathrm{uu}}+2} \oplus \cdots \oplus E_d,$$

and $\chi_{d^{\mathrm{uu}}}(\mu) > 0 > \chi_{d^{\mathrm{uu}}+2}(\mu)$. Let $\chi_{d^{\mathrm{uu}}+1}(\mu) \stackrel{\text{def}}{=} \chi_{\mathrm{c}}(\mu)$, we say that $\chi_{\mathrm{c}}(\mu)$ is the *central exponent* of μ. In this partially hyperbolic setting the *logarithm of the center derivative* map

$$\mathrm{J}_f^{\mathrm{c}}(x) \stackrel{\text{def}}{=} \log |Df_x|_{E^{\mathrm{c}}(x)}| \tag{1.5}$$

[7]A Df-invariant splitting $T_\Lambda M = F \oplus E$ is *dominated* if there are constants $C > 0$ and $\lambda < 1$ such that $||Df^{-n} F_{f^n(x)}|| \, ||Df^n E_x|| < C\lambda^n$ for all $x \in \Lambda$ and $n \in \mathbb{N}$. In our case domination means that the bundles $E^{\mathrm{uc}} \oplus E^{\mathrm{ss}}$ and $E^{\mathrm{uu}} \oplus E^{\mathrm{cs}}$ are both dominated, where $E^{\mathrm{uc}} = E^{\mathrm{uu}} \oplus E^{\mathrm{c}}$ and $E^{\mathrm{cs}} = E^{\mathrm{c}} \oplus E^{\mathrm{ss}}$.

is well defined and continuous, therefore the central Lyapunov exponent of the measure is given by the integral

$$\chi_{\mathrm{c}}(\mu) = \int \mathrm{J}^{\mathrm{c}}_f \, d\mu.$$

This equality allows to use the methods in the previous sections to construct and control nonhyperbolic ergodic measures.

Let us explain some relevant points of our study. A (new) difficulty, compared with Theorem 5, is that the logarithm of the center derivative $\mathrm{J}^{\mathrm{c}}_f$ cannot take values with different signs at homoclinically related periodic points (by definition, such points have the same indices and thus the sign of $\mathrm{J}^{\mathrm{c}}_f$ is the same). To recover this signal property we consider chain recurrence classes containing saddles of different indices.

Theorem 7. *Let M be a boundaryless compact manifold and $\mathcal{U}$ a C^1-open set of* $\mathrm{Diff}^1(M)$ *such that every $f \in \mathcal{U}$ has hyperbolic periodic orbits p_f and q_f such that:*

- *they have different indices and depend continuously on $f \in \mathcal{U}$,*
- *their chain recurrence classes $C(p_f, f)$ and $C(q_f, f)$ are equal and have a partially hyperbolic splitting with one-dimensional center.*

Then there is a C^1-open and dense subset $\mathcal{V} \subset \mathcal{U}$ such that every diffeomorphism $f \in \mathcal{V}$ has a nonhyperbolic ergodic measure μ_f whose support is the homoclinic class $H(p_f, f) = H(q_f, f)$.

Let us first observe that Theorem 7 can be rephrased in terms of robust cycles instead of periodic points in the same chain recurrence class. For that we need to review the definition of a *robust cycle*. Recall that a hyperbolic set Λ_f of $f \in \mathrm{Diff}^1(M)$ has a well defined hyperbolic continuation Λ_g for every g close to f. Two transitive hyperbolic basic sets Λ_f and Γ_f of a diffeomorphism f have a *C^1-robust (heterodimensional) cycle* if these sets have different indices and if there is a C^1-neighbourhood $\mathcal{U}_f$ of f such that for every $g \in \mathcal{U}$ the invariant sets of Λ_g and Γ_g intersect cyclically. As discussed in [BBD2], the dynamical scenarios of "dynamics with C^1-robust cycles" and "dynamics with chain recurrence classes containing C^1-robustly saddles of different indices" are essentially equivalent (they coincide in a C^1-open and dense subset of $\mathrm{Diff}^1(M)$).

We now describe explicitly the open and dense subset $\mathcal{V}$ of $\mathcal{U}$ in Theorem 7 using *dynamical blenders* and *flip-flop configurations* introduced in [BBD2], see Remark 1.4. Naively, a *dynamical blender* is a hyperbolic and partially hyperbolic set together with a *strictly invariant family of discs* (i.e., the image of any disc of the family contains another disc of the family) almost tangent to its strong unstable direction, see Definition 6.3. In very rough terms, a *flip-flop configuration* of a diffeomorphism f and a continuous function φ is a C^1-robust cycle associated to a hyperbolic periodic point q and a dynamical blender Λ such that φ is bigger than $\alpha > 0$ in the blender Λ and smaller than $-\alpha < 0$ on the orbit of q. Important properties of flip-flop configurations are their C^1-robustness, that they occur C^1-open and densely in the set $\mathcal{U}$ in Theorem 7, and that they yield flip-flop families. The latter allows to apply our criterion for zero averages. The set $\mathcal{V}$ in Theorem 7 is described in the remark below.

Remark 1.4 (The set $\mathcal{V}$ in Theorem 7)**.** The set $\mathcal{V}$ is the subset of $\mathcal{U}$ of diffeomorphisms with flip-flop configurations "containing" the saddle q_g.

To state our next result recall that a *filtrating region* of a diffeomorphism f is the intersection of an attracting region and a repelling region of f. Let U be a filtrating region of f endowed with a strictly forward invariant unstable cone field of index i and a strictly backward invariant cone field of index $\dim(M) - i - 1$, see Section 6.1.2 for the precise definitions. Then the maximal f-invariant set in U has a partially hyperbolic splitting $E^{\mathrm{uu}} \oplus E^{\mathrm{c}} \oplus E^{\mathrm{ss}}$, with $\dim(E^{\mathrm{c}}) = 1$. As above this allows us to define the logarithm of the center derivative $\mathrm{J}^{\mathrm{c}}_f$ of f. We have the following "variation" of Theorem 7.

Theorem 8. *Let M be a boundaryless compact manifold. Consider $f \in \mathrm{Diff}^1(M)$ with a a filtrating region U endowed with a strictly Df-invariant unstable cone field of index i and a strictly Df^{-1}-invariant cone field of index $\dim(M) - i - 1$.*

Assume that f has a flip-flop configuration associated to a dynamical blender and a hyperbolic periodic point q both contained in U.

Then there is a C^1-neighbourhood $\mathcal{V}_f$ of f such that every $g \in \mathcal{V}_f$ has a non-hyperbolic ergodic measure whose support is the whole homoclinic class $H(q_g, g)$ of the continuation q_g of q.

The hypothesis in this theorem imply that the blender and the saddle in the flip-flop configuration are in the same chain recurrence class. With the terminology of robust cycles, they have a C^1-robust cycle.

Note that Theorem 8 is not a perturbation result: it holds for every diffeomorphism with such a flip-flop configuration. Moreover, and more important, the hypotheses in Theorem 8 are open (the set U is also a filtrating set for every g sufficiently close to f, hence the homoclinic class $H(q_g, g)$ is contained in U and partially hyperbolic, and flip-flop configurations are robust). Thus Theorem 8 holds for the homoclinic class of the continuation of q for diffeomorphisms g close to f.

Remark 1.5. Theorem 8 does not require the continuous variation of the homoclinic class $H(q_g, g)$ with respect to g. Note also that, in general, homoclinic classes only depend lower semi-continuously on the diffeomorphism. As a consequence, the partial hyperbolicity of a homoclinic class is not (in general) a robust property. The relevant assumption is that the homoclinic classes are contained in a partially hyperbolic filtrating neighbourhood which guaranteed the robust partial hyperbolicity of the homoclinic class.

We can change the hypotheses in the theorem, omitting that U is a filtrating neighbourhood and considering homoclinic classes depending continuously on the diffeomorphism (this occurs in a residual subset of diffeomorphisms). Then, by continuity, the class is robustly contained in the partially hyperbolic region and we can apply the previous arguments.

1.6. **Applications to robustly nonhyperbolic transitive diffeomorphisms.** A diffeomorphism $f \in \mathrm{Diff}^1(M)$ is *transitive* if it has a dense orbit. The diffeomorphism f is *C^1-robustly transitive* if any diffeomorphism g that is C^1-close to f is also transitive. In other words, a diffeomorphism is C^1-robustly transitive if it belongs to the C^1-interior of the set of transitive diffeomorphisms.

We denote by $\mathcal{RT}(M)$ the (C^1-open) subset of $\mathrm{Diff}^1(M)$ consisting of diffeomorphisms f such that:

- f is robustly transitive,
- f has a pair of hyperbolic periodic points of different indices,

- f has a partially hyperbolic splitting $TM = E^{\mathrm{uu}} \oplus E^{\mathrm{c}} \oplus E^{\mathrm{ss}}$, where E^{uu} is uniformly expanding, E^{ss} is uniformly contracting, and E^{c} is one-dimensional.

Note that the last condition implies that the hyperbolic periodic points of f have either u-index $\dim(E^{\mathrm{uu}})$ or $\dim(E^{\mathrm{uu}}) + 1$. Note also that our assumptions imply that $\dim(M) \geq 3$ (in lower dimensions $\mathcal{RT}(M) = \varnothing$, see [PS]).

In dimension ≥ 3 and depending on the type of manifold M, the set $\mathcal{RT}(M)$ contains interesting examples. Chronologically, the first examples of such partially hyperbolic robustly transitive diffeomorphisms were obtained in [S] considering diffeomorphisms in $\mathbb{T}^4$ obtained as skew products of Anosov diffeomorphisms on $\mathbb{T}^2$ and derived from Anosov on $\mathbb{T}^2$ ($\mathbb{T}^i$ stands for the i-dimensional torus). Later, [M1] provides examples in $\mathbb{T}^3$ considering derived from Anosov diffeomorphisms. Finally, [BD1] gives examples that include perturbations of time-one maps of transitive Anosov flows and perturbations of skew products of Anosov diffeomorphisms and isometries.

Theorem 9. *There is a C^1-open and dense subset $\mathcal{Z}(M)$ of $\mathcal{RT}(M)$ such that every $f \in \mathcal{Z}(M)$ has an ergodic nonhyperbolic measure whose support is the whole manifold M.*

Let us mention some related results. First, by [BBD2], there is a C^1-open and dense subset of $\mathcal{RT}(M)$ formed by diffeomorphisms with an ergodic nonhyperbolic measure with positive entropy, but the support of these measures is not the whole ambient. By [BDG], there is a residual subset of $\mathcal{RT}(M)$ of diffeomorphism with an ergodic nonhyperbolic measure with full support. Finally, a statement similar to our theorem is stated in [BZ], see Footnote 4.

Recall that given a periodic point p of f the measure $\mu_{\mathcal{O}(p)}$ is the unique f-invariant measure supported on the orbit of p.

Corollary 10. *Consider a continuous map $\varphi\colon M \to \mathbb{R}$. Suppose that $f \in \mathcal{RT}(M)$ has two hyperbolic periodic orbits p and q such that*

$$\mu_{\mathcal{O}(p)}(\varphi) > 0 > \mu_{\mathcal{O}(q)}(\varphi).$$

Then there are a C^1-neighbourhood $\mathcal{V}_f$ of f and a C^1-open and dense subset $\mathcal{O}_f$ of $\mathcal{V}_f$ such that every $g \in \mathcal{O}_f$ has an ergodic measure μ_g with full support on M such that

$$\int \varphi \, d\mu_g = 0.$$

Remark 11. By [C, Proposition 1.4], for diffeomorphisms in $\mathcal{Z}(M)$ every hyperbolic ergodic measure μ is the weak$*$ limit of periodic measures supported on points whose orbits tend (in the Hausdorff topology) to the support of the measure μ. Thus, Corollary 10 holds after replacing the hypothesis $\mu_{\mathcal{O}(p)}(\varphi) > 0 > \mu_{\mathcal{O}(q)}(\varphi)$ by the existence of two hyperbolic ergodic measures ν^+ and ν^- such that $\int \varphi \, d\nu^+ > 0 > \int \varphi \, d\nu^-$.

1.7. **Organization of the paper.** In Section 2 we introduce the concepts involved in the criterion of control at any scale with a long sparse tail and prove Theorem 1. In Section 3 we introduce the notion of a pattern and see how they are induced by long tails of scales. We study the *concatenations of plaques* of flip-flop families (associated to a map φ) and the control of the averages of φ corresponding to these concatenations, see Theorem 3.9. In Section 4 we prove Theorem 2, Corollary 3, and Proposition 4. In Section 5 we prove Theorem 5 involving flip-flop families and homoclinic relations. In Section 6 we review some key ingredients as dynamical blenders and flip-flop configurations and prove Theorems 7 and 8. Finally, in Section 7 we apply our methods to construct nonhyperbolic ergodic measures with full support for some robustly transitive diffeomorphisms, proving Theorem 9 and Corollary 10.

2. A criterion for zero averages: control at any scale up to a long sparse tail

The construction that we present for controlling averages is probably too rigid but it is enough to achieve our goals and certain constraints perhaps could be relaxed. However, at this state of the art, we do not aim for full generality but prefer to present the ingredients of the construction in a simple as possible way. One may aim to extract a general conceptual principle behind the construction, but this is beyond the focus of this paper. In Sections 2.1 and 2.2 we introduce the concepts involved in the criterion for controlling averages and in Section 2.3 we prove Theorem 1.

2.1. Scales and long sparse tails.

In what follows we introduce the definitions of scales and long sparse tails.

Definition 2.1 (Scale)**.** *A sequence $\mathcal{T}=(T_n)_{n\in\mathbb{N}}$ of strictly positive natural numbers is called* a scale *if there is a sequence $\bar\kappa=(\kappa_n)_{n\geq 1}$ (the* sequence of factors *of the scale) of natural numbers with $\kappa_n\geq 3$ for every n such that*

- *$T_n=\kappa_n\,T_{n-1}$ for every $n\geq 1$;*
- *$\kappa_{n+1}/\kappa_n\to\infty$.*

We assume that the number T_0, and hence every T_n, is a multiple of 3*.*

We now introduce some notation. In what follows, given $a,b\in\mathbb{R}$ we let

$$[a,b]_{\mathbb{N}}\stackrel{\text{def}}{=}[a,b]\cap\mathbb{N}.$$

Given a subset $\mathbb{M}$ of $\mathbb{N}$ a *component* of $\mathbb{M}$ is an interval of integers $[a,b]_{\mathbb{N}}\subset\mathbb{M}$ such that $a,b\in\mathbb{M}$ and $a-1,b+1\notin\mathbb{M}$.

Definition 2.2 (Controling sequence)**.** *Let $\bar\varepsilon=(\varepsilon_n)_{n\in\mathbb{N}}$ be a sequence of positive numbers converging to* 0*. We say that $\bar\varepsilon$ is a* controlling sequence *if*

$$\sum_n \varepsilon_n<+\infty \quad \textit{and} \quad \prod_n (1-\varepsilon_n)>0.$$

Remark 2.3. For a sequence $(\varepsilon_n)_{n\in\mathbb{N}}$ of numbers with $\epsilon_n\in(0,1)$ one has

$$\sum_n \varepsilon_n<+\infty \quad\Longleftrightarrow\quad \prod_n (1-\varepsilon_n)>0.$$

Remark 2.4. Let $\mathcal{T}=(T_n)_{n\in\mathbb{N}}$ be a scale, $\kappa_{n+1}=\frac{T_{n+1}}{T_n}$, and $\varepsilon_n=\frac{2}{\kappa_n}$. Then the sequence $(\varepsilon_n)_{n\in\mathbb{N}}$ is a controlling one.

Definition 2.5 (Long sparse tail)**.** *Consider a scale $\mathcal{T}=(T_n)_{n\in\mathbb{N}}$ and a controlling sequence $\bar\varepsilon=(\varepsilon_n)_{n\in\mathbb{N}}$. A set $R_\infty\subset\mathbb{N}$ is a* $\mathcal{T}$-long $\bar\varepsilon$-sparse tail *if the following properties hold:*

a) *Every component of R_∞ is of the form $[k\,T_n,(k+1)\,T_n-1]_{\mathbb{N}}$, for some k and n (we say that such a component has size T_n).*

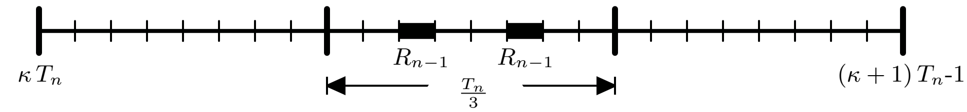

FIGURE 2. A long sparse tail

Let R_n be the union of the components of R_∞ of size T_n and let

$$R_{n,\infty} \stackrel{\text{def}}{=} \bigcup_{i\geq n} R_i,$$

the union of the components of R_∞ of size larger than or equal to T_n.

b) $0 \notin R_\infty$, in particular $[0, T_n - 1]_{\mathbb{N}} \not\subset R_n$.

c) Consider an interval I of natural numbers of the form

$$I = [kT_n, (k+1)T_n - 1]_{\mathbb{N}}, \quad \textit{for some } n \geq 1 \textit{ and } k \geq 0,$$

that is not contained in any component of R_∞ then the following properties hold:

- center position:

$$\left[kT_n, kT_n + \frac{T_n}{3}\right] \cap R_{n-1} = \varnothing = \left[\big((k+1)T_n - 1\big) - \frac{T_n}{3}, \big((k+1)T_n - 1\big)\right] \cap R_{n-1}.$$

- $\bar{\varepsilon}$-sparseness:

$$0 < \frac{\#(R_{n-1} \cap I)}{T_n} < \varepsilon_n.$$

The conditions in Definition 2.5 are depicted in Figure 2.

Definition 2.6 (Good and bad intervals)**.** *With the notation of Definition 2.5, an interval I of the form $I = [kT_n, (k+1)T_n - 1]_{\mathbb{N}}$ is called n-*bad *if $I \subset R_{n,\infty} \subset R_\infty$. The interval is called n-*good *if $I \cap R_{n,\infty} = \varnothing$.*

Remark 2.7 (On the definition of a $\mathcal{T}$-long $\bar{\epsilon}$-sparse tail)**.**

a) It is assumed that $0 \notin R_\infty$. This implies that for every $n \geq 0$ the initial interval $[0, T_n - 1]_{\mathbb{N}}$ is not a component of R_∞ of size T_n. Therefore $[0, T_n - 1]$ is disjoint from R_n and thus from R_m for every $m \geq n$. In other words, the interval $[0, T_n - 1]$ is n-good, that is,

$$[0, T_n - 1] \cap R_{n,\infty} = \varnothing.$$

b) Let $I = [kT_n, (k+1)T_n - 1]$ be an interval as in Item (c) of Definition 2.5. By Item (a) the interval I is either contained in a component of R_∞ whose size is larger than or equal to T_n or is disjoint from $R_{n,\infty}$. Thus, Item (c) considers the case where the interval is disjoint from $R_{n,\infty}$.

Now any component of R_∞ of size less than T_n is either disjoint from I or contained in I: just note that such a component has length T_m, $m < n$, and starts at a multiple of T_m and T_n is a multiple of T_m.

Item (c) describes the position and quantity of the components of size T_{n-1} in the interval I. For that, one splits the interval I into tree parts of equal length $\frac{T_n}{3}$. The following properties are required:

- Every component of size T_{n-1} contained in I is contained in the middle third interval;

- The middle third interval contains at least one component of size T_{n-1}. Thus the intersection $I \cap R_{n-1}$ is not empty, but the the set R_{n-1} has a small density in the interval I that is upper bounded by ϵ_n.

c) Item (c) does not consider the case $n = 0$. For $n = 0$ and an interval I of the form $[kT_0, (k+1)T_0 - 1]$ there are two possibilities: either I is a component of R_∞ (i.e., contained in R_0) or I is disjoint from R_∞.

d) Given any interval I of the form $I = [kT_n, (k+1)T_n - 1]_{\mathbb{N}}$ there are two possibilities:
   - either $I \subset R_\infty$ and then $I \subset R_{n,\infty}$ and I is n-bad;
   - or $I \cap R_{n,\infty} = \varnothing$ and then I is n-good.

The definition of a long sparse tail involves many properties and conditions, thus its existence it is not obvious. We solve this difficulty in the next lemma.

Lemma 2.8 (Existence of long sparse tails)**.** *Consider a scale $\mathcal{T} = (T_n)_{n\in\mathbb{N}}$ and its sequence of factors $\bar{\kappa} = (k_n)_{n\geq 1}$. Write $\varepsilon_n = \frac{2}{\kappa_n}$ and let $\bar{\epsilon} = (\epsilon_n)_{n\geq 1}$. Then there is a $\mathcal{T}$-long $\bar{\varepsilon}$-sparse tail R_∞.*

Proof. First note that, by Remark 2.4, the sequence $\bar{\varepsilon}$ is a controlling one.

The construction of the set R_∞ is done inductively. For each $n \in \mathbb{N}$ we define the intersection of the set R_∞ with the intervals $[0, T_n - 1]$. We denote such an intersection by $R_\infty(T_n)$.

For $n = 0$, we let $R_\infty(T_0) = \varnothing$. Fix now $n > 0$ and suppose that the sets $R_\infty(T_{n-1})$ has been constructed satisfying (in restriction to the interval $[0, T_{n-1}-1]$) the properties in Definition 2.5. We now proceed to define the set $R_\infty(T_n)$.

For any $i \leq n-1$ we denote by $R_{i,n-1}$ the union of the components of $R_\infty(T_{n-1})$ of length T_i. We next define the family of subsets $\{R_{j,n},\, j = 0, \dots, n\}$ of $[0, T_n - 1]$ by decreasing induction on j as follows. We let

$$R_{n,n} = \varnothing \qquad \text{and} \qquad R_{n-1,n} = \left[\frac{T_n}{3}, \frac{T_n}{3} + T_{n-1} - 1\right]_{\mathbb{N}}.$$

Let $j < n$ and assume that the sets $R_{i,n}$ are defined for every $n \geq i > j$. The set $R_{j,n}$ is defined as follows:

- if $[kT_{j+1}, (k+1)T_{j+1} - 1]_{\mathbb{N}} \subset \bigcup_{i>j} R_{i,n}$ then
$$R_{j,n} \cap [kT_{j+1}, (k+1)T_{j+1} - 1] = \varnothing,$$
- Otherwise we let

$$R_{j,n} \cap [kT_{j+1}, (k+1)T_{j+1} - 1]_{\mathbb{N}} = \left[\Big(k + \frac{1}{3}\Big)T_{j+1}, \Big(k + \frac{1}{3}\Big)T_{j+1} + T_j - 1\right]_{\mathbb{N}}. \tag{2.1}$$

Note that by construction,

$$R_\infty(T_n) = \bigcup_{i=0}^{n} R_{i,n}.$$

Claim 2.9. The set $R_\infty(T_n)$ satisfies (in restriction to the interval $[0, T_n - 1]$) the conditions of Definition 2.5.

Proof. Property (a) in the definition follows from the construction: the components of $R_{i,n}$ have size T_i and have no adjacent points with the components of $\bigcup_{j>i}^{n} R_{j,n}$.

For Property (b) one checks inductively that $O \notin R_{i,n}$ for every i and n.

Property (c) is a consequence of (2.1). If the set $R_{i,n}$ intersects a segment $[kT_{i+1}, (k+1)T_{i+1} - 1]$ then it is contained in its middle third interval, implying the center position condition. For the sparseness note that by construction and the definition of ε_i, for each i it holds

$$0 < \frac{\#(R_{i-1} \cap I)}{T_i} = \frac{T_{i-1}}{T_i} = \frac{1}{\kappa_i} < \varepsilon_j.$$

This completes the proof of the claim. $\square$

Our construction also provides immediately the following properties: For every $i < n$ it holds:

- if $m \geq n$ then $R_{i,m} \cap [0, T_n - 1] = R_{i,n}$,
- if $m \geq n$ the $R_\infty(T_m) \cap [0, T_n - 1] = R_\infty(T_n)$, and
- $R_{i,n} \subset R_{i,n+1}$.

The tail is now defined by

$$R_\infty \stackrel{\text{def}}{=} \bigcup_{i=0}^{\infty} R_i, \quad \text{where} \quad R_i = \bigcup_{n>i} R_{i,n}. \tag{2.2}$$

By construction, the set R_∞ is an ε-sparse tail of $\mathcal{T}$. $\square$

2.2. **Control at any scale up a long sparse tail.** In this section we give the definition of controlled points.

Definition 2.10. *Let X be a compact set, $f\colon X \to X$ a homeomorphism, and $\varphi\colon X \to \mathbb{R}$ a continuous map. Consider*

- *a scale $\mathcal{T}$, a controlling sequence $\bar\varepsilon$, and a $\mathcal{T}$-long $\bar\varepsilon$-sparse tail R_∞;*
- *decreasing sequences of positive numbers $\bar\delta = (\delta_n)_{n\in\mathbb{N}}$ and $\bar\alpha = (\alpha_n)_{n\in\mathbb{N}}$, converging to 0.*

The f-orbit of a point $x \in X$ is $\bar\delta$-dense along the tail R_∞ *if for every component I of R_∞ of length T_n the segment of orbit $\{f^i(x),\, i \in I\}$ is δ_n-dense in X.*

The Birkhoff averages of φ along the orbit of x are $\bar\alpha$-controlled for the scale $\mathcal{T}$ with the tail R_∞ *if for every interval*

$$I = [k\,T_n, (k+1)\,T_n - 1]_{\mathbb{N}}$$

such that $I \not\subset R_{n+1,\infty}$ (i.e., I is either n-good or is a component of R_n) it holds

$$\varphi_I(x) \stackrel{\text{def}}{=} \frac{1}{T_n} \sum_{i\in I} \varphi(f^i(x)) \in [-\alpha_n, \alpha_n].$$

Definition 2.11. *Let X be a compact set, $f\colon X \to X$ a homeomorphism, and $\varphi\colon X \to \mathbb{R}$ a continuous map.*

A point $x \in X$ is controlled at any scale with a long sparse tail with respect to X and φ *if there are a scale $\mathcal{T}$, a controlling sequence $\bar\varepsilon$, a $\mathcal{T}$-long $\bar\varepsilon$-sparse tail R_∞, and sequences of positive numbers $\bar\delta$ and $\bar\alpha$ converging to 0, such that*

- *the f-orbit of x is $\bar\delta$-dense along the tail R_∞ and*
- *the Birkhoff averages of φ along the orbit of x are $\bar\alpha$-controlled for the scale $\mathcal{T}$ with the tail R_∞.*

In this definition we say that $\bar\delta$ is the density forcing sequence, *$\bar\alpha$ is the* average forcing sequence, *and the point x is* $(\bar\delta, \bar\alpha, \bar\epsilon, \mathcal{T}, R_\infty)$-controlled.

2.3. **Proof of Theorem 1.** In this section we prove Theorem 1, thus we use the assumptions and the notations in its statement. Consider a point $x_0 \in X$ that is controlled at any scale with a long sparse tail for X and φ. Let

- $\mathcal{T} = (T_n)_{n\in\mathbb{N}}$ be the scale;
- R_∞ the $\mathcal{T}$-long $\bar\varepsilon$-sparse tail; and
- $\bar\delta = (\delta_n)_{n\geq 1}$ the density forcing sequence and $\bar\alpha = (\alpha_n)_{n\geq 1}$ the average forcing sequence.

Let μ be a measure that is a weak$*$ limit of the empirical measures $(\mu_n(x_0))_{n\in\mathbb{N}}$. As x_0 remains fixed let us write $\mu_n \stackrel{\text{def}}{=} \mu_n(x_0)$. We need to prove that for μ-almost every point x it holds:

a) the forward orbit of x is dense in X
b) the Birkhoff averages of x satisfy $\lim_{n\to\infty} \frac{1}{n}\sum_{i=0}^{n-1} \varphi(f^i(x)) = 0$.

Proposition 2.12 below immediately implies item (a) (item (b) follows from Proposition 2.17).

Proposition 2.12. *Under the assumptions above, for every k the (forward) orbit of μ-almost every point is $2\,\delta_k$-dense in X.*

Proof. Fix k. For any given $t > 0$ and $\delta > 0$ consider the set

$$X(t,\delta) \stackrel{\text{def}}{=} \Big\{x \in X\colon \{x,\dots,f^t(x)\} \text{ is } \delta\text{-dense in } X\Big\}$$

and let

$$P_{\infty,t} \stackrel{\text{def}}{=} \liminf_{n\to\infty} P_{n,t}, \quad \text{where} \quad P_{n,t} \stackrel{\text{def}}{=} \mu_n(X(t,\delta_k)).$$

Lemma 2.13. $\lim_{t\to\infty} P_{\infty,t} = 1$.

We postpone the proof of this lemma and deduce the proposition from it. Just note that the interior of $X(t,2\delta_k)$ contains the closure of $X(t,\delta_k)$ for every t. Thus $\mu(X(t,2\delta_k)) \geq P_{\infty,t}$. Taking the limit when $t\to\infty$ we prove the proposition. □

Proof of Lemma 2.13. Fixed k take $t > T_{k+1}$.

Claim 2.14. The set of times $i\in\mathbb{N}$ such that $f^i(x_0) \notin X(t,\delta_k)$ is contained in the set

$$\bigcup_{j=0}^{\infty} \Big(R_{m_t+j} \cup (R_{m_t+j} - T_{k+1})\Big),$$

where

- $m_t \stackrel{\text{def}}{=} \inf\{m \geq k+1\colon T_m + 2\,T_{k+1} > t\}$ and
- $R_{k+j} - T_{k+1} \stackrel{\text{def}}{=} \{\ell = i - T_{k+1},\ \text{where } i \in R_{k+j}\}$.

Proof. Take i such that $f^i(x_0) \notin X(t,\delta_k)$. Then the set $\{f^i(x_0),\dots,f^{i+t}(x_0)\}$ is not δ_k-dense. Let $I = [i, i+t]_{\mathbb{N}}$. Recalling Definition 2.10, we have that I can not contain any component of R_∞ of size T_k or greater than T_k. This implies that

- the interval I does not contain any ℓ-bad interval for $\ell \geq k$,
- as a consequence of the sparseness property in item (c) of Definition 2.5, the interval I does not contain any $(\ell+1)$-good interval for $\ell\geq k$ (i.e., disjoint from or $R_{\ell+1,\infty}$).

Thus necessarily the interval I intersects some bad interval $J = [r_m^-, r_m^+]_{\mathbb{N}} \subset R_m$, $m > k$, such that

$$I \subset [r_m^- - T_{k+1}, r_m^+ + T_{k+1}].$$

Otherwise I must contain a $(k+1)$-good interval. Observe that this implies that

$$T_m + 2\,T_{k+1} > t,$$

otherwise the segment of orbit $\{f^{i+j}(x_0)\}_{j=0}^{t}$ would be δ_k-dense, a contradiction. Hence $m \geq m_t$.

Recall that $T_{k+1} < t$, hence $i \in [r_m^- - T_{k+1}, r_m^+]$. Thus

$$i \in J \cup (J - T_{k+1}) \subset R_m \cup (R_m - T_{k+1})$$

for some $m \geq m_t$. This ends the proof of the claim. $\square$

In view of Claim 2.14, to prove the lemma it is enough to see the following:

Claim 2.15.

$$\lim_{t\to+\infty} \lim_{n\to+\infty} \frac{1}{n} \# \left([0,n] \cap \bigcup_{j=0}^{\infty} \Big(R_{m_t+j} \cup (R_{m_t+j} - T_{k+1}) \Big) \right) = 0.$$

Proof. Note that the components of the set $R_{m_t+j} \cup (R_{m_t+j} - T_{k+1})$ are intervals of length $T_{m_t+j} + T_{k+1} < 2\,T_{m_t+j}$. Thus the claim is a direct consequence of next fact (recall the definition of $R_{m_t,\infty}$ in Definition 2.5).

Fact 2.16.

$$\lim_{t\to\infty} \limsup_{n\to\infty} \frac{1}{n} \# \big(R_{m_t,\infty} \cap [0,n] \big) \to 0.$$

Proof. We need to estimate the proportion

$$\varrho(m,n) \stackrel{\text{def}}{=} \frac{\# \big(R_m \cap [0,n]\big)}{n}$$

of the set R_m in $[0,n]_{\mathbb{N}}$. We claim that $\varrho(m,n) < 3\,\varepsilon_{m+1}$. There are three cases:

- $T_{m+1} \leq n$: Let $k\,T_{m+1} \leq n < (k+1)\,T_{m+1}$, where $k \in \mathbb{N}$ and $k \geq 1$. By the sparseness condition we have

  $$\#\big(R_m \cap [0, (k+1)\,T_{m+1}]\big) < (k+1)\,\varepsilon_{m+1}.$$

  Therefore

  $$\frac{\# \big(R_m \cap [0,n]\big)}{n} \leq \frac{\# \big(R_m \cap [0,n]\big)}{k\,T_{m+1}} \leq \frac{(k+1)\varepsilon_{m+1}}{k} < 2\,\varepsilon_{m+1}.$$

- $T_m \leq n < T_{m+1}$: Since $[0, T_{m+1} - 1]$ is an $(m+1)$-good interval we have that $[0, T_{m+1}/3]$ and R_m are disjoint. If the proportion is 0 we are done. Otherwise, by the center position condition, $n > \frac{T_{m+1}}{3}$. Therefore

  $$\frac{\# \big(R_m \cap [0,n]\big)}{n} < 3\, \frac{\# \big(R_m \cap [0,T_{m+1}]\big)}{T_{m+1}} < 3\,\varepsilon_{m+1},$$

  where the last inequality follows from the sparseness condition.
- $n < T_m$: In this case, by condition (b) in Definition 2.5, $R_m \cap [0,n] = \varnothing$.

Since $\varrho(m,n) < 3\,\varepsilon_{m+1}$ for every n we get

$$\frac{1}{n}\#\left(R_{m_t,\infty}\cap[0,n]\right) < 3\sum_{m=m_t}^{\infty}\varepsilon_m.$$

Since, by definition, $\sum_{m=0}^{\infty}\varepsilon_m < +\infty$ this implies

$$\lim_{t\to\infty}\limsup_{n\to\infty}\frac{1}{n}\#(R_{m_t,\infty}\cap[0,n]) \le 3\lim_{t\to\infty}\sum_{m=m_t}^{\infty}\varepsilon_m = 0,$$

proving the fact. □

This ends the proof of Claim 2.15 □

The proof of Lemma 2.13 is now complete. □

Proposition 2.12 gives the density of orbits in Theorem 1. To end the proof of the theorem it remains to prove the part relative to the averages. This is an immediate consequence of next proposition. Recall the notation of finite Birkhoff averages $\varphi_n(x)$ and of limit averages $\varphi_\infty(x)$ of a function φ in (1.2) and (1.3). Recall also that μ is a weak$*$ limit of the empirical measures $\mu_n = \mu_n(x_0)$.

Proposition 2.17. *Fix $k\in\mathbb{N}$. For μ-almost every point x the limite average $\varphi_\infty(x)$ is well defined and belongs to $[-3\alpha_k, 3\alpha_k]$.*

Proof. Let B be the set of points such that the limit average $\varphi_\infty(x)$ is well defined. By Birkhoff theorem it holds $\mu(B)=1$. Therefore it is enough to prove that for every $x\in B$ there is a sequence $n_j = n_j(x)\to\infty$ such that $\varphi_{n_j}(x)\in[-3\,\alpha_k, 3\,\alpha_k]$ for every j. For $t\in\mathbb{N}$ define the number

$$q_t \stackrel{\text{def}}{=} \liminf_{n\to+\infty} q_{t,n}, \quad \text{where} \quad q_{t,n} \stackrel{\text{def}}{=} \mu_n\big(\{x\colon \varphi_t(x)\in[-2\,\alpha_k, 2\,\alpha_k]\}\big).$$

Lemma 2.18. $\lim_{t\to\infty} q_t = 1$.

Let us postpone the proof of this lemma and conclude the proof of the proposition assuming it. By definition of μ

$$\mu(\{x\colon \varphi_t(x)\in[-3\,\alpha_k, 3\,\alpha_k]\}) \ge q_t.$$

By Lemma 2.18, $q_t\to 1$, thus there is a subsequence (q_{t_i}) such that

$$\sum_0^\infty (1-q_{t_i}) < +\infty.$$

Fix the sequence $(q_{t_i})_i$ and define the sets

$$Y_N \stackrel{\text{def}}{=} \bigcap_{j>N}\{x\colon \varphi_{t_j}(x)\in[-2\,\alpha_k, 2\,\alpha_k]\}.$$

By definition

$$\mu(Y_N) \ge 1 - \left(\sum_{j=N}^{\infty}(1-q_{t_j})\right)$$

and hence

$$\mu(Y) = 1, \quad \text{where} \quad Y \stackrel{\text{def}}{=} \bigcup_N Y_N.$$

We have that $\mu(B \cap Y) = 1$ and that every point $x \in Y \cap B$ has Birkhoff average $\varphi_\infty(x) \in [-3\,\alpha_k, 3\,\alpha_k]$, this ends the proof of the proposition (assuming Lemma 2.18).

Proof of Lemma 2.18. Recall that $x_0 \in X$ is controlled at any scale with a long sparse tail for X and φ. Recall also the choices of $\mathcal{T}$, R_∞, $\bar\varepsilon$, and $\bar\alpha$.

Fix k and take a number $t > 2\,T_k$. Pick a time interval $I = [i, i+t-1]_{\mathbb{N}}$. For each $j \geq k$ consider the intervals H of the form $[rT_j, (r+1)T_j - 1]_{\mathbb{N}}$ contained in I which are either j-good or components of the set R_j. By definition of the average forcing sequence $\bar\alpha$ (Definition 2.10) in each of these intervals H the average satisfies

$$\varphi_H(x_0) \in [-\alpha_j, \alpha_j] \subset [-\alpha_k, \alpha_k].$$

We call these subintervals H of I *α_k-controlled.*

Claim 2.19. For every i and t the union of the α_k-controlled intervals contained in $I = [i, i+t-1]_{\mathbb{N}}$ is a (possibly empty) interval $J = J_I$ such that $\varphi_J(x_0) \in [-\alpha_k, \alpha_k]$.

Proof. Let us first define auxiliary intervals $A = A_I$ and $B = B_I$. The interval A is defined as follows:

- If $i \in R_m$ for some $m \geq k$ then A is the intersection of the component of R_m containing i and I;
- otherwise we let $A = [i, jT_k - 1]$, where j is the infimum of the numbers r with $i \leq r\,T_k$ (note that A is empty if $i = jT_k$).

The interval B is symmetrically defined as follows:

- if $i + t \in R_m$ for some $m \geq k$ then B is the intersection of the component of R_m containing $i + t$ and I;
- otherwise we let $B = [\ell T_k, i + t]$, where ℓ is the maximum of the numbers r $rT_k \leq i + t + 1$ (note that B is empty if $i + t = jT_k$).

Fact 2.20. $J = [i, i + t] \setminus (A \cup B)$.

Proof. Just note that by construction every component of R_m intersecting J is contained in J. A similar inclusion holds for every m-good interval intersecting J. These two inclusions imply the fact. $\square$

It remains to see that J is the union of pairwise disjoint α_k-controlled intervals. By Fact 2.20 and construction, the components of $\bigcup_{m \geq k} R_m$ intersecting J are contained in J. These components are pairwise disjoint and their complement is a union of T_k-intervals which are good. This implies that J is a disjoint union of intervals H where the average satisfies $\varphi_H(x_0) \in [-\alpha_k, \alpha_k]$. This implies that the average $\varphi_J(x_0)$ of φ in J belongs to $[-\alpha_k, \alpha_k]$, ending the proof of the claim. $\square$

Claim 2.21. Fix k. Given any $m \in \mathbb{N}$ there is t_m such that for every $t \geq t_m$ and for every $i \in \mathbb{N}$ such that

$$\frac{1}{t} \sum_{j=0}^{t-1} \varphi(f^{i+j}(x_0)) \notin [-2\,\alpha_k, 2\,\alpha_k],$$

then either i or $i + t$ belongs to $\bigcup_{\ell \geq m} R_\ell$.

Proof. Pick and interval $I = [i, i + t]_{\mathbb{N}}$ and associate to it the interval $J = J_I$ in Claim 2.19 and the intervals $A = A_I$ and $B = B_I$ in its proof. As $\varphi_J(x_0) \in [-\alpha_k, \alpha_k]$, in order to have $|\varphi_I(x_0)| > 2\,\alpha_k$ the set $I \setminus J = A \cup B$ must fill a relatively large proportion (depending on α_k and $\sup|\varphi|$ but independent of t) of the interval I. In other words, there is a constant $C>0$ such that

$$\frac{\#(A\cup B)}{t} > C. \tag{2.3}$$

Fixed m, let

$$t_m \stackrel{\text{def}}{=} \frac{2\,T_m}{C}+1.$$

Take any $t\geq t_m$. The proof is by contradiction, if $i, i+t \notin \bigcup_{\ell\geq m} R_\ell$ then $|A|,|B|\leq T_m$ and therefore

$$\frac{\#(A\cup B)}{t} \leq \frac{2\,T_m}{t} < C,$$

a contradicting (2.3). The proof of the claim is complete. □

We are now ready to conclude the proof of the lemma. Claim 2.21 implies that for $t>t_m$ the number $(1-q_t)$ is less than twice the density of the set $\bigcup_{\ell\geq m} R_\ell$ in $[0,t]$. Fact 2.16 implies that this density goes to 0 as $t\to\infty$, proving the lemma. □

The proof of Proposition 2.17 is now complete. □

3. Patterns, concatenations, flip-flop families, and control of averages

In this section we introduce the notion of a pattern (Section 3.1) and explain its relations with the scales and tails in the previous section. In Section 3.2 we see that a $\mathcal{T}$-long tail of a scale $\mathcal{T}$ induces patterns in its good intervals. Patterns will be used to codify certain orbits (the orbit follows some distribution pattern). This naive idea is formalised in the notion of a concatenation of sets following a pattern, see Section 3.3. We are interested in concatenations of plaques of a flip-flop family (associated to a map φ) and in the control of the averages of φ corresponding to these concatenations, see Section 3.4. The main result in this section is Theorem 3.9 that gives the control of averages for concatenations, see Section 3.5. In the sequel we will make more precise these vague notions.

3.1. Patterns.

A scale $\mathcal{T}=(T_n)_{n\geq 0}$ induces, for each n, a partition of $\mathbb{N}$ consisting of intervals of the form $[\ell\,T_n,(\ell+1)\,T_n-1]_{\mathbb{N}}$. A pattern is a partition of these intervals respecting some compatibility rules given by the scale.

Definition 3.1 (Pattern)**.** *Let $\mathcal{T}=(T_n)_{n\geq 0}$ be a scale and $I\subset\mathbb{N}$ an interval of the form $I=[\ell\,T_n,(\ell+1)\,T_n-1]_{\mathbb{N}}$ for some $\ell\in\mathbb{N}$.*

A T_n-pattern $\mathfrak{P}=\mathfrak{P}(I)$ of the interval I consists of a partition $\mathcal{P}=\{I_i\}_{i=1}^r$ of I into intervals $I_i=[k\,T_{\ell(i)},(k+1)\,T_{\ell(i)-1}]_{\mathbb{N}}$, where $\ell(i)\in\{0,\dots,n\}$, and a map $\iota\colon\mathcal{P}\to\{r,w\}$ such that

- *either $\ell(i)\neq 0$ and then $\iota(I_i)=w$,*
- *or $\ell(i)=0$ and then $\iota(I_i)\in\{r,w\}$.*

We write $\mathfrak{P}=(\mathcal{P},\iota)$.

A subinterval of $I=[kT_i,(k+1)T_i-1]$ that is not strictly contained in an interval of the partition $\mathcal{P}$ is called $\mathfrak{P}$-admissible (of length T_i)*.*

In this definition, the script w refers to "walk" and r to "rest".

Remark 3.2.

a) If $[kT_i, (k+1)T_i - 1]$ is a $\mathfrak{P}$-admissible subinterval of I, then the restriction of the partition $\mathcal{P}$ and the restriction of the map ι induces a T_i-pattern in $[kT_i, (k+1)T_i - 1]$, which is called a *subpattern* of $\mathfrak{P}$.
b) A T_n-pattern consists either of a unique interval of w-type or is a "concatenation" of T_{n-1}-patterns.

Consider an interval I and a pattern $\mathfrak{P}$ of it as in Definition 3.1. A point $j = k\,T_i \in \mathbb{N}$ is *i-initial for the pattern* $\mathfrak{P}$ if the interval $[j, j+T_i-1]_{\mathbb{N}}$ is admissible. A point $j \in \mathbb{N}$ is *$\mathfrak{P}$-initial* if it is i-initial for some i. We denote the set of initial points of $\mathfrak{P}$ by $I(\mathfrak{P})$. The set of *$\mathfrak{P}$-marked points* of the T_n-pattern $\mathfrak{P}$, denoted by $M(\mathfrak{P})$, is the union of the point $\{(\ell+1)T_n\}$ and the set of all initial points of $\mathfrak{P}$.

3.2. **Tails and patterns.** We now see that given a scale $\mathcal{T}$ and a $\mathcal{T}$-long sparse tail R_∞, the tail induces patterns in its good intervals I (i.e., $I \cap R_{n,\infty} = \varnothing$, recall Definition 2.6). In this subsection the sparseness of the tail is not relevant.

Lemma 3.3 (Pattern induced by a tail)**.** *Let $\mathcal{T} = (T_n)_{n\in\mathbb{N}}$ be a scale and R_∞ a $\mathcal{T}$-long sparse tail. Let*

$$I = [\ell\, T_n, (\ell+1)\, T_n - 1]_{\mathbb{N}}$$

be an n-good interval of R_∞ and consider the partition $\mathcal{P}$ of I and the map $\iota\colon \mathcal{P} \to \{r, w\}$ defined as follows:

- *the intervals J of $\mathcal{P}$ with $\iota(J) = w$ are the components of R_∞ contained in $[\ell\, T_n, (\ell+1)\, T_n - 1]$;*
- *the complement of R_∞ in $[\ell\, T_n, (\ell+1)\, T_n - 1]_{\mathbb{N}}$ can be written as the union of intervals J of the type $[k\, T_0, (k+1)\, T_0 - 1]$, these intervals are the elements of the partition $\mathcal{P}$ with $\iota(J) = r$.*

Then $\mathfrak{P} = (\mathcal{P}, \iota)$ defines a T_n-pattern in I.

Proof. To prove the lemma it is enough to recall that, by definition of a n-good interval, there is no component of R_∞ containing the interval $[\ell\, T_n, (\ell+1)\, T_n - 1]_{\mathbb{N}}$ and that every T_k is a multiple of T_0. $\square$

The pattern $\mathfrak{P}$ in Lemma 3.3 is called *the pattern induced by the tail R_∞* in the good interval $[\ell\, T_n, (\ell+1)\, T_n - 1]$ and is denoted by $\mathfrak{P}_{n,\ell}$, or by $\mathfrak{P}_{n,\ell}(R_\infty)$ (for emphasising the role of the tail).

The next remark associates a sequence of patterns to the tail R_∞.

Remark 3.4. [Initial patterns for a long tail] With the notation of Lemma 3.3, by definition of the tail R_∞, the initial interval of length T_n, $[0, T_n - 1]_{\mathbb{N}}$, is a good interval. We let $\mathfrak{P}_n = \mathfrak{P}_{n,0}$ and call it *the initial T_n-pattern of R_∞*.

The set of initial points of $\mathfrak{P}_n$ consists of the following points:

$$\Big(\{\text{origins of the components of } R_\infty\} \cup \{kT_0 \notin R_\infty\}\Big) \cap [0, T_n - 1].$$

Remark 3.5. [Compatibility of induced patterns] For every $i < n$ the restriction of the initial pattern $\mathfrak{P}_n$ to the interval $[0, T_i - 1]_{\mathbb{N}}$ is the initial pattern $\mathfrak{P}_i$. In other words, $\mathfrak{P}_i$ is the *initial T_i-subpattern* of $\mathfrak{P}_n$.

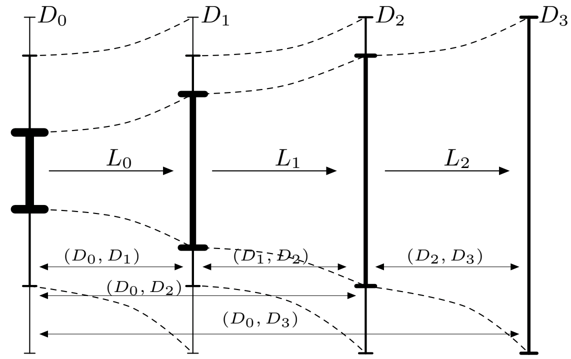

FIGURE 3. Concatenations

3.3. **Concatenations and controlled plaque-segments.** Consider a compact metric space (X, d), a homeomorphism $f\colon X \to X$, and an open set U of X. Consider a family $\mathcal{D}$ of compact sets contained in U. We call the elements in $\mathcal{D}$ plaques[8].

Given a pair of plaques $D_0, D_1 \in \mathcal{D}$ we say (D_0, D_1) is a *plaque-segment of size T relative to U and $\mathcal{D}$* if:

- $f^{-i}(D_1) \subset U$ for every $i \in \{0, \dots, T\}$ and
- $f^{-T}(D_1) \subset D_0$.

We say that D_0 is the *origin of the segment* and D_1 is the *end of the segment.*

Let (D_0, D_1) and (D_1, D_2) be two plaque-segments of lengths L_0 and L_1, respectively, relative to U and $\mathcal{D}$. Then (D_0, D_2) is a plaque-segment of length $L_0 + L_1$, called *the concatenation of* (D_0, D_1) *and* (D_1, D_2). We use the notation $(D_0, D_2) \stackrel{\text{def}}{=} (D_0, D_1) * (D_1, D_2)$. See Figure 3.

Definition 3.6. *Let (X, d) be compact metric space, $f\colon X \to X$ a homeomorphism, U an open set of X, $\varphi\colon U \mapsto \mathbb{R}$ a continuous map, and $\mathcal{D}$ a family of plaques contained in U.*

*Consider $T \in \mathbb{N}$ and a subset $J \subset \mathbb{R}$. A plaque-segment (D_0, D_1) of length $T > 0$ relative to U and $\mathcal{D}$ is called (J, T)-*controlled *if*

$$\varphi_T(x) = \frac{1}{T} \sum_{i=0}^{T-1} \varphi(f^i(x)) \in J, \quad \textit{for every } x \in f^{-T}(D_1) \subset D_0.$$

When there is no ambiguity on the pair U and $\mathcal{D}$ the dependence on these sets will be omitted.

Definition 3.7. *Let (X, d) be compact metric space, $f\colon X \to X$ a homeomorphism, U an open set of X, $\varphi\colon U \mapsto \mathbb{R}$ a continuous map, and $\mathcal{D}$ a family of plaques contained in U.*

Consider a a scale $\mathcal{T} = (T_n)_{n \in \mathbb{N}}$, a T_n-pattern $\mathfrak{P}$ of $[\ell\, T_n, (\ell+1)\, T_n - 1]_{\mathbb{N}}$, $n > 0$, the set $M(\mathfrak{P})$ of its marked points, and a family of subsets $\mathcal{J} = (J_i)_{i \in \mathbb{N}}$ of $\mathbb{R}$.

*A family $\{D_i\}_{i \in M(\mathfrak{P})}$ of plaques of $\mathcal{D}$ is called $(\mathcal{J}, \mathfrak{P})$-*controlled *(relatively to U and $\mathcal{D}$) if:*

[8] In our applications, the elements of $\mathcal{D}$ are sets in a flip-flop family, recall Definition 1.1.

- *For every $\mathfrak{P}$-admissible interval $[k\,T_i,(k+1)\,T_i-1]$ the pair $(D_{kT_i},D_{(k+1)T_i})$ is a plaque-segment of length T_i that is (J_i,T_i)-controlled (relative to U and $\mathcal{D}$).*
- *For any $i,j\in M(\mathfrak{P})$, $i<j$, the pair (D_i,D_j) is a plaque-segment of length $j-i$ (relative to U and $\mathcal{D}$).*

3.4. **Distortion of Birkhoff averages and concatenations in flip-flop families.** The following result is a translation of [BBD2, Lemma 2.4] to the context of flip-flop families with sojourns. Recall the notation for Birkhoff averages in (1.2).

Lemma 3.8 (Small distortion of Birkhoff averages over long concatenations)**.** *Let $f\colon X\to X$ be a homeomorphism, $\varphi\colon X\to\mathbb{R}$ a continuous function, and $\mathfrak{F}$ a flip-flop family associated to φ and f with sojourns in a compact set Y.*

Then for every $\alpha>0$ there exists $t=t(\alpha)\in\mathbb{N}$ with the following property: Consider any $T\geq t$ and any family of plaques $\{D_i\}_{0\leq i\leq T}$ of $\mathfrak{F}$ such that for every $i=0,\dots,T-1$. (D_i,D_{i+1}) is a plaque-segment of length L_i. Then the plaque-segment

$$(D_0,D_T)=(D_0,D_1)*(D_1,D_2)*\cdots*(D_{T-1},D_T)$$

satisfies

$$|\varphi_L(x)-\varphi_L(y)|<\alpha,\quad\textit{where}\quad L\stackrel{\text{def}}{=}\sum_{i=0}^{T-1}L_i,$$

for every pair of points $x,y\in D_0$ such that

$$f^{L'_{i-1}}(x),f^{L'_{i-1}}(y)\in D_i,\quad\textit{where}\quad L'_{i-1}\stackrel{\text{def}}{=}\sum_{j=0}^{i-1}L_j$$

for every $i=0,\dots,T-1$.

The proof is the same as the one of [BBD2, Lemma 2.4] and the key ingredient is the expansion properties in item (c) in Definitions 1.1 and 1.2. We omit this proof and refer to [BBD2].

3.5. **Flip-flop families and concatenations.** The aim of this section is to prove the following theorem.

Theorem 3.9. *Let (X,d) be a compact metric space, Y a compact subset of X, U an open subset of X, $f\colon X\to X$ a homeomorphism, $\varphi\colon U\to\mathbb{R}$ a continuous function, and $\mathfrak{F}$ a flip-flop family with sojourns along Y associated to φ and f whose plaques are contained in U.*

Consider sequences $(\delta_n)_{n\in\mathbb{N}}$, $(\alpha_n)_{n\in\mathbb{N}}$, and $(\beta_n)_{n\in\mathbb{N}}$ of positive numbers such that:

$$(\delta_n)_n\to 0\quad\textit{and}\quad \alpha_{n+1}<\frac{\alpha_n}{4}<\beta_n<\frac{\alpha_n}{2}.$$

Then there is a scale $\mathcal{T}=(T_n)_{n\in\mathbb{N}}$ satisfying the following properties: For every plaque $D\in\mathfrak{F}$, every T_n-pattern $\mathfrak{P}=(\mathcal{P},\iota)$, and every $\omega\in\{+,-\}$ there is a family of plaques $\mathcal{D}_{\mathfrak{P}}=\{D_a\}_{a\in M(\mathfrak{P})}$ of $\mathfrak{F}$ such that :

(I1) $D_0=D$;

(I2) the family $\{D_a\}_{a\in M(\mathfrak{P})}$ is $(\mathcal{J}_n,\mathfrak{P})$-controlled (relatively to U and $\mathfrak{P}$) where $\mathcal{J}_n=\{J_i\}_{i\in\{0,\dots,n\}}$ and

$$J_i\stackrel{\text{def}}{=}\left[-\alpha_i,-\frac{\alpha_i}{2}\right]\cup\left[\frac{\alpha_i}{2},\alpha_i\right]\ \textit{for } i<n \textit{ and } J_n\stackrel{\text{def}}{=}\omega\left[\frac{\alpha_n}{2},\alpha_n\right];$$

(I3) if $[a, a+T_i-1]$ is an interval of the partition $\mathcal{P}$ of w-type then for every point $x \in f^{-T_i}(D_{a+T_i})$ the segment of orbit $\{x, \dots, f^{T_i}(x)\}$ is δ_i-dense in Y.

We say that the family of plaques $\mathcal{D}_{\mathfrak{P}} = \{D_a\}_{a\in M(\mathfrak{P})}$ in Theorem 3.9 is $((\mathcal{J})_n, \mathfrak{P})$-*controlled and starts at* D_0.

Proof. The construction of the scales $\mathcal{T}$ in the theorem is done by induction on n (assuming that T_i is defined for $i \le n$ we will define T_{n+1}). The proof considers two cases: either the pattern is trivial or it is a concatenations of T_n-patterns. Proposition 3.11 deals with trivial patterns while Proposition 3.14 deals with concatenations.

Note that the Theorem 3.9 claims the existence of a scale $\mathcal{T} = (T_n)_{n\in\mathbb{N}}$ that holds for every disk $D \in \mathfrak{F}$. In the proofs of Propositions 3.11 and 3.14 we get such a number depending on the plaque $D \in \mathfrak{F}$ and uniformly bounded. To uniformize this number for every plaque in $\mathfrak{F}$ we will use Lemma 3.10 below. Recall the definition of the distortion time number $t(\alpha)$ in Lemma 3.8 associated to $\alpha > 0$.

Lemma 3.10 (Uniformization)**.** *Take a scale $(T_n)_{n\in\mathbb{N}}$ and a sequence $(\alpha_n)_{n\in\mathbb{N}}$ as in Theorem 3.9. Then for every plaque $D_0 \in \mathfrak{F}$, every $\omega_0 \in \{-,+\}$, and every*

$$t \ge \tau_n \stackrel{\text{def}}{=} \max\left\{ t\left(\frac{\alpha_{n+1}}{6}\right), 6\,\frac{\alpha_n\, T_n}{\alpha_{n+1}} \right\}$$

the following property holds:

Let $D_1 \in \mathfrak{F}$ be a plaque such that (D_0, D_1) is a $(\omega_0\,[\frac{\alpha_{n+1}}{2}, \alpha_{n+1}], t)$-controlled. plaque-segment.

Then there is $\omega \in \{-,+\}$ such that if (D_1, D_2) is a plaque-segment that is $(\omega\,[\frac{\alpha_n}{2}, \alpha_n], T_n)$-controlled then the concatenation $(D_0, D_2) = (D_0, D_1) * (D_1, D_2)$ is $(\omega_0\,[\frac{\alpha_{n+1}}{2}, \alpha_{n+1}], t + T_n)$-controlled.

Proof. We prove the lemma for $\omega_0 = +$, the case $\omega_0 = -$ is analogous. Note first that the choice of t implies that the distortion of φ_t in $f^{-t}(D_1)$ is bounded by $\frac{\alpha_{n+1}}{6}$. Moreover, as the orbit segment (D_0, D_1) is $([\frac{\alpha_{n+1}}{2}, \alpha_{n+1}], t)$-controlled, we have

$$\varphi_t(f^{-t}(D_1)) \subset \left[\frac{\alpha_{n+1}}{2}, \alpha_{n+1}\right].$$

Now there are two cases:

a) if $\displaystyle\max_{x\in f^{-t}(D_1)} \varphi_t(x) \le \frac{5\,\alpha_{n+1}}{6}$, we choose $\omega = +$,

b) otherwise $\displaystyle\min_{x\in f^{-t}(D_1)} \varphi_t(x) \ge \frac{4\,\alpha_{n+1}}{6}$ and we choose $\omega = -$.

In the first case, as $\omega = +$ one can easily check that

$$\varphi_{t+T_n}(x) > \varphi_t(x) \quad \text{for every } x \in f^{-(t+T_n)}(D_2). \tag{3.1}$$

Moreover, as $t > 6\,\frac{\alpha_n\, T_n}{\alpha_{n+1}}$ and (D_1, D_2) is $([\frac{\alpha_n}{2}, \alpha_n], T_n)$-controlled we have

$$\varphi_{t+T_n}(x) < \frac{\frac{5}{6}\,\alpha_{n+1} t + T_n \alpha_n}{t + T_n} < \frac{t\,\alpha_{n+1}}{t + T_n} < \alpha_{n+1} \quad \text{for every } x \in f^{-(t+T_n)}(D_2).$$

Finally, recalling that (D_0, D_1) is $([\frac{\alpha_{n+1}}{2}, \alpha_{n+1}], t)$-controlled and (3.1)

$$\frac{\alpha_{n+1}}{2} < \varphi_t(x) < \varphi_{t+T_n}(x) < \alpha_{n+1}$$

ending the proof in the first case.

Case (b) is analogous and hence omitted. The proof of the lemma is now complete. □

Proposition 3.11 (Trivial patterns)**.** *Under the assumptions of Theorem 3.9, assume that for every $i = 0, \dots, n$ there are defined natural numbers T_i such that the conclusions in the theorem hold for T_i-patterns.*

Then there is $\widetilde{T}_{n+1}$ such that for every $T > \widetilde{T}_{n+1}$, every plaque $D \in \mathfrak{F}$, and every $\omega \in \{+,-\}$ there is a plaque $D_0 \in \mathfrak{F}$ such that

- *$f^{-T}(D_0) \subset D$,*
- *for every $x \in f^{-T}(D_0)$ the set $\{x, f(x), \dots, f^T(x)\}$ is δ_{n+1}-dense in Y, and*
- *(D_0, D) is $(\omega\,[\frac{\alpha_{n+1}}{2}, \alpha_{n+1}], T)$-controlled.*

Proof. We only present the proof for the case $\omega = +$, the case $\omega = -$ is analogous and thus omitted. Let $N = N_{\delta_{n+1}}$ as in Definition 1.2. Then for every plaque $D \in \mathfrak{F}$ there is a $\widetilde{D}_0 \in \mathfrak{F}$ such that

- $f^{-i}(\widetilde{D}_0) \subset U$ for all $i = 0, \dots, N$,
- $f^{-N}(\widetilde{D}_0) \subset D$, and
- for every $x \in f^{-N}(\widetilde{D}_0)$ the set $\{x, f(x), \dots, f^N(x)\}$ is δ_{n+1}-dense in Y.

In what follows we will focus on the control of Birkhoff averages. Note that the average of φ_N in $f^{-N}(\widetilde{D}_0)$ is uniformly upper bounded by the maximum of $|\varphi|$ in X denoted by $\max|\varphi|$.

Recall the definition of $t(\alpha)$ in Lemma 3.8 and fix k_0 large enough such that

$$k_0 > t\left(\frac{\alpha_{n+1}}{6}\right)\frac{1}{T_n}. \tag{3.2}$$

Since $\beta_n < \frac{\alpha_n}{2}$ we have that if k_0 is large enough then for every $k \geq k_0$ it holds

$$\beta_n < \frac{-N\max|\varphi| + k\,T_n\,\frac{\alpha_n}{2}}{N + k\,T_n} < \frac{N\max|\varphi| + k\,T_n\,\alpha_n}{N + k\,T_n} < 3\,\frac{\alpha_n}{2}. \tag{3.3}$$

Claim 3.12. Consider k_0 satisfying equations (3.2) and (3.3). Then for every $k \geq k_0$ there is plaque $\widetilde{D}_1 \in \mathfrak{F}$ such that

a) $f^{-kT_n}(\widetilde{D}_1) \subset \widetilde{D}_0$;

b) for every $x \in f^{-k\,T_n - N}(\widetilde{D}_1)$ it holds
$$\frac{1}{N + k\,T_n}\sum_{j=0}^{N+k\,T_n-1}\varphi(f^j(x)) \in \left[\beta_n, \frac{3\,\alpha_n}{2}\right];$$

c) for every $x, y \in f^{-kT_n - N}(\widetilde{D}_1)$.
$$\frac{1}{N + k\,T_n}\sum_{j=0}^{N+k\,T_n-1}|\varphi(f^j(x)) - \varphi(f^j(y))| < \frac{\alpha_{n+1}}{6}.$$

Proof. To prove the first item we consider the concatenation of k orbit-segments $(\widehat{D}_i, \widehat{D}_{i+1})$, $i = 0, \dots, k-1$, of size T_n associated to $\omega_i = +$ given by the induction hypothesis and such that $\widehat{D}_0 = \widetilde{D}_0$. These pairs are obtained inductively: assumed defined the pair $(\widehat{D}_i, \widehat{D}_{i+1})$ we apply induction hypothesis to the final plaque $\widehat{D}_{i+1}$. To conclude it is enough to take $\widetilde{D}_1 = \widehat{D}_k$. See Figure 4.

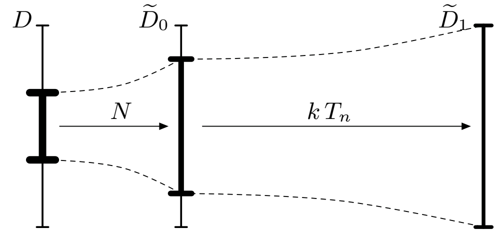

FIGURE 4. Proof of Claim 3.12

To prove item (b) note that each plaque-segment $(\widehat{D}_i, \widehat{D}_{i+1})$ is $([\frac{\alpha_n}{2}, \alpha_n], T_n)$-controlled. Hence the averages in these segments belong to $[\frac{\alpha_n}{2}, \alpha_n]$. Now the control of averages follows from the choice of k_0 in equation (3.3).

Item (c) follows from the distortion Lemma 3.8 and the choice of k_0 in equation (3.2). This ends the proof of the claim. $\square$

The proof of the proposition now follows arguing exactly as in the concatenation result [BBD2, Lemma 2.12]. For completeness, we recall the main arguments involved in this proof.

Note that in Claim 3.12 we can assume that the constant k_0 is such that

$$k_0 > \max\left\{\frac{6\,\alpha_n}{\alpha_{n+1}}, t\left(\frac{\alpha_{n+1}}{6}\right)\frac{1}{T_n}\right\}. \tag{3.4}$$

Lemma 3.13. *For every $k \geq k_0$ there are i_0 and a plaque $D_1 \in \mathfrak{F}$ such that (D, D_1) is $\left(\left[\frac{1}{2}\alpha_{n+1}, \alpha_{n+1}\right], \widetilde{T}_{n+1}\right)$-controlled, where*

$$\widetilde{T}_{n+1} = N + (k + i_0) T_n.$$

Proof. We just describe the main steps of the proof. Take $\widehat{D}_0 \overset{\text{def}}{=} \widetilde{D}_1$, where $\widetilde{D}_1$ is the plaque given by Claim 3.12. By the induction hypothesis, given any $i \in \{0, \dots, n\}$ there is a family of $([-\alpha_n, -\frac{\alpha_n}{2}], T_n)$-controlled orbit-segments $(\widehat{D}_j, \widehat{D}_{j+1})$ for $j = 0, \dots, k + i - 1$. This implies that

$$\varphi_{T_n}\big(f^{-T_n}(\widehat{D}_{j+1})\big) \subset \left[-\alpha_n, -\frac{\alpha_n}{2}\right].$$

Consider the orbit-segment $(D, \widehat{D}_{k+i})$ of length $N + (k+i)\,T_n$ obtained concatenating $(D, \widehat{D}_0 = \widetilde{D}_1)$ and $(\widehat{D}_j, \widehat{D}_{j+1})$, $j = 0, \dots, k + i - 1$, that is,

$$(D, \widehat{D}_{k+i}) = (D, \widehat{D}_0) * (\widehat{D}_0, \widehat{D}_1) * \dots * (\widehat{D}_{k+i-1}, \widehat{D}_{k+i}).$$

The choice of k_0 in (3.4) ($k_0\,\alpha_{n+1} > 6\,\alpha_n$) immediately implies that every $x \in f^{-N-(k+i)T_n}(\widehat{D}_{k+i})$ satisfies the following implication,

$$\varphi_{N+(k+i)T_n}(x) \geq \frac{5}{6}\alpha_{n+1} \quad \Longrightarrow \quad \varphi_{N+(k+i+1)T_n}(x) \geq \frac{4}{6}\alpha_{n+1}. \tag{3.5}$$

Let

$$\ell_0 \overset{\text{def}}{=} \frac{3\,(N + k\,T_n)\,\alpha_n}{T_n\,(\alpha_n + \alpha_{n+1})}.$$

By item (b) in Claim 3.12 we have that

$$\beta_n \leq \varphi_{N+k\,T_n}(x) \leq \frac{3\,\alpha_n}{2} \quad \text{for every } x \in f^{-N-k\,T_n}(\widehat{D}_0).$$

A simple calculation gives that for every $\ell > \ell_0$ it holds

$$\varphi_{N+(k+\ell)\,T_n}(x) < \frac{1}{2}\,\alpha_{n+1} \quad \text{for every} \quad x \in f^{-N-(k+\ell)\,T_n}(\widehat{D}_{k+\ell}). \tag{3.6}$$

Equation (3.6) implies that there is a first i_0 with $\varphi_{N+(k+i_0)\,T_n}(\bar{x}) \le \frac{5}{6}\,\alpha_{n+1}$ for some $\bar{x} \in f^{-N-(k+i_0)\,T_n}(\widehat{D}_{k+i_0})$. From Equation (3.5) we get

$$\varphi_{N+(k+i_0)\,T_n}(\bar{x}) \in \left[\frac{4}{6}\alpha_{n+1}, \frac{5}{6}\alpha_{n+1}\right]. \tag{3.7}$$

By the choice of k_0 in (3.2), the distortion of $\varphi_{N+(k+i_0)T_n}$ in $f^{-N-(k+i_0)\,T_n}(\widehat{D}_{k+i_0})$ is strictly less than $\frac{1}{6}\alpha_{n+1}$. Equation (3.7) now implies that

$$\varphi_{N+(k+i_0)\,T_n}(x) \in \left[\frac{1}{2}\alpha_{n+1}, \alpha_{n+1}\right] \quad \text{for every } x \in f^{-N-(k+i_0)\,T_n}(\widehat{D}_{k+i_0}).$$

The lemma follows taking $D_1 = \widehat{D}_{k+i_0}$. □

Take i_0 as in Lemma 3.13, k_0 as in (3.4), $k \ge k_0$ sufficiently large, and define

$$\widetilde{T}_{n+1} \stackrel{\text{def}}{=} N + (k+i_0)\,T_n > \tau_n,$$

where τ_n is as in Lemma 3.10. Consider the plaque D_1 given by Lemma 3.13 such that (D, D_1) is $(\left[\frac{1}{2}\alpha_{n+1}, \alpha_{n+1}\right], \widetilde{T}_n)$-controlled. Using the induction hypothesis, consider a plaque D_2 such that (D_1, D_2) is $(\left[\frac{1}{2}\alpha_n, \alpha_n\right], T_n)$-controlled. As $\widetilde{T}_{n+1} > \tau_n$, Lemma 3.10 implies that (D_0, D_2) is $(\left[\frac{1}{2}\alpha_{n+1}, \alpha_{n+1}\right], \widetilde{T}_{n+1} + T_n)$-controlled. Repeating this last argument j times (any j) we get a plaque $D_2(j)$ such that $(D, D_2(j))$ is $(\left[\frac{1}{2}\alpha_{n+1}, \alpha_{n+1}\right], \widetilde{T}_{n+1} + j\,T_n)$-controlled. This completes the average control in the proposition and ends its proof. □

Given a pattern $\mathfrak{P}$ and its set of marked points $M(\mathfrak{P})$ we let $e_{M(\mathfrak{P})}$ the right extreme of $M(\mathfrak{P})$.

Proposition 3.14 (Concatenation of patterns)**.** *Under the assumptions of Theorem 3.9, assume that for every $i = 0, \dots, n$ there is defined $T_i \in \mathbb{N}$ satisfying the conclusions in the theorem.*

Then there is a constant k_0 such that for every $k \ge k_0$, every family $\{\mathfrak{P}_i\}_{i=1}^k$ of T_n-patterns, every $\omega \in \{-,+\}$, and every plaque $D \in \mathfrak{F}$ there is a sequence of symbols $(\omega_i)_{i=1,\dots,k}$, $\omega_i \in \{-,+\}$, satisfying the following property:

Consider the family of sets $\mathcal{J}_i = \{J_{i,j}\}$, $i = 1, \dots, k$ and $j = 1, \dots, n$, defined by

- $J_{i,j} \stackrel{\text{def}}{=} \left[-\alpha_j, \frac{-1}{2}\,\alpha_j\right] \cup \left[\frac{1}{2}\,\alpha_j, \alpha_j\right]$ *if $j < n$ and*
- $J_{i,n} \stackrel{\text{def}}{=} \omega_i \left[\frac{1}{2}\,\alpha_n, \alpha_n\right]$.

Let $\mathcal{D}_{\mathfrak{P}_1} = \{D_{1,j}\}_{j \in M(\mathfrak{P}_1)}$ be a family of $(\mathcal{J}_1, \mathfrak{P}_1)$-controlled plaques given by the induction hypothesis associated to the plaque $D_{1,0} = D$.

Let $\mathcal{D}_{\mathfrak{P}_i} = \{D_{i,j}\}_{j \in M(\mathfrak{P}_i)}$ be the family of $(\mathcal{J}_i, \mathfrak{P}_i)$-controlled plaques associated to the final plaque $D_{i-1, e_{M(\mathfrak{P}_{i-1})}} = D_{i,0}$ of the family $\mathcal{D}_{\mathfrak{P}_{i-1}} = \{D_{i-1,j}\}_{j \in M(\mathfrak{P}_{i-1})}$ given by the induction hypothesis.

Then the plaque-segment $(D, D_{k, e_{M(\mathfrak{P}_k)}})$ is $(\omega\,[\frac{1}{2}\,\alpha_{n+1}, \alpha_{n+1}], k\,T_n)$-controlled.

Proof. The proof of follows arguing as in [BBD2, Lemma 2.12]. We now recall the strategy for $\omega = +$ (the case $\omega = -$ is analogous and thus omitted). We start by taking a sequence of signals $\omega_i = +$ for every i. This sequence is sufficiently large to guarantee distortion smaller than $\frac{1}{6}\alpha_{n+1}$ in the pre-images of the plaques. We stop for some large i_0. Note that the averages are in $\left[\frac{1}{2}\alpha_n, \alpha_n\right]$. Thereafter for $i > i_0$ we consider a sequence of $\omega_i = -$ as in the proof of the previous lemma. In this way the averages of φ start to decrease. We stop when this average at some point of the plaque belongs to $\left[\frac{4}{6}\alpha_{n+1}, \frac{5}{6}\alpha_{n+1}\right]$ (the key point is that the averages can not jump from above $\frac{5}{6}\alpha_{n+1}$ to below $\frac{4}{6}\alpha_{n+1}$, this is because for large i the cone of size $\left[\frac{4}{6}\alpha_{n+1}, \frac{5}{6}\alpha_{n+1}\right]$ is very width, this is exactly the argument in [BBD2]). The distortion control implies that the average of φ in the whole pre-image of the plaque is contained in $\left[\frac{1}{2}\alpha_{n+1}, \alpha_{n+1}\right]$.

We conclude the proof using Lemma 3.10: we can continue concatenating plaque-segments keeping the averages in $\left[\frac{1}{2}\alpha_{n+1}, \alpha_{n+1}\right]$. This completes the sketch of the proof of the proposition. $\square$

End of the proof of Theorem 3.9. The definition of the scale $(T_n)_n$ is done inductively on n. Assuming that T_i is defined for $i \leq n$, we define T_{n+1} as follows. Take $\widetilde{T}_{n+1}$ as in Lemma 3.13 and k_0 as in Claim 3.12. Then define T_{n+1} as a multiple of T_n such that

$$T_{n+1} \geq \max\{\widetilde{T}_{n+1}, k_0\, T_n\} \quad \text{and} \quad \frac{T_{n+1}}{T_n} \geq (n+1)\,\frac{T_n}{T_{n-1}}.$$

Take now a T_{n+1}-pattern $\mathfrak{P} = (\mathcal{P}, \iota)$, $\omega \in \{+, -\}$, and a plaque $D \in \mathfrak{F}$. There are two cases to consider. If $\mathcal{P}$ is the trivial T_{n+1}-pattern we just need to apply Proposition 3.11. Otherwise, the T_{n+1}-pattern $\mathfrak{P}$ is a concatenation of a sequence of at least k_0 T_n-patterns. The theorem follows by applying Proposition 3.14 to this family of T_n-patterns. $\square$

Bearing in mind Remark 3.5, we are interested to get plaque-segments associated to T_n-patterns respecting the plaque-segments associated to its initial subpatterns. A slight variation of the proof of Proposition 3.14 implies the following addendum:

Addendum 3.15 (Extension of initial subpatterns)**.** *With the hypotheses of Theorem 3.9, the scale $\mathcal{T} = (T_n)_{n\in\mathbb{N}}$ can be chosen satisfying the following additional property:*

Let $\mathfrak{P}_1$ be a T_n-pattern and $\mathfrak{P}_2$ be a T_{n+1}-pattern such that $\mathfrak{P}_1$ is the initial T_n-subpattern of $\mathfrak{P}_2$.

Consider a flip-flop family $\mathfrak{F}$ and D_0 a plaque $\mathfrak{F}$. Let $\{D_a\}_{a\in M(\mathfrak{P}_1)}$ be a family of plaques associated to the pattern $\mathfrak{P}_1$ starting at D_0 given by Theorem 3.9.

Then for every $\omega \in \{+, -\}$, there is a family $\{\widetilde{D}_a\}_{a\in M(\mathfrak{P}_2)}$ of plaques associated to the pattern $\mathfrak{P}_2$ satisfying the conclusion of Theorem 3.9, starting at D_0 and such that

$$\widetilde{D}_a = D_a, \quad \textit{for every } a \in M(\mathfrak{P}_1).$$

This result allows us to choose the family of plaques associated to a pattern extending the ones associated to its initial subpatterns.

4. Flip-flop families with sojourns: proof of Theorem 2

In this section we prove Theorem 2, Corollary 3, and Proposition 4.

4.1. **Proof of Theorem 2.** Consider a homeomorphism $f\colon X \to X$ defined on a compact metric space (X, d), a continuous function $\varphi\colon X \to \mathbb{R}$, and a flip-flop family $\mathfrak{F} = \mathfrak{F}^+ \bigsqcup \mathfrak{F}^-$ with sojourns along a compact subset Y of X associated to φ and f. We need to see that every plaque $D \in \mathfrak{F}$ contains a point x_D that is controlled at any scale with a long sparse tail with respect to φ and Y.

Fix a sequences of strictly positive numbers $(\alpha_n)_n$ and $(\beta_n)_n$ such that

$$0 < \alpha_{n+1} < \frac{\alpha_n}{4} < \beta_n < \frac{\alpha_n}{2}.$$

Associated to these sequences we consider the family of control intervals $\mathcal{J}_n = \{J_i\}_{i\in\{0,\dots,n\}}$ defined as in Theorem 3.9. We also take an arbitrary sequence of positive numbers $(\delta_n)_n$ converging to 0. Denote by $\mathcal{T} = (T_n)_{n\in\mathbb{N}}$ the scale associated to these sequences given by Theorem 3.9.

By Lemma 2.8 there are a sequence $\bar{\epsilon} = (\varepsilon_n)_n$ and a $\mathcal{T}$-long $\bar{\varepsilon}$-sparse tail R_∞. Let $\mathfrak{P}_n$ be the sequence of initial patterns associated to the tail R_∞ given by Remark 3.4. Let $M(R_\infty)$ be the set of marked points of the components of $\mathfrak{P}_n$,

$$M(R_\infty) \stackrel{\text{def}}{=} \bigcup_0^\infty M(\mathfrak{P}_n).$$

Lemma 4.1. *For every plaque $D_0 \in \mathfrak{F}$ there is a sequence $(D_a)_{a\in M(R_\infty)}$ of plaques of $\mathfrak{F}$ such that for every n the subfamily $(D_a)_{a\in M(\mathfrak{P}_n)}$ is $(\mathcal{J}_n, \mathfrak{P}_n)$-controlled.*

Proof. Apply first Theorem 3.9 to construct the family associated to the pattern $\mathfrak{P}_0$. Thereafter inductively apply Addendum 3.15 to construct the family of sets associated to $\mathfrak{P}_{n+1}$ extending the family constructed for $\mathfrak{P}_n$. □

By the expansion property (c) in Definition 1.2 of a flip-flop family with sojourns we have that

$$\bigcap_{a\in M(R_\infty)} f^{-a}(D_a) = \{x_D\} \subset D_0. \tag{4.1}$$

By construction, the point x_D is controlled at any scale with long sparse tail R_∞ with respect to φ and Y, proving Theorem 2. □

4.2. **Proof of Corollary 3.** Let μ be any weak$*$-accumulation point of the family of empirical measures $(\mu_n(x_D))_n$. By Theorem 1, for μ-almost every point x its Birkhoff average $\varphi_\infty(x)$ is zero its orbit is dense in X. This immediately implies that almost every component ν of the ergodic decomposition of μ has full support and $\int \varphi\, d\nu = 0$. □

4.3. **Proof Proposition 4.** Given $t \in (\alpha, \beta)$ consider $\alpha < \alpha_t < t < \beta_t < \beta$. Consider small cylinders $C(\alpha_t)$ and $C(\beta_t)$ where the map φ is less than α_t and bigger than β_t, respectively. Consider now unstable subsets of these cylinders (i.e., the intersection of the cylinders with unstable sets). For sufficiently large m we have that these sets are a flip-flop family relative to f^m. Now it is enough to apply either the criterion in [BBD2] (to get item (a)) or to apply Corollary 3 (to get item (b)).

5. Proof of Theorem 5: Flip-flop families and homoclinic relations

The goal of this section is to prove Theorem 5. Consider $f \in \mathrm{Diff}^1(M)$ and a pair of hyperbolic periodic points p and q of f that are homoclinically related and a continuous function $\varphi\colon M \to \mathbb{R}$ such that $\int \varphi\, d\mu_{\mathcal{O}(p)} < t < \int \varphi\, d\mu_{\mathcal{O}(q)}$ (recall that $\mu_{\mathcal{O}(p)}$ and $\mu_{\mathcal{O}(q)}$ are the unique f-invariant probability measures supported on the orbits p and q, respectively). For notational simplicity, let us assume that the periodic points p and q are fixed points. In this case the assumption above just means $\varphi(p) < t < \varphi(q)$. After replacing φ by the map $\varphi_t = \varphi - t$, to prove the theorem it is enough to get an ergodic measure μ_t whose support is $H(p,f)$ such that $\int \varphi_t\, d\mu_t = 0$. Thus, in what follows, we can assume that $t = 0$ and hence $\varphi(p) < 0 < \varphi(q)$.

5.1. **Flip-flop families obtained from homoclinic relations.** To prove the theorem we construct a flip-flop family associated to φ and f^n for some $n > 0$. We begin by recalling some constructions from [BBD2].

5.1.1. *The space of discs $\mathcal{D}^i(M)$.* Recall that M is a closed and compact Riemannian manifold, let $\dim(M) = d$. For each $i \in \{1, \dots, d-1\}$ denote by $\mathcal{D}^i(M)$ the set of i-dimensional (closed) discs C^1-embedded in M. In the space $\mathcal{D}^i(M)$ the standard C^1-topology is defined as follows, given a disc $D \in \mathcal{D}^i(M)$ its neighbourhoods are of the form $\{f(D)\colon f \in \mathcal{W}\}$, where $\mathcal{W}$ is a neighbourhood of the identity map in $\mathrm{Diff}^1(M)$. In [BBD2] it is introduced the following metric $\mathfrak{d}$ on the space $\mathcal{D}^i(M)$,

$$(D_1, D_2) \mapsto \mathfrak{d}(D_1, D_2) \stackrel{\text{def}}{=} d_{\text{Haus}}(TD_1, TD_2) + d_{\text{Haus}}(T\partial D_1, T\partial D_2),$$

where $D_1, D_2 \in \mathcal{D}^i(M)$, the tangent bundles TD_i and $T\partial D_i$ are considered as compact subsets of the corresponding Grassmannian bundles, and d_{Haus} denotes the corresponding Hausdorff distances. The distance $\mathfrak{d}$ behaves nicely for the composition of diffeomorphisms: if D and D' are close the same holds for $f(D)$ and $f(D')$, see [BBD2, Proposition 3.1].

Given a family of discs $\mathfrak{D} \subset \mathcal{D}^i(M)$ and $\eta > 0$, we denote by $\mathcal{V}^{\mathfrak{d}}_\eta(\mathfrak{D})$ the open η-neighbourhood of $\mathfrak{D}$ with respect to the distance $\mathfrak{d}$,

$$\mathcal{V}^{\mathfrak{d}}_\eta(\mathfrak{D}) \stackrel{\text{def}}{=} \{D \in \mathcal{D}^i(M)\colon \mathfrak{d}(D, \mathfrak{D}) < \eta\}.$$

5.1.2. *Proof of Theorem 5.* Since $\varphi(p) < 0 < \varphi(q)$, there are small local unstable manifolds $W^{\rm u}_{loc}(p,f)$ and $W^{\rm u}_{loc}(q,f)$ of p and q such that φ is strictly negative in $W^{\rm u}_{loc}(p,f)$ and strictly positive in $W^{\rm u}_{loc}(q,f)$. Similarly for the local stable manifolds $W^{\rm s}_{loc}(p,f)$ and $W^{\rm s}_{loc}(q,f)$ of p and q.

Since p and q are homoclinically related there $\ell_0 \geq 0$ and small discs $\Delta^{\rm s}_p \subset W^{\rm s}(p)$ and $\Delta^{\rm s}_q \subset W^{\rm s}(q,f)$ such that the intersections $\Delta^{\rm s}_p \cap f^{\ell_0}(W^{\rm u}_{loc}(q,f))$ and $\Delta^{\rm s}_q \cap f^{\ell_0}(W^{\rm u}_{loc}(p,f))$ are both transverse and consist of just a point.

For $\varrho > 0$ consider the ϱ-neighbourhoods $\mathcal{V}^{\mathfrak{d}}_\varrho(p) \stackrel{\text{def}}{=} \mathcal{V}^{\mathfrak{d}}_\varrho(W^{\rm u}_{loc}(p,f))$ and $\mathcal{V}^{\mathfrak{d}}_\varrho(q) \stackrel{\text{def}}{=} \mathcal{V}^{\mathfrak{d}}_\varrho(W^{\rm u}_{loc}(q,f))$ of the local unstable manifolds of p and q for the distance $\mathfrak{d}$. For ϱ small enough, every disc in $\mathcal{V}^{\mathfrak{d}}_\varrho(p)$ intersects transversely $\Delta^{\rm s}_p$. We also have that φ is uniformly negative (say less than $-\alpha < 0$) in every disc in this neighbourhood. Finally, the derivative of Df is uniformly expanding in restriction to this family of discs. There are similar assertions for the discs in $\mathcal{V}^{\mathfrak{d}}_\varrho(q)$: every disc of this neighbourhood meets transversely $\Delta^{\rm s}_q$, φ is larger than $\alpha > 0$ in the discs, and Df is a uniform expansion.

Remark 5.1. If $\varrho > 0$ is small enough then there is ℓ_1 such that for every $\ell > \ell_1$ the image $f^\ell(D)$ of any disc $D \in \mathcal{V}^{\mathfrak{d}}_\varrho(p)$ contains discs in $\mathcal{V}^{\mathfrak{d}}_\varrho(p)$ and in $\mathcal{V}^{\mathfrak{d}}_\varrho(q)$. The same holds (with the same constant) for discs in $\mathcal{V}^{\mathfrak{d}}_\varrho(q)$. This is a well known fact and is the ground of the proof of the existence of unstable manifolds using a graph transformation. In what follows we assume that ϱ satisfies this property.

We consider the following family $\mathfrak{F} = \mathfrak{F}^+ \bigsqcup \mathfrak{F}^-$ of discs:

- $\mathfrak{F}^-$ is the family of discs in $\mathcal{V}^{\mathfrak{d}}_\varrho(p)$ contained in $W^{\mathrm{u}}(p,f) \cup W^{\mathrm{u}}(q,f)$;
- $\mathfrak{F}^+$ is the family of discs in $\mathcal{V}^{\mathfrak{d}}_\varrho(q)$ contained in $W^{\mathrm{u}}(p,f) \cup W^{\mathrm{u}}(q,f)$.

Note that as q are homoclinically related these two families are both infinite.

Proposition 5.2. *There is n such that the family $\mathfrak{F}$ is a flip-flop family associated to φ and f^n and has sojourns (for f) along the homoclinic class $H(p,f)$.*

We postpone the proof of Proposition 5.2 and prove the theorem.

Proof of Theorem 5. Consider the flip-flop family $\mathfrak{F}$ with sojourns along $H(p,f)$ given by Proposition 5.2. Exactly as in the proof of Theorem 2 in Section 4 we use Theorem 3.9, Addendum 3.15, and Lemma 2.8 to construct a scale $\mathcal{T}$, a tail R_∞, a sequence of increasing patterns $\mathfrak{P}_n$, and a family of discs D_a, $a \in M(R_\infty)$ such that the restriction of this family to the marked sets $M(\mathfrak{P}_n)$ is controlled by the pattern $\mathfrak{P}_n$ for every n (exactly as in Lemma 4.1). The expansion property in the flip-flop family implies that (recall equation (4.1))

$$\bigcap_{a \in M(R_\infty)} f^{-a}(D_a) = x_\infty.$$

Claim 5.3. $x_\infty \in H(p,f)$.

Proof. Every disc D_a belongs to $\mathfrak{F}$, hence it is contained in $W^{\mathrm{u}}(p,f) \cup W^{\mathrm{u}}(q,f)$ and intersects transversely $W^{\mathrm{s}}(p,f) \cup W^{\mathrm{s}}(q,f)$. Thus D_a contains a point of $H(p,f)$. The f-invariance of $H(p,f)$ implies that the same holds for $f^{-a}(D_a)$ The compactness of $H(p,f)$ implies that $x_\infty \in H(p,f)$. $\square$

By construction, the point x_∞ is controlled at any scale with a long sparse tail for φ and f (the ambient space here is $H(p,f)$). The theorem now follows from Theorem 1. $\square$

5.2. **Proof of Proposition 5.2.** We split the proof of the proposition into two parts:

5.2.1. $\mathfrak{F} = \mathfrak{F}^+ \cup \mathfrak{F}^-$ *is a flip-flop family.* By construction, the map φ is less than $-\alpha < 0$ in the discs of $\mathcal{V}^{\mathfrak{d}}_\varrho(p)$ and bigger than $\alpha > 0$ in the discs of $\mathcal{V}^{\mathfrak{d}}_\varrho(q)$. The definition of $\mathfrak{F}^\pm$ implies (a) in Definition 1.1.

Recall the choice of ℓ_0 above and that, by construction, the image $f^{\ell_0}(D)$ of any disc $D \in \mathfrak{F}$ intersects transversely the compact parts Δ^{s}_p of $W^{\mathrm{s}}(p,f)$ and Δ^{s}_q of $W^{\mathrm{s}}(q,f)$. Thus the λ-lemma (inclination lemma) and the invariance of $W^{\mathrm{u}}(p,f) \cup W^{\mathrm{u}}(q,f)$ imply the existence of $n_0 > 0$ such that for every $n > n_0$ and every disc $D \in \mathfrak{F}$ the set $f^n(D)$ contains a disc $D^+ \in \mathfrak{F}^+$ and a disc $D^- \in \mathfrak{F}^-$. This proves item (b) in Definition 1.1.

It remains to get the expansion property in item (c) of Definition 1.1. We need to get n such that for every $D \in \mathfrak{F}$ the disc $f^n(D)$ contains a disc D' such that $f^n \colon f^{-n}(D') \to D'$ is a uniform expansion. For that it is enough to take sufficiently large n (independent of D). To see why this is so recall first that $f^{n_0}(D)$ contains a disc $D_{n_0} \in \mathfrak{F}^+$. Now Remark 5.1 provides a sequence of discs $D_{n_0+i\ell_0}$ in $\mathfrak{F}^\pm$ such that $D_{n_0+(i+1)\ell_0} \subset f^{\ell_0}(D_{n_0+i})$ and $f^{\ell_0}\colon f^{-\ell_0}(D_{n_0+(i+1)\ell_0}) \to D_{n_0+(i+1)\ell_0}$ is a uniform expansion. This implies that for i large enough (independent of D) we get the announced expansion for $f^{n_0+i\ell_0}\colon f^{-(n_0+i\ell_0)}(D_{n_0+i\ell_0}) \to D_{n_0+i\ell_0}$, just note that the $i\,\ell_0$ additional iterates in the "expanding part" compensate any contraction introduced by the first n_0 iterates.

5.2.2. *The family $\mathfrak{F}$ sojourns along the homoclinic class $H(p,f)$.* Consider any $\delta > 0$. We need prove that there is $N > 0$ such that every disc $D \in \mathfrak{F}$ contains a pair of discs $\widehat{D}^+$, $\widehat{D}^-$ such that for every $x \in \widehat{D}^\pm$ the segment of orbit $\{x, \dots, f^N(x)\}$ is δ-dense in $H(p,f)$ (item (a)), $f^N(\widehat{D}^+) \in \mathfrak{F}^+$ and $f^N(\widehat{D}^-) \in \mathfrak{F}^-$ (item (b)), and $f^N\colon \widehat{D}^\pm \to f^N(\widehat{D}^\pm)$ is expanding (item (c)).

We need the following property of $H(p,f)$ that is a direct consequence of the density of transverse homoclinic intersection points of $W^{\mathrm{u}}(p,f) \cap W^{\mathrm{s}}(p,f)$ in $H(p,f)$ and the existence of (hyperbolic) horseshoes associated to these points.

Remark 5.4. For every $\epsilon > 0$ there is a hyperbolic periodic point $r_\epsilon \in H(p,f)$ that is homoclinically related to p and q whose orbit is $\epsilon/2$-dense in $H(p,f) = H(q,f)$.

To prove item (a) consider the point $r = r_{\frac{\delta}{2}} \in H(p,f)$ given by Remark 5.4. As the points r, p, and q are pairwise homoclinically related, the stable manifold of the orbit of r, $W^{\mathrm{s}}(\mathcal{O}(r), f)$ accumulates the ones of p and q. Hence there are compact discs $\Delta^{\mathrm{s}}_{r,p}, \Delta^{\mathrm{s}}_{r,q} \subset W^{\mathrm{s}}(\mathcal{O}(r), f)$ such that any disc in $\mathcal{V}^{\mathfrak{d}}_\varrho(p) \cup \mathcal{V}^{\mathfrak{d}}_\varrho(q)$ meets transversely $\Delta^{\mathrm{s}}_{r,p} \cup \Delta^{\mathrm{s}}_{r,q}$.

Let π be the period of r. As in Remark 5.1, for each $i = 0, \dots, \pi - 1$, we fix a small local unstable manifold $W^{\mathrm{u}}_{loc}(f^i(r), f)$ and a small C^1-neighbourhood $\mathcal{V}^{\mathfrak{d}}_\eta(f^i(r)) \stackrel{\text{def}}{=} \mathcal{V}^{\mathfrak{d}}_\eta(W^{\mathrm{u}}_{loc}(f^i(r), f))$ such that the image $f(D)$ of any disc $D \in \mathcal{V}^{\mathfrak{d}}_\eta(f^i(r))$ contains a disc in $\mathcal{V}^{\mathfrak{d}}_\eta(f^{i+1}(r))$ (for $\pi - 1$ we take "$\pi = 0$").

Take now $D = D_0$ any disc in $\mathcal{V}^{\mathfrak{d}}_\eta(r)$, let D_1 be a sub-disc of $f(D_0)$ in $\mathcal{V}^{\mathfrak{d}}_\eta(f(r))$, and inductively define D_{i+1} as a disc in $\mathcal{V}^{\mathfrak{d}}_\eta(f^{i+1}(r))$ contained in $f(D_i)$. Assuming thta the local unstable manifolds and their neighbourhoods are small enough we have that every point in a disc of $\mathcal{V}^{\mathfrak{d}}_\eta(f^i(r))$, $i = 0, \dots, \pi - 1$, is at distance less that $\frac{\delta}{2}$ from the orbit of r. Since the orbit of r is $\frac{\delta}{2}$-dense in $H(p,f)$ for every $x \in f^{-\pi}(D_\pi) \subset D$, we have that the segment of orbit $\{x, \dots, f^\pi(x)\}$ is δ-dense in $H(p,f)$.

Consider now a disc $D \in \mathfrak{F}$. By construction, this disc intersects transversely $W^{\mathrm{s}}(\mathcal{O}(r), f)$ in some point of $\Delta^{\mathrm{s}}_{r,p} \cup \Delta^{\mathrm{s}}_{r,q}$. By the λ-lemma there is j_0 (independent of D) such that $f^{j_0}(D)$ contains a disc D_0 in $\mathcal{V}^{\mathfrak{d}}_\eta(r)$. The argument above provides a sequence of discs $D_i \in \mathcal{V}^{\mathfrak{d}}_\eta(f^i(r))$, $j \in \{0, \dots, \pi - 1\}$, with $D_{i+1} \subset f(D_i)$ and such that for every $x \in f^{-\pi}(D_\pi) \subset D_0 \subset f^{j_0}(D)$ its orbit segment $\{x, \dots, f^\pi(x)\}$ is δ-dense in $H(p,f)$. A new application of the λ-lemma provides a uniform $j_1 > 0$ such that $f^{j_1}(D_\pi)$ contains discs $\widetilde{D}^\pm \in \mathfrak{F}^\pm$ (recall that the initial $D \in \mathfrak{F}$ and therefore it is contained in $W^{\mathrm{u}}(p,f) \cup W^{\mathrm{u}}(q,f)$).

Now it is enough to take

$$N \stackrel{\text{def}}{=} j_0 + \pi + j_1 \quad \text{and} \quad \widehat{D}^\pm \stackrel{\text{def}}{=} f^{-j_0-\pi-j_1}(\widetilde{D}^\pm) \subset D.$$

By construction the orbit segment $\{y, \dots, f^N(y)\}$ of any point $y \in \widehat{D}^{\pm}$ is δ-dense in $H(p,f)$, proving item (a) in Definition 1.2. By construction, $f^N(\widehat{D}^{\pm}) = \widetilde{D}^{\pm} \in \mathfrak{F}^{\pm}$ proving item (b) in Definition 1.2

Note that the discs $\widehat{D}^{\pm} \subset D$ satisfy the density in $H(p,f)$ and return to $\mathfrak{F}^{\pm}$ properties, however they can fail to satisfy the expansion property in (c) in Definition 1.2. To get additionally such an expansion one considers further iterates of the disc in a "expanding" region nearby p or q. The expansion is obtained using Remark 5.1 and arguing exactly as in Section 5.2.1. The proof of Proposition 5.2 is now complete. □

5.3. **Proof of Corollary 6.** By hypothesis, the saddles p_f and q_f have different u-indices (say i and j, $i < j$) that depend continuously on f and whose chain recurrence classes coincide for every diffeomorphism f in a C^1-open set $\mathcal{U}$. As in the proof of Theorem 5, let us assume that $t = 0$ and hence the Birkhoff average of φ is negative in $\mathcal{O}(p_f)$ and positive in $\mathcal{O}(q_f)$.

According to [ABCDW], up to restrict to a C^1-open and dense subset of $\mathcal{U}$, we can assume that for every $k \in [i,j]$ every diffeomorphism $f \in \mathcal{U}$ has a periodic point r_f of u-index k that is C^1-robustly in $C(p_f, f)$. Therefore, after replacing p_f and q_f by other periodic points, we can assume that the u-indices of p_f and q_f are consecutive.

Following Propositions 3.7 and 3.10 in [ABCDW], an arbitrarily small C^1-perturbation of f gives a diffeomorphism h with a periodic point r_h having a (unique) center eigenvalue equal to 1 that is robustly in $C(p_h, h)$. This means that this (non-hyperbolic) periodic point r_h admits a continuation $r_g \in C(p_g, g) = C(q_g, g)$ for some g arbitrarily close to f.

Consider the average of φ along the orbit of r_h and assume first that it is different from zero, for example negative. Then, after an arbitrarily small perturbation, we can transform r_h in a hyperbolic point r_g of g of the same index as q_g and homoclinically related to q_g (for this last step we use the version of Hayashi's connecting lemma [H] for chain recurrence classes in [BC][9]). The diffeomorphism g belongs to $\mathcal{U}$, the saddles r_g and q_g are homoclinically related, and the averages of φ in these orbits have different signals. The corollary now follows from Theorem 5.

In the case when the average of φ throughout the orbit of r_g is zero one needs a slight modification of the previous argument. Let us sketch this construction, arguing as above, we can assume that, after an arbitrarily small perturbation, the point r_g is hyperbolic of the same index as p_g (with center derivative arbitrarily close to one) and that r_g and p_g are homoclinically related. Using the homoclinic relation between r_g and p_g we get a point $\bar{r}_g$ with some center eigenvalue arbitrarily close to one and with negative average for φ. Next, arguing as above and after a small perturbation, we change the index of the point $\bar{r}_g$ and generate transverse cyclic intersections between $\bar{r}_g$ and q_g (i.e., we put the saddle $\bar{r}_g$ in the homoclinic

[9]This result guarantees that given two saddles in the same chain recurrence there is an arbitrarily small C^1-perturbation of the diffeomorphism that gives an intersection between the invariant manifolds of these saddles. If the saddles belong C^1-robustly to the same class then one can repeat the previous argument, interchanging the roles of the saddles, to get that the invariant manifolds of these saddles meet cyclically. Finally, if the saddles have the same index one can turn these intersections into transverse ones, thus the two saddles are homoclinically related and hence they are C^1-robustly in the same homoclinic class.

class of q_g). We are now in the previous case and prove the corollary using the saddles $\bar{r}_g$ and q_g. $\square$

6. Flip-flop families in partially hyperbolic dynamics

In this section we prove Theorems 7 and 8. For that we borrow and adapt some constructions in [BBD2]. In Section 6.1 we recall the definition of a dynamical blender and its main properties. Section 6.2 is dedicated to the study of flip-flop configurations. In Section 6.3 we see how flip-flop configurations yield flip-flop families with sojourns. In Section 6.4 we analyse the control of averages in flip-flop configurations. Finally, in Section 6.5 we conclude the proofs of Theorems 7 and 8.

6.1. **Dynamical blenders.** The definition of a *dynamical blender* in [BBD2] involves three main ingredients: the distance on the space of C^1-discs (Section 5.1.1), strictly invariant families of discs (Section 6.1.1), and invariant cone fields (Section 6.1.2). We now describe succinctly these ingredients.

6.1.1. *Strictly invariant families of discs.* Recall the notation $\mathcal{D}^i(M)$ for the set of i-dimensional (closed) discs C^1-embedded in M and the definitions of the distance $\mathfrak{d}$ and the open neighbourhood $\mathcal{V}^{\mathfrak{d}}_\eta(\mathfrak{D})$ of a family of discs $\mathfrak{D}$ with respect to $\mathfrak{d}$ in Section 5.1.1.

Definition 6.1 (Strictly f-invariant families of discs)**.** *Let $f \in \mathrm{Diff}^1(M)$. A family of discs $\mathfrak{D} \subset \mathcal{D}^i(M)$ is* strictly f-invariant *if there is $\varepsilon > 0$ such that for every disc $D_0 \in \mathcal{V}^{\mathfrak{d}}_\varepsilon(\mathfrak{D})$ there is a disc $D_1 \in \mathfrak{D}$ with $D_1 \subset f(D_0)$.*

The existence of a strictly invariant family of discs is a C^1-robust property: If the family $\mathfrak{D}$ is strictly f-invariant then there are $\mu, \eta > 0$ such that the family $\mathfrak{D}_\mu = \mathcal{V}^{\mathfrak{d}}_\mu(\mathfrak{D})$ is strictly g-invariant for every $g \in \mathrm{Diff}^1(M)$ that is η-C^1-close to f, see [BBD2, Lemma 3.8].

6.1.2. *Invariant cone fields.* Given a vector space of finite dimension E, we say that a subset C of E is a *cone of index i* if there are a splitting $E = E_1 \oplus E_2$ with $\dim(E_1) = i$ and a norm $\|\cdot\|$ defined on E such that

$$C = \{v = v_1 + v_2 \colon v_i \in E_i,\ \|v_2\| \leq \|v_1\|\}.$$

A cone C' is *strictly contained* in the cone C above if there is $\alpha > 1$ such that

$$C' \subset C_\alpha = \{v_1 + v_2 \colon v_i \in E_i,\ \|v_2\| \leq \alpha^{-1}\|v_1\|\} \subset C.$$

A *cone field of index i* defined on a subset V of a compact manifold M is a continuous map $x \mapsto \mathcal{C}(x) \subset T_xM$ that associates to each point $x \in V$ a cone $\mathcal{C}(x)$ of index i . We denote this cone field by $\mathcal{C} = \{\mathcal{C}(x)\}_{x \in V}$.

Given a diffeomorphism $f \in \mathrm{Diff}^1(M)$ and a cone field $\mathcal{C} = \{\mathcal{C}(x)\}_{x\in V}$ we say that $\mathcal{C}$ is *strictly Df-invariant* if $Df(x)(\mathcal{C}(x))$ is strictly contained in $\mathcal{C}(f(x))$ for every $x \in V \cap f^{-1}(V)$.

The following result is a standard lemma about persistence of invariant cone fields (see for instance [BBD2, Lemma 3.9]).

Lemma 6.2. *Let $f \in \mathrm{Diff}^1(M)$, V a compact subset of M, and $\mathcal{C}$ a strictly Df-invariant cone field defined on V. Then there is a C^1-neighbourhood $\mathcal{U}$ of f such that $\mathcal{C}$ is strictly Dg-invariant for every $g \in \mathcal{U}$.*

6.1.3. *Dynamical blenders.* We are now ready to define a *dynamical blender* and recall its main properties.

Definition 6.3 (Dynamical blender, [BBD2])**.** *Let $f \in \mathrm{Diff}^1(M)$. A compact f-invariant set $\Gamma \subset M$ is a* dynamical cu-blender of uu-index i *if the following properties hold:*

a) there is an open neighbourhood V of Γ such that

$$\Gamma = \bigcap_{n\in\mathbb{Z}} f^n(\overline{V});$$

b) the set Γ is transitive;
c) the set Γ is (uniformly) hyperbolic with u*-index strictly larger than i;*
d) there is a strictly Df-invariant cone field $\mathcal{C}^{\mathrm{uu}}$ of index i defined on $\overline{V}$; and
e) there are a strictly f-invariant family of discs $\mathfrak{D} \subset \mathcal{D}^i(M)$ and $\varepsilon > 0$ such that every disc in $\mathcal{V}^{\mathfrak{d}}_\varepsilon(\mathfrak{D})$ is contained in V and tangent to $\mathcal{C}^{\mathrm{uu}}$.

We say that V is the domain of the blender*, $\mathcal{C}^{\mathrm{uu}}$ is its* strong unstable cone field*, and $\mathfrak{D}$ is its* strictly invariant family of discs*. To emphasise the role of these objects we write $(\Gamma, V, \mathcal{C}^{\mathrm{uu}}, \mathfrak{D})$.*

Remark 6.4. Let Γ be a hyperbolic set of u-index j that is also a cu-blender of uu-index i. By definition, the set Γ has a partially hyperbolic splitting (recall (1.4)) of the form

$$T_\Gamma M = E^{\mathrm{uu}} \oplus E^{\mathrm{cu}} \oplus E^{\mathrm{s}},$$

where $\dim(E^{\mathrm{uu}}) = i$, $\dim(E^{\mathrm{cu}}) = j - i \geq 1$, and $E^{\mathrm{u}} = E^{\mathrm{uu}} \oplus E^{\mathrm{cu}}$. Here E^{s} and E^{u} are the stable and unstable bundles of Γ. We also define the bundle $E^{\mathrm{cs}} \stackrel{\mathrm{def}}{=} E^{\mathrm{cu}} \oplus E^{\mathrm{ss}}$.

Next lemma claims that blenders have well defined continuations.

Lemma 6.5 (Lemma 3.8 and Scholium 3.14 in [BBD2])**.** *Let $(\Gamma, V, \mathcal{C}^{\mathrm{uu}}, \mathfrak{D})$ be a dynamical blender of $f \in \mathrm{Diff}^1(M)$. Then there are a C^1-neighbourhood $\mathcal{U}$ of f and $\varepsilon > 0$ such that for every diffeomorphism $g \in \mathcal{U}$ the* 4*-tuple $(\Gamma_g, V, \mathcal{C}^{\mathrm{uu}}, \mathcal{V}^{\mathfrak{d}}_\epsilon(\mathfrak{D}))$ is a dynamical blender, where Γ_g is the hyperbolic continuation of Γ for g.*

Moreover, every disc $D \in \mathcal{V}^{\mathfrak{d}}_\varepsilon(\mathfrak{D})$ meets the local stable manifold of Γ_g defined by

$$W^{\mathrm{s}}_{loc}(\Gamma_g) \stackrel{\mathrm{def}}{=} \{x \in V \colon f^i(x) \in V \text{ for every } i \geq 0\}.$$

6.2. **Flip-flop configurations and partial hyperbolicity.** We now recall the definition of a flip-flop configuration and borrow some results from [BBD2].

Definition 6.6 (Flip-flop configuration)**.** *Consider $f \in \mathrm{Diff}^1(M)$ having a dynamical* cu*-blender $(\Gamma, V, \mathcal{C}^{\mathrm{uu}}, \mathfrak{D})$ of* uu*-index i and a hyperbolic periodic point q of* u*-index i. We say that $(\Gamma, V, \mathcal{C}^{\mathrm{uu}}, \mathfrak{D})$ and q form a* flip-flop configuration *if there are:*

- *a disc Δ^{u} contained in the unstable manifold $W^{\mathrm{u}}(q, f)$ and*
- *a compact submanifold with boundary $\Delta^{\mathrm{s}} \subset V \cap W^{\mathrm{s}}(q, f)$*

such that:

a) The disc Δ^{u} belongs to the interior of the family $\mathfrak{D}$.
b) $f^{-n}(\Delta^{\mathrm{u}}) \cap \overline{V} = \varnothing$ for all $n > 0$.
c) There is $N > 0$ such that $f^n(\Delta^{\mathrm{s}}) \cap \overline{V} = \varnothing$ for every $n > N$. Moreover, if $x \in \Delta^{\mathrm{s}}$ and $j > 0$ are such that $f^j(x) \notin V$ then $f^i(x) \notin \overline{V}$ for every $i \geq j$.
d) $T_y W^{\mathrm{s}}(q, f) \cap \mathcal{C}^{\mathrm{uu}}(y) = \{0\}$ for every $y \in \Delta^{\mathrm{s}}$.

e) *There are a compact set K in the relative interior of Δ^{s} and $\epsilon > 0$ such that for every $D \in \mathfrak{D}$ there exists x such that $K \cap D = \{x\}$ and $d(x, \partial D) > \epsilon$.*

The sets Δ^{u} and Δ^{s} are called the unstable and stable connecting sets *of the flip-flop configuration, respectively.*

[BBD2, Proposition 4.2] asserts that flip-flop configuration are C^1-robust. Next lemma claims that flip-flop configurations yield partially hyperbolic dynamics. Recall Remark 6.4 and the definition of the center unstable bundle E^{cu} of a blender.

Lemma 6.7 (Lemma 4.6 in [BBD2])**.** *Consider $f \in \mathrm{Diff}^1(M)$ having a hyperbolic periodic point q and a dynamical blender $(\Gamma, V, \mathcal{C}^{\mathrm{uu}}, \mathfrak{D})$ in a flip-flop configuration with connecting sets $\Delta^{\mathrm{u}} \subset W^{\mathrm{u}}(q, f)$ and $\Delta^{\mathrm{s}} \subset W^{\mathrm{s}}(q, f)$. Consider the closed set*

$$\Delta \stackrel{\text{def}}{=} \mathcal{O}(q) \cup \overline{V} \cup \bigcup_{k \geq 0} f^k(\Delta^{\mathrm{s}}) \cup \bigcup_{k \leq 0} f^k(\Delta^{\mathrm{u}}).$$

Then there is a compact neighbourhood U of Δ, called a partially hyperbolic neighbourhood of the flip-flop configuration*, such that the maximal invariant set $\Gamma(U)$ of f in U*

$$\Gamma(U) \stackrel{\text{def}}{=} \bigcap_{i \in \mathbb{Z}} f^i(U)$$

has a partially hyperbolic splitting

$$T_{\Gamma(U)} M = \widetilde{E}^{\mathrm{uu}} \oplus \widetilde{E}^{\mathrm{cs}},$$

where $\widetilde{E}^{\mathrm{uu}}$ is uniformly expanding and $\widetilde{E}^{\mathrm{uu}}$ and $\widetilde{E}^{\mathrm{cu}}$ extend the bundles E^{uu} and E^{cs}, respectively, defined over Γ.

Moreover, there is a strictly Df-invariant cone field over U that extends the cone field $\mathcal{C}^{\mathrm{uu}}$ defined on $\overline{V}$ (we also denote this cone field by $\mathcal{C}^{\mathrm{uu}}$) whose vectors are uniformly expanded by Df.

6.3. **Flip-flop families with sojourns in homoclinic classes.** [BBD2, Proposition 4.9] claims that flip-flop configurations yield flip-flop families. These configurations are enough to construct measures with controlled averages. However they do not provide control of the support of the obtained measure. In this paper, we want to get measures with "full support". Bearing this in mind we defined flip-flop families with sojourns (Definition 1.2). These "sojourns" guarantee "density" of orbits in the ambient space.

Theorem 6.8. *Consider $f \in \mathrm{Diff}^1(M)$ with a hyperbolic periodic point q and a dynamical blender Γ in a flip-flop configuration. Let $\varphi \colon M \to \mathbb{R}$ be a continuous function that is positive on the blender Γ and negative on the orbit of q.*

Then there are $N \geq 1$ and a flip-flop family $\mathfrak{F}$ with respect to φ_N and f^N which sojourns along the homoclinic class $H(q, f)$ (for f).

Moreover, given any $\delta > 0$ the flip-flop family $\mathfrak{F}$ can be chosen such that:

- *every $D \in \mathfrak{F}$ is contained in a δ-neighbourhood of $\Gamma \cup \{\mathcal{O}(q)\}$,*
- *every $D \in \mathfrak{F}$ transversely intersects $W^{\mathrm{s}}(q, f^N)$, and*
- *there is $D \in \mathfrak{F}$ contained in $W^{\mathrm{u}}_{loc}(q, f^N)$.*

To prove this theorem we need to recall the construction of flip-flop families in [BBD2]. As the families in [BBD2] do not have sojourns we need to adapt this construction to our context bearing in mind this fact.

6.3.1. *Flip-flop families associated to flip-flop configurations.* We now borrow the following result from [BBD2] and recall some steps of its proof.

Proposition 6.9 (Proposition 4.9 in [BBD2])**.** *Let $f \in \mathrm{Diff}^1(M)$ be a diffeomorphism with a hyperbolic periodic point q and a dynamical blender Γ in a flip-flop configuration. Let U be a partially hyperbolic neighbourhood of this configuration and $\varphi\colon U \to \mathbb{R}$ a continuous function that is positive on the blender and negative on the orbit of q.*

Then there are $N \geq 1$ and a flip-flop family $\mathfrak{F} = \mathfrak{F}^+ \bigsqcup \mathfrak{F}^-$ with respect to φ_N and f^N.

Moreover, given any $\delta > 0$ the flip-flop family can be chosen such that the plaques in $\mathfrak{F}^+$ are contained in a δ-neighbourhood of Γ and the plaques in $\mathfrak{F}^-$ are contained in a δ-neigbourhood of q.

Note that Theorem 6.8 is just the proposition above with additional sojourns. Observe also that the map φ_N is only defined on $\bigcap_0^{N-1} f^{-i}(U)$ and that the plaques of $\mathfrak{F}$ are contained in that set.

We now review the construction in [BBD2]. For simplicity let us suppose that q is a fixed point. The definition of the family $\mathfrak{F}$ in Proposition 6.9 involves a preliminary family of discs $\mathfrak{D}_q$ satisfying the following properties (see [BBD2, Lemma 4.11]):

(p1) The family of discs $\mathfrak{D}_q$ form a small C^1-neighbourhood (in the metric $\mathfrak{d}$) of the local unstable manifold $W^{\mathrm{u}}_{loc}(q,f)$. This neighbourhood can be taken arbitrarily small.
(p2) The sets of the family $\mathfrak{F}^-$ are contained in discs in $\mathfrak{D}_q$.
(p3) Each disc in $\mathfrak{D}_q$ contains a plaque of $\mathfrak{F}^-$.
(p4) The image by f^N of any plaque of $\mathfrak{F}$ contains a disc of $\mathfrak{D}_q$.

We have the following direct consequences of the properties above:

(p5) As $\mathfrak{D}_q$ can be taken contained in an arbitrarily small neighbourhood of $W^{\mathrm{u}}_{loc}(q,f)$, we can assume that $W^{\mathrm{s}}_{loc}(q,f)$ meets transversely every disc in $\mathfrak{D}_q$.
(p6) As $W^{\mathrm{u}}_{loc}(q,f)$ can be chosen arbitrarily small, we can assume that f^N expands uniformly the vectors tangent to the discs in $\mathfrak{D}_q$ (see also Remark 5.1).
(p7) As a consequence of items (p2),(p3), and (p4), the image by f^N of any disc in $\mathfrak{D}_q$ contains a disc in $\mathfrak{D}_q$.

We say that the flip-flop family $\mathfrak{F}$ is *prepared with and adapted family* $\mathfrak{D}_q$ if $\mathfrak{F}$ and $\mathfrak{D}_q$ satisfy properties (p1)–(p7) above.

6.3.2. *Proof of Theorem 6.8.* Since a flip-flop family yields a prepared flip-flop family, Theorem 6.8 is a consequence of the following proposition.

Proposition 6.10. *Let $f \in \mathrm{Diff}^1(M)$ be a diffeomorphism with a hyperbolic periodic point q and a dynamical blender Γ in a flip-flop configuration, U be a partially hyperbolic neighbourhood of this configuration, and $\varphi\colon U \to \mathbb{R}$ a continuous function that is positive on the blender and negative on the orbit of q.*

Let $N \geq 1$ and $\mathfrak{F}$ be a prepared flip-flop family with an adapted family of discs $\mathfrak{D}_q$ with respect to φ_N and f^N.

Then the flip-flop family $\mathfrak{F}$ sojourns along the homoclinic class $H(q, f^N)$.

Proof. We need the following lemma whose proof is similar to the one of Proposition 5.2 and follows using the partially hyperbolicity in the set U.

Lemma 6.11. *For every $\delta > 0$ there is $L \in \mathbb{N}$ such that every disc $D \in \mathfrak{D}_q$ contains a sub-disc $\widehat{D}$ such that*

- *for every $x \in \widehat{D}$ the segment of orbit $\{x, \dots, f^L(x)\}$ is δ-dense in $H(q, f^N)$,*
- *$f^L(\widehat{D})$ contains a disc of $\mathfrak{D}_q$.*
- *for every $i \in \{0, \dots, L\}$ and every pair of points $x, y \in \widehat{D}$ it holds*
$$d(f^{L-i}(x), f^{L-i}(y)) \leq \zeta\, \alpha^i\, d(f^L(x) f^L(y)),$$
for some constants $\zeta > 0$ and $0 < \alpha < 1$ (independent of the points and the discs).

Proof. Consider a hyperbolic periodic point $r_\delta \in H(q, f^N)$ homoclinically related to q and whose orbit is δ-dense in $H(q, f^N)$ (recall Remark 5.4). The λ-lemma implies that a compact part of $W^{\mathrm{s}}(r_\delta, f)$ intersects transversely every disc in $\mathfrak{D}_q$. Thus, again by the λ-lemma, iterations of any disc $D \in \mathfrak{D}_q$ accumulate to $W^{\mathrm{u}}_{loc}(\mathcal{O}(r_\delta), f)$. Again the λ-lemma and (p7) in the definition of a prepared family imply that further iterations of D contains a disc in $\mathfrak{D}_q$. Since the number of iterates involved can be taken uniform, considering the corresponding pre-image one gets the disc $\widehat{D}$ satisfying the first two items of the lemma. Finally, exactly as in the end of the proof of Proposition 5.2 further iterations provides the uniform expansion property. This ends the proof of the lemma. $\square$

To end the proof of the proposition recall that, by condition (p4), the image of any plaque $D \in \mathfrak{F}$ contains a disc in $\mathfrak{D}_q$. This provides the "sojourns property" for $\mathfrak{F}$ (may be one needs to add some extra additional "final" iterates for recovering the expansion). $\square$

6.4. **Control of averages in flip-flop configurations.** As a first consequence of Theorem 6.8 we get measures with controlled averages and full support in a homoclinic class.

Theorem 6.12. *Let $f \in \mathrm{Diff}^1(M)$ be a diffeomorphism and $\varphi\colon M \to \mathbb{R}$ be a continuous map. Suppose that f has a dynamical blender Γ and hyperbolic periodic point q that are in flip-flop configuration with respect to φ and f. Then there is an ergodic measure ν whose support is the whole homoclinic class $H(q, f)$ such that*
$$\int \varphi d\,\nu = 0.$$

Proof. Under the hypotheses of the theorem, Theorem 6.8 provides a flip-flop family $\mathfrak{F}$ associated to f^N and φ_N which sojourns in $H(q, f^N)$. By Theorem 2 every plaque $D \in \mathfrak{F}$ contains a point x_D that is controlled at any scale with a long sparse tail with respect φ_N and $H(q, f^N)$. Note that $W^{\mathrm{u}}_{loc}(q, f)$ contains a plaque $\Delta \in \mathfrak{F}$. Let $x_\Delta \in \Delta$ be the controlled point given by Theorem 2.

Claim 6.13. $x_\Delta \in H(q, f^N)$.

Proof. By construction, there is a sequence of discs D_k and numbers $n_k \to \infty$ such that $D_{k+1} \subset f^{n_k}(D_k)$ and $D_0 = \Delta$. Thus $f^{-n_k}(D_k) \subset \Delta$ is a decreasing nested sequence of compact sets that satisfies
$$x_\Delta = \bigcap_{k \in \mathbb{N}} f^{-n_k}(D_k).$$

The fact that this intersection is just a point follows from the expansion property in the definition of a flip-flop family. Since $D_k \subset W^{\mathrm{u}}(q, f^N)$ and intersects transversely $W^{\mathrm{s}}(q, f)$ we have that D_k contains a point $x_k \in H(q, f^N)$. Hence $y_k = f^{-n_k}(x_k) \in H(q, f^N) \cap f^{-n_K}(D_k)$. Thus $y_k \to x_\Delta$ and $x_\Delta \in H(q, f^N)$. $\square$

Take any measure μ that is an accumulation point of the measures

$$\mu_n \stackrel{\text{def}}{=} \frac{1}{n} \sum_0^{n-1} \delta(f^{iN}(x_\Delta)).$$

As the point x_Δ is controlled at any scale with a long sparse tail with respect to φ and $H(q, f^N)$, Theorem 1 implies that μ-almost every point y has a dense orbit in $H(q, f^N)$ and the average of φ_N along the f^N-orbit of y is zero.

To conclude the proof of the proposition consider the measure

$$\eta \stackrel{\text{def}}{=} \frac{1}{N} \sum_0^{N-1} f_*^i(\mu).$$

Now the f-orbit of η-almost every point y is dense in $H(q, f)$ and satisfies $\varphi_\infty(y) = 0$. By construction, any ergodic component ν of η satisfies the conclusion in the theorem. $\square$

Remark 6.14. Let us compare Theorem 6.12 with Corollary 6. In both cases there is a continuous map φ with a "positive" and a "negative region". Corollary 6 is a perturbation result while Theorem 6.12 does not involve perturbations. The corollary provides a (locally) open and dense subset of diffeomorphisms f having an ergodic measure μ_f with $\int \varphi \, d\mu_f = 0$. In the theorem the mere existence of the flip-flop configuration for f and φ gives an ergodic measure μ_f of f with $\int \varphi \, d\mu_f = 0$.

In the corollary the support of the ergodic measure is not completely determined (either $H(p, f)$ or $H(q, f)$) while in the theorem the support is $H(q, f)$ (q is the saddle in the flip-flop configuration).

6.5. **Ergodic non-hyperbolic measures with full support.** In this section we conclude the proofs of Theorems 7 and 8.

6.5.1. *Proof of Theorem 7.* Consider a C^1-open set $\mathcal{U} \subset \mathrm{Diff}^1(M)$ consisting of diffeomorphisms f having hyperbolic periodic points p_f and q_f, depending continuously on f, of different indices whose chain recurrence classes $C(p_f, f)$ and $C(q_f, f)$ coincide and have a partially hyperbolic splitting with one-dimensional center. This implies that the u-indices of p_f and q_f are $j+1$ and j for some j.

Consider $f \in \mathcal{U}$. We prove that there are a neighbourhood $\mathcal{V}_f$ of f and an open and dense subset $\mathcal{Z}_f$ of $\mathcal{V}_f$ where the conclusion of the theorem holds (every $g \in \mathcal{Z}_f$ has a nonhyperbolic ergodic measure μ_g whose support is $H(p_g, g) = H(q_g, g)$). The theorem follows considering the set $\mathcal{V} = \bigcup_{f \in \mathcal{U}} \mathcal{Z}_f$ that is, by construction, open and dense in $\mathcal{U}$. Thus, in what follows, we fix $f \in \mathcal{U}$ and study a local problem in a neighbourhood of f.

The partial hyperbolicity of $C(p_f, f)$ gives neighbourhoods V of $C(p_f, f)$ and $\mathcal{V}_f$ of f such that for every $g \in \mathcal{V}_g$ the maximal invariant set of g in V

$$\Upsilon_g \stackrel{\text{def}}{=} \bigcap_{i \in \mathbb{Z}} g^i(V) \tag{6.1}$$

has a partially hyperbolic splitting with one-dimensional center. Since chain recurrence classes depend upper semi-continuously, after shrinking $\mathcal{V}_f$, if necessary, we can assume that $C(p_g, g) \subset V$ for every $g \in \mathcal{V}_f$. Thus $H(p_g, g) = H(q_g, g) \subset C(p_g, g) \subset \Upsilon_g$ and these sets are partially hyperbolic with one-dimensional center. Hence [BDPR, Theorem E] (see Remark 1.3) gives an open and dense subset $\mathcal{W}_f$ of $\mathcal{V}_f$ such that for every $f \in \mathcal{W}_f$ the homoclinic classes of p_f and q_f are equal (here the hypothesis on the one-dimensional center is essential). To prove the theorem it is enough to get an open an dense subset $\mathcal{Z}_f$ of $\mathcal{W}_f$ (thus of $\mathcal{V}_f$) consisting of diffeomorphisms g having a nonhyperbolic ergodic measure μ_g whose support is $H(p_g, g) = H(q_g, g)$.

By [BD2] there is an open and dense subset $\mathcal{C}_f$ of $\mathcal{W}_f$ such that every $g \in \mathcal{C}_f$ has transitive hyperbolic sets Λ^p_g and Λ^q_g, with $p_g \in \Lambda^p_g$ and $q_g \in \Lambda^q_g$, having a robust cycle (i.e., there is a neighbourhood of g consisting of diffeomorphisms h such that the invariant sets of $\Lambda^p_h \ni p_h$ and $\Lambda^q_h \ni q_h$ meet cyclically). Now [BBD2, Proposition 5.2] (Robust cycles yield spawners) and [BBD2, Proposition 5.3] (Spawners yield split flip-flop configurations) gives an open and dense subset $\mathcal{Z}_f$ of $\mathcal{C}_f$ such that every $g \in \mathcal{Z}_f$ has a dynamical blender $\Gamma_g \subset C(p_g, g)$ that is in a flip-flop configuration with q_g. Moreover, the blender Γ_g and the saddle p_g are *homoclinically related* (their invariant manifolds intersect cyclically and transversely). As in the case of a homoclinic relation between periodic points this implies that $\Gamma_g \subset C(p_g, g)$.

For $g \in \mathcal{Z}_f$ consider a continuous function J^c_g defined on the partially hyperbolic set Υ_g as the logarithm of the center derivative of g, recall (1.5). Note that $\Gamma_g \cup \mathcal{O}(q_g) \subset \Upsilon_g$. Up considering an adapted metric, we can assume the map J^c_g is positive on the blender Γ_g and negative on the orbit of q_g. We extend the map J^c_g to a continuous map defined on the whole manifold (with a slight abuse of notation, we denote this new map also by J^c_g). For diffeomorphisms $g \in \mathcal{Z}_g$ Theorem 6.12 gives an ergodic measure μ_g whose support is $H(q_g, g) = H(p_g, g)$ and such that $\int \mathrm{J}^c_g \, d\mu_g = 0$. As μ_g is supported on $H(q_g, g) \subset \Upsilon_g$, the function J^c_g coincides with the logarithm of the center derivative of g in the support of μ_g. Thus $\int \mathrm{J}^c_g \, d\mu_g = 0$ is the center Lyapunov exponent of μ_g. This concludes the proof of the theorem. $\square$

6.5.2. *Proof of Theorem 8.* In the previous sections we dealt with averages of continuous functions. For the analysis of Lyapunov exponents let us recall that in [BBD2] the partial hyperbolicity of the set guarantees the continuity of central (one-dimensional) derivatives in a locally maximal invariant set (these maps are continuously extended to a neighbourhood of the set). Here we we argue as in previous sections keeping in mind the following three facts: (1) The existence of a flip-flop configuration is a hypothesis. (2) The filtrating neighbourhood implies that it contains the homoclinic classes. (3) The existence of invariant cone fields in the filtrating neighbourhood gives the partial hyperbolicity with one-dimensional center of the maximal invariant set in U and thus of the homoclinic classes.

7. Applications to robust transitive diffeomorphisms

In this section we prove Theorem 9 and Corollary 10. Recall that $\mathcal{RT}(M)$ is the (open) subset of $\mathrm{Diff}^1(M)$ of diffeomorphisms that are robustly transitive and have a pair of hyperbolic periodic points of different indices and a partially hyperbolic splitting with one-dimensional center. We prove the following proposition:

Proposition 7.1. *There is a C^1-open and dense subset $\mathcal{I}(M)$ of $\mathcal{RT}(M)$ such that for every $f \in \mathcal{I}(M)$ there are hyperbolic periodic points p_f and q_f of different indices such that*

$$H(p_f, f) = H(q_f, f) = M.$$

In view of this proposition, Theorem 9 is a direct consequence of Theorem 7 (in $\mathcal{RT}(M)$ the unique chain recurrence class is the whole manifold M) and Corollary 10 is a direct consequence of Corollary 6.

Proof of Proposition 7.1. The diffeomorphisms $f \in \mathcal{RT}(M)$ have a partially hyperbolic splitting with one-dimensional center $TM = E^{\mathrm{uu}} \oplus E^{\mathrm{c}} \oplus E^{\mathrm{ss}}$, where E^{uu} is uniformly expanding, E^{ss} is uniformly contracting, and $\dim(E^{\mathrm{c}}) = 1$. This implies that there is $j \in \{1, \dots, \dim(M) - 2\}$ such that every hyperbolic periodic point p of f has s-index either j or $j+1$, where $j = \dim(E^{\mathrm{ss}})$.

We define $\mathcal{RT}_j(M)$ as the open subset of $\mathcal{RT}(M)$ consisting of diffeomorphism whose saddles have s-indices either j or $j+1$. The next lemma is a consequence of the ergodic closing lemma in [M2], for an explicit formulation of this result see [DPU, Theorem in page 4].

Lemma 7.2. *The set $\bigcup_{j=1}^{\dim(M)-2} \mathcal{RT}_j(M)$ is open and dense in $\mathcal{RT}(M)$.*

In view of this lemma the proposition is a consequence of the following result:

Lemma 7.3. *Let $j \in \{1, \dots, \dim(M) - 2\}$. There is an open an dense subset $\mathcal{I}_j(M)$ of $\mathcal{RT}_j(M)$ such that every $f \in \mathcal{I}_j(M)$ has hyperbolic periodic points p_f and q_f of different indices such that*

$$H(p_f, f) = H(q_f, f) = M.$$

Proof. For the diffeomorphisms $f \in \mathcal{RT}_j(M)$ there are defined the strong stable foliation $\mathcal{F}^{\mathrm{ss}}_f$ of dimension j and the strong unstable foliation $\mathcal{F}^{\mathrm{uu}}_f$ of dimension $\dim(M) - j - 1$. Recall that $\mathcal{F}^{\mathrm{ii}}_f$, $\mathrm{i} = \mathrm{s}, \mathrm{u}$, is the only Df-invariant foliation of dimension $\dim(E^{\mathrm{ii}})$ tangent to E^{ii}.

The foliation $\mathcal{F}^{\mathrm{ii}}_f$ is *minimal* if every leaf $F^{\mathrm{ii}}_f(x)$ of $\mathcal{F}^{\mathrm{ii}}_f$ is dense in M. The foliation $\mathcal{F}^{\mathrm{ii}}_f$ is *C^1-robustly minimal* if there is a C^1-neighbourhood $\mathcal{V}_f$ of f such that for every $g \in \mathcal{V}_f$ the foliation $\mathcal{F}^{\mathrm{ii}}_g$ is minimal. We denote by $\mathcal{M}^{\mathrm{i}}_j(M)$, $i = \mathrm{s}, \mathrm{u}$, the open subset of $\mathcal{RT}_j(M)$ of diffeomorphisms such that $\mathcal{F}^{\mathrm{ii}}_f$ is robustly minimal. Let

$$\mathcal{M}_j(M) \stackrel{\text{def}}{=} \mathcal{M}^{\mathrm{s}}_j(M) \cup \mathcal{M}^{\mathrm{u}}_j(M).$$

Lemma 7.4 ([BDU, RH^{2}U])**.** *The set $\mathcal{M}_j(M)$ is open and dense in $\mathcal{RT}_j(M)$.*

We need the following property.

Claim 7.5.

- Let $f \in \mathcal{M}^{\mathrm{u}}_j(M)$. Then $H(q, f) = M$ for every saddle q of s-index $j+1$.
- Let $f \in \mathcal{M}^{\mathrm{s}}_j(M)$. Then $H(q, f) = M$ for every saddle q of s-index j.

Proof. We prove the first item, the second one is analogous and thus omitted. Fix any hyperbolic periodic point q of s-index $j+1$. Then the unstable manifold of q is a leaf of $\mathcal{F}^{\mathrm{uu}}_f$, hence it is dense in M.

The minimality of $\mathcal{F}^{\mathrm{uu}}_f$ and the fact that $W^{\mathrm{s}}(q, f)$ contains a disc of dimension $j+1$ transverse to $\mathcal{F}^{\mathrm{uu}}_f$ imply that there is $K > 0$ such that $W^{\mathrm{s}}(q, f)$ intersects transversely every strong unstable disc of radius larger than K.

Take now any point $x \in M$ and any $\epsilon > 0$. We see that the ball $B_\epsilon(x)$ intersects $H(q,f)$. Since this holds for any $x \in M$ and $\epsilon > 0$ and $H(q,f)$ is closed this implies $H(q,f) = M$.

The density of $W^{\mathrm{u}}(q,f)$ implies that there is a disc $\Delta \subset W^{\mathrm{u}}(q,f)$ of dimension $\dim(E^{\mathrm{uu}})$ contained in $B_\epsilon(x)$. Since Df expands the vectors tangent to E^{uu} there is $n > 0$ such that $f^n(\Delta)$ has radius at least K. Thus $f^n(\Delta)$ meets transversely $W^{\mathrm{s}}(q,f)$. Thus Δ contains a point of the homoclinic class of q. This implies the claim. □

By [BDPR, Theorem E] (see also Remark 1.3) in this partially hyperbolic setting with one-dimensional center, there is an open and dense subset $\mathcal{P}_j(M)$ of $\mathcal{RT}_j(M)$ such that for every pair of saddles p_f and q_f of f it holds $H(p_f,f) = H(q_f,f)$. Note that this claim is only relevant when the saddles have different indices.

In view of Claim 7.5 to prove Lemma 7.3 it is enough to take

$$\mathcal{I}_j(M) = \mathcal{P}_j(M) \cap \mathcal{M}_j(M)$$

that is open and dense in $\mathcal{RT}_j(M)$ (recall Lemma 7.4). □

The proof of Proposition 7.1 is now complete. □

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