Notes on diagonals of the product and symmetric variety of a surface

Luca Scala

Abstract

Let X be a smooth quasi-projective algebraic surface and let Δ_n the big diagonal in the product variety X^n. We study cohomological properties of the ideal sheaves $\mathcal{I}^k_{\Delta_n}$ and their invariants $(\mathcal{I}^k_{\Delta_n})^{\mathfrak{S}_n}$ by the symmetric group, seen as ideal sheaves over the symmetric variety S^nX. In particular we obtain resolutions of the sheaves of invariants $(\mathcal{I}_{\Delta_n})^{\mathfrak{S}_n}$ for $n = 3, 4$ in terms of invariants of sheaves over X^n whose cohomology is easy to calculate. Moreover, we relate, via the Bridgeland-King-Reid equivalence, powers of determinant line bundles over the Hilbert scheme to powers of ideals of the big diagonal Δ_n. We deduce applications to the cohomology of double powers of determinant line bundles over the Hilbert scheme with 3 and 4 points and we give universal formulas for their Euler-Poincaré characteristic. Finally, we obtain upper bounds for the regularity of the sheaves $\mathcal{I}^k_{\Delta_n}$ over X^n with respect to very ample line bundles of the form $L \boxtimes \cdots \boxtimes L$ and of their sheaves of invariants $(\mathcal{I}^k_{\Delta_n})^{\mathfrak{S}_n}$ on the symmetric variety S^nX with respect to very ample line bundles of the form $\mathcal{D}_L$.

Introduction

Let X be a smooth quasi-projective algebraic surface and consider the product variety X^n, for $n \geq 2$. The big diagonal Δ_n is the closed subscheme of X^n defined as the scheme-theoretic union of all pairwise diagonals Δ_I, where I is a cardinality 2 subset of $\{1, \dots, n\}$. The aim of this article is the study of of the ideal sheaf $\mathcal{I}_{\Delta_n}$ of the big diagonal Δ_n of the product variety X^n and its invariants $(\mathcal{I}_{\Delta_n})^{\mathfrak{S}_n}$, seen as an ideal sheaf over the symmetric variety S^nX.

The main reason for studying diagonal ideals is that their geometry is tightly intertwined with the geometry of the Hilbert scheme of points $X^{[n]}$ and that of the isospectral Hilbert scheme B^n. As an example of this close interplay, we mention that in [Sca15b] we related the singularities of the isospectral Hilbert scheme in terms of the singularities of the pair $(X^n, \mathcal{I}_{\Delta_n})$ and it is by studying the latter that we could prove that the singularities of B^n are canonical if $n \leq 5$, log-canonical if $n \leq 7$ and not log-canonical if $n \geq 9$.

In this work we are concerned with cohomological properties of the ideal sheaf $\mathcal{I}_{\Delta_n}$ and its invariants $(\mathcal{I}_{\Delta_n})^{\mathfrak{S}_n}$. A first motivation comes from the study of symmetric powers $S^kL^{[n]}$ of tautological bundles over the Hilbert scheme of points. Let $\mu : X^{[n]} \longrightarrow S^nX$ be the Hilbert-Chow morphism. In [Sca15a] we built a natural filtration $\mathcal{W}^\lambda \mu_* S^k L^{[n]}$ of the push-forward $\mu_* S^k L^{[n]}$, indexed by partitions λ of k of length at most n, whose graded sheaves, at least for k low — but we believe it is a general fact —, are given, up to tensorization by some line bundles, by invariants of diagonal ideals by certain subgroups of $\mathfrak{S}_n$: we indicate them with $\mathcal{L}^\lambda(-2\lambda\Delta)$. The sheaves $\mathcal{L}^\lambda(-2\lambda\Delta)$ are in general more complicated than the invariants $(\mathcal{I}_{\Delta_n})^{\mathfrak{S}_n}$ of the big diagonal; however, for $\lambda = 1^k$ (in exponential notation), the sheaf $\mathcal{L}^\lambda(-2\lambda\Delta)$ is directly related to the sheaf of invariants $(\mathcal{I}_{\Delta_n})^{\mathfrak{S}_n}$, as we shall see. Therefore, understanding the invariants $(\mathcal{I}_{\Delta_n})^{\mathfrak{S}_n}$ is instrumental for the investigation of symmetric powers of tautological bundles.

There is, moreover, another and more direct source of interest for the cohomological study of diagonal ideals. We recall that the Bridgeland-King-Reid transform

$$\mathbf{\Phi} : \mathbf{R}p_* \circ q^* : \mathbf{D}^b(X^{[n]}) \longrightarrow D^b_{\mathfrak{S}_n}(X^n) ,$$

where $p : B^n \longrightarrow X^n$ is the blow-up along the diagonal Δ_n and where $q : B^n \longrightarrow X^{[n]}$ is the (flat) quotient map by the symmetric group $\mathfrak{S}_n$, is an equivalence of derived categories between the derived category of coherent sheaves over the Hilbert scheme $X^{[n]}$ and the $\mathfrak{S}_n$-equivariant derived category of the product variety X^n. Since the ideal $\mathcal{I}_{\Delta_n}$, or its powers $\mathcal{I}^k_{\Delta_n}$, are $\mathfrak{S}_n$-equivariant sheaves over X^n, they need to correspond, under the BKR-equivalence, to some remarkable object over the Hilbert scheme of points: indeed, it turns out that they are related, up to tensorization by the alternating representation ε_n of the symmetric group, to powers of determinant line bundles on $X^{[n]}$. Considering slightly more general $\mathfrak{S}_n$-equivariant objects on X^n we proved the following

Theorem 1.8 *Let F be a vector bundle of rank r and A be a line bundle over the smooth quasi-projective surface X. Consider, over the Hilbert scheme of points $X^{[n]}$, the rank nr tautological bundle $F^{[n]}$ associated to F and the natural line bundle $\mathcal{D}_A$ associated to A. Then, in $\mathbf{D}^b_{\mathfrak{S}_n}(X^n)$ we have:*

$$\mathbf{\Phi}((\det F^{[n]})^{\otimes k} \otimes \mathcal{D}_A) \simeq^{qis} \mathcal{I}^{rk}_{\Delta_n} \otimes \big(((\det F)^{\otimes k} \otimes A) \boxtimes \cdots \boxtimes ((\det F)^{\otimes k} \otimes A)\big) \otimes \varepsilon_n^{rk} \,. \qquad (*)$$

Taking $\mathfrak{S}_n$-invariants, the previous theorem yields

$$\mathbf{R}\mu_*((\det F^{[n]})^{\otimes k} \otimes \mathcal{D}_A) \simeq \pi_*(\mathcal{I}^{rk}_{\Delta_n} \otimes \varepsilon_n^{rk})^{\mathfrak{S}_n} \otimes \mathcal{D}^k_{\det F} \otimes \mathcal{D}_A \,. \qquad (**)$$

Therefore, by means of the BKR-equivalence $\mathbf{\Phi}$, or of the derived push forward $\mathbf{R}\mu_*$ of the Hilbert-Chow morphism, one might use facts about diagonal ideals and their invariants to deduce properties of determinants line bundles over Hilbert schemes; on the other hand, one can use known results about determinants on Hilbert schemes of points to enlighten properties of the ideal $\mathcal{I}_{\Delta_n}$, its powers and their invariants. This is precisely what happens, as we explain below.

In order to obtain resolutions of invariants $(\mathcal{I}_{\Delta_n})^{\mathfrak{S}_n}$ in terms of simpler sheaves, at least from the point of view of cohomology computations, we consider, for cardinality 2 subsets $I \subseteq \{1, \dots, n\}$, complexes

$$\mathcal{K}^\bullet_I : \mathcal{O}_{X^n} \longrightarrow \mathcal{O}_{\Delta_I} \longrightarrow 0$$

which we take as right resolutions of the ideals $\mathcal{I}_{\Delta_I}$. Being the ideal $\mathcal{I}_{\Delta_n}$ the intersection of the ideals $\mathcal{I}_{\Delta_I}$, the former is isomorphic to the zero-cohomology sheaf $\mathcal{H}^0(\otimes_I \mathcal{K}^\bullet_I)$, where I runs among cardinality 2 subsets of $\{1, \dots n\}$. However, the complex $\otimes_I \mathcal{K}^\bullet_I$ is far from being exact, because the partial diagonals are not transverse: we are then led to consider the derived tensor product $\otimes^L_I \mathcal{K}^\bullet_I$ of the complexes $\mathcal{K}^\bullet_I$ and its associated spectral sequence

$$E_1^{p,q} := \bigoplus_{i_1+\cdots+i_m=p} \mathrm{Tor}_{-q}(\mathcal{K}^{i_1}_{I_1}, \dots, \mathcal{K}^{i_m}_{I_m}) \qquad (\star)$$

abutting to the sheaf cohomology $\mathcal{H}^{p+q}(\otimes^L_I \mathcal{K}^\bullet_I)$. Here $m = \binom{n}{2}$ and $\{I_1, \dots, I_m\}$ are all cardinality 2 subsets of $\{1, \dots, n\}$. It is now clear that the ideal $\mathcal{I}_{\Delta_n}$ of the big diagonal is given by the term $E_2^{0,0}$ of the spectral sequence above. In order to deal with the latter we now face two difficulties. The first is the understanding of the multitors appearing in $(\star)$, which are of the form $\mathrm{Tor}_{-q}(\mathcal{O}_{\Delta_{I_1}}, \dots, \mathcal{O}_{\Delta_{I_l}})$, for some cardinality 2 multi-indexes $I_1, \dots, I_l$, $l \leq m$. The second is the handling of combinatorial possibilities given by the multi-indexes $I_1, \dots, I_l$. As for the first, we establish in section 2.1 the following general formula for multitors of structural sheaves of smooth subvarieties $Y_1, \dots, Y_l$ of a smooth variety M intersecting in a smooth variety $Z = Y_1 \cap \cdots \cap Y_l$

$$\mathrm{Tor}_j(\mathcal{O}_{Y_1}, \dots, \mathcal{O}_{Y_l}) \simeq \Lambda^j(\oplus_{i=1}^l N_{Y_i}\big|_Z / N_Z)^*$$

in terms of normal bundles N_{Y_i} and N_Z of Y_i and Z in M, respectively. For the second, we think of the multi-indexes I_i as edges of a subgraph Γ of the complete graph K_n with n vertices, such that no vertex of Γ is isolated. Classifying all possible multitors appearing in $(\star)$ — up to isomorphisms — is then reduced to classifying all possible graphs Γ of this kind. The usefulness of the graph-theoretic approach extends further since several interesting properties of the multitor $\mathrm{Tor}_q(\mathcal{O}_{\Delta_{I_1}}, \dots, \mathcal{O}_{\Delta_{I_l}})$ can be understood in graph-theoretic terms. A fundamental fact is that if $I_1, \dots, I_l$ identify a graph Γ with c independent cycles, then the associated multitor $\mathrm{Tor}_q(\Delta, \Gamma) := \mathrm{Tor}_q(\mathcal{O}_{\Delta_{I_1}}, \dots, \mathcal{O}_{\Delta_{I_l}})$ is isomorphic to the exterior power $\Lambda^q(Q^*_\Gamma)$, where Q_Γ is a rank $2c$ vector bundle over the intersection $\Delta_\Gamma = \Delta_{I_1} \cap \cdots \cap \Delta_{I_l}$. Resorting to the associated graphs is very helpful also when considering the $\mathfrak{S}_n$-action on the naturally equivariant spectral sequence $E_1^{p,q}$. The $\mathfrak{S}_n$-action on $E_1^{p,q}$ induces a $\mathrm{Stab}_{\mathfrak{S}_n}(\Gamma)$-action on the multitor $\mathrm{Tor}_q(\Delta,\Gamma)\simeq\Lambda^q(Q_\Gamma^*)$; the group $\mathrm{Stab}_{\mathfrak{S}_n}(\Gamma)$ acts fiberwise on the vector bundle Q_Γ via the representation $\mathbb{C}^2\otimes q_\Gamma$, where q_Γ is the vector space generated over $\mathbb{C}$ by independent cycles. These facts allow us to classify, for $n=3,4$ all isomorphism classes of multitors appearing in $(\star)$ and thoroughly study the spectral sequence of invariants $(E_1^{p,q})^{\mathfrak{S}_n}$. As a consequence, we deduce the following right resolutions of $(\mathcal{I}_{\Delta_n})^{\mathfrak{S}_n}$ for $n=3,4$. Denote with w_k the map $X\times S^{n-k}X\longrightarrow S^nX$ sending (x,y) to $kx+y$. Then we have

Theorems 3.14–3.20. *We have the following resolutions of ideals of invariants $(\mathcal{I}_{\Delta_3})^{\mathfrak{S}_3}$ and $\mathcal{I}_{\Delta_4}^{\mathfrak{S}_4}$ over the symmetric variety S^3X and S^4X, respectively.*

$$0\longrightarrow\big(\mathcal{I}_{\Delta_3}\big)^{\mathfrak{S}_3}\longrightarrow\mathcal{O}_{S^3X}\xrightarrow{r}w_{2*}(\mathcal{O}_{X\times X})\xrightarrow{D}w_{3*}(\Omega^1_X)\longrightarrow0$$
$$0\longrightarrow\big(\mathcal{I}_{\Delta_4}\big)^{\mathfrak{S}_4}\longrightarrow\mathcal{O}_{S^4X}\xrightarrow{r}w_{2*}(\mathcal{O}_X\boxtimes\mathcal{O}_{S^2X})_0\xrightarrow{D}w_{3*}(\Omega^1_X\boxtimes\mathcal{I}_{\Delta_2})_0\xrightarrow{C}w_{4*}(S^3\Omega^1_X)\longrightarrow0$$

The maps r,D,C are explicit. Here we indicated with $w_{2*}(\mathcal{O}_X\boxtimes\mathcal{O}_{S^2X})_0$ and $w_{3*}(\Omega^1_X\boxtimes\mathcal{I}_{\Delta_2})_0$ particular subsheaves of $w_{2*}(\mathcal{O}_X\boxtimes\mathcal{O}_{S^2X})$ and $w_{3*}(\Omega^1_X\boxtimes\mathcal{I}_{\Delta_2})$ which will be made precise in subsection 3.3. The terms appearing in the resolutions do not present any difficulty from the point of view of cohomology computations. Therefore, by theorems 3.14, 3.20 and by formula $(**)$ we deduce, for $n=3,4$, *a spectral sequence $E_1^{p,q}$ abutting to the cohomology* $H^{p+q}(X^{[n]},(\det L^{[n]})^{\otimes 2}\otimes\mathcal{D}_A)$ and *universal formulas for the Euler-Poincaré characteristic $\chi(X^{[n]},(\det L^{[n]})^{\otimes 2}\otimes\mathcal{D}_A)$ of twisted double powers of determinant line bundles over Hilbert schemes of points* . These facts have a direct application to the sheaves $\mathcal{L}^\lambda(-2\lambda\Delta)$, when $\lambda=(r,\dots,r)$, $|\lambda|=rl$, since the sheaf $\mathcal{L}^\lambda(-2\lambda\Delta)$ is isomorphic to

$$\mathcal{L}^\lambda(-2\lambda\Delta)\simeq v_{l*}\big((\mathcal{I}_{\Delta_l}^{2r})^{\mathfrak{S}_l}\otimes\mathcal{D}_L^{\otimes r}\boxtimes\mathcal{O}_{S^{n-l}X}\big)\ ,$$

where v_l is the finite map $v_l:S^lX\times S^{n-l}X\longrightarrow S^nX$ sending $(x,y)\longmapsto x+y$. We also obtain a right resolution of the sheaf $\mathcal{L}^{2,1,1}(-2\Delta)$ over S^nX, which is more difficult to treat, since it is not directly related to determinant line bundles over Hilbert schemes.

Finally, as anticipated, we can use properties of determinant line bundles over Hilbert scheme to deduce important facts about diagonal ideals and their invariants. We use formulas $(*)$ and $(**)$ and the positivity properties of $\det L^{[n]}$ when L is n-very ample to study vanishing theorems and regularity for the ideal sheaves $\mathcal{I}_{\Delta_n}^k$ over X^n and their invariants $(\mathcal{I}_{\Delta_n}^k)^{\mathfrak{S}_n}$ over S^nX. In particular, we proved the following result.

Theorem 4.6. *Let X be a smooth projective surface and L be a line bundle over X. Suppose that, for a certain $m\in\mathbb{N}^*$, $L^m\otimes K_X^{-1}=\otimes_{i=1}^{2[(k+1)/2]}B_i$, with B_1 n-very ample and the other B_i, $i\neq1$, $(n-1)$-very ample. Then we have the vanishing*

$$H^i(S^nX,(\mathcal{I}_{\Delta_n}^k)^{\mathfrak{S}_n}\otimes\mathcal{D}_L^m)=0\qquad\qquad \textit{for } i>0.$$

If, moreover, L is very ample over X, then the ideal $(\mathcal{I}_{\Delta_n}^k)^{\mathfrak{S}_n}$ is $(m+2n)$-regular with respect to $\mathcal{D}_L$. *In particular the regularity* $\mathrm{reg}((\mathcal{I}_{\Delta_n}^k)^{\mathfrak{S}_n})$ *of the ideal $(\mathcal{I}_{\Delta_n}^k)^{\mathfrak{S}_n}$ with respect to the line bundle $\mathcal{D}_L$ is bounded above by*

$$\mathrm{reg}((\mathcal{I}_{\Delta_n}^k)^{\mathfrak{S}_n})\leq m_0+2n$$

where m_0 is the minimum of all m satisfying the condition above.

We proved a similar statement (theorem 4.8) about *vanishing and regularity for $\mathcal{I}_{\Delta_n}^k$ with respect to the line bundle $L\boxtimes\cdots\boxtimes L$ on X^n* for $2\leq n\leq7$. These results can be written in a nicer way when the surface X has Picard number one. Indeed, suppose that $\mathrm{Pic}(X)=\mathbb{Z}B$ and let r be the minimum positive power of B such that B^r is very ample and write $K_X=B^w$, for some integer w. Then the regularities $\mathrm{reg}(\mathcal{I}_{\Delta_n}^k)$ and $\mathrm{reg}((\mathcal{I}_{\Delta_n}^k)^{\mathfrak{S}_n})$, with respect to the line bundle $B^r\boxtimes\cdots\boxtimes B^r$ on X^n and $\mathcal{D}_{B^r}$ on S^nX, respectively, are bounded above by

$$\mathrm{reg}(\mathcal{I}_{\Delta_n}^k)\leq(k+3)n-k+\lceil w/r\rceil\qquad\qquad \text{for } 2\leq n\leq7$$
$$\mathrm{reg}((\mathcal{I}_{\Delta_n}^k)^{\mathfrak{S}_n})\leq 2n([(k+1)/2]+1)-2[(k+1)/2]+1+\lceil w/r\rceil\qquad \text{for all } n\in\mathbb{N},\ n\geq2.$$

Conventions. i). We work over the field of complex numbers. By point we will always mean closed point. ii). Let A a $\mathbb{C}$-algebra and M an A-module. For $n \in \mathbb{N} \setminus \{0\}$, consider the symmetric power $S^n M$ of the module M. We consider $S^n M$ as the space of $\mathfrak{S}_n$-invariants of $M^{\otimes n}$ for the action of $\mathfrak{S}_n$ permutating the factors in the tensor product. Throughout this article, we will use the following convention for the symmetric product $u_1. \cdots .u_n$ of elements $u_i \in M$:

$$u_1. \cdots .u_n := \sum_{\sigma \in \mathfrak{S}_n} u_{\sigma(1)} \otimes \cdots \otimes u_{\sigma(n)} \ ,$$

where the right hand side is seen in $M^{\otimes n}$. We use an analogous convention for the exterior product: $u_1 \wedge \cdots \wedge u_n := \sum_{\sigma \in \mathfrak{S}_n} (-1)^\sigma u_{\sigma(1)} \otimes \cdots \otimes u_{\sigma(n)}$, where $(-1)^\sigma$ is the signature of the of permutation σ and where we see $\Lambda^n M$ as the space of anti-invariants for the action of $\mathfrak{S}_n$ over $M^{\otimes n}$.

1 The Bridgeland-King-Reid transform of diagonal ideals

Consider a smooth quasi-projective algebraic surface X. Denote with $X^{[n]}$ the Hilbert scheme of n points over X and with B^n the isospectral Hilbert scheme [Hai99, Hai01], that is, the blow-up of X^n along the big diagonal Δ_n. We indicate with $p : B^n \longrightarrow X^n$ the blow-up map, with $q : B^n \longrightarrow X^{[n]}$ the quotient projection by the symmetric group $\mathfrak{S}_n$ and with $\mu : X^{[n]} \longrightarrow S^n X$ the Hilbert-Chow morphism. The Bridgeland-King-Reid equivalence [BKR01, Hai01, Hai02], in the case of the action of the symmetric group $\mathfrak{S}_n$ over the product variety X^n, provides an equivalence of derived categories

$$\mathbf{\Phi} := \mathbf{R}p_* \circ q^* : \mathbf{D}^b(X^{[n]}) \longrightarrow \mathbf{D}^b_{\mathfrak{S}_n}(X^n) \tag{1.1}$$

from the derived category of coherent sheaves over the Hilbert scheme of n points over X and the $\mathfrak{S}_n$-equivariant derived category of the product variety X^n. Any power $\mathcal{I}^m_{\Delta_n}$ of the ideal $\mathcal{I}_{\Delta_n}$ is naturally a $\mathfrak{S}_n$-equivariant coherent sheaf over X^n: it is then natural to ask what is the corresponding complex of coherent sheaves over the Hilbert schemes of points for the equivalence (1.1). In general we can twist the ideal $\mathcal{I}^m_{\Delta_n}$ with the line bundle $L \boxtimes \cdots \boxtimes L$ (n-factors) and ask the same question for $\mathcal{I}^m_{\Delta_n} \otimes L \boxtimes \cdots \boxtimes L$. To give a general statement, we need to introduce the line bundle $\mathcal{D}_L$.

Remark 1.1. If L is a line bundle on X, the line bundle $L \boxtimes \cdots \boxtimes L$ (n-factors) on X^n *descends* to a line bundle $\mathcal{D}_L$ on $S^n X$, in the sense that $\pi^* \mathcal{D}_L = L \boxtimes \cdots \boxtimes L$ [DN89, Thm 2.3]. As a consequence, the line bundle $\mathcal{D}_L$ coincides with the sheaf of $\mathfrak{S}_n$-invariants, on $S^n X$, of the line bundle $L \boxtimes \cdots \boxtimes L$. Pulling-back the line bundle $\mathcal{D}_L$ via the Hilbert-Chow morphism $\mu : X^{[n]} \longrightarrow S^n X$ we get a line bundle $\mu^* \mathcal{D}_L$ on the Hilbert scheme, called the *natural line bundle* on $X^{[n]}$ associated to the line bundle L on X. For brevity's sake, we will denote it again with $\mathcal{D}_L$.

We need as well a technical lemma about the local cohomology of ideal sheaves $\mathcal{I}^s_{\Delta_n}$, $s \in \mathbb{N}^*$ with respect to the closed subscheme W given by the intersection of double diagonals in X^n. More precisely, we define W as the scheme-theoretic intersection of pairwise diagonals

$$W := \bigcap_{\substack{|I|=|J|=2 \\ I,J \subseteq \{1,\dots,n\}, I \neq J}} \Delta_I \cap \Delta_J \ .$$

It is a closed subscheme of X^n of codimension 4.

Notation 1.2. We denote with X^n_{**} the open subset $X^n \setminus W$ and with B^n_{**}, $S^n_{**}X$, $X^{[n]}_{**}$ the open subsets $B^n_{**} := p^{-1}(X^n_{**})$, $S^n_{**}X := \pi(X^n_{**})$, $X^{[n]}_{**} := \mu^{-1}(S^n_{**}X)$. We denote moreover, with $j : X^n_{**} \longrightarrow X^n$ the open immersion of X^n_{**} into X^n. We also denote with j the open immersion of each of the open sets B^n_{**}, $S^n_{**}X$, $X^{[n]}_{**}$ into their closure B^n, $S^n X$, $X^{[n]}$, respectively; it will be clear from the context over which variety we are working.

Remark 1.3. The following is an important result about powers of the ideal sheaf of the diagonal Δ_n in X^n, and it has been proven by Haiman in [Hai01, Corollary 3.8.3]. Over X^n, for all $s \in \mathbb{N}$ one has:

$$\bigcap_{\substack{I \subseteq \{1,\dots,n\} \\ |I|=2}} \mathcal{I}^s_{\Delta_I} = \left(\bigcap_{\substack{I \subseteq \{1,\dots,n\} \\ |I|=2}} \mathcal{I}_{\Delta_I} \right)^s = \mathcal{I}^s_{\Delta_n} \tag{1.2}$$

The local cohomology property of the ideal sheaves we want to prove is the following.

Lemma 1.4. *Let* $\ell : \{(i,j) \mid i,j \in \mathbb{N}\ ,\ 1 \leq i < j \leq n\} \longrightarrow \mathbb{N}$ *be a function. Then*

$$j_*j^* \bigcap_{1\leq i<j\leq n} \mathcal{I}_{\Delta_{ij}}^{\ell(i,j)} = \bigcap_{1\leq i<j\leq n} \mathcal{I}_{\Delta_{ij}}^{\ell(i,j)} \ .$$

Proof. We begin by proving recursively that, for any $s \in \mathbb{N}$ and for any fixed natural numbers i, j, $1 \leq i < j \leq n$, we have

$$\mathcal{H}^l_W(\mathcal{I}^s_{\Delta_{i,j}}) = 0 \qquad \text{for all } 0 \leq l \leq 2\ . \tag{1.3}$$

Indeed it is true for $s = 0$, since $\mathcal{I}^0_{\Delta_{i,j}} \simeq \mathcal{O}_{X^n}$, X^n is normal, and W is of codimension 4 in X^n. Consider now $s \in \mathbb{N}$. The local cohomology long exact sequence applied to the short exact sequence

$$0 \longrightarrow \mathcal{I}^{s+1}_{\Delta_{ij}} \longrightarrow \mathcal{I}^s_{\Delta_{ij}} \longrightarrow \mathcal{I}^s_{\Delta_{ij}}/\mathcal{I}^{s+1}_{\Delta_{ij}} \longrightarrow 0$$

yields:

$$\begin{aligned}0 \longrightarrow \mathcal{H}^0_W(\mathcal{I}^{s+1}_{\Delta_{ij}}) \longrightarrow \mathcal{H}^0_W(\mathcal{I}^s_{\Delta_{ij}}) \longrightarrow \mathcal{H}^0_W(\mathcal{I}^s_{\Delta_{ij}}/\mathcal{I}^{s+1}_{\Delta_{ij}}) \longrightarrow&\\ \longrightarrow \mathcal{H}^1_W(\mathcal{I}^{s+1}_{\Delta_{ij}}) \longrightarrow \mathcal{H}^1_W(\mathcal{I}^s_{\Delta_{ij}}) \longrightarrow \mathcal{H}^1_W(\mathcal{I}^s_{\Delta_{ij}}/\mathcal{I}^{s+1}_{\Delta_{ij}}) \longrightarrow&\\ \longrightarrow \mathcal{H}^2_W(\mathcal{I}^{s+1}_{\Delta_{ij}}) \longrightarrow \mathcal{H}^2_W(\mathcal{I}^s_{\Delta_{ij}})\ .&\end{aligned} \tag{1.4}$$

Note that $\mathcal{H}^l_W(\mathcal{I}^s_{\Delta_{ij}}/\mathcal{I}^{s+1}_{\Delta_{ij}}) = 0$ for $l = 0, 1$ because the sheaf $\mathcal{I}^s_{\Delta_{ij}}/\mathcal{I}^{s+1}_{\Delta_{ij}}$ is a vector bundle over the smooth subvariety Δ_{ij}, in which W is of codimension 2 and because of [Sca09, Lemma 3.1.9]. Now, if $\mathcal{H}^l_W(\mathcal{I}^s_{\Delta_{ij}}) = 0$ for $l = 0, 1, 2$, the local cohomology long exact sequence above yields the vanishing for $\mathcal{I}^{s+1}_{\Delta_{ij}}$ and $l = 0, 1, 2$. Induction on s then yields (1.3) for any $s \in \mathbb{N}$.

Since $\mathcal{H}^l_W(\mathcal{I}^s_{\Delta_{ij}}) = 0$ for $l = 0, 1, 2$ and for any $s \in \mathbb{N}$, an analogous argument via the long exact sequence in local cohomology applied to the short exact sequence

$$0 \longrightarrow \mathcal{I}^s_{\Delta_{ij}} \longrightarrow \mathcal{O}_{X^n} \longrightarrow \mathcal{O}_{X^n}/\mathcal{I}^s_{\Delta_{ij}} \longrightarrow 0$$

proves that $\mathcal{H}^l_W(\mathcal{O}_{X^n}/\mathcal{I}^s_{\Delta_{ij}}) = 0$ for $l = 0, 1$ and for any s; this last fact is equivalent to the isomorphism

$$j_*j^*\mathcal{O}_{X^n}/\mathcal{I}^s_{\Delta_{ij}} \simeq \mathcal{O}_{X^n}/\mathcal{I}^s_{\Delta_{ij}}\ .$$

The above isomorphism, together with the following commutative diagram

$$\begin{tikzcd} 0 \arrow[r] & \bigcap\limits_{1\leq i<j\leq n} \mathcal{I}_{\Delta_{ij}}^{\ell(i,j)} \arrow[r] \arrow[d] & \mathcal{O}_{X^n} \arrow[r] \arrow[d, "\simeq"'] & \bigoplus\limits_{1\leq i<j\leq n} \mathcal{O}_{X^n}/\mathcal{I}_{\Delta_{ij}}^{\ell(i,j)} \arrow[d, "\simeq"'] \\ 0 \arrow[r] & j_*j^* \bigcap\limits_{1\leq i<j\leq n} \mathcal{I}_{\Delta_{ij}}^{\ell(i,j)} \arrow[r] & j_*j^*\mathcal{O}_{X^n} \arrow[r] & \bigoplus\limits_{1\leq i<j\leq n} j_*j^*\mathcal{O}_{X^n}/\mathcal{I}_{\Delta_{ij}}^{\ell(i,j)} \end{tikzcd}$$

where the second and third vertical arrows are isomorphisms, because we proved so above, yields the statement of the lemma. □

Lemma 1.5. *Let* $k \in \mathbb{N}^*$*. We have the isomorphism of sheaves of invariants over* S^nX*:*

$$\big(\mathcal{I}^{2k-1}_{\Delta_n}\big)^{\mathfrak{S}_n} \simeq \big(\mathcal{I}^{2k}_{\Delta_n}\big)^{\mathfrak{S}_n}\ .$$

Proof. The statement follows using (1.2), taking $\mathfrak{S}_n$-invariant in the exact sequence

$$0 \longrightarrow \mathcal{I}^{2k-1}_{\Delta_n} \longrightarrow \mathcal{O}_{X^n} \longrightarrow \oplus_{i<j}\, \mathcal{O}_{X^n}/\mathcal{I}^{2k-1}_{\Delta_{ij}}$$

and noting that $\pi_*\big(\oplus_{i<j}\, \mathcal{O}_{X^n}/\mathcal{I}^{2k-1}_{\Delta_{ij}}\big)^{\mathfrak{S}_n} \simeq \pi_*\big(\mathcal{O}_{X^n}/\mathcal{I}^{2k-1}_{\Delta_{12}}\big)^{\mathrm{Stab}_{\mathfrak{S}_n}(\{1,2\})} \simeq \pi_*\big(\mathcal{O}_{X^n}/\mathcal{I}^{2k}_{\Delta_{12}}\big)^{\mathrm{Stab}_{\mathfrak{S}_n}(\{1,2\})} \simeq \pi_*\big(\oplus_{i<j}\, \mathcal{O}_{X^n}/\mathcal{I}^{2k}_{\Delta_{ij}}\big)^{\mathfrak{S}_n}$, since $\pi_*(\mathcal{I}^{2k-1}_{\Delta_{1,2}})^{\mathfrak{S}(\{1,2\})} \simeq \pi_*(\mathcal{I}^{2k}_{\Delta_{1,2}})^{\mathfrak{S}(\{1,2\})}$. □

Remark 1.6. Indicate now with E the exceptional divisor (or the boundary) of $X^{[n]}$: it is the exceptional divisor for the Hilbert-Chow morphism and the branching divisor for the map $q : B^n \longrightarrow X^{[n]}$. It is well known [Leh99, Lemma 3.7] that

$$\mathcal{O}_{X^{[n]}}(-E) \simeq (\det \mathcal{O}_X^{[n]})^{\otimes 2} .$$

As a consequence there exists a divisor e on the Hilbert scheme $X^{[n]}$ such that $E = 2e$, and such that $\mathcal{O}_{X^{[n]}}(-e) = \det \mathcal{O}_X^{[n]}$. It is also well known that $\det L^{[n]} \simeq \mathcal{D}_L \otimes \det \mathcal{O}_X^{[n]}$, which can be rewritten, with the notations just explained, as

$$\det L^{[n]} \simeq \mathcal{D}_L(-e) . \tag{1.5}$$

Denote now with E_B the exceptional divisor over the isospectral Hilbert scheme, that is, the exceptional divisor for the blow-up map $p : B^n \longrightarrow X^n$. We have $\mathcal{O}_{B^n}(-E_B) \simeq q^*\mathcal{O}_{X^{[n]}}(-e)$.

Denoting with ε_n the alternating representation of $\mathfrak{S}_n$, we can now prove

Theorem 1.7. *Let X be a smooth quasi-projective algebraic surface. Then*

$$\mathbf{\Phi}(\mathcal{O}_{X^{[n]}}(-le)) \simeq \mathbf{\Phi}((\det \mathcal{O}_X^{[n]})^{\otimes l}) \simeq \mathcal{I}^l_{\Delta_n} \otimes \varepsilon_n^l$$
$$\mathbf{R}\mu_*\mathcal{O}_{X^{[n]}}(-le) \simeq \mathbf{R}\mu_*(\det \mathcal{O}_X^{[n]})^{\otimes l}) \simeq \pi_*^{\mathfrak{S}_n}(\mathcal{I}^l_{\Delta_n} \otimes \varepsilon_n^l) .$$

The proof of theorem 1.7 is a consequence of the following more general result.

Theorem 1.8. *Let F be a vector bundle of rank r and A be a line bundle over the smooth quasi-projective surface X. Consider, over the Hilbert scheme of points $X^{[n]}$, the rank nr tautological bundle $F^{[n]}$ associated to F and the natural line bundle $\mathcal{D}_A$ associated to A. Then, in $\mathbf{D}^b_{\mathfrak{S}_n}(X^n)$ we have:*

$$\mathbf{\Phi}((\det F^{[n]})^{\otimes k} \otimes \mathcal{D}_A) \simeq^{qis} \mathcal{I}^{rk}_{\Delta_n} \otimes (((\det F)^{\otimes k} \otimes A) \boxtimes \cdots \boxtimes ((\det F)^{\otimes k} \otimes A)) \otimes \varepsilon_n^{rk} .$$

Proof. Step 1. Case A trivial. Let's prove the formula for A trivial first. It is sufficient to prove the formula for $k = 1$; for arbitrary k it follows by applying the formula for $k = 1$ to the sheaf $F' = F^{\oplus k}$. Note first that the vanishing

$$R^ip_*(q^* \det F^{[n]}) = 0 \qquad \text{for all } i > 0$$

is a consequence of the more general vanishing [Sca09, Thm. 2.3.1, Cor. 4.13], [Sca09, Prop. 21]:

$$R^ip_*(q^*S^\lambda F^{[n]}) = 0 \qquad \text{for all } i > 0$$

for any Schur functor $S^\lambda F^{[n]}$ of any tautological bundle $F^{[n]}$ associated to a vector bundle F on X. Hence we just have to prove

$$p_*q^* \det F^{[n]} \simeq \mathcal{I}^r_{\Delta_n} \otimes (\det F \boxtimes \cdots \boxtimes \det F) \otimes \varepsilon_n^r$$

as $\mathfrak{S}_n$-equivariant sheaves on X^n. Consider the open set B^n_{**} defined in notation 1.2; it is the complementary of the closed subscheme $p^{-1}(W)$, which is of *codimension* 2 *in* B^n. Recall that we indicate with j both the open immersions $j : B^n_{**} \hookrightarrow B^n$ and $j : X^n_{**} \hookrightarrow X^n$. The projection $p|_{B^n_{**}} : B^n_{**} \longrightarrow X^n_{**}$ coincides with the smooth blow up of the (now disjoint) pairwise diagonals Δ_{ij} in X^n_{**}; we denote with E_{ij} the irreducible component of the exceptional divisor E_B dominating Δ_{ij}. Over B^n_{**} we have the exact sequence [Dan01] of $\mathfrak{S}_n$-equivariant sheaves:

$$0 \longrightarrow q^*F^{[n]} \longrightarrow \oplus_i p^*F_i \longrightarrow \oplus_{i<j} p^*F_i\big|_{E_{ij}} \longrightarrow 0 .$$

Over B^n_{**} the Weil divisors E_{ij} are Cartier: hence, over B^n_{**} we get

$$q^* \det F^{[n]} \simeq \det q^*F^{[n]} \simeq (\det(\oplus_i p^*F_i)) \otimes \det(\oplus_{i<j} p^*F_i\big|_{E_{ij}})^{-1} .$$

Let's compute first the second factor; for each sheaf $p^*F_i\big|_{E_{ij}}$, we have, over B^n_{**}:

$$0 \longrightarrow p^*F_i(-E_{ij}) \longrightarrow p^*F_i \longrightarrow p^*F_i\big|_{E_{ij}} \longrightarrow 0 .$$

Hence, over B^n_{**}

$$\det(p^*F_i\big|_{E_{ij}}) \simeq (\det p^*F_i) \otimes (\det p^*F_i(-E_{ij}))^{-1} \simeq \mathcal{O}_{B^n}(rE_{ij})$$

and

$$\det(\oplus_{i<j} p^* F_i\big|_{E_{ij}}) \simeq \otimes_{i<j} \det(p^* F_i\big|_{E_{ij}}) \simeq \otimes_{i<j} \mathcal{O}_{B^n}(rE_{ij}) \simeq \mathcal{O}_{B^n}(rE_B) \ . \tag{1.6}$$

As for the first factor, we have, *just as coherent sheaves, without considering the $\mathfrak{S}_n$-action*:

$$\det(\oplus_i p^* F_i) \simeq p^*(\otimes_i \det F_i) \simeq p^*(\det F \boxtimes \cdots \boxtimes \det F) \ .$$

However, it is clear that a consecutive transposition $\tau_{i,i+1}$ acts on the sheaf on the left hand side with the sign $(-1)^r$, while it acts trivially on the right hand side: hence, to have an isomorphism *as $\mathfrak{S}_n$-equivariant sheaves* we have to correct the previous formula by the representation ε_n^r; that is, as $\mathfrak{S}_n$-sheaves:

$$\det(\oplus_i p^* F_i) \simeq p^*(\det F \boxtimes \cdots \boxtimes \det F) \otimes \varepsilon_n^r \ . \tag{1.7}$$

From (1.7) and (1.6) we get that, as $\mathfrak{S}_n$-equivariant sheaves, over B^n_{**}:

$$q^* \det F^{[n]} \simeq p^*(\det F \boxtimes \cdots \boxtimes \det F) \otimes \mathcal{O}_{B^n}(-rE_B) \otimes \varepsilon_n^r \ .$$

Since this is an isomorphism of vector bundles, since B^n is normal and since the complementary of B^n_{**} in B^n is a closed subscheme of codimension 2, the previous isomorphism extends to the whole variety B^n as an isomorphism of $\mathfrak{S}_n$-equivariant vector bundles. Therefore, by projection formula:

$$\mathbf{R}p_* q^* \det F^{[n]} \simeq (\det F \boxtimes \cdots \boxtimes \det F) \otimes p_* \mathcal{O}_{B^n}(-rE_B) \otimes \varepsilon_n^r \ .$$

To finish the proof we just have to show that $p_* \mathcal{O}_{B^n}(-rE_B) \simeq \mathcal{I}^r_{\Delta_n}$. Since $\mathcal{O}_{B^n}(-rE_B)$ is a line bundle, since B^n is normal and B^n_{**} is the complementary of a closed of codimension 2, we have $\mathcal{O}_{B^n}(-rE_B) \simeq j_* j^* \mathcal{O}_{B^n}(-rE_B)$; hence

$$\begin{aligned} p_* \mathcal{O}_{B^n}(-rE_B) &\simeq p_* j_* j^* \mathcal{O}_{B^n}(-rE_B) \simeq \ j_*(p\big|_{B^n_{**}})_* j^* \mathcal{O}_{B^n}(-rE_B) \\ &\simeq j_*(p\big|_{B^n_{**}})_* \mathcal{O}_{B^n}(-rE_B)\big|_{B^n_{**}} \simeq j_* j^* \mathcal{I}^r_{\Delta_n} \simeq \mathcal{I}^r_{\Delta_n} \end{aligned}$$

since, over X^n_{**}, p is a smooth blow-up and thanks to lemma 1.4.

Step 2. Arbitrary A. If A is non trivial we write:

$$\begin{aligned} \mathbf{\Phi}((\det F^{[n]})^{\otimes k} \otimes \mathcal{D}_A) &\simeq \mathbf{R}p_* q^*((\det F^{[n]})^{\otimes k} \otimes \mathcal{D}_A) \simeq \mathbf{R}p_* \big(q^*(\det F^{[n]})^{\otimes k} \otimes q^* \mathcal{D}_A\big) \\ &\simeq \mathbf{R}p_* \big(q^*(\det F^{[n]})^{\otimes k} \otimes p^*(A \boxtimes \cdots \boxtimes A)\big) \\ &\simeq \big(\mathbf{R}p_* q^*(\det F^{[n]})^{\otimes k}\big) \otimes (A \boxtimes \cdots \boxtimes A) \end{aligned}$$

where in the third isomorphism we used that $q^* \mathcal{D}_A \simeq p^*(A \boxtimes \cdots \boxtimes A)$, and in the third we used projection formula. Now the formula follows immediately from the previous case. ☐

Corollary 1.9. *Let F be a vector bundle of rank r and A a line bundle on the smooth quasi-projective surface X. Then*

$$\mathbf{R}\mu_*((\det F^{[n]})^{\otimes k} \otimes \mathcal{D}_A) \simeq \pi_* \Big(\mathcal{I}^k_{\Delta_n} \otimes \varepsilon_n^k\Big)^{\mathfrak{S}_n} \otimes \mathcal{D}^{\otimes k}_{\det F} \otimes \mathcal{D}_A \ .$$

2 Multitors of pairwise diagonals in X^n

Let X be a smooth algebraic variety and let $n \in \mathbb{N}$, $n \geq 2$. In this section we will study multitors of the form $\mathrm{Tor}_q^{\mathcal{O}_{X^n}}(\mathcal{O}_{\Delta_{I_1}}, \dots, \mathcal{O}_{\Delta_{I_l}})$, where Δ_{I_j} are pairwise diagonals in X^n. This study will be useful in order to prove the resolutions of $\mathfrak{S}_n$-invariants of diagonal ideals of subsections 3.2 and 3.3

2.1 A general formula for multitors

In this section we will prove a general formula for multitors $\mathrm{Tor}_q^{\mathcal{O}_M}(\mathcal{O}_{Y_1}, \dots, \mathcal{O}_{Y_l})$ of structural sheaves of smooth subvarieties Y_i of a smooth algebraic variety M.

Theorem 2.1. *Let M be a smooth algebraic variety and $Y_1, \ldots, Y_l$ be smooth subvarieties of M such that the intersection $Z := Y_1 \cap \cdots \cap Y_l$ is smooth. Then:*

$$\mathrm{Tor}_q^{\mathcal{O}_M}(\mathcal{O}_{Y_1}, \ldots, \mathcal{O}_{Y_l}) = \Lambda^q(\oplus_{i=1}^l N_{Y_i}\big|_Z / N_Z)^* \ , \tag{2.1}$$

where N_{Y_i} and N_Z denote the normal bundles of Y_i and Z in M, respectively.

Proof. Let x be a point of Z, and U an affine open neighbourhood of x. Restricting U if necessary, we can find generators $f_{j_1}, \ldots, f_{j_{c_j}}$ of $\mathcal{I}_{Y_j}(U)$, such that $c_j = \operatorname{codim} Y_j$. It is possible to find them since Y_j is complete intersection in U; we can moreover find $g_1, \ldots, g_c$ generators of $\mathcal{I}_Z(U)$, with $c = \operatorname{codim} Z \leq \sum_j c_j = d$. Denote simply with g the vector $(g_1, \ldots, g_c) \in \mathcal{O}_M(U)^{\oplus c}$, with f the vector $(f_1, \ldots, f_{c_1}, \ldots, f_{l_1}, \ldots, f_{l_{c_l}}) \in \mathcal{O}_M(U)^{\oplus d}$ and with f_j the vector $(f_{j_1}, \cdots, f_{j_{c_j}}) \in \mathcal{O}_M(U)^{c_j}$. Denote with E the vector bundle $\mathcal{O}_M^c$, with F_j the bundle $\mathcal{O}_M^{c_j}$ and with F the bundle $\mathcal{O}_M^d$. It is clear that, over U, g defines a section of E, f_j defines a section of F_j and f a section of F. We can identify the conormal bundle N_Z^* with the restriction $E^*\big|_Z$ and $N_{Y_j}^*$ with $F_j^*\big|_{Y_j}$.

Step 1. Since all the varieties Y_i and Z are smooth, by the exactness of the conormal sequence, we can identify conormal bundles $N_{Y_i}^*$ and N_Z^* with differentials in Ω_M^1 vanishing over Y_i and over Z, respectively. Since both f_{j_i} and g_h are generators of $I_Z(U)$, there is a $d \times c$-matrix $B \in M_{d\times c}(\mathcal{O}_M(U))$ and a $c \times d$-matrix $B \in M_{c\times d}(\mathcal{O}_M(U))$ such that $g = Bf$, $f = Ag$. This means that, taking differentials: $dg = (dB)f + Bdf$ and $df = (dA)g + Adg$. On points $y \in Z \cap U$ we have just: $dg = Bdf$, $df = Adg$ and $dg = BAdg$. Now, since Z, over U, is smooth and complete intersection of $Z(g_1), \ldots, Z(g_c)$, we have that $[g_i]$ are a basis of $I_Z(U)/I_Z(U)^2$, that is dg_i are a local frame for N_Z^* over $Z \cap U$ and df_{j_i} are a local frame for $N_{Y_j}^*$. Hence dg_i are linearly indipendent in $\Omega_M^1\big|_Z(y) \supseteq N_Z^*(y)$ for any $y \in Z \cap U$. Now on $\Omega_M^1\big|_Z(y)$ we have the relation $dg = B(y)A(y)dg$, meaning that the matrix $B(y)A(y)$ takes linearly independent into linearly independent, which implies that for any $y \in Z \cap U$, the matrix $B(y)$ is surjective and $A(y)$ is injective. Hence, $B(z)$ and $A(z)$ have to be surjective and injective, respectively, in a neighbourhood of x. Restricting U, we can suppose that A and B are injective and surjective, respectively, in any point of U.

Step 2. Let E, F_i, F, g, f, U built as in the previous step. The matrix A allows to define an injective morphism of vector bundles over U:

$$0 \longrightarrow E \xrightarrow{A} F_1 \oplus \cdots \oplus F_l \xrightarrow{p_Q} Q \longrightarrow 0$$

whose cokernel we call Q. It is a locally free sheaf on U of rank $d - c$. Note that A takes the section g into the section f, and hence defines an injective morphism of pairs $A : (E, g) \longrightarrow (F, f)$. Since we are on an affine open set, the sequence splits; hence we have a morphism $p_E : F \longrightarrow E$ such that $p_E \circ A = \mathrm{id}_E$; the splitting yields an isomorphism: $F \longrightarrow E \oplus Q$, given by (p_E, p_Q). Under this isomorphism the section f of F is carried onto the section $g \oplus 0$ of Q, since $(p_E, p_Q)f = p_E f \oplus p_Q f = p_E Ag \oplus p_Q Ag = g \oplus 0$. Hence we have an isomorphism of pairs $(F, f) \simeq (E \oplus Q, g \oplus 0)$.

Step 3. The previous step yields an isomorphism of Koszul complexes: $K^\bullet(F, f) \simeq K^\bullet(E \oplus Q, g \oplus 0) \simeq K^\bullet(E, g) \otimes K^\bullet(Q, 0)$. Note that

$$Q\big|_Z \simeq (\oplus_j F_j/E)\big|_Z \simeq (\oplus_j F_j\big|_Z)/E\big|_Z \simeq \oplus_j N_{Y_j}\big|_Z / N_Z \ .$$

Hence:

$$\begin{aligned}
\mathrm{Tor}_q^{\mathcal{O}_M}(\mathcal{O}_{Y_1}, \ldots, \mathcal{O}_{Y_l}) &\simeq H^{-q}(K^\bullet(F, f)) \simeq H^{-q}(K^\bullet(E \oplus Q, g \oplus 0)) \\
&\simeq H^{-q}(K^\bullet(E, g) \otimes K^\bullet(Q, 0)) \\
&\simeq \bigoplus_{-q = r+s} H^r(K^\bullet(E, g)) \otimes K^s(Q, 0) \\
&\simeq H^0(K^\bullet(E, g)) \otimes \Lambda^q Q^* \\
&\simeq \Lambda^q Q^*\big|_Z \\
&\simeq \Lambda^q(\oplus_{i=1}^l N_{Y_i}\big|_Z / N_Z)^* \ ,
\end{aligned}$$

where in the fourth isomorphism we used the fact that $K(E \oplus Q, g \oplus 0)$ is a tensor product of $K(E, g)$ with $K(Q, 0)$, which is a complex of locally free sheaves with zero differentials.

Step 4. We obtained the wanted isomorphism locally. That these local isomorphisms glue to a global one is an easy exercise and we leave it to the reader. $\square$

Notation 2.2. Let $k_1, \dots, k_l$ be positive integers. Let M be an algebraic variety and $F_1, \dots, F_l$ coherent sheaves over M. We denote with $\mathrm{Tor}_q^{k_1,\dots,k_l}(F_1, \dots, F_l)$ the multitor $\mathrm{Tor}_q^{\mathcal{O}_M}(F_1, \dots, F_1, \dots, F_l, \dots, F_l)$, where, for all i, the factor F_i is repeated k_i times.

The product of symmetric groups $\mathfrak{S}_{k_1} \times \cdots \times \mathfrak{S}_{k_l}$ obvioulsy acts on multitors of the form $\mathrm{Tor}_q^{k_1,\dots,k_l}(F_1, \dots, F_l)$, defined above. Details of this action are described in [Sca09, Appendix B]. Therefore one can study them as $\mathfrak{S}_{k_1} \times \cdots \times \mathfrak{S}_{k_l}$-representations. We have the following.

Proposition 2.3. *Let M be a smooth algebraic variety and $Y_1, \dots, Y_l$ locally complete intersection subvarieties of M such that the intersection $Z = Y_1 \cap \cdots \cap Y_l$ is locally complete intersection. Then for $k_1, \dots, k_l$ positive integers we have, as $\mathfrak{S}_{k_1} \times \cdots \times \mathfrak{S}_{k_l}$ representations:*

$$\mathrm{Tor}_q^{k_1,\dots,k_l}(\mathcal{O}_{Y_1}, \dots, \mathcal{O}_{Y_l}) \simeq \bigoplus_{q_1+q_2=q} \mathrm{Tor}_{q_1}^{\mathcal{O}_M}(\mathcal{O}_{Y_1}, \dots, \mathcal{O}_{Y_l}) \otimes \Lambda^{q_2}(\oplus_{i=1}^l N^*_{Y_i/M} \otimes \rho_{k_i}) ,$$

where ρ_{k_i} is the standard representation of the symmetric group $\mathfrak{S}_{k_i}$.

Proof. We solve locally, on adequate open affine subsets U_i, the structural sheaves $\mathcal{O}_{Y_i}$ with Koszul complex $K^\bullet(F_i, s_i)$, where F_i is a vector bundle of rank $\mathrm{codim}_{U_i} Y_i$ and s_i is a section of F_i transverse to the zero section. Then, over $U = U_1 \cap \cdots \cap U_l$, we have:

$$\mathrm{Tor}_q^{k_1,\dots,k_l}(\mathcal{O}_{Y_1}, \dots, \mathcal{O}_{Y_l}) = H^{-q}(\otimes_{i=1}^l \otimes_{j=1}^{k_i} K^\bullet(F_i, s_i)) = H^{-q}(\otimes_{i=1}^l K^\bullet(F_i \otimes R_{k_i}, \sigma_{k_i} \otimes s_i))$$

where $R_{k_i} \simeq \mathbb{C}^{k_i}$ is the natural representation of $\mathfrak{S}_{k_i}$, with canonical basis e_i, and σ_{k_i} is its invariant element, that is $\sigma_{k_i} = \sum_{h=1}^{k_i} e_h \in R_{k_i}$. Then, since $K^\bullet(F_i \otimes R_{k_i}, \sigma_{k_i} \otimes s_i) = K^\bullet(F_i \otimes \rho_{k_i}, 0) \otimes K^\bullet(F_i, s_i)$ by [Sca15a, Remark B.5], we have:

$$\begin{aligned} \mathrm{Tor}_q^{k_1,\dots,k_l}(\mathcal{O}_{Y_1}, \dots, \mathcal{O}_{Y_l}) &= H^{-q}(\otimes_{i=1}^l K^\bullet(F_i, s_i) \otimes \otimes_{i=1}^l K^\bullet(F_i \otimes \rho_{k_i}, 0)) \\ &= H^{-q}(\otimes_{i=1}^l K^\bullet(F_i, s_i) \otimes K^\bullet(\oplus_{i=1}^l F_i \otimes \rho_{k_i}, 0)) \\ &= \bigoplus_{q_1+q_2=q} \mathrm{Tor}_{q_1}^{\mathcal{O}_M}(\mathcal{O}_{Y_1}, \dots, \mathcal{O}_{Y_l}) \otimes \Lambda^{q_2}(\oplus_{i=1}^l N^*_{Y_i/X} \otimes \rho_{k_i}) \end{aligned}$$

as $\times_{i=1}^l \mathfrak{S}_{k_i}$-representations. Now the open sets of the form U cover the algebraic variety M. It is an easy exercise to prove that the local isomorphism shown above glue to give a global isomorphism over M. $\square$

Corollary 2.4. *Let M be a smooth algebraic variety and $Y_1, \dots, Y_l$ smooth subvarieties of M such that the intersection $Z := Y_1 \cap \cdots \cap Y_l$ is smooth. Let $k_1, \dots, k_l$ positive integers. Then we have, as $\mathfrak{S}_{k_1} \times \cdots \times \mathfrak{S}_{k_l}$-representations*

$$\mathrm{Tor}_q^{k_1,\dots,k_l}(\mathcal{O}_{Y_1}, \dots, \mathcal{O}_{Y_l}) \simeq \Lambda^q([(\oplus_{i=1}^l N_{Y_i}\big|_Z)/N_Z]^* \bigoplus \oplus_{i=1}^l N^*_{Y_i} \otimes \rho_{k_i}) .$$

2.2 Multitors of pairwise diagonals

In this subsection we apply the previous general formula to the case of pairwise diagonals. We are concerned with multitors of the form $\mathrm{Tor}_q^{\mathcal{O}_{X^n}}(\mathcal{O}_{\Delta_{I_1}}, \dots, \mathcal{O}_{\Delta_{I_l}})$, where X is a smooth algebraic variety, and where I_j are subsets of $\{1, \dots, n\}$ of cardinality 2 such that $I_i \neq I_j$ if $i \neq j$. It is therefore convenient to think of the multi-indexes I_j as *edges of a simple graph*. We refer to [Die10] for basic concepts in graph theory. More precisely, given l distinct multi-indexes $I_1, \dots, I_l$ of cardinality 2, we can build the simple graph Γ, whose set of vertices V_Γ is defined as the set $I_1 \cup \cdots \cup I_l$ and whose edges are $E_\Gamma = \{I_1, \dots, I_l\}$. All the vertices of the graph Γ are non isolated, that is, they have degree greater or equal than 1. On the other hand, given a simple graph Γ, such that its vertices V_Γ are a subset of $\{1, \dots, n\}$ and are non isolated, its edges are a set of l distinct cardinality-2 multi-indexes $\{I_1, \dots, I_l\}$ such that $V_\Gamma = I_1 \cup \cdots \cup I_l$. For such a graph Γ, we denote with Δ_Γ the intersection of diagonals Δ_I, $I \in E_\Gamma$ and with $\mathrm{Tor}_q(\Delta, \Gamma)$ the multitor $\mathrm{Tor}_q^{\mathcal{O}_{X^n}}(\Delta_{I_1}, \dots, \Delta_{I_l})$. The isomorphism class of this multitor does not depend on the order in which the edges I_j are taken; however, the order of the diagonals is important when dealing with permutation of factors in a multitor: therefore, in the following section, we will always consider l-uples of cardinality 2-multi-indexes $(I_1, \dots, I_l)$, ordered via the lexicographic order.

Remark 2.5. Let Γ be a simple graph. Let v the number of vertices, l the number of edges and k the number of connected components. The *number of independent cycles* c of the graph Γ is given by $c = l - v + k$.

Remark 2.6. Let X be a smooth algebraic variety of dimension d. Let Γ be a simple graph without isolated vertices. The subvariety Δ_Γ of X^n, intersection of the distinct pairwise diagonals Δ_I, $I \in E_\Gamma$ is smooth of codimension $d(v-k)$, where $v = |V_\Gamma|$. This fact, together the possibility of using formula (2.1), since all varieties Δ_I, $I \in E_\Gamma$ and Δ_Γ are smooth, allows us to translate properties of the graph Γ into properties of the multitor $\mathrm{Tor}_q(\Delta, \Gamma)$. In particular, it is clear that

- *the pairwise diagonals* Δ_I, $I \in E_\Gamma$, *intersect transversely in the subvariety* Δ_Γ if and only if $d(v-k) = \mathrm{codim}_{X^n} \Delta_\Gamma = \sum_{I \in E_\Gamma} \mathrm{codim}_{X^n} \Delta_I = dl$, that is, if and only if $c = l - v + k = 0$, that is, *if and only if the graph* Γ *is acyclic*; in this case $\mathrm{Tor}_q(\Delta, \Gamma) = 0$ for all $q > 0$.
- the sheaf $Q_\Gamma := \big[\oplus_{I \in E_\Gamma} N_{\Delta_I}\big|_{\Delta_\Gamma} \big] / N_{\Delta_\Gamma}$ is a vector bundle over Δ_Γ of rank dc; hence $\mathrm{Tor}_q(\Delta, \Gamma) = 0$ for $q > dc$

For $A \subseteq \{1, \dots, n\}$ denote with $\mathfrak{S}(A)$ the symmetric group of the set A and with ρ_A its the standard representation. Let $\mathfrak{S}_\Gamma := \mathrm{Stab}_{\mathfrak{S}_n}(\Gamma)$ be the subgroup of $\mathfrak{S}_n$ transforming the graph Γ into itself. It is a subgroup of $\mathfrak{S}(V_\Gamma) \times \mathfrak{S}(\overline{V_\Gamma})$. Indicate with $\widehat{\mathfrak{S}_\Gamma}$ the subgroup $\mathfrak{S}(V_\Gamma) \cap \mathfrak{S}_\Gamma$ of $\mathfrak{S}_\Gamma$.

Suppose now that $\Gamma_1, \dots, \Gamma_k$ are the connected components of the graph Γ; let now $S_1, \dots, S_t$ be the partition of $\{1, \dots, k\}$ induced by the equivalence relation defined by $i \sim j$ if and only if Γ_i is isomorphic to Γ_j. Denote with $\mathfrak{S}_k^\Gamma$ the subgroup $\mathfrak{S}(S_1) \times \cdots \times \mathfrak{S}(S_t)$ of $\mathfrak{S}_k$ and with $\widetilde{\mathfrak{S}}_{\Gamma_i} = \mathrm{Stab}_{\mathfrak{S}_n}(\Gamma_i) \cap \mathfrak{S}(V_{\Gamma_i})$, where $\mathfrak{S}(V_{\Gamma_i})$ is naturally seen as a subgroup of $\mathfrak{S}_n$. Then there is a split exact sequence

$$1 \longrightarrow \widetilde{\mathfrak{S}}_{\Gamma 1} \times \cdots \times \widetilde{\mathfrak{S}}_{\Gamma_k} \longrightarrow \widehat{\mathfrak{S}_\Gamma} \longrightarrow \mathfrak{S}_k^\Gamma \longrightarrow 1 \,. \tag{2.2}$$

In other words, the subgroup $\widehat{\mathfrak{S}_\Gamma}$ of the stabilizer $\mathfrak{S}_\Gamma$ is a semi-direct product $(\widetilde{\mathfrak{S}}_{\Gamma 1} \times \cdots \times \widetilde{\mathfrak{S}}_{\Gamma_k}) \rtimes \mathfrak{S}_k^\Gamma$; the proof of this fact is analogous to [Sca15a, Lemma 2.12]. The full stabilizer $\mathfrak{S}_\Gamma$ is isomorphic to

$$\mathfrak{S}_\Gamma \simeq \widehat{\mathfrak{S}_\Gamma} \times \mathfrak{S}(\overline{V_\Gamma}) \simeq \Big((\widetilde{\mathfrak{S}}_{\Gamma 1} \times \cdots \times \widetilde{\mathfrak{S}}_{\Gamma_k}) \rtimes \mathfrak{S}_k^\Gamma \Big) \times \mathfrak{S}(\overline{V_\Gamma}) \,.$$

The multitor $\mathrm{Tor}_q(\Delta, \Gamma)$ is naturally $\mathfrak{S}_\Gamma$-linearized. Consider the standard representation ρ_{V_Γ} of $\mathfrak{S}(V_\Gamma)$. It can naturally be seen as a $\mathfrak{S}(V_\Gamma) \times \mathfrak{S}(\overline{V_\Gamma})$-representation, since the second factor acts trivially on V_Γ. Denote as $\rho_\Gamma := \mathrm{Res}_{\mathfrak{S}_\Gamma} \rho_{V_\Gamma}$ the restriction of ρ_{V_Γ} to $\mathfrak{S}_\Gamma$.

Notation 2.7. If Γ is a subgraph of K_n without isolated points and with k connected components $\Gamma_1, \dots, \Gamma_k$, we denote with $i_\Gamma : X^k \hookrightarrow X^{V_{\Gamma_1}} \times \cdots \times X^{V_{\Gamma_k}}$ the immersion defined by embedding each factor X in the factor $X^{V_{\Gamma_i}}$ diagonally; for any connected component Γ_i we indicate with $p_{\Gamma_i} : X^n \longrightarrow X^{V_{\Gamma_i}}$ the projection onto the factors in V_{Γ_i}; the morphism $p_\Gamma : X^n \longrightarrow X^{V_{\Gamma_1}} \times \cdots \times X^{V_{\Gamma_k}}$ is defined as $p_\Gamma := p_{\Gamma_1} \times \cdots \times p_{\Gamma_k}$. Finally, If F is a sheaf over X^k, we indicate with F_Γ the sheaf over X^n defined as $p_\Gamma^* i_{\Gamma *} F$. If Γ has a single edge, say $E_\Gamma = \{I\}$, we will denote, for brevity's sake F_Γ with F_I.

Notation 2.8. We will indicate with W_Γ and q_Γ the representations of $\mathfrak{S}_\Gamma$ defined by $W_\Gamma := \oplus_{I \in E_\Gamma} \rho_I$ and by the exact sequence

$$0 \longrightarrow \rho_\Gamma \longrightarrow W_\Gamma \longrightarrow q_\Gamma \longrightarrow 0 \,,$$

respectively. It is clear that, if Γ_i are the connected components of the graph Γ, the vector space $\oplus_{i=1}^k q_{\Gamma_i}$ is naturally a $\mathfrak{S}_\Gamma$-representation isomorphic to q_Γ.

Notation 2.9. Let Γ be a simple graph without isolated vertices, and let γ be an oriented cycle in Γ. If I is an edge of γ, $I = \{i, j\}$, $i < j$, then we define the sign $\eta_{I,\gamma}$ to be $+1$ if the vertex j is the immediate successor of i in γ, and $\eta_{I,\gamma} = -1$ otherwise.

Remark 2.10. The representation q_Γ is generated over $\mathbb{C}$ by independent cycles. More precisely, we can consider q_Γ as a subrepresentation of $\oplus_{I\in E_\Gamma} R_I$, where $R_I \simeq \mathbb{C}^I$, with basis $e_i, i \in I$, is the natural $\mathfrak{S}(I)$-representation. If $I = \{i,j\}$, $i < j$, let's indicate with e_I the vector $e_i - e_j$. If γ is an oriented cycle in Γ, then we can consider the vector $e_\gamma := \sum_{I\in\gamma} \eta_{I,\gamma} e_I$ in $W_\Gamma = \oplus_{I\in E_\Gamma} \rho_I \subseteq \oplus_{I\in E_\Gamma} R_I$. Now $\gamma_1, \dots, \gamma_c$ are independent cycles in Γ if and only if $e_{\gamma_1} \dots, e_{\gamma_c}$ are independent in W_Γ and project to a basis in q_Γ. Hence we can identify q_Γ with the subspace $\mathbb{C}e_{\gamma_1} \oplus \cdots \oplus \mathbb{C}e_{\gamma_c}$ of W_Γ.

Remark 2.11. Consider the vector bundles $\boxplus_{i=1}^k TX \otimes q_{\Gamma_i}$ and $\boxplus_{i=1}^k \Omega^1_X \otimes q_{\Gamma_i}$ over X^k. They are naturally $\mathfrak{S}_\Gamma$-equivariant; indeed the $\mathfrak{S}_\Gamma$ acts on the variety X^k via the surjective composition $\mathfrak{S}_\Gamma \longrightarrow \widehat{\mathfrak{S}_\Gamma} \longrightarrow \mathfrak{S}_k^\Gamma$; this action lifts to the tangent and cotangent bundle. On the other hand the action of $\mathfrak{S}_\Gamma$ over the representations q_{Γ_i} is induced by its action on q_Γ.

We have the following interpretation of a multitor of pairwise diagonals $\mathrm{Tor}_q(\Delta, \Gamma)$ in terms of data attached to the graph Γ.

Proposition 2.12. *As $\mathfrak{S}_\Gamma$-sheaves, we have that* $\mathrm{Tor}_q(\Delta,\Gamma) \simeq \Lambda^q(\boxplus_{i=1}^k \Omega^1_X \otimes q_{\Gamma_i})_\Gamma$.

Proof. Consider first the case in which Γ is connected. Then $\Delta_\Gamma \simeq X \times X^{\overline{V_\Gamma}}$. It is then sufficient to prove the proposition when $n = v$, since the case $n > v$ can be then obtained by flat base change. The morphism $i_\Gamma : X \longrightarrow X^n$ is an isomorphism over the image Δ_Γ. Hence pulling back the exact sequence over Δ_Γ

$$0 \longrightarrow N_{\Delta_\Gamma} \longrightarrow \oplus_{I\in E_\Gamma} N_I\big|_{\Delta_\Gamma} \longrightarrow Q_\Gamma \longrightarrow 0$$

via i_Γ, we obtain over X, by definition of ρ_Γ and q_Γ, a $\mathfrak{S}_\Gamma$-equivariant exact sequence

$$0 \longrightarrow TX \otimes \rho_\Gamma \longrightarrow TX \otimes W_\Gamma \longrightarrow TX \otimes q_\Gamma \longrightarrow 0\,.$$

Hence the vector bundle $Q_\Gamma = \big[\oplus_{I\in E_\Gamma} N_{\Delta_I}\big|_{\Delta_\Gamma}\big]/N_{\Delta_\Gamma}$ over Δ_Γ is isomorphic to $i_{\Gamma *}(TX \otimes q_\Gamma) = (TX \otimes q_\Gamma)_\Gamma$ and we conclude by formula (2.1).

Consider now a general Γ. We have that $\Delta_\Gamma \simeq X^k \times X^{\overline{V_\Gamma}}$: it is then sufficient to consider the case $n = v$ for the same reasone as above. In this case $i_\Gamma : X^k \longrightarrow X^n$ is an isomorphism over the image Δ_Γ. Consider the connected components Γ_i, $i = 1, \dots, k$, of the graph Γ. We have $N_{\Delta_\Gamma} \simeq \oplus_{i=1}^k N_{\Delta_{\Gamma_i}}\big|_{\Delta_\Gamma}$, since the partial diagonals Δ_{Γ_i} intersect transversely. For each $i = 1, \dots, k$, we have sequences

$$0 \longrightarrow N_{\Delta_{\Gamma_i}} \longrightarrow \oplus_{I\in E_{\Gamma_i}} N_I\big|_{\Delta_{\Gamma_i}} \longrightarrow Q_{\Gamma_i} \longrightarrow 0\,.$$

Hence over Δ_Γ we have an exact sequence

$$0 \longrightarrow N_{\Delta_\Gamma} \longrightarrow \oplus_{I\in E_\Gamma} N_I\big|_{\Delta_I} \longrightarrow \oplus_{i=1}^k Q_{\Gamma_i} \longrightarrow 0$$

and hence pulling everything back to X^k we get the exact sequence

$$0 \longrightarrow \boxplus_{i=1}^k TX \otimes \rho_{\Gamma_i} \longrightarrow \boxplus_{i=1}^k TX \otimes W_{\Gamma_i} \longrightarrow \boxplus_{i=1}^k TX \otimes q_{\Gamma_i} \longrightarrow 0\,.$$

Hence, as $\mathfrak{S}_\Gamma$-vector bundles over Δ_Γ, $Q_\Gamma \simeq i_{\Gamma *}(\boxplus_{i=1}^k TX \otimes q_{\Gamma_i}) \simeq (\boxplus_{i=1}^k TX \otimes q_{\Gamma_i})_\Gamma$, and we conclude by formula (2.1). ∎

Notation 2.13. Let Γ a simple graph without isolated vertices, such that $V_\Gamma \subseteq \{1, \dots, n\}$. If $J \subseteq \{1, \dots, n\}$, $|J| = 2$, and $J \notin E_\Gamma$, we will indicate with $\Gamma \cup J$ the graph obtained by Γ adding the edge J, that is, the graph defined by $V_{\Gamma\cup J} := V_\Gamma \cup J$, $E_{\Gamma\cup J} := E_\Gamma \cup \{J\}$.

Proposition 2.14. *Let X a smooth algebraic variety of dimension d. Let Γ be a simple graph without isolated vertices such that $V_\Gamma \subseteq \{1, \dots, n\}$ and with edges $E_\Gamma = \{I_1, \dots, I_l\}$. Let $J \subseteq \{1, \dots, n\}$, $|J| = 2$, and $J \notin E_\Gamma$. Identifying $\mathrm{Tor}_q(\Delta, \Gamma)$ with $\mathrm{Tor}_q(\Delta_{I_1}, \dots, \Delta_{I_l}, \mathcal{O}_{X^n})$, the $\mathfrak{S}_\Gamma \cap \mathfrak{S}_{\Gamma\cup J}$-equivariant map*

$$i_{\Gamma,\Gamma\cup J} : \mathrm{Tor}_q(\Delta, \Gamma) \longrightarrow \mathrm{Tor}_q(\Delta, \Gamma \cup J)\,,$$

induced by the restriction $\mathcal{O}_{X^n} \longrightarrow \Delta_J$, can be identified with

- *the restriction $\Lambda^q(Q^*_\Gamma) \longrightarrow \Lambda^q(Q^*_\Gamma)\big|_{\Delta_{\Gamma\cup J}}$ if $J \not\subseteq V_\Gamma$*

- *the natural injection $\Lambda^q(Q^*_\Gamma) \longrightarrow \Lambda^q(Q^*_{\Gamma\cup J})$, induced by the injection of vector bundles $Q^*_\Gamma \longrightarrow Q^*_{\Gamma\cup J}$ over Δ_Γ, if $J \subseteq V_\Gamma$.*

Proof. Let x be a point in Δ_Γ. On a small affine open neighbourhood of x, we can find vector bundles F_I, $I \in E_\Gamma$, of rank d, such that Δ_I are the zero locus of sections s_I of F_I transverse to the zero section. The same is true for Δ_J. The structural sheaves $\mathcal{O}_{\Delta_I}$ can then be resolved with Koszul complexes $K^\bullet(F_I, s_I)$. Denote with F_Γ the vector bundle $\oplus_{I\in E_\Gamma} F_I$ and with s_Γ the section $\oplus_I s_I$. Therefore

$$\mathrm{Tor}_q(\Delta,\Gamma) \simeq H^{-q}(\otimes_{I\in E_\Gamma} K^\bullet(F_I, s_I)) \simeq H^{-q}(K^\bullet(F_\Gamma, s_\Gamma)) \simeq \Lambda^q(Q^*_\Gamma)$$
$$\mathrm{Tor}_q(\Delta,\Gamma\cup J) \simeq H^{-q}(K^\bullet(F_{\Gamma\cup J}, s_{\Gamma\cup J})) \simeq \Lambda^q(Q^*_{\Gamma\cup J}) \ .$$

The morphism $\mathrm{Tor}_q(\Delta,\Gamma) \longrightarrow \mathrm{Tor}_q(\Delta,\Gamma\cup J)$ of the statement is induced by the injection of vector bundles $i_J : F^*_\Gamma = \oplus_{I\in E_\Gamma} F^*_I \longrightarrow \oplus_{I\in E_{\Gamma\cup J}} F^*_I = F^*_{\Gamma\cup J}$; one then sees, as in the proof of theorem 2.1 that the injection i_J induces the natural map:

$$Q^*_\Gamma = [\oplus_{I\in E_\Gamma} N_{\Delta_I}\big|_{\Delta_\Gamma}/N_{\Delta_\Gamma}]^* \longrightarrow [(\oplus_{I\in E_\Gamma} N_{\Delta_I} \oplus N_{\Delta_J})\big|_{\Delta_{\Gamma\cup J}}/N_{\Delta_{\Gamma\cup J}}]^* = Q^*_{\Gamma\cup J} \ .$$

Now if $J \not\subseteq V_\Gamma$, we just have that $N_{\Delta\cup J} \simeq N_{\Delta_\Gamma} \oplus N_{\Delta_J}$, and hence $Q^*_{\Gamma\cup J} \simeq Q^*_\Gamma\big|_{\Delta_{\Gamma\cup J}}$ and the previous map is the restriction; on the other hand, if $J \subseteq V_\Gamma$, $Q^*_{\Gamma\cup J}$ is a vector bundle over $\Delta_{\Gamma\cup J} = \Delta_\Gamma$ and the previous map is the natural injection. This proves the statement. □

Proposition 2.15. *Let K_v the complete graph on v vertices and suppose that $V_{K_v} \subseteq \{1,\dots,n\}$. Then, as $\mathfrak{S}_{K_v}$-representations,*

$$\mathrm{Tor}_q(\Delta, K_v) \simeq \Lambda^q(\Omega^1_X \otimes \Lambda^2\rho_v)_{K_v} \ ,$$

where ρ_v denotes the standard representation of $\widehat{\mathfrak{S}_{K_v}} \simeq \mathfrak{S}_v$.

Proof. It is sufficient to consider the case $v = n$, since the case $v < n$ follows by flat base change. By propositon 2.12, it is sufficient to prove that the representation q_{K_n} is isomorphic to $\Lambda^2\rho_n$. Since $\rho_{K_n} \simeq \rho_n$, it is sufficient to prove that the representation $W_{K_n} = \oplus_{|I|=2,\, I\subseteq\{1,\dots,n\}}\rho_I \simeq \rho_n \oplus \Lambda^2\rho_n$. In order to achieve this, it is sufficient to prove that the characters of the two representations are the same. Let $\mathbf{i}$ the n-uple $(i_1,\dots,i_n)$, with $\sum_j j i_j = n$. Denote with $C_\mathbf{i}$ the conjugacy class in $\mathfrak{S}_n$ of permutations having i_j j-cycles. By Frobenius formula [FH91, exercise 4.15], the character of $\rho_n \oplus \Lambda^2\rho_n$ is valued, on $C_\mathbf{i}$:

$$(\chi_{\rho_n} + \chi_{\Lambda^2\rho_n})(C_\mathbf{i}) = i_1 - 1 + \frac{1}{2}(i_1-1)(i_1-2) - i_2 = \binom{i_1}{2} - i_2 \ .$$

On the other hand, a basis of the representation W_{K_n} is given by vectors e_J, $J \subseteq \{1,\dots,n\}$, $|J| = 2$. If $I = \{i,j\}$, we have that $(ij)e_I = -e_I$ and $(ij)e_{ih} = e_{jh}$ if $h \notin I$. Hence, any cycle γ_j of length $j \geq 3$ will act with trace zero. The 1-cycles $(j_1)\dots(j_{i_1})$ act trivially on a $\binom{i_1}{2}$-dimensional space, and hence with trace $\binom{i_1}{2}$. Since the cycles are disjoint, the traces add up. Hence

$$\chi_{W_{K_n}}(C_\mathbf{i}) = \binom{i_1}{2} - i_2 = (\chi_{\rho_n} + \chi_{\Lambda^2\rho_n})(C_\mathbf{i}) \ .$$

□

2.3 The cases $n = 3, 4$.

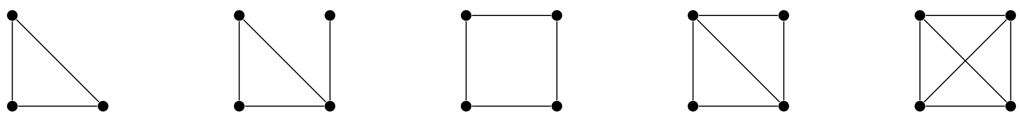
We analyse here in detail the representations q_Γ with Γ a non-acyclic subgraph of K_n, $n = 3, 4$, with non isolated vertices.

Non acyclic graphs. There is a unique non-acyclic subgraph of K_3 without isolated vertices, the complete graph K_3 itself. On the other hand, up to isomorphism, the non-acyclic subgraphs of K_4 without isolated vertices are the following:

- for $l = 3$, the complete graph K_3 with 3 vertices; its stabilizer is the group $\mathfrak{S}_3$;

- for $l = 4$, the 4-cycle C_4 and the graph $K_3 \cup J$, obtained by K_3 adding an edge $J \not\subseteq E_{K_3}$; since C_4 has dihedral symmetry, $\mathfrak{S}_{C_4}$ is isomorphis to the dihedral group D_4; on the other hand, adding an edge to the graph K_3 reduces its symmetries to $\mathfrak{S}_2$.
- for $l = 5$ the graph $C_4 \cup L$, obtained adding an edge $L \not\subseteq E_{C_4}$ to the 4-cycle C_4; its stabilizer is $\mathfrak{S}_{C_4 \cup J} \simeq \mathfrak{S}_2 \times \mathfrak{S}_2$;
- for $l = 6$, the complete graph K_4; its stabilizer is the full symmetric group $\mathfrak{S}_4$.

The non-acyclic subgraphs of K_4 without isolated vertices are represented in the figure below.



K_3 $\quad$ $K_3 \cup J$ $\quad$ C_4 $\quad$ $C_4 \cup L$ $\quad$ K_4

Classification of representations q_Γ. In order to classify the representations q_Γ we need the following lemma

Lemma 2.16. *Let $n = r + s$, with $r, s \in \mathbb{N} \setminus \{0\}$. Consider the symmetric groups $\mathfrak{S}_{r+s}$, $\mathfrak{S}_r$, $\mathfrak{S}_s$. Then, as $\mathfrak{S}_r \times \mathfrak{S}_s$-representations, $\rho_{r+s} \simeq \rho_r \oplus \rho_s \oplus 1$. As a consequence, if $n = \sum_{i=1}^r k_i$, as $\mathfrak{S}_{k_1} \times \cdots \times \mathfrak{S}_{k_r}$-representations, we have $\rho_{k_1+\cdots+k_r} \simeq \rho_{k_1} \oplus \cdots \oplus \rho_{k_r} \oplus 1^{r-1}$, where 1^{r-1} is the $(r-1)$-dimensional trivial representation.*

Proof. The natural $\mathfrak{S}_n$-representation R_n splits as $R_n \simeq R_r \oplus R_s$ when seen as as $\mathfrak{S}_r \times \mathfrak{S}_s$-represenation. The inclusion $\rho_{r+s} \hookrightarrow R_n = R_r \oplus R_s$ is $\mathfrak{S}_r \times \mathfrak{S}_s$-equivariant. Then, as $\mathfrak{S}_r \times \mathfrak{S}_s$-representations $R_n \simeq R_r \oplus R_s = (\rho_r \oplus 1) \oplus (\rho_s \oplus 1) \simeq \rho_r \oplus \rho_s \oplus 1^2$. Hence the statement: $\rho_{r+s} \simeq \rho_r \oplus \rho_s \oplus 1$. $\square$

Remark 2.17. The dihedral group D_4, generated by the reflection σ and the rotation ρ, has 4 finite dimensional irreducible representations: the trivial, the standard representation θ, the determinantal det := $\det\theta$, the linear $\ell(1_\sigma, -1_\rho)$, the linear $\ell(-1_\sigma, -1_\rho)$. A description and a character table for these representation is given in [Ser77].

We already know, by proposition 2.15, that $q_{K_3} \simeq \Lambda^2\rho_3 \simeq \varepsilon$ and that $q_{K_4} \simeq \Lambda^2\rho_4 \simeq \rho_4 \otimes \varepsilon$. As for the remaining representations q_Γ for Γ a subgraph of K_4 without isolated vertices we have the following.

- $\Gamma = K_3 \cup J$. Up to isomorphism we can think that $E_{K_3 \cup J} = \{\{1,2\},\{1,3\},\{2,3\},\{3,4\}\}$. Hence $\mathfrak{S}_{K_3 \cup J} \simeq \mathfrak{S}(\{1,2\})$ and $W_{K_3 \cup J} = \rho_{12} \oplus \rho_{13} \oplus \rho_{23} \oplus \rho_{34}$. The character $\chi_{W_{K_3 \cup J}}$ is easily $(4,0)$ according to the conjugacy classes of $1, (12)$; but $(4,0) = (1,-1) + (1,-1) + (1,0) + (1,0) = 2\chi_\varepsilon + 2\chi_1$. Hence $\chi_{q_{K_3 \cup J}} = \chi_{W_{K_3 \cup J}} - \chi_{\rho_{K_3 \cup J}} = 2\chi_\varepsilon + 2\chi_1 - \chi_{\mathrm{Res}^{\mathfrak{S}_4}_{\mathfrak{S}_2}\rho_4} = 2\chi_\varepsilon + 2\chi_1 - \chi_\varepsilon - 2\chi_1 = \chi_\varepsilon$ by lemma 2.16. Hence $q_{K_3 \cup J} \simeq \varepsilon$.
- $\Gamma = C_4$. Up to isomorphism, we can think thak $E_{C_4} = \{\{1,2\},\{1,4\},\{2,3\},\{3,4\}\}$. We easily have that the stabilizer $\mathfrak{S}_{C_4}$ is isomorphic to the dihedral group D_4, where the reflection σ and the rotation ρ are identified with $\sigma = (24)$ and $\rho = (1234)$, for example. Then $\chi_{W_{C_4}}$, according to conjugacy classes, $1, \sigma, \sigma\rho, \rho, \rho^2$, is given by $\chi_{W_{C_4}} = (4,0,-2,0,0)$, and hence W_{C_4} is isomorphic to $\det \oplus \ell(1_\sigma, -1_\rho) \oplus \theta$. Computing characters we get that $\chi_{\rho_{C_4}} = (3,1,-1,-1,-1)$ and hence ρ_{C_4} is isomorphic to $\ell(1_\sigma, -1_\rho) \oplus \theta$ as D_4-representation. Hence $q_{C_4} \simeq \det$ as D_4-representation.
- $\Gamma = C_4 \cup L$. Up to isomorphism, suppose that the graph $C_4 \cup L$ has edges $E_{C_4 \cup L} = \{\{1,2\},\{1,4\},\{2,3\},\{3,4\},\{1,3\}\}$, so that $\mathfrak{S}_{C_4 \cup L} \simeq \mathfrak{S}(1,3) \times \mathfrak{S}(2,4)$. Then, according to conjugacy classes $1, (13), (24), (13)(24)$, the character $\chi_{W_{C_4 \cup L}}$ is given by $(5,-1,1,-1)$. By lemma 2.16, $\rho_{C_4 \cup L} \simeq \mathrm{Res}^{\mathfrak{S}_4}_{\mathfrak{S}_2 \times \mathfrak{S}_2}\rho_5 \simeq (\varepsilon \otimes 1) \oplus (1 \otimes \varepsilon) \oplus 1$ and hence $\chi_{\rho_{C_4 \cup L}} = (3,1,1,-1)$. Hence $\chi_{q_{C_4 \cup L}} = (2,-2,0,0)$, which yields $q_{C_4 \cup L} \simeq (\varepsilon \otimes 1) \oplus (\varepsilon \otimes \varepsilon)$.

Multitors as $\mathfrak{S}_\Gamma$-representations. For $n = 3, 4$, all non-acyclic graph are connected, hence the corresponding $\mathrm{Tor}_q(\Delta, \Gamma)$ is isomorphic to $\Lambda^q(\Omega^1_X \otimes q_\Gamma)_\Gamma$. In the following table we summarize the results obtained in this section, in the case X is a surface, expliciting the representations in the vector bundle $\Lambda^q(\Omega^1_X \otimes q_\Gamma)$. We indicate with V^λ the irreducible representation of $\mathfrak{S}_n$ associated to the partition λ of n.

	$\Gamma = K_3$	$\Gamma = K_3 \cup J$	$\Gamma = C_4$	$\Gamma = C_4 \cup L$	$\Gamma = K_4$
$q = 1$	$\Omega^1_X \otimes \varepsilon$	$\Omega^1_X \otimes \varepsilon$	$\Omega^1_X \otimes \det$	$\Omega^1_X \otimes (\varepsilon \otimes 1) \oplus (\varepsilon \otimes \varepsilon)$	$\Omega^1_X \otimes \rho_4 \otimes \varepsilon$
$q = 2$	K_X	K_X	K_X	$K_X \otimes (1^2 \oplus (1 \otimes \varepsilon)) \oplus S^2\Omega^1_X \otimes (1 \otimes \varepsilon)$	$K_X \otimes (1 \oplus \rho_4 \oplus V^{2,2}) \oplus S^2\Omega^1_X \otimes \rho_4 \otimes \varepsilon$
$q = 3$	0	0	0	$K_X \otimes \Omega^1_X \otimes [(\varepsilon \otimes 1) \oplus (\varepsilon \otimes \varepsilon)]$	$S^3\Omega^1_X \oplus \Omega^1_X \otimes K_X \otimes (\rho_4 \oplus \rho_4 \otimes \varepsilon \oplus V^{2,2})$
$q = 4$	0	0	0	K_X^2	$S^2\Omega^1_X \otimes K_X \otimes \rho_4 \otimes \varepsilon \oplus K_X^2 \otimes (1 \oplus \rho_4 \oplus V^{2,2})$
$q = 5$	0	0	0	0	$\Omega^1_X \otimes K_X^2 \otimes \rho_4 \otimes \varepsilon$
$q = 6$	0	0	0	0	K_X^3

Table 1

where we used the formula for the exterior power of a tensor product

$$\Lambda^q(\Omega^1_X \otimes \rho_4 \otimes \varepsilon) = \bigoplus_\lambda S^\lambda \Omega^1_X \otimes S^{\lambda'}(\rho_4 \otimes \varepsilon)$$

where the direct sum is taken on partitions λ of q such that λ has at most 2 rows and at most 3 columns [FH91, Exercise 6.11], and where we used the next lemma.

Lemma 2.18. *We have the following isomorphisms of $\mathfrak{S}_4$-representations:*

$$S^2\rho_4 \simeq S^2V^{2,1,1} \simeq 1 \oplus \rho_4 \oplus V^{2,2}$$
$$\Lambda^3(\rho_4 \otimes \varepsilon) \simeq 1$$
$$S^{2,1}(\rho_4 \otimes \varepsilon) \simeq \rho_4 \oplus (\rho_4 \otimes \varepsilon) \oplus V^{2,2}$$
$$S^{2,1,1}(\rho_4 \otimes \varepsilon) \simeq \rho_4 \otimes \varepsilon$$
$$S^{2,2}(\rho_4 \otimes \varepsilon) \simeq S^{2,2}\rho_4 \simeq 1 \oplus \rho_4 \oplus V^{2,2}$$
$$S^{2,2,1}(\rho_4 \otimes \varepsilon) \simeq \rho_4 \otimes \varepsilon$$
$$S^{2,2,2}(\rho_4 \otimes \varepsilon) \simeq 1$$

Proof. To compute $S^2\rho_4$, we remark that $\chi_{S^2\rho_4} = \chi^2_{\rho_4} - \chi_{\Lambda^2\rho_4} = \chi^2_{\rho_4} - \chi_{\rho_4 \otimes \varepsilon}$, which is $(6, 2, 0, 0, 2)$, according to the conjugacy classes $1, (12), (123), (1234), (12)(34)$. Hence the isomorphism in the statement.

Moreover we have that $S^{2,1}V^{2,1,1}$ is the kernel of the surjective homomorphism $\rho_4 \otimes \rho_4 \simeq \Lambda^2V^{2,1,1} \otimes V^{2,1,1} \longrightarrow \Lambda^3V^{2,1,1} \simeq 1$ (see [FH91, page 76]). Hence it is easy to compute the character and deduce the isomorphism $S^{2,1}(\rho_4 \otimes \varepsilon) \simeq \rho_4 \oplus (\rho_4 \otimes \varepsilon) \oplus V^{2,2}$ in the statement.

As for $S^{2,2}V^{2,1,1}$, we use the Schur polynomial $S_{2,2}(x_1, x_2, x_3) = x_1^2x_2^2 + x_1^2x_3^2 + x_2^2x_3^2 + x_1^2x_2x_3 + x_1x_2^2x_3 + x_1x_2x_3^2$ in order to compute explicitely the character $\chi_{S^{2,2}V_{2,1,1}}$ in terms of the eigenvalues of elements of $\mathfrak{S}_4$ acting on $V^{2,1,1} \simeq \rho_4 \otimes \varepsilon$; we have that $\chi_{S^{2,2}V^{2,1,1}}$ is again $(6, 2, 0, 0, 2)$: hence $S^{2,2}V^{2,1,1} \simeq S^2\rho_4 \simeq 1 \oplus \rho_4 \oplus V^{2,2}$. The remaining isomorphism are routine verifications and are left to the reader. □

3 Invariants of diagonal ideals for low n.

Let X be a smooth algebraic variety. The ideal $\mathcal{I}_{\Delta_n}$ of the big diagonal Δ_n is the intersection $\mathcal{I}_{\Delta_n} = \cap_{I \subseteq \{1,\dots,n\}, |I|=2} \mathcal{I}_{\Delta_I}$ of ideals of pairwise diagonals Δ_I, $I \subseteq \{1, \dots, n\}$, $|I| = 2$: it is then isomorphic to the kernel of the natural morphism

$$\mathcal{O}_{X^n} \longrightarrow \bigoplus_{I \subseteq \{1,\dots,n\}, |I|=2} \mathcal{O}_{\Delta_I} \; . \tag{3.1}$$

Hence it is useful to consider, for each multi-index I of cardinality 2, a right resolution $\mathcal{K}^\bullet_I$ of the ideal sheaf $\mathcal{I}_{\Delta_I}$:

$$\mathcal{K}^\bullet_I : \qquad 0 \longrightarrow \mathcal{O}_{X^n} \longrightarrow \mathcal{O}_{\Delta_I} \longrightarrow 0 \,,$$

concentrated in degree 0 and 1: indeed, the first nontrivial map of the $\mathfrak{S}_n$-equivariant complex $\bigotimes_I \mathcal{K}_I^\bullet$ is indeed exactly (3.1); here the order in which the tensor product is taken is always the lexicographic order on the cardinality 2-multi-indexes. However, the complex $\bigotimes_I \mathcal{K}_I^\bullet$ is not exact, and in order to to deal with this problem, it is better to consider the derived tensor product of complexes

$$\bigotimes_I^L \mathcal{K}_I^\bullet \,.$$

Indeed, let $r = n(n+1)/2$ and consider the spectral sequence

$$E_1^{p,q} := \bigoplus_{i_1+\cdots+i_r=p} \mathrm{Tor}_{-q}(\mathcal{K}_{1,2}^{i_1}, \cdots, \mathcal{K}_{n_1,n}^{i_r}) \,. \tag{3.2}$$

It abuts to the cohomology $\mathcal{H}^{p+q}(\bigotimes_I^L \mathcal{K}_I^\bullet)$ and the term $E_2^{0,0}$ is clearly isomorphic to the ideal $\mathcal{I}_{\Delta_n}$.

Remark 3.1. Since $\mathcal{I}_{\Delta_I}$ are sheaves and the complexes $\mathcal{K}_I^\bullet$ are their resolutions, $\mathcal{H}^{p+q}(\bigotimes_I^L \mathcal{K}_I^\bullet) = \mathrm{Tor}_{-p-q}(\mathcal{I}_{\Delta_{12}}, \cdots, \mathcal{I}_{\Delta_{n-1,n}}) = 0$ for $p+q > 0$. Consequently, the abutment of the spectral sequence is zero for $p+q>0$.

We plan to get information on the sheaf of invariants $\mathcal{I}_{\Delta_n}^{\mathfrak{S}_n} = (E_2^{0,0})^{\mathfrak{S}_n}$ from the vanishing of the abutment in positive degree and from studying the spectral sequence of invariants in detail.

3.1 The comprehensive $\mathfrak{S}_n$-action on the spectral sequence $E_1^{p,q}$.

We will here briefly explain the action of the symmetric group $\mathfrak{S}_n$ on the derived tensor product $\bigotimes_I^L \mathcal{K}_I^\bullet$ or, equivalently, on the spectral sequence $E_1^{p,q}$. This is analogous to what done in [Sca09] for the derived tensor power of a complex of sheaves $\mathcal{C}^\bullet$. The point here is that, when considering the multitors $\mathrm{Tor}_{-q}(\mathcal{K}_{1,2}^{i_1}, \cdots, \mathcal{K}_{n_1,n}^{i_r})$, the terms $\mathcal{K}_I^h$ are not just sheaves, but terms of a complex $\mathcal{K}_I^\bullet$. In the following remark we will recall what we explained in detail in [Sca09, Section 4.1] and [Sca09, Appendix B].

Remark 3.2. Let $\mathcal{C}_1^\bullet$, $\mathcal{C}_2^\bullet$ complexes of sheaves over a variety M. If we have a tensor product of complexes of sheaves $\mathcal{C}_1^\bullet \otimes \mathcal{C}_2^\bullet$ the permutation of factors $\tau_{12} : \mathcal{C}_1^\bullet \otimes \mathcal{C}_2^\bullet \longrightarrow \mathcal{C}_2^\bullet \otimes \mathcal{C}_1^\bullet$ is a morphism of complexes if and only if τ_{12} acts on the term of degree h, that is the term $(\mathcal{C}_1^\bullet \otimes \mathcal{C}_2^\bullet)^h := \oplus_{i+j=h} \mathcal{C}_1^i \otimes \mathcal{C}_2^j$, exchanging the terms $\mathcal{C}_1^i \otimes \mathcal{C}_2^j \longrightarrow \mathcal{C}_2^j \otimes \mathcal{C}_1^i$ and *twisting by the sign* $(-1)^{ij}$. The same argument can be applied to a tensor product of complexes $\mathcal{C}_1^\bullet \otimes \cdots \otimes \mathcal{C}_r^\bullet$. Indeed, in order to understand how a general permutation of factors operate on a tensor product of complexes, it is sufficient to understand how a consecutive transposition acts, and this is completely analogous to the case $r=2$.

If now we want to understand the effect of permutating factors in a *derived tensor product* $\mathcal{C}_1^\bullet \otimes^L \cdots \otimes^L \mathcal{C}_r^\bullet$, we have to resolve each of the complexes $\mathcal{C}_j^\bullet$ with a complex of locally free $R_j^\bullet$ (at least locally), and apply the previous reasoning to $R_1^\bullet \otimes \cdots \otimes R_r^\bullet$. To be more explicit, at the level of spectral sequences, consider a consecutive transposition $\tau_{j,j+1} \in \mathfrak{S}_r$ and consider the spectral sequences

$$E_1^{p,q} = \oplus_{i_1+\cdots+i_r=p} \mathrm{Tor}_{-q}(\mathcal{C}_1^{i_1}, \dots, \mathcal{C}_j^{i_j}, \mathcal{C}_{j+1}^{i_{j+1}}, \dots, \mathcal{C}_r^{i_r})$$
$$E'^{p,q}_1 = \oplus_{h_1+\cdots+h_r=p} \mathrm{Tor}_{-q}(\mathcal{C}_1^{h_1}, \dots, \mathcal{C}_{j+1}^{h_{j+1}}, \mathcal{C}_j^{h_j}, \dots, \mathcal{C}_r^{h_r}) \,,$$

abutting to $\mathcal{H}^{p+q}(\mathcal{C}_1^\bullet \otimes^L \cdots \otimes^L \mathcal{C}_j^\bullet \otimes^L \mathcal{C}_{j+1}^\bullet \otimes^L \cdots \otimes^L \mathcal{C}_r^\bullet)$ and $\mathcal{H}^{p+q}(\mathcal{C}_1^\bullet \otimes^L \cdots \otimes^L \mathcal{C}_{j+1}^\bullet \otimes^L \mathcal{C}_j^\bullet \otimes^L \cdots \otimes^L \mathcal{C}_r^\bullet)$. The consecutive transposition $\tau_{j,j+1}$ induces an isomorphism $E_1^{p,q} \longrightarrow E'^{p,q}_1$ and hence isomorphisms

$$\widehat{\tau_{j,j+1}} : \mathrm{Tor}_{-q}(\mathcal{C}_1^{i_1}, \dots, \mathcal{C}_j^{i_j}, \mathcal{C}_{j+1}^{i_{j+1}}, \dots, \mathcal{C}_r^{i_r}) \longrightarrow \mathrm{Tor}_{-q}(\mathcal{C}_1^{i_1}, \dots, \mathcal{C}_{j+1}^{i_{j+1}}, \mathcal{C}_j^{i_j}, \dots, \mathcal{C}_r^{i_r}) \,.$$

Now, considering the $\mathcal{C}_l^{i_l}$ just sheaves, and not as terms of a complex $\mathcal{C}_l^\bullet$, one has the standard permutation of factors in a multitor

$$\widetilde{\tau_{j,j+1}} : \mathrm{Tor}_{-q}(\mathcal{C}_1^{i_1}, \dots, \mathcal{C}_j^{i_j}, \mathcal{C}_{j+1}^{i_{j+1}}, \dots, \mathcal{C}_r^{i_r}) \longrightarrow \mathrm{Tor}_{-q}(\mathcal{C}_1^{i_1}, \dots, \mathcal{C}_{j+1}^{i_{j+1}}, \mathcal{C}_j^{i_j}, \dots, \mathcal{C}_r^{i_r}) \,.$$

We proved in [Sca09, section 4.1] that the two isomorphisms $\widetilde{\tau_{j,j+1}}$ and $\widehat{\tau_{j,j+1}}$ are related by the sign

$$\widehat{\tau_{j,j+1}} = (-1)^{i_j i_{j+1}} \widetilde{\tau_{j,j+1}} \,. \tag{3.3}$$

Remark 3.3. When we say that the complex $\mathcal{K}^\bullet_{1,2} \otimes^L \cdots \otimes^L \mathcal{K}^\bullet_{n-1,n}$ is $\mathfrak{S}_n$-equivariant, what we mean is that $\mathfrak{S}_n$ *acts up to permutation of factors.* More precisely, we can interpret the $\mathfrak{S}_n$-action in the following way. For brevity's sake, denote with $m = n(n+1)/2$ and set $E_{K_n} := \{J_1, \dots, J_m\}$ in lexicographic order. Consider the monomorphism $\theta : \mathfrak{S}_n \hookrightarrow \mathfrak{S}(E_{K_n})$ induced by the natural action of $\mathfrak{S}_n$ on E_{K_n}: if $\sigma \in \mathfrak{S}_n$, we will briefly indicate with $\tilde{\sigma}$ its image in $\mathfrak{S}(E_{K_n})$. Denote with $\Delta_{\mathfrak{S}_n}$ the subgroup of $\mathfrak{S}_n \times \mathfrak{S}(E_{K_n})^{\mathrm{op}}$ given by the set of couples $(\sigma, \tilde{\sigma}^{-1})$, such that $\sigma \in \mathfrak{S}_n$. The complex $\mathcal{K}^\bullet_{J_1} \otimes^L \cdots \otimes^L \mathcal{K}^\bullet_{J_m}$ is now $\Delta_{\mathfrak{S}_n}$*-equivariant*; any element $(\sigma, \tilde{\sigma}^{-1}) \in \Delta_{\mathfrak{S}_n}$ acts via a composition

$$\mathcal{K}^\bullet_{J_1} \otimes^L \cdots \otimes^L \mathcal{K}^\bullet_{J_m} \xrightarrow{\lambda_\sigma} \sigma^*\big(\mathcal{K}^\bullet_{\sigma(J_1)} \otimes^L \cdots \otimes^L \mathcal{K}^\bullet_{\sigma(J_m)}\big) \xrightarrow{\lambda_{\tilde{\sigma}^{-1}}} \sigma^*\big(\mathcal{K}^\bullet_{J_1} \otimes^L \cdots \otimes^L \mathcal{K}^\bullet_{J_m}\big)$$

where λ_σ is the *geometric action* and is induced by isomorphisms

$$\mathcal{K}^\bullet_{J_1} \otimes^L \cdots \otimes^L \mathcal{K}^\bullet_{J_m} \longrightarrow \sigma^*\mathcal{K}^\bullet_{\sigma(J_1)} \otimes^L \cdots \otimes^L \sigma^*\mathcal{K}^\bullet_{\sigma(J_m)} \simeq \sigma^*\big(\mathcal{K}^\bullet_{\sigma(J_1)} \otimes^L \cdots \otimes^L \mathcal{K}^\bullet_{\sigma(J_m)}\big) \ ,$$

while $\lambda_{\tilde{\sigma}^{-1}}$ is the *permutation of factors* induced by $\tilde{\sigma}^{-1}$: it operates the same way as the permutations described in remark 3.2. At the level of the spectral sequence (3.2) and in the identification of $\mathrm{Tor}_{-q}(\mathcal{K}^{i_1}_{J_1}, \dots, \mathcal{K}^{i_m}_{J_m})$ with $\mathrm{Tor}_{-q}(\Delta, \Gamma)$ for some subgraph Γ of K_n without isolated vertices, the action of an element $(\sigma, \tilde{\sigma})$ is expressed through compositions of isomorphisms

$$\mathrm{Tor}_{-q}(\Delta, \Gamma) \xrightarrow{\lambda_\sigma} \sigma^*\mathrm{Tor}_{-q}(\Delta, \sigma(\Gamma)) \xrightarrow{\lambda_{\tilde{\sigma}^{-1}}} \sigma^*\mathrm{Tor}_{-q}(\Delta, \sigma(\Gamma))$$

where λ_σ is described by the geometric action seen throughout subsection 2.2 and where $\lambda_{\tilde{\sigma}^{-1}}$ is the derived permutative action described in remark 3.2: in other words, after formula 3.3, $\lambda_{\tilde{\sigma}^{-1}}$ operates with the sign $\varepsilon_{E_{\sigma(\Gamma)}}(\tilde{\sigma}^{-1}) = \varepsilon_{E_{\sigma(\Gamma)}}(\tilde{\sigma})$, where $\tilde{\sigma}$ is naturally seen in $\mathfrak{S}(E_{\sigma(\Gamma)})$ and where $\varepsilon_{E_{\sigma(\Gamma)}}$ is the alternating representation of $\mathfrak{S}(E_{\sigma(\Gamma)})$. In particular, if $\sigma \in \mathfrak{S}_\Gamma$, for the comprehensive $\mathfrak{S}_\Gamma$ action, we have the isomorphism of $\mathfrak{S}_\Gamma$-representations:

$$\mathrm{Tor}_{-q}(\Delta, \Gamma) \simeq \Lambda^q(\boxplus_{i=1}^k \Omega^1_X \otimes q_{\Gamma_i})_{\Gamma_i} \otimes \mathrm{Res}_{\mathfrak{S}_\Gamma} \varepsilon_{E_\Gamma} \ .$$

From now on, we wil omit talking about the group $\Delta_{\mathfrak{S}_n}$ and, for brevity's sake, when considering the $\mathfrak{S}_n$-action on the derived tensor product $\mathcal{K}^\bullet_{1,2} \otimes^L \cdots \otimes^L \mathcal{K}^\bullet_{n-1,n}$, we will always tacitly intend the $\Delta_{\mathfrak{S}_n}$-action explained here above.

Remark 3.4. Denote with $\mathcal{G}_{l,n}$ the set of subgraphs of the complete graph K_n without isolated vertices and with l edges. We can form $\mathfrak{S}_n$-equivariant complexes of $\mathcal{O}_{X^n}$-modules $(\mathit{\Gamma}^\bullet_q, \partial^\bullet)$ on X^n by setting $\mathit{\Gamma}^0_0 := \mathcal{O}_{X^n}$, $\mathit{\Gamma}^p_q := \oplus_{\Gamma \in \mathcal{G}_{p,n}} \Lambda^q(Q^*_\Gamma)$ and where the differential $\partial^p : \mathit{\Gamma}^p_q \longrightarrow \mathit{\Gamma}^{p+1}_q$ is defined, over the component $\Lambda^q(Q^*_{\Gamma'})$, $\Gamma' \in \mathcal{G}_{p+1,n}$, as the alternating sum:

$$\partial^p(x)_{\Gamma'} = \sum_{\substack{\Gamma \in \mathcal{G}_{p,n} \\ \Gamma \subseteq \Gamma'}} \varepsilon_{\Gamma,\Gamma'} i_{\Gamma,\Gamma'}(x)$$

over the subgraphs of Γ' with p-edges of the inclusions $i_{\Gamma,\Gamma'}$. The sign $\varepsilon_{\Gamma,\Gamma'}$ is defined as $(-1)^{a-1}$, where a is the position in Γ' — according to the lexicographic order — of the only edge in Γ' which is not in Γ. It is now immediate, using proposition 2.14, to show that the complexes $\mathit{\Gamma}^\bullet_q$ are $\mathfrak{S}_n$-equivariant and isomorphic to $E_1^{\bullet,q}$. The complexes $\mathit{\Gamma}^\bullet_q$ are not exact in general, as we will see in the sequel; however, they seem to arise in a pretty natural way as combinatorial objects, without the need to be linked to multitors; they might have an interest on their own. On the other hand it seems difficult to describe a general pattern for their cohomology.

Notation 3.5. In what follows, if $H \subseteq \{1, \dots, n\}$ is a cardinality 3 multi-index, we will indicate with $K_3(H)$ the complete graph with vertices in H, which is a 3-cycle. Sometimes, for brevity's sake, and when there is no risk of confusion, we will indicate this 3-cycle directly with H, instead of $K_3(H)$.

Notation 3.6. For each $r \in \mathbb{N}$, we will write the sheaf $(\Omega^1_X \boxtimes \mathcal{O}_{X^{n-3}}) \otimes \mathcal{I}^r_{\Delta_{n-2}}$ over the variety $X \times X^{n-3}$ just with $\Omega^1_X \boxtimes \mathcal{I}^r_{\Delta_{n-2}}$, or with $\Omega^1_X(-r\Delta_{n-2})$. We will also indicate with $\Omega^1_X \boxtimes \mathcal{O}_{\Delta_{n-2}}$ the sheaf $(\Omega^1_X \boxtimes \mathcal{O}_{X^{n-3}}) \otimes \mathcal{O}_{\Delta_{n-2}}$.

Lemma 3.7. *The kernel of the first differential* $d_1 : E_1^{1,0} \longrightarrow E_1^{2,0}$ *of the spectral sequence* $E_1^{p,q}$ *is given by*

$$\ker d_1 \simeq \Big(\oplus_{|I|=2} \mathcal{O}_{\Delta_I} \Big)_0 := \Big\{ (f_I)_I \in \oplus_{|I|=2} \mathcal{O}_{\Delta_I} \mid (f_J - f_K)\big|_{\Delta_J \cap \Delta_K} = 0, \ \forall\, J, K,\ J \neq K \Big\} \ .$$

The term $E_2^{3,-1}$ *is given by:* $E_2^{3,-1} \simeq \bigoplus_{|H|=3} Q^*_{K_3(H)} \otimes \bigcap_{|J|=2, J \not\subseteq H} \mathcal{I}_{\Delta_J} \simeq \bigoplus_{|H|=3} (\Omega^1_X \boxtimes \mathcal{I}_{\Delta_{n-2}})_{K_3(H)}$.

Proof. The first statement is a consequence of the fact that the map $\partial^1 : \mathit{\Gamma}^1_0 \longrightarrow \mathit{\Gamma}^2_0$ in remark 3.4 is given by

$$(\partial^1 (f_I)_I)_\Gamma = (\varepsilon_{J,\Gamma} f_J + \varepsilon_{K,\Gamma} f_K)\big|_{\Delta_J \cap \Delta_K} = \varepsilon_{J,\Gamma} (f_J - f_K)\big|_{\Delta_J \cap \Delta_K} \ ,$$

where Γ is the graph with two edges J and K.

The second statement follows in a similar way, considering that $E_1^{3,1} \simeq \mathit{\Gamma}^3_1 \simeq \bigoplus_{|H|=3} Q^*_{K_3(H)}$ and that the differential $\partial^3 : \mathit{\Gamma}^3_1 \longrightarrow \mathit{\Gamma}^4_1$ is induced by restrictions

$$\partial^3 ((x_H)_H)_{K_3(L) \cup J} = \varepsilon_{K_3(L), K_3(L) \cup J}\, x_L \big|_{\Delta_L \cap \Delta_J} \ ,$$

where H and L are cardinality 3 multi-indexes and J is a cardinality 2 multi-index. Hence $(x_H)_H \in \bigoplus_{|H|=3} Q^*_{K_3(H)}$ belongs to $E_2^{3,1}$ if and only each restriction $x_H\big|_{\Delta_H \cap \Delta_J}$ is zero. But this means exactly it belongs to $\bigoplus_{|H|=3} Q^*_{K_3(H)} \otimes \cap_{|J|=2, J \not\subseteq H} \mathcal{I}_{\Delta_J}$. Note that each sheaf $Q^*_{K_3(H)} \otimes \cap_{|J|=2, J \not\subseteq H} \mathcal{I}_{\Delta_J}$ is isomorphic to $(\Omega^1_X \boxtimes \mathcal{I}_{\Delta_{n-2}})_{K_3(H)}$. ☐

Remark 3.8. By Danila's lemma, we have the isomorphism of sheaves of invariants over $S^n X$

$$(E_1^{p,q})^{\mathfrak{S}_n} \simeq \bigoplus_{[\Gamma] \in \mathcal{G}_{l,n} / \mathfrak{S}_n} \pi_* (\mathrm{Tor}_{-q}(\Delta, \Gamma))^{\mathfrak{S}_\Gamma} \simeq \bigoplus_{[\Gamma] \in \mathcal{G}_{l,n} / \mathfrak{S}_n} \pi_* (\Lambda^q (Q^*_\Gamma))^{\mathfrak{S}_\Gamma}$$

where on the right hand sides we consider the comprehensive $\mathfrak{S}_n$-action. More in general, we consider a subgroup G of $\mathfrak{S}_n$. Hence the sheaves of G-invariants over the symmetric variety $S^n X$

$$\pi_* (E_1^{p,q})^G \simeq \bigoplus_{[\Gamma] \in \mathcal{G}_{l,n} / G} \pi_* (\mathrm{Tor}_{-q}(\Delta, \Gamma))^{\mathrm{Stab}_G(\Gamma)} \simeq \bigoplus_{[\Gamma] \in \mathcal{G}_{l,n} / G} \pi_* (\Lambda^q (Q^*_\Gamma))^{\mathrm{Stab}_G(\Gamma)}$$

The following table lists the groups $\mathfrak{S}_\Gamma$ and the representation $\mathrm{Res}_{\mathfrak{S}_\Gamma} \varepsilon_{E_\Gamma}$ for all isomorphisms classes of non empty graphs $\Gamma \subseteq K_4$ without isolated vertices. Here we indicate with A_1 the graph with a single edge, with A_2 a graph with two intersecting edges, with B_2 a graph with two non-intersecting edges, with A_3 and B_3 the acyclic subgraphs of K_4 with three edges and with, respectively, no vertex of degree 3 and a single vertex of degree 3.

	$\Gamma = A_1$	$\Gamma = A_2$	$\Gamma = B_2$	$\Gamma = A_3$	$\Gamma = B_3$	$\Gamma = K_3$	$\Gamma = K_3 \cup J$	$\Gamma = C_4$	$\Gamma = C_4 \cup L$	$\Gamma = K_4$
$\mathfrak{S}_\Gamma$	$\mathfrak{S}_2 \times \mathfrak{S}_2$	$\mathfrak{S}_2$	$\mathfrak{S}_2 \times \mathfrak{S}_2 \times \mathfrak{S}_2$	$\mathfrak{S}_2$	$\mathfrak{S}_2$	$\mathfrak{S}_3$	$\mathfrak{S}_2$	D_4	$\mathfrak{S}_2 \times \mathfrak{S}_2$	$\mathfrak{S}_4$
$\mathrm{Res}_{\mathfrak{S}_\Gamma} \varepsilon_{E_\Gamma}$	1	ε	$1 \otimes 1 \otimes \varepsilon$	ε	ε	ε	ε	$\ell(-1_\rho, 1_\sigma)$	1	1

Table 2

The case of the graph $\Gamma = B_2$ needs a line of explanation. We can suppose that Γ is the graph consisting of the edges $\{1,2\}, \{3,4\}$; hence $\mathfrak{S}_\Gamma = \langle (12) \rangle \times \langle (34) \rangle \times \langle (13)(24) \rangle$. The subgroup $\langle (13)(24) \rangle$ is isomorphic to $\mathfrak{S}_2^\Gamma$ and acts nontrivially in the representation $\mathrm{Res}_{\mathfrak{S}_2 \times \mathfrak{S}_2 \times \mathfrak{S}_2} \varepsilon_{E_\Gamma}$.

Lemma 3.9. *The invariants* $(E_1^{2,0})^{\mathfrak{S}_n}$ *of the term* $E_1^{2,0}$ *of the spectral sequence* $E_1^{p,q}$ *are isomorphic to the sheaf* $\mathcal{A}_4(\mathcal{O}_X) := \pi_* \Big((\mathcal{O}_{\Delta_{12}} \otimes \mathcal{O}_{\Delta_{34}}) \otimes \varepsilon \otimes 1 \Big)^{\langle (13)(34) \rangle \otimes \mathfrak{S}(\{5, \dots, n\})}$. *Over an affine open set* $S^n U$ *its module of sections is isomorphic to* $\Lambda^2 H^0(\mathcal{O}_U) \otimes S^{n-4} H^0(\mathcal{O}_U)$.

Proof. We first remark that, for any $n \geq 4$, the types A_2 and B_2 are the only isomorphism classes of subgraphs of K_n with 2 edges and without isolated vertices. By remark 3.8, we have that $\pi_* (E_1^{2,0})^{\mathfrak{S}_n} \simeq \pi_* (\mathcal{O}_{\Delta_{\Gamma_1}} \otimes \mathrm{Res}_{\mathfrak{S}_{\Gamma_1}} \varepsilon_{\Gamma_1})^{\mathfrak{S}_{\Gamma_1}} \oplus \pi_* (\mathcal{O}_{\Gamma_2} \otimes \mathrm{Res}_{\mathfrak{S}_{\Gamma_2}} \varepsilon_{\Gamma_2})^{\mathfrak{S}_{\Gamma_2}}$, where Γ_1 is a graph of type A_2, and Γ_2 is a graph of type B_2. Now the first summand is zero and the second identifies to the one in the statement, when taken Γ_2 to be the graph with edges $\{1,2\}$, $\{3,4\}$. ☐

Notation 3.10. Let $n, l \in \mathbb{N}^*$, $l < n$. We denote with w_l the morphism $X \times S^{n-l}X \longrightarrow S^n X$ sending (x, y) to the 0-cycle $lx + y$. It is a finite morphism if $l = 1$ and a closed immersion if $l \geq 2$.

Proposition 3.11. *In the identifications $(E_1^{1,0})^{\mathfrak{S}_n} \simeq w_{2*}(\mathcal{O}_X \boxtimes \mathcal{O}_{S^{n-2}X})$ and $(E_1^{2,0})^{\mathfrak{S}_n} \simeq \mathcal{A}_4(\mathcal{O}_X)$, the invariant differential $d_1^{\mathfrak{S}_n} : (E_1^{1,0})^{\mathfrak{S}_n} \longrightarrow (E_1^{2,0})^{\mathfrak{S}_n}$ of the spectral sequence $(E_1^{p,q})^{\mathfrak{S}_n}$ is determined locally, over an affine open set of the form $S^n U = \operatorname{Spec} S^n A$, by the formula*

$$d_1^{\mathfrak{S}_n}(a \otimes b_1 . \dots . b_{n-2}) = \sum_{1 \leq i < j \leq n-2} (a \otimes b_i b_j - b_i b_j \otimes a) \otimes \widehat{b_{ij}} \ ,$$

where $a, b_i \in A$.

Proof. The expression is obtained — over an affine open set of the form $S^n U$ as in the statement — by identifying $\pi_*(E_1^{2,0})^{\mathfrak{S}_n}$ with $\pi_*(\mathcal{O}_{\Delta_{\Gamma_2}})^{\mathfrak{S}_{\Gamma_2}}$ where Γ_2 is the graph with edges $\{1,2\}$, $\{3,4\}$ we considered in the proof of lemma 3.9. Hence the map of invariants $d_1^{\mathfrak{S}_n}$ can be identified with the morphism

$$w_{2*}(\mathcal{O}_X \boxtimes \mathcal{O}_{S^{n-2}X}) \simeq \pi_*(\mathcal{O}_{\Delta_{12}})^{\mathfrak{S}(\{3,\dots,n\})} \simeq [\pi_*(\mathcal{O}_{\Delta_{12}}) \oplus \pi_*(\mathcal{O}_{\Delta_{34}})]^{\mathfrak{S}_{\Gamma_2}} \longrightarrow \pi_*(\mathcal{O}_{\Delta_{\Gamma_2}})^{\mathfrak{S}_{\Gamma_2}} \simeq \mathcal{A}_4(\mathcal{O}_X)$$

given by

$$\begin{aligned} d_1^{\mathfrak{S}_n}(a \otimes b_1 \dots b_{n-2}) &= d_1(a \otimes b_1 \dots b_{n-2} + (13)(24)_*(a \otimes b_1 \dots b_{n-2})) \\ &= d_1(a \otimes b_1 \dots b_{n-2}) + (13)(24)_* d_1(a \otimes b_1 \dots b_{n-2}) \\ &= \sum_{1 \leq i < j \leq n-2} (a \otimes b_i b_j - b_i b_j \otimes a) \otimes \widehat{b_{ij}} \end{aligned}$$

where we saw the element $b_1 . \dots . b_{n-2}$ as a $\mathfrak{S}(\{3,\dots,n\})$-invariant element in $H^0(U)^{\otimes n-2}$. ☐

Notation 3.12. We denote the kernel $\ker d_1^{\mathfrak{S}_n}$ with $w_{2*}(\mathcal{O}_X \boxtimes \mathcal{O}_{S^{n-2}X})_0$.

The next lemma is immediate from lemma 3.7.

Lemma 3.13. *The invariants $(E_2^{3,-1})^{\mathfrak{S}_n}$ of the term $E_2^{3,-1}$ of the spectral sequence $E_1^{p,q}$ are isomorphic to the sheaf $w_{3*}((\Omega_X^1 \boxtimes \mathcal{I}_{\Delta_{n-2}})^{\mathfrak{S}_{n-3}})$, where $\mathfrak{S}_{n-3}$ acts on the factor X^{n-3} of the product $X \times X^{n-3}$.*

3.2 The case $n = 3$.

From now on we will place ourselves over a smooth algebraic surface X.

Theorem 3.14. *Let X be a smooth algebraic surface. The complex of coherent sheaves over $S^3 X$*

$$\mathbb{I}_3^\bullet : \qquad 0 \longrightarrow \mathcal{O}_{S^3 X} \xrightarrow{r} w_{2*}(\mathcal{O}_{X \times X}) \xrightarrow{D} w_{3*}(\Omega_X^1) \longrightarrow 0$$

— where r is the restriction and d is given locally by $D(a \otimes b) = 2adb - bda$ — is a resolution of the sheaf of invariants $(\mathcal{I}_{\Delta_3})^{\mathfrak{S}_3}$.

Proof. All subgraphs $\Gamma \subseteq K_3$ are connected. Hence the multitors $\operatorname{Tor}_q(\Delta, \Gamma)$ are isomorphic to

$$\Lambda^q(\Omega_X^1 \otimes q_\Gamma)_\Gamma \otimes \operatorname{Res}_{\mathfrak{S}_\Gamma} \varepsilon_{E_\Gamma}$$

for the comprehensive $\mathfrak{S}_\Gamma$-action. Combining table 1 with table 2 we see immediately that the term $(E_1^{3,-2})^{\mathfrak{S}_3} \simeq \operatorname{Tor}_2(\Delta, K_3)^{\mathfrak{S}_3} \simeq w_{3*}(K_X \otimes \varepsilon)^{\mathfrak{S}_3}$ vanishes. Moreover, $(E_1^{2,0})^{\mathfrak{S}_3} \simeq \pi_*(\mathcal{O}_{\Delta_{123}} \otimes \varepsilon)^{\mathfrak{S}_2}$, $(E_1^{3,0})^{\mathfrak{S}_3} \simeq \pi_*(\mathcal{O}_{\Delta_{123}} \otimes \varepsilon)^{\mathfrak{S}_3}$: hence there are no $\mathfrak{S}_3$-invariants for $q = 0$, $p = 2, 3$. The only nonzero terms in the spectral sequence of invariants are of the form $(E_1^{0,0})^{\mathfrak{S}_3}$, $(E_1^{1,0})^{\mathfrak{S}_3}$ and $(E_1^{3,-1})^{\mathfrak{S}_3}$. The first two are easily proven to be isomorphic to the sheaves $\mathcal{O}_{S^3 X}$ and $w_{2*}(\mathcal{O}_{X \times X})$, respectively. The last one is $(E_1^{3,-1})^{\mathfrak{S}_3} \simeq \operatorname{Tor}_1(\Delta, K_3)^{\mathfrak{S}_3} \simeq w_{3*}(\Omega_X^1)$. Hence we have the resolution of the statement where the map $D : w_{2*}(\mathcal{O}_{X \times X}) \longrightarrow w_{3*}(\Omega_X^1)$ is induced by the second differential $d_2^{\mathfrak{S}_3}$ of the spectral sequence of invariants; the precise local expression of the map D follows from proposition A.12 in the appendix. ☐

Let $G = \mathfrak{S}(\{23\})$. In section 4.3 we will need the following result about the invariants $\pi_*(\mathcal{I}_{\Delta_3})^G$.

Proposition 3.15. *Let X be a smooth algebraic surface. Over S^3X, the sheaf of invariants $\pi_*(\mathcal{I}_{\Delta_3})^G$ is resolved by the complex*

$$0 \longrightarrow w_{1*}(\mathcal{O}_X \boxtimes \mathcal{O}_{S^2X}) \longrightarrow \big[w_{2*}(\mathcal{O}_X \boxtimes \mathcal{O}_X)^{\oplus 2}\big]_0 \longrightarrow w_{3*}(\Omega^1_X) \longrightarrow 0\,.$$

Here the sheaf $\big[w_{2*}(\mathcal{O}_X \boxtimes \mathcal{O}_X)^{\oplus 2}\big]_0$ is the kernel of the map $w_{2*}(\mathcal{O}_X \boxtimes \mathcal{O}_X)^{\oplus 2} \longrightarrow w_{2*}(\mathcal{O}_{\Delta_2})$ given locally by $(a \otimes u, b \otimes v) \longmapsto au - bv$. The first map of the complex is locally defined as $a \otimes u.v \longmapsto (au \otimes v + av \otimes u, 2uv \otimes a)$, while the second is determined by $(a \otimes u, b \otimes v) \longmapsto 2adu - vdb$.

Proof. We consider the invariants $\pi_*(E_1^{p,q})^G$ of the spectral sequence $E_1^{p,q}$ by the group $G = \mathfrak{S}(\{23\})$. By proposition 2.12, remark 3.3 and remark 3.8, the terms $\pi_*(E_1^{p,q})$, as G-representations, are

$$\pi_*(E_1^{p,q}) \simeq \bigoplus_{[\Gamma] \in \mathcal{G}_{p,3}\,/G} \Lambda^{-q}(\Omega^1_X \otimes \mathrm{Res}_{\mathrm{Stab}_G(\Gamma)}\, q_\Gamma)_\Gamma \otimes \mathrm{Res}_{\mathrm{Stab}_G(\Gamma)}\, \varepsilon_{E_\Gamma}\,.$$

It is then immediate to prove that $\pi_*(E_1^{0,0})^G \simeq \pi_*(\mathcal{O}_{X^3})^G \simeq w_{1*}(\mathcal{O}_X \boxtimes \mathcal{O}_{S^2X})$, $\pi_*(E_1^{1,0})^G \simeq \pi_*(\mathcal{O}_{\Delta_{12}}) \oplus \pi_*(\mathcal{O}_{\Delta_{23}}) \simeq w_{2*}(\mathcal{O}_X \boxtimes \mathcal{O}_X)^{\oplus 2}$, $\pi_*(E_1^{2,0})^G \simeq \pi_*(\mathcal{O}_{\Delta_{123}}) \simeq w_{2*}(\mathcal{O}_{\Delta_2})$. For $q < 0$ the only nontrivial terms is

$$\pi_*(E_1^{3,-1})^G \simeq w_{3*}(\Omega^1_X)$$

since $\pi_*(E_1^{3,-2})^G \simeq \big[w_{3*}(\Lambda^2(\Omega^1_X \otimes \varepsilon)) \otimes \varepsilon\big]^G = 0$. It is now easy to see that $\pi_*(E_2^{1,0})^G \simeq \big[w_{2*}(\mathcal{O}_X \boxtimes \mathcal{O}_X)^{\oplus 2}\big]_0$; hence, drawing the page E_2 of the spectral sequence, we get the complex in the statement. To prove that the maps are the ones mentioned above, one sees immediately that the first is induced by restrictions, while for the second one has just to track down the higher differential d_2^G, but this is done easily taking G-invariants in the statement of corollary A.11. □

Remark 3.16. Theorem 3.14 and proposition 3.15 continue to hold when X is an arbitrary smooth algebraic variety, with essentially the same proof; we just used that the only nontrivial terms in the spectral sequences of invariants $(E_1^{p,q})^{\mathfrak{S}_3}$ and $(E_1^{p,q})^G$, for $p + q > 0$, happen for $(p, q) = (0, 0), (1, 0), (3, -1)$; this is true in any dimension.

3.3 The case $n = 4$.

Let X be a smooth algebraic surface. In order to understand the $\mathfrak{S}_4$-invariants of the sheaf $\mathcal{I}_{\Delta_4}$, we have to work out the spectral sequence $\pi_*(E_1^{p,q})^{\mathfrak{S}_4}$; the first step, by virtue of remark 3.8, is to compute, for each class $[\Gamma] \in \Gamma \in \mathcal{G}_{p,4}\,/\mathfrak{S}_4$, the invariants $\pi_*(\mathrm{Tor}_{-q}(\Delta, \Gamma) \otimes \mathrm{Res}_{\mathfrak{S}_\Gamma}\, \varepsilon_{E_\Gamma})^{\mathfrak{S}_\Gamma}$. For $q < 0$, we are just interested in graphs with at least one cycle, which are all connected: the above sheaves then have the form

$$\pi_*(\mathrm{Tor}_{-q}(\Delta, \Gamma) \otimes \mathrm{Res}_{\mathfrak{S}_\Gamma}\, \varepsilon_{E_\Gamma})^{\mathfrak{S}_\Gamma} \simeq \pi_*(\Lambda^{-q}(\Omega^1_X \otimes q_\Gamma)_\Gamma \otimes \mathrm{Res}_{\mathfrak{S}_\Gamma}\, \varepsilon_{E_\Gamma})^{\mathfrak{S}_\Gamma}$$

and hence can be computed easily by combining table 1 with table 2. For convenience of the reader we present the computation of the invariants $\pi_*(\mathrm{Tor}_{-q}(\Delta, \Gamma) \otimes \mathrm{Res}_{\mathfrak{S}_\Gamma}\, \varepsilon_{E_\Gamma})^{\mathfrak{S}_\Gamma}$ in the following table.

	$\Gamma = K_3$	$\Gamma = K_3 \cup J$	$\Gamma = C_4$	$\Gamma = C_4 \cup L$	$\Gamma = K_4$
$q = -1$	$w_{3*}(\Omega^1_X \boxtimes \mathcal{O}_X)$	$w_{4*}(\Omega^1_X)$	0	0	0
$q = -2$	0	0	0	$w_{4*}(K_X \oplus K_X)$	$w_{4*}(K_X)$
$q = -3$	0	0	0	0	$w_{4*}(S^3\Omega^1_X)$
$q = -4$	0	0	0	$w_{4*}(K_X^2)$	$w_{4*}(K_X^2)$
$q = -5$	0	0	0	0	0
$q = -6$	0	0	0	0	$w_{4*}(K_X^3)$

Table 3

As for $q = 0$, it is clear that

$$\pi_*(E_1^{p,0})^{\mathfrak{S}_4} \simeq \pi_*(\mathit{\Gamma}_0^p)^{\mathfrak{S}_n} \simeq \oplus_{[\Gamma] \in \mathcal{G}_{p,4}\,/\mathfrak{S}_4} \pi_*(\mathcal{O}_{\Delta_\Gamma} \otimes \mathrm{Res}_{\mathfrak{S}_\Gamma}\, \varepsilon_{E_\Gamma})^{\mathfrak{S}_\Gamma}\,.$$

We have the following lemmas

Lemma 3.17. *The complex $\pi_*(E_1^{\bullet,0})^{\mathfrak{S}_4} \simeq \pi_*(\varGamma_0^\bullet)^{\mathfrak{S}_n}$ is quasi isomorphic to the complex*

$$0 \longrightarrow \mathcal{O}_{S^4X} \xrightarrow{r} w_{2*}(\mathcal{O}_X \boxtimes \mathcal{O}_{S^2X}) \xrightarrow{d_1^{\mathfrak{S}_n}} \mathcal{A}_4(\mathcal{O}_X) \longrightarrow 0\ ,$$

which is exact in degree greater or equal than 2. The first map is the restriction, while the second is given locally by $a \otimes b.c \longmapsto a \wedge bc$.

Proof. With the help of table 2, we immediately have that the complex $(\pi_* E_1^{\bullet,0})^{\mathfrak{S}_4}$ is quasi-isomorphic to the complex

$$0 \longrightarrow \mathcal{O}_{S^4X} \xrightarrow{r} w_{2*}(\mathcal{O}_X \boxtimes \mathcal{O}_{S^2X}) \xrightarrow{d_1^{\mathfrak{S}_n}} \mathcal{A}_4(\mathcal{O}_X) \longrightarrow 0 \longrightarrow 0 \longrightarrow \pi_*(\mathcal{O}_{\Delta_{1234}}) \longrightarrow \pi_*(\mathcal{O}_{\Delta_{1234}}) \longrightarrow 0\ .$$

The map $\pi_*(\mathcal{O}_{\Delta_{1234}}) \longrightarrow \pi_*(\mathcal{O}_{\Delta_{1234}})$ is immediately an isomorphism, being induced by the identity on the sheaf $\mathcal{O}_{\Delta_{1234}}$; the map $d_1^{\mathfrak{S}_n} : w_{2*}(\mathcal{O}_X \boxtimes \mathcal{O}_{S^2X}) \longrightarrow \mathcal{A}_4(\mathcal{O}_X)$ is surjective, by proposition 3.11, since it is given locally, on an affine open subset of the form S^nU, $U = \mathrm{Spec}(A)$, by the map $A \otimes S^2A \longrightarrow \Lambda^2 A$, sending $a \otimes b.c$ to $a \wedge bc = a \otimes bc - bc \otimes a$. □

Lemma 3.18. *Consider a graph Γ of the kind $C_4 \cup L$. Then the vector space $\Lambda^4(\mathbb{C}^2 \otimes q_\Gamma)$ is completely $\mathfrak{S}_\Gamma$-invariant. Moreover, the composition*

$$c : \Lambda^4(\mathbb{C}^2 \otimes q_\Gamma) \longrightarrow \Lambda^4(\mathbb{C}^2 \otimes q_{K_4}) \longrightarrow \Lambda^4(\mathbb{C}^2 \otimes q_{K_4})^{\mathfrak{S}_4}$$

where the first map is the injection i_{Γ,K_4} and the second is the projection onto the invariants, is an isomorphism.

Proof. The first statement is a consequence of table 1, case in which Γ is of the kind $C_4 \cup L$, $q = 4$, and $X = \mathbb{C}^2$. By the same table we can also deduce that the vector space $\Lambda^4(\mathbb{C}^2 \otimes q_{K_4})^{\mathfrak{S}_4}$ is one dimensonal. Therefore, to prove the second statement, we just have to prove that the map c is nonzero. It is not restrictive to suppose that Γ is defined by edges $\{1,2\}, \{1,3\}, \{1,4\}, \{2,3\}, \{3,4\}$. Indicating an oriented 3-cycle with a sequence of its vertices, the basis of $\mathbb{C}^2 \otimes q_\Gamma$ is then given by vectors $e_i \otimes e_{123}$ and and $e_i \otimes e_{134}$, $i = 1, 2$. Denote more briefly, for an oriented 3-cycle H, with $\gamma_H = e_1 \otimes e_H$ and $\delta_H = e_2 \otimes e_H$, where we write the elements in the set H in an order according to the given orientation. Hence we can take as a basis of $\mathbb{C}^2 \otimes q_\Gamma$ the vectors γ_{123}, δ_{123}, γ_{134}, δ_{134}; similarly, as a basis of $\mathbb{C}^2 \otimes q_{K_4}$ we take the previous vectors, to which we add $\gamma_{124}, \delta_{124}$. For brevity's sake, denote with $\alpha_H := \gamma_H \wedge \delta_H$. A basis of $\Lambda^4(\mathbb{C}^2 \otimes q_\Gamma)$, which is fully invariant, is then given by $a = \alpha_{123} \wedge \alpha_{134}$. Since this element is invariant by $\mathfrak{S}_\Gamma \simeq \langle(13)\rangle \times \langle(24)\rangle$, we have that the map c is, up to a constant, given by

$$c(a) = \sum_{[\sigma] \in \mathfrak{S}_4/\mathfrak{S}_\Gamma} \sigma_* i_{\Gamma,K_4} a\ .$$

Now the cosets $\mathfrak{S}_4/\mathfrak{S}_\Gamma$ can be represented by the set $\{\mathrm{id}, (12), (14), (23), (34), (12)(34)\}$, hence $c(a)$ is given, up to a constant by

$$\begin{aligned} c(a) &= \alpha_{123} \wedge \alpha_{134} + \alpha_{213} \wedge \alpha_{234} + \alpha_{423} \wedge \alpha_{431} + \alpha_{132} \wedge \alpha_{124} + \alpha_{124} \wedge \alpha_{143} + \alpha_{214} \wedge \alpha_{243} \\ &= \alpha_{123} \wedge \alpha_{134} + \alpha_{123} \wedge \alpha_{234} + \alpha_{234} \wedge \alpha_{134} + \alpha_{123} \wedge \alpha_{124} + \alpha_{124} \wedge \alpha_{134} + \alpha_{124} \wedge \alpha_{234} \end{aligned}$$

Now, using that $\alpha_{234} = \alpha_{123} + \alpha_{134} - \alpha_{124}$, we get easily that, up to a constant,

$$c(a) = 3\big(\alpha_{123} \wedge \alpha_{124} + \alpha_{123} \wedge \alpha_{134} + \alpha_{124} \wedge \alpha_{134}\big)$$

which is a nonzero element of $\Lambda^4(\mathbb{C}^2 \otimes q_{K_4})^{\mathfrak{S}_4} \subseteq \Lambda^4(\mathbb{C}^2 \otimes q_{K_4})$. □

Notation 3.19. Consider now the second invariant differential

$$A := d_2^{\mathfrak{S}_4} : w_{3*}(\Omega^1_X \boxtimes \mathcal{I}_{\Delta_2}) \simeq \pi_*(E_2^{3,-1})^{\mathfrak{S}_4} \simeq \ \longrightarrow \pi_*(E_2^{5,-2})^{\mathfrak{S}_4}\ .$$

Its precise expression is determined in the appendix, corollary A.15. We denote with $w_{3*}(\Omega^1_X \boxtimes \mathcal{I}_{\Delta_2})_0$ its kernel.

Recall notation 3.12. We have the following result.

Theorem 3.20. *The sheaf of invariants $(\mathcal{I}_{\Delta_4})^{\mathfrak{S}_4}$ admits a right resolution given by a natural complex*

$$\mathbb{I}_4^\bullet := 0 \longrightarrow \mathcal{O}_{S^4X} \xrightarrow{r} w_{2*}(\mathcal{O}_X \boxtimes \mathcal{O}_{S^2X})_0 \xrightarrow{D} w_{3*}(\Omega^1_X \boxtimes \mathcal{I}_{\Delta_2})_0 \xrightarrow{C} w_{4*}(S^3\Omega^1_X) \longrightarrow 0\ ,$$

where r is the restriction, D is defined locally as $D(a \otimes u.v) = (2adu - uda) \otimes v + (2adv - vda) \otimes u$ and C is determined, up to a constant, by $C(\omega \otimes f) = -\operatorname{sym}(\omega \otimes d^2_\Delta f)$, where $\omega \in \Omega^1_X$ and $f \in \mathcal{I}_{\Delta_2}$.

Proof. The abutment of the spectral sequence of invariants $\pi_*(E_1^{p,q})^{\mathfrak{S}_4}$ is zero in positive degree, after remark 3.1. The complex $\pi_*(E_1^{\bullet,0})^{\mathfrak{S}_4}$ has been studied in the previous lemma. At level $q = -1$, recalling table 3 and notation 3.6, we just have the complex $w_{3*}(\Omega^1_X \boxtimes \mathcal{O}_X) \longrightarrow w_{3*}(\Omega^1_X \boxtimes \mathcal{O}_{\Delta_2}) \simeq w_{4*}(\Omega^1_X)$, in degree 3 and 4, and at level $q = -2$ we have the complex $w_{4*}(K_X \oplus K_X) \longrightarrow w_{4*}(K_X)$, in degree 5 and 6. Moreover, at level $q = -4$, we have the complex $w_{4*}(K_X^2) \longrightarrow w_{4*}(K_X^2)$, in degree 5 and 6. The only other nonzero terms for $q < 0$ are $\pi_*(E_1^{6,-3})^{\mathfrak{S}_4} \simeq w_{4*}(S^3\Omega^1_X)$ and $\pi_*(E_1^{6,-6})^{\mathfrak{S}_4} \simeq w_{4*}(K_X^3)$.

We now prove that the map $w_{3*}(\Omega^1_X \boxtimes \mathcal{O}_X) \longrightarrow w_{4*}(\Omega^1_X)$ is surjective. Indeed it can be seen as the map of $\mathfrak{S}_4$-invariants

$$(E_1^{3,-1})^{\mathfrak{S}_4} \simeq \pi_*((\Omega^1_X)_{K_3(123)})^{\mathfrak{S}_3} \longrightarrow \pi_*((\Omega^1_X)_{K_3(123)\cup\{34\}})^{\mathfrak{S}_2}\ .$$

This map is naturally the composition

$$\pi_*((\Omega^1_X)_{K_3(123)})^{\mathfrak{S}_3} \hookrightarrow \pi_*((\Omega^1_X)_{K_3(123)})^{\mathfrak{S}_2} \longrightarrow \pi_*((\Omega^1_X)_{K_3(123)\cup\{34\}})^{\mathfrak{S}_2}$$

where the first map is the natural inclusion and the second is the map of $\mathfrak{S}_2$-invariants of the map $\pi_*((\Omega^1_X)_{K_3(123)}) \longrightarrow \pi_*((\Omega^1_X)_{K_3(123)\cup\{34\}})$. But this last map is surjective, since the map $(\Omega^1_X)_{K_3(123)} \longrightarrow (\Omega^1_X)_{K_3(123)\cup\{34\}}$ is a restriction and hence surjective, and because π is finite. Moreover, the natural inclusion $\pi_*((\Omega^1)_{K_3(123)})^{\mathfrak{S}_3} \hookrightarrow \pi_*((\Omega^1_X)_{K_3(123)})^{\mathfrak{S}_2}$ is an isomorphism, since both terms coincide with $\pi_*((\Omega^1_X)_{K_3(123)})$. This means in particular that $\pi_*(E_2^{4,-1})^{\mathfrak{S}_4} = 0$.

Let's now look at the map $w_{4*}(K_X \oplus K_X) \longrightarrow w_{4*}(K_X)$. Its cokernel is isomorphic to $\pi_*(E_2^{6,-2})^{\mathfrak{S}_4}$. Now, because of the form of the complexes $\pi_*(E_1^{\bullet,q})^{\mathfrak{S}_4}$ for $q = 0, -1$, and because we proved that $\pi_*(E_2^{4,-1})^{\mathfrak{S}_4} = 0$, there are no nonzero higher invariant differentials $d_l^{\mathfrak{S}_4}$ targeting $\pi_*(E_2^{6,-2})^{\mathfrak{S}_4}$: hence this term has to live till the ∞-page, as a graded sheaf of the abutment, but the abutment in degree 4 is zero. Hence $\pi_*(E_2^{6,-2})^{\mathfrak{S}_4}$ has to vanish and the map $w_{4*}(K_X \oplus K_X) \longrightarrow w_{4*}(K_X)$ has to be surjective. Hence $\pi_*(E_2^{5,-2})^{\mathfrak{S}_4} \simeq w_{4*}(K_X)$.

Finally, we look at the map $\pi_*(E_1^{5,-4})^{\mathfrak{S}_4} \simeq w_{4*}(K_X^2) \longrightarrow w_{4*}(K_X^2) \simeq \pi_*(E_1^{6,-4})^{\mathfrak{S}_4}$ at level $q = -4$. It coincides with the map of invariants

$$\pi_*(\Lambda^4(Q^*_\Gamma))^{\mathfrak{S}_\Gamma} \longrightarrow \pi_*(\Lambda^4(Q^*_{K_4}))^{\mathfrak{S}_4}\ , \tag{3.4}$$

induced by the inclusion $i_{\Gamma,K_4} : Q^*_\Gamma \hookrightarrow Q^*_{K_4}$, where Γ is a graph of the kind $C_4 \cup L$. Now, since both $\mathfrak{S}_\Gamma$ and $\mathfrak{S}_4$ act trivially over $\Delta_{1234} = \Delta_\Gamma$, and $\pi|_{\Delta_{1234}}$ is a closed immersion, the map (3.4) is an isomorphism if and only if the map of line bundles $i_{\Gamma*}K_X^2 \simeq \Lambda^4(Q^*_\Gamma)^{\mathfrak{S}_\Gamma} \longrightarrow \Lambda^4(Q^*_{K_4})^{\mathfrak{S}_4} \simeq i_{\Gamma*}K_X^2$ over Δ_{1234} is. Since $\Lambda^4(Q^*_\Gamma)$ is fully $\mathfrak{S}_\Gamma$-invariant, the previous map coincide, up to constants, with the composition

$$\Lambda^4(Q^*_\Gamma) \longrightarrow \Lambda^4(Q^*_{K_4}) \longrightarrow \Lambda^4(Q^*_{K_4})^{\mathfrak{S}_4}\ ,$$

where the first is the inclusion and the second is the projection onto the invariants. Now, on the fibers, this map is precisely the map of lemma 3.18. Hence it is an isomorphism.

Consequently, at level E_2, the only nonzero terms are

$$\begin{aligned}
&\pi_*(E_2^{0,0})^{\mathfrak{S}_4} \simeq (\mathcal{I}_{\Delta_4})^{\mathfrak{S}_4} && \pi_*(E_2^{1,0})^{\mathfrak{S}_4} \simeq \operatorname{coker}\big(\mathcal{O}_{S^4X} \longrightarrow w_{2*}(\mathcal{O}_X \boxtimes \mathcal{O}_{S^2X})_0\big)\\
&\pi_*(E_2^{3,-1})^{\mathfrak{S}_4} \simeq w_{3*}(\Omega^1_X \boxtimes \mathcal{I}_{\Delta_2})\ , && \pi_*(E_2^{5,-2})^{\mathfrak{S}_4} \simeq w_{4*}(K_X)\\
&\pi_*(E_2^{6,-3})^{\mathfrak{S}_4} \simeq w_{4*}(S^3\Omega^1_X)\ , && \pi_*(E_2^{6,-6})^{\mathfrak{S}_4} \simeq w_{4*}(K_X^3)\ .
\end{aligned}$$

Therefore, drawing the page E_2 of the spectral sequence $\pi_*(E_1^{p,q})^{\mathfrak{S}_4}$, we deduce a complex $0 \longrightarrow \pi_*(E_2^{1,0})^{\mathfrak{S}_4} \longrightarrow w_{3*}(\Omega^1_X \boxtimes \mathcal{I}_{\Delta_2}) \xrightarrow{A} w_{4*}(K_X) \longrightarrow 0$ which is non exact only in the middle, with cohomology isomorphic to $w_{4*}(S^3\Omega^1_X)$; henceforth we have an exact sequence

$$0 \longrightarrow \pi_*(E_2^{1,0})^{\mathfrak{S}_4} \longrightarrow w_{3*}(\Omega^1_X \boxtimes \mathcal{I}_{\Delta_2})_0 \xrightarrow{C} w_{4*}(S^3\Omega^1_X) \longrightarrow 0 .$$

The statement of the theorem follows, since one has as well an exact sequence

$$0 \longrightarrow (\mathcal{I}_{\Delta_4})^{\mathfrak{S}_4} \longrightarrow \mathcal{O}_{S^4X} \longrightarrow w_{2*}(\mathcal{O}_X \times \mathcal{O}_{S^2X})_0 \longrightarrow \pi_*(E_2^{1,0})^{\mathfrak{S}_4} \longrightarrow 0 .$$

The precise expression of map D and C will be determined in the appendix, proposition A.12 and proposition A.19. $\square$

To finish this subsection, we present two immediate byproducts of the proof of theorems 3.14 and 3.20.

Corollary 3.21. *For n equal to 3 or 4, the $\mathfrak{S}_n$-invariants of the product ideal $\prod_{I \in E_{K_n}} \mathcal{I}_{\Delta_I} = \mathcal{I}_{\Delta_{12}} \cdot \dots \cdot \mathcal{I}_{\Delta_{n-1,n}}$ and of the ideal $\mathcal{I}_{\Delta_n}$ coincide:*

$$\Big(\mathcal{I}_{\Delta_{12}} \cdot \dots \cdot \mathcal{I}_{\Delta_{n-1,n}}\Big)^{\mathfrak{S}_n} \simeq (\mathcal{I}_{\Delta_n})^{\mathfrak{S}_n} = \Big(\mathcal{I}_{\Delta_{12}} \cap \dots \cap \mathcal{I}_{\Delta_{n-1,n}}\Big)^{\mathfrak{S}_n} .$$

Proof. The invariants $\big(\mathcal{I}_{\Delta_{12}} \cdot \dots \cdot \mathcal{I}_{\Delta_{n-1,n}}\big)^{\mathfrak{S}_n}$ of the product of ideals coincide with the term $\pi_*(E_\infty^{0,0})^{\mathfrak{S}_n}$ of the spectral sequence $\pi_*(E_1^{p,q})^{\mathfrak{S}_n}$ above, since the abutment in degree 0 is the tensor product $\mathcal{I}_{\Delta_{12}} \otimes \dots \otimes \mathcal{I}_{\Delta_{n-1,n}}$ and since $\pi_*(E_\infty^{0,0})$, being the first graded sheaf for the natural filtration on the abutment, is the image of the natural morphism $(\mathcal{I}_{\Delta_{12}} \otimes \dots \otimes \mathcal{I}_{\Delta_{n-1,n}})^{\mathfrak{S}_n} \longrightarrow (\mathcal{O}_{X^n})^{\mathfrak{S}_n} \simeq \mathcal{O}_{S^nX}$. But it is evident from the proof of theorems 3.14 and 3.20 that $\pi_*(E_\infty^{0,0})^{\mathfrak{S}_n} \simeq \pi_*(E_2^{0,0})^{\mathfrak{S}_n} \simeq (\mathcal{I}_{\Delta_n})^{\mathfrak{S}_n}$. $\square$

Remark 3.22. It seems to us an interesting question whether the statement of corollary 3.21 is true for general n; in some contexts (for example when taking inverse images) the product of ideals is better behaved that the intersection; therefore, knowing that, at least at level of invariants, the two coincide, might turn out useful in some applications.

Remark 3.23. When taking the tensor product of ideals, things are clearly different. While it is true that over S^3X, we have the isomorphism $\big(\mathcal{I}_{\Delta_{12}} \otimes \mathcal{I}_{\Delta_{13}} \otimes \mathcal{I}_{\Delta_{23}}\big)^{\mathfrak{S}_3} \simeq \big(\mathcal{I}_{\Delta_{12}} \cdot \mathcal{I}_{\Delta_{13}} \cdot \mathcal{I}_{\Delta_{23}}\big)^{\mathfrak{S}_3}$, over S^4X the two sheaves are definitely not isomorphic; their difference is measured by the sheaf $w_{4*}(K_X^3)$, as the next exact sequence proves:

$$0 \longrightarrow w_{4*}(K_X^3) \longrightarrow \Big(\mathcal{I}_{\Delta_{12}} \otimes \dots \otimes \mathcal{I}_{\Delta_{34}}\Big)^{\mathfrak{S}_4} \longrightarrow (\mathcal{I}_{\Delta_4})^{\mathfrak{S}_4} \longrightarrow 0 .$$

3.4 Twisting by the line bundle $\mathcal{D}_L$

Let now F be a $\mathfrak{S}_n$-equivariant coherent sheaf over X^n. By definition of the line bundle $\mathcal{D}_L$ on the symmetric variety S^nX (see remark 1.1), using projection formula and taking $\mathfrak{S}_n$-invariants we have the following equation:

$$(\pi_*^{\mathfrak{S}_n} F) \otimes \mathcal{D}_L \simeq \pi_*^{\mathfrak{S}_n}(F \otimes L^{\boxtimes n}) .$$

Because of this fact, all results proved in this section continue to work when we tensorize the sheaf $\mathcal{I}_{\Delta_n}$ with a line bundle of the form $L \boxtimes \dots \boxtimes L$, or its invariants by $\mathcal{D}_L$. In particular we have that for n equal to 3 or 4, the sheaf of invariants $\pi_*(\mathcal{I}_{\Delta_n} \otimes L^{\boxtimes n})^{\mathfrak{S}_n} \simeq (\mathcal{I}_{\Delta_n})^{\mathfrak{S}_n} \otimes \mathcal{D}_L$ is resolved by the complex $\mathbb{I}_n^\bullet \otimes \mathcal{D}_L$; in other words

Corollary 3.24. *Over S^3X and S^4X, respectively, we have resolutions*

$$0 \longrightarrow \pi_*(\mathcal{I}_{\Delta_3} \otimes L^{\boxtimes 3})^{\mathfrak{S}_3} \longrightarrow \mathcal{D}_L \longrightarrow w_{2*}(L^2 \boxtimes L) \longrightarrow w_{3*}(\Omega^1_X \otimes L^3) \longrightarrow 0$$

$$\begin{aligned} 0 \longrightarrow \pi_*(\mathcal{I}_{\Delta_4} \otimes L^{\boxtimes 4})^{\mathfrak{S}_3} \longrightarrow \mathcal{D}_L \longrightarrow w_{2*}(L^2 \boxtimes \mathcal{D}_L)_0 \longrightarrow \\ \longrightarrow w_{3*}((\Omega^1_X \otimes L^3 \boxtimes L) \otimes \mathcal{I}_{\Delta_2})_0 \longrightarrow w_{4*}(S^3\Omega^1_X \otimes L^4) \longrightarrow 0 \end{aligned}$$

4 Applications

4.1 Cohomology of $(\det L^{[n]})^2$ for low n.

Theorem 4.1. *Let X be a smooth quasi-projective surface and L and A two line bundles over X. Then, for $n=3$ or $n=4$, the cohomology $H^*(X^{[n]}, (\det L^{[n]})^{\otimes 2} \otimes \mathcal{D}_A)$ is computed by the spectral sequence*

$$E_1^{p,q} := H^q(S^nX, \mathbb{I}^p_n \otimes \mathcal{D}_L^{\otimes 2} \otimes \mathcal{D}_A) \ .$$

Proof. By corollary 1.9 and by lemma 1.5, we have

$$\mathbf{R}\mu_*((\det L^{[n]})^{\otimes 2} \otimes \mathcal{D}_A) \simeq \big(\mathcal{I}^2_{\Delta_n}\big)^{\mathfrak{S}_n} \otimes \mathcal{D}_L^{\otimes 2} \otimes \mathcal{D}_A \simeq \big(\mathcal{I}_{\Delta_n}\big)^{\mathfrak{S}_n} \otimes \mathcal{D}_L^{\otimes 2} \otimes \mathcal{D}_A \ .$$

Hence, applying the functor $\mathbf{R}\Gamma$ on both sides, we get

$$H^*(X^{[n]}, (\det L^{[n]})^{\otimes 2} \otimes \mathcal{D}_A) \simeq \mathbf{R}\Gamma\mathbf{R}\mu_*((\det L^{[n]})^{\otimes 2} \otimes \mathcal{D}_A) \simeq H^*(S^nX, (\mathcal{I}_{\Delta_n})^{\mathfrak{S}_n} \otimes \mathcal{D}_L^{\otimes 2} \otimes \mathcal{D}_A) \ .$$

The spectral sequence in the statement computes the hypercohomology of the complex $\mathbb{I}^\bullet_n \otimes \mathcal{D}_L^{\otimes 2} \otimes \mathcal{D}_A$, which is a resolution of the sheaf $\big(\mathcal{I}_{\Delta_n}\big)^{\mathfrak{S}_n} \otimes \mathcal{D}_L^{\otimes 2} \otimes \mathcal{D}_A$, by theorems 3.14 and 3.20. ☐

An immediate application yields the computation of the Euler-Poincaré characteristic of $(\det L^{[n]})^2 \otimes \mathcal{D}_A$.

Corollary 4.2. *Let X a smooth projective surface and L and A line bundles over X. For $n=3$ and $n=4$ we have the following formulas for the Euler-Poincaré characteristic of $(\det L^{[n]})^2 \otimes \mathcal{D}_A$ over the HIlbert scheme $X^{[n]}$:*

$$\begin{aligned}
\chi(X^{[3]}, (\det L^{[3]})^2 \otimes \mathcal{D}_A) &= \binom{\chi(L^2 \otimes A)+2}{3} - \chi(L^4 \otimes A^2)\chi(L^2 \otimes A) + \chi(\Omega^1_X \otimes L^6 \otimes A^3) \\
\chi(X^{[4]}, (\det L^{[4]})^2 \otimes \mathcal{D}_A) &= \binom{\chi(L^2 \otimes A)+3}{4} - \chi(L^4 \otimes A^2)\binom{\chi(L^2 \otimes A)+1}{2} + \binom{\chi(L^4 \otimes A^2)}{2} + \\
&\quad + \chi(\Omega^1_X \otimes L^6 \otimes A^3)\chi(L^2 \otimes A) - \chi(\Omega^1_X \otimes L^8 \otimes A^4) \\
&\quad - \chi(K_X \otimes L^8 \otimes A^4) - \chi(S^3\Omega^1_X \otimes L^8 \otimes A^4)
\end{aligned}$$

We now mention an effective vanishing result for the cohomology of $(\det L^{[n]})^k \otimes \mathcal{D}_A$, for any n and k.

Remark 4.3. We recall that a line bundle L on a smooth projective surface X is called m-very ample if, for any $\xi \in X^{[m+1]}$, the restriction map $H^0(L) \longrightarrow H^0(L_\xi)$ is surjective. The property of being m-very ample generalizes tha fact of being very ample, since "1-very ample" means exactly "very ample". After results of [BS91] and [CG90], one can prove that $\det L^{[n]}$ is globally generated if L is $(n-1)$-very ample, and that it is actually very ample if L is n-very ample (see also [Sca15a, Cor. 5.10]).

Corollary 4.4. *Let X be a smooth projective surface and L and A two line bundles over X such that $L^k \otimes A \otimes K_X^{-1}$ is a product $\otimes_{i=1}^k B_i$ of line bundles B_i, with B_1 n-very ample and B_j $(n-1)$-very ample, for $j=2,\dots,k$. Then we have the vanishing*

$$H^i(X^{[n]}, (\det L^{[n]})^{\otimes k} \otimes \mathcal{D}_A) = 0 \qquad \textit{for } i > 0 \ .$$

In particular the statement is true if $L^k \otimes A \otimes K_X^{-1}$ is a product of $k(n-1)+1$ very ample line bundles over X.

Proof. One has just to note that, in the hypothesis of the corollary, and since $\mathcal{D}_{K_X} \simeq K_{X^{[n]}}$, $(\det L^{[n]})^{\otimes k} \otimes \mathcal{D}_A \simeq \big(\otimes_{i=1}^k \det B_i^{[n]}\big) \otimes K_{X^{[n]}}$. Then one uses Kodaira vanishing, since all line bundles $\det B_i^{[n]}$ are nef and $\det B_1^{[n]}$ is very ample. The last statement follows from the fact that a product of l 1-very ample line bundles is l-very ample [BS91]. ☐

4.2 Regularity of $\mathcal{I}^k_{\Delta_n}$ and $(\mathcal{I}^k_{\Delta_n})^{\mathfrak{S}_n}$

Notation 4.5. If $s \in \mathbb{Q}$ is a rational number, we denote with $[s]$ its integral part, or round-down, and with $\lceil s \rceil$ its round-up.

Theorem 4.6. *Let X be a smooth projective surface and L be a line bundle over X. Let $n \in \mathbb{N}$, $n \geq 2$. Let $m \in \mathbb{N}$ be an integer with the property*

a) $L^m \otimes K_X^{-1} = \otimes_{i=1}^{2[(k+1)/2]} B_i$, with B_1 n-very ample and with B_j $(n-1)$-very ample, for $j > 1$.

Then we have the vanishing

$$H^i(S^n X, (\mathcal{I}^k_{\Delta_n})^{\mathfrak{S}_n} \otimes \mathcal{D}^m_L) = 0 \qquad \text{for } i > 0 \,.$$

If, moreover, L is very ample on X, then $(\mathcal{I}^k_{\Delta_n})^{\mathfrak{S}_n}$ is $(m+2n)$-regular with respect to $L \boxtimes \cdots \boxtimes L$. Therefore, if m_0 is the minimum of m such that condition a) is true, we have the upper bound

$$\operatorname{reg}((\mathcal{I}^k_{\Delta_n})^{\mathfrak{S}_n}) \leq m_0 + 2n$$

for the regularity of the ideal sheaf $(\mathcal{I}^k_{\Delta_n})^{\mathfrak{S}_n}$ with respect to $\mathcal{D}_L$.

Remark 4.7. We note that condition a) is true in particular if $L^m \otimes K_X^{-1}$ is a tensor product $2n[(k+1)/2] - 2[(k+1)/2] + 1$ very ample line bundles over X. If this holds and, additonaly, the line bundle L is very ample, then the ideal $(\mathcal{I}^k_{\Delta_n})^{\mathfrak{S}_n}$ is $(m+2n)$-regular with respect to $\mathcal{D}_L$.

Proof of theorem 4.6. For the first statement, by lemma 1.5 and by corollary 1.9, we have that

$$(\mathcal{I}^k_{\Delta_n})^{\mathfrak{S}_n} \otimes \mathcal{D}^m_L \simeq (\mathcal{I}^{2[(k+1)/2]}_{\Delta_n})^{\mathfrak{S}_n} \otimes \mathcal{D}^m_L = \mathbf{R}\mu_*((\det \mathcal{O}^{[n]}_X)^{\otimes 2[(k+1)/2]} \otimes \mathcal{D}^m_L) \,.$$

Hence

$$H^i(S^n X, (\mathcal{I}^k_{\Delta_n})^{\mathfrak{S}_n} \otimes \mathcal{D}^m_L) \simeq H^i(X^{[n]}, \mathcal{D}_{L^m}(-2[(k+1)/2]e))$$

and we conclude by [Sca15a, Propositon 5.14].

For the second part, from the first statement and from the fact that, under the hypothesis, $\mathcal{D}_L$ is very ample, we immediately have that $H^i(S^n X, (\mathcal{I}^k_{\Delta_n})^{\mathfrak{S}_n} \otimes \mathcal{D}^{\otimes m+l}_L) = 0$ for $i > 0$ and for $l \geq 0$. Consequently, $(\mathcal{I}^k_{\Delta_n})^{\mathfrak{S}_n}$ has to be $(m+2n)$-regular with respect to $\mathcal{D}_L$. $\square$

Theorem 4.8. *Let X be a smooth projective surface and L be a line bundle over X. Let $n \in \mathbb{N}$, $2 \leq n \leq 7$. Let $m \in \mathbb{N}$ be an integer with the property*

b) $L^m \otimes K_X^{-1} = \otimes_{i=1}^{k+1} B_i$, with B_1 n-very ample and with B_j $(n-1)$-very ample, for $j > 1$.

Then we have the vanishing

$$H^i(X^n, \mathcal{I}^k_{\Delta_n} \otimes (L^m \boxtimes \cdots \boxtimes L^m)) = 0 \qquad \text{for } i > 0 \,.$$

If, moreover, L is very ample, then the ideal sheaf $\mathcal{I}^k_{\Delta_n}$ is $(m+2n)$-regular with respect to $L \boxtimes \cdots \boxtimes L$. Therefore, if m_0 is the minimum of m such that condition b) is true, we have the upper bound

$$\operatorname{reg}(\mathcal{I}^k_{\Delta_n}) \leq m_0 + 2n$$

for the regularity of the ideal sheaf $\mathcal{I}^k_{\Delta_n}$ with respect to $L \boxtimes \cdots \boxtimes L$.

Remark 4.9. We note that condition b) is true if $L^m \otimes K_X^{-1}$ is a tensor product of $(k+1)n - k$ very ample line bundles over X. If this holds and, additionaly, L is very ample, then, for $2 \leq n \leq 7$, the ideal sheaf $\mathcal{I}^k_{\Delta_n}$ is $(m+2n)$ regular with respect to $L \boxtimes \cdots \boxtimes L$.

Proof of theorem 4.8. The first statement follows immediately by [Sca15a, Propositon 5.15] and by the fact that, by [Sca15b, Theorem 2.12], B^n has log-canonical singularities for $n \leq 7$. As for the second, its proof is analogous to the proof of the similar statement for the regularity of $(\mathcal{I}^k_{\Delta_n})^{\mathfrak{S}_n}$ in theorem 4.6. $\square$

The previous regularity results are nicer to state when X has Picard number one.

Corollary 4.10. *Let X be a smooth projective surface with Picard group* $\mathrm{Pic}(X) \simeq \mathbb{Z}B$*, where B is the ample generator. Let r be the minimum positive power of B such that B^r is very ample. Suppose, moreover, that $K_X \simeq B^w$, for some integer w. Then we have the following.*

- *The sheaf $(\mathcal{I}^k_{\Delta_n})^{\mathfrak{S}_n}$ is $(m+2n)$-regular with respect to $\mathcal{D}_{B^r}$, if $m \geq 2n[(k+1)/2]-2[(k+1)/2]+1+w/r$. Hence, with respect to $\mathcal{D}_{B^r}$,*

$$\mathrm{reg}((\mathcal{I}^k_{\Delta_n})^{\mathfrak{S}_n}) \leq 2n([(k+1)/2]+1) - 2[(k+1)/2] + 1 + \lceil w/r \rceil \ .$$

- *If $2 \leq n \leq 7$, the sheaf $\mathcal{I}^k_{\Delta_n}$ is $(m+2n)$-regular with respect to $B^r \boxtimes \cdots \boxtimes B^r$, if $m \geq (k+1)n-k+w/r$. Hence, with respect to $B^r \boxtimes \cdots \boxtimes B^r$,*

$$\mathrm{reg}(\mathcal{I}^k_{\Delta_n}) \leq (k+3)n - k + \lceil w/r \rceil \ .$$

Remark 4.11. If $X = \mathbb{P}_2$, taking $B = \mathcal{O}_{\mathbb{P}_2}(1)$, we can say, more simply, that, $\mathrm{reg}((\mathcal{I}^k_{\Delta_n})^{\mathfrak{S}_n}) \leq 2n([(k+1)/2]+1) - 2[(k+1)/2] - 2$ and, for $2 \leq n \leq 7$, $\mathrm{reg}(\mathcal{I}^k_{\Delta_n}) \leq (k+3)(n-1)$.

Remark 4.12. The results proven in this subsection for $\mathcal{I}^k_{\Delta_n}$ are valid for $2 \leq n \leq 7$. We expect them to hold also for $n = 8$, since B^n should have log-canonical singularities also in this case [Sca15b, Conjecture 2]. However, we don't know, and it seems to us an interesting question, if the previous bound is still a good upper bound for the regularity of $\mathcal{I}^k_{\Delta_n}$, for general n, or, if not, what would be a good one. The proof we gave here can't go through in general since we proved that B^n does not have log-canonical singularities for $n \geq 9$ [Sca15b, Theorem 2.12].

4.3 The sheaves $\mathcal{L}^\mu(-2\mu\Delta)$.

Let $n, k \in \mathbb{N}$, $n \geq 2$, and let μ be a partition of k of length $l(\mu) \leq n$. The symmetric group $\mathfrak{S}_n$ acts naturally on the set of compositions of k supported in $\{1, \ldots, n\}$. Indicate with L^μ the line bundle on X^n defined by $L^\mu := \otimes_{i=1}^n p_i^* L^{\otimes \mu_i}$, where $p_i : X^n \longrightarrow X$ is the projecton onto the i-th factor. In [Sca15a] we defined sheaves $\mathcal{L}^\mu(-2\mu\Delta)$ over the symmetric variety $S^n X$ as

$$\mathcal{L}^\mu(-2\mu\Delta) := \pi_* \Big(L^\mu \otimes \bigcap_{1 \leq i < j \leq l(\mu)} \mathcal{I}^{2\mu_j}_{\Delta_{i,j}} \Big)^{\mathrm{Stab}_{\mathfrak{S}_n}(\mu)}$$

where we see μ as a composition of k supported in $\{1, \ldots, n\}$. If $\mu_2 = \cdots = \mu_{l(\mu)} = l$, we denote more simply this sheaf with $\mathcal{L}^\mu(-2l\Delta)$. We also use sheaves $\mathcal{L}^\mu(-m\Delta)$, for an integer $m \in \mathbb{N}$, whose definition is analogous. The interest in such sheaves comes from the fact that we believe they could be in all generality the graded sheaves for a natural filtration on the direct image $\mu_*(S^k L^{[n]})$ of symmetric powers of tautological bundles on $X^{[n]}$ via the Hilbert-Chow morphism.

Remark 4.13. The sheaf $\mathcal{L}^\mu(-2\mu\Delta)$ over the symmetric variety $S^n X$ is closely related to the same sheaf over $S^{l(\mu)} X$; more precisely, if v_l is the finite morphism $v_l : S^l X \times S^{n-l} X \longrightarrow S^n X$, sending the couple of 0-cycles (x, y) to $x + y$, we can write the isomorphism of sheaves over $S^n X$:

$$\mathcal{L}^\mu(-2\mu\Delta) \simeq {v_{l(\mu)}}_* (\mathcal{L}^\mu(-2\mu\Delta) \boxtimes \mathcal{O}_{S^{n-l(\mu)} X}) \ ,$$

and, in general, if A is a line bundle over X, then $\mathcal{L}^\mu(-2\mu\Delta) \otimes \mathcal{D}_A \simeq {v_{l(\mu)}}_* (\mathcal{L}^\mu(-2\mu\Delta) \otimes \mathcal{D}_A \boxtimes \mathcal{D}_A)$.

Remark 4.14. Let $\lambda = (r, \ldots, r)$ and set $l = l(\lambda)$. Then it is immediate to see that the sheaf $\mathcal{L}^\lambda(-2r\Delta)$ over $S^l X$ is isomorphic to

$$\mathcal{L}^\lambda(-2r\Delta) \simeq (\mathcal{I}^{2r}_{\Delta_l})^{\mathfrak{S}_l} \otimes \mathcal{D}_{L^r} \simeq \mu_*((\det \mathcal{O}_X^{[l]})^{\otimes 2r}) \otimes \mathcal{D}_{L^r} \ .$$

If $n \geq l$, over $S^n X$, in general we have that $\mathcal{L}^\lambda(-2r\Delta) \otimes \mathcal{D}_A \simeq v_{l*}((\mathcal{I}^{2r}_{\Delta_l})^{\mathfrak{S}_l} \otimes \mathcal{D}_{L^r \otimes A} \boxtimes \mathcal{D}_A) \simeq v_{l*}(\mu_*((\det \mathcal{O}_X^{[l]})^{\otimes 2r}) \otimes \mathcal{D}_{L^r \otimes A} \boxtimes \mathcal{D}_A)$.

We come to our results. Denote the partition $(1,\dots,1)$ of l with 1^l (in exponential notation). As a consequence of the previous remark, as well as of theorems 3.14, 3.20, and of subsection 3.4, we obtain the following

Corollary 4.15. *For $l=3,4$, and $n \geq l$, the sheaf $\mathcal{L}^{1^l}(-2\Delta)\otimes\mathcal{D}_A$ over S^nX is resolved by the complex $v_{l*}(\mathbb{I}_l^\bullet \otimes \mathcal{D}_{L\otimes A}\boxtimes\mathcal{D}_A)$:*

$$\mathcal{L}^{1^l}(-2\Delta)\otimes\mathcal{D}_A \simeq^{\mathrm{qis}} v_{l*}(\mathbb{I}_l^\bullet \otimes \mathcal{D}_{L\otimes A}\boxtimes\mathcal{D}_A)\ .$$

Similarly, thanks to proposition 3.15 we obtain

Corollary 4.16. *The sheaf $\mathcal{L}^{2,1,1}(-\Delta)\otimes\mathcal{D}_A$ is resolved over S^nX by the complex*

$$\begin{aligned}0 \longrightarrow\ & w_{3*}(v_{1*}(L^2\otimes A\boxtimes\mathcal{D}_{L\otimes A})\boxtimes\mathcal{D}_A)\longrightarrow\\ &\longrightarrow w_{3*}([v_{2*}(L^3\otimes A^2\boxtimes L\otimes A)\oplus v_{2*}(L^2\otimes A^2\boxtimes L^2\otimes A)]_0\boxtimes\mathcal{D}_A)\longrightarrow\\ &\longrightarrow w_{3*}(v_{3*}(\Omega^1_X\otimes L^4\otimes A^3)\boxtimes\mathcal{D}_A)\longrightarrow 0\ .\end{aligned} \tag{4.1}$$

It would be now immediate to give a formula for the Euler-Poincaré characteristic of $\mathcal{L}^{1^l}(-2\Delta)\otimes\mathcal{D}_A$, using corollary 3.24 or corollary 4.2, and of $\mathcal{L}^{2,1,1}(-\Delta)\otimes\mathcal{D}_A$. We leave this to the reader.

The sheaf $\mathcal{L}^{2,1,1}(-2\Delta)$ To finish this section we will describe the sheaf $\mathcal{L}^{2,1,1}(-2\Delta)$, which is important for the work [Sca15a]. If $A\subseteq\{1,\dots,n\}$ and if μ is a composition of some integer l supported in $\{1,\dots,n\}$, we define μ_A as the compositon coinciding with μ over the set A, and with 0 over $\{1,\dots,n\}\setminus A$. We define with $|\mu_A|=\sum_{i=1}^n\mu_A(i)$. In [Sca15a, Remark 4.6] we defined a natural $\mathrm{Stab}_{\mathfrak{S}_n}(\mu)$-equivariant differential

$$d^l_\Delta : L^\mu\otimes\mathcal{I}^l_{\Delta_{l(\mu)}} \longrightarrow \bigoplus_{\substack{|I|=2\\ I\subseteq\{1,\dots,l(\mu)\}}} L^\mu\otimes\mathcal{I}^l_{\Delta_I}/\mathcal{I}^{l+1}_{\Delta_I}\simeq \bigoplus_{\substack{|I|=2\\ I\subseteq\{1,\dots,l(\mu)\}}}(S^l\Omega^1_X\otimes L^{|\mu_I|})_I\otimes L^{\mu_{\bar I}}$$

and an invariant version over S^nX:

$$d^l_\Delta:\mathcal{L}^\mu(-l\Delta):=\pi_*(L^\mu\otimes\mathcal{I}^l_{\Delta_{l(\mu)}})^{\mathrm{Stab}_{\mathfrak{S}_n}(\mu)}\longrightarrow \pi_*\Big(\bigoplus_{\substack{|I|=2\\ I\subseteq\{1,\dots,l(\mu)\}}}(S^l\Omega^1_X\otimes L^{|\mu_I|})_I\otimes L^{\mu_{\bar I}}\Big)^{\mathrm{Stab}_{\mathfrak{S}_n}(\mu)}$$

whose kernel is $\mathcal{L}^\mu(-(l+1)\Delta)$. Denote with $\mathcal{K}^1_{(1)(1)}(-l\Delta)$ the sheaf

$$\mathcal{K}^1_{(1)(1)}(-l\Delta):=\pi_*((\Omega^1_X\otimes L^3)_{\{12\}}\otimes p_3^*L\otimes\mathcal{I}^l_{\Delta_{23}})^{\mathfrak{S}(4,\dots,n)}\ ;$$

it will be denoted it just with $\mathcal{K}^1_{(1)(1)}$ if $l=0$. It is clear that, for $\mu=(2,1,1)$,

$$\begin{aligned}\pi_*\Big(\bigoplus_{\substack{|I|=2\\ I\subseteq\{1,\dots,l(\mu)\}}}(\Omega^1_X\otimes L^{|\mu_I|})_I\otimes L^{\mu_{\bar I}}\Big)^{\mathrm{Stab}_{\mathfrak{S}_n}(\mu)} &\simeq \mathcal{K}^1_{(1)(1)}\oplus\pi_*\big((\Omega^1_X\otimes L^2)_{\{23\}}\otimes p_1^*L^2\big)^{\mathfrak{S}(2,3)\times\mathfrak{S}(4,\dots,n))}\\ &\simeq\mathcal{K}^1_{(1)(1)}\ ,\end{aligned}$$

since $\mathfrak{S}(2,3)$ acts with a sign on the sheaf $(\Omega^1_X\otimes L^2)_{\{23\}}$. With these notations, we can prove the following fact.

Proposition 4.17. *We have the exact sequence over S^nX:*

$$\begin{aligned}0\longrightarrow\mathcal{L}^{2,1,1}(-2\Delta)\longrightarrow\mathcal{L}^{2,1,1}(-\Delta)\longrightarrow\mathcal{K}^1_{(1)(1)}(-2\Delta)\longrightarrow&\\ \longrightarrow w_{3*}((S^3\Omega^1_X\otimes L^4)\boxtimes\mathcal{O}_{S^{n-3}X})\longrightarrow 0\ ,&\end{aligned}\tag{4.2}$$

where the third map is the differential d^1_Δ and the fourth one is the composition:

$$\begin{aligned}\mathcal{K}^1_{(1)(1)}(-2\Delta)=\pi_*\Big[((\Omega^1_X\otimes L^3)_{\{12\}}\otimes p_3^*L\otimes\mathcal{I}^2_{\Delta_{23}}\Big]^{\mathfrak{S}(\{4,\dots,n\})}&\longrightarrow\\ \longrightarrow\pi_*\Big[((\Omega^1_X\otimes L^3)_{\{12\}}\otimes p_3^*L\otimes\mathcal{I}^2_{\Delta_{23}}/\mathcal{I}^3_{\Delta_{23}}\Big]^{\mathfrak{S}(\{4,\dots,n\})}&\simeq\\ \simeq w_{3*}\big((\Omega^1_X\otimes S^2\Omega^1_X\otimes L^4)\boxtimes\mathcal{O}_{S^{n-3}X}\big)\longrightarrow w_{3*}\big((S^3\Omega^1_X\otimes L^4)\boxtimes\mathcal{O}_{S^{n-3}X}\big)\ .&\end{aligned}\tag{4.3}$$

Proof. It is clear that, by construction, the differential d^1_Δ takes values in $\mathcal{K}^1_{(1)(1)}(-2\Delta)$; it is also clear that $\mathcal{L}^{2,1,1}(-2\Delta)$ is the kernel of the third map. Moreover, by construction, the map $\mathcal{K}^1_{(1)(1)}(-2\Delta) \longrightarrow w_{3*}((S^3\Omega^1_X \otimes L^4) \boxtimes \mathcal{O}_{S^{n-3}X})$ is surjective. Hence it remains to prove that the sequence

$$\mathcal{L}^{2,1,1}(-\Delta) \longrightarrow \mathcal{K}^1_{(1)(1)}(-2\Delta) \longrightarrow w_{3*}((S^3\Omega^1_X \otimes L^4) \boxtimes \mathcal{O}_{S^{n-3}X}) \tag{4.4}$$

is exact. By GAGA principle it is actually sufficient to prove the exactness of the sequence (4.4) for $X = \mathbb{C}^2$ and L trivial. More precisely, let $(S^nX)_{\mathrm{an}}$ be the complex analytic space associated to the complex algebraic variety S^nX. Then the natural morphism $((S^nX)_{\mathrm{an}}, \mathcal{O}_{(S^nX)_{\mathrm{an}}}) \longrightarrow (S^nX, \mathcal{O}_{S^nX})$ is faithfully flat, by GAGA principle. This implies, in particular, that the sequence (4.4) is exact over S^nX if and only if the induced sequence of complex analytic sheaves

$$\mathcal{L}^{2,1,1}(-\Delta)_{\mathrm{an}} \longrightarrow \mathcal{K}^1_{(1)(1)}(-2\Delta)_{\mathrm{an}} \longrightarrow w_{3*}((S^3\Omega^1_X \otimes L^4) \boxtimes \mathcal{O}_{S^{n-3}X})_{\mathrm{an}} \tag{4.5}$$

is exact over $(S^nX)_{\mathrm{an}}$. But this is holds if and only if it holds for an arbitrary small[1] open set of $(S^nX)_{\mathrm{an}}$ in the complex topology. Now a sufficiently small open set of $(S^nX)_{\mathrm{an}}$ in the complex topology is always biholomorphic to a sufficiently small open set of $(S^n\mathbb{C}^2)_{\mathrm{an}}$, in the complex topology. Hence it is sufficient to prove that the sequence (4.5) is exact analytically over $(S^n\mathbb{C}^2)_{\mathrm{an}}$ and L trivial, but this is equivalent, invoking GAGA principle again, to proving the same fact algebraically over $S^n\mathbb{C}^2$ and L trivial.

It is also easy to see that it is sufficient to prove the statement for $n = 3$. In this case set $A = \mathbb{C}[x, y]$, $A^{\otimes 3} = \mathbb{C}[x_1, x_2, x_3, y_1, y_2, y_3]$. Identifying coherent sheaves with modules, it is sufficient to prove that the sequence of S^3A-modules

$$(\mathcal{I}_{\Delta_3})^{\mathfrak{S}(\{2,3\})} \xrightarrow{d^1_\Delta} (\Omega^1_A \otimes_{\mathbb{C}} A)(-2\Delta) \longrightarrow S^3\Omega^1_A \tag{4.6}$$

is exact, where we wrote briefly $(\Omega^1_A \otimes_{\mathbb{C}} A)(-2\Delta)$ for $(\Omega^1_A \otimes_{\mathbb{C}} A) \otimes_{A\otimes A} \mathcal{I}^2_{\Delta_2}$, where $\mathcal{I}_{\Delta_2}$ is the ideal of the diagonal in $A \otimes A$ and where $A \otimes A$ acts componentwise on $\Omega^1_A \otimes_{\mathbb{C}} A$.

The ideal $\mathcal{I}_{\Delta_3}$ of the big diagonal in X^3 equals the ideal $\langle \mathcal{I}_{\Delta_{12}}\mathcal{I}_{\Delta_{13}}\mathcal{I}_{\Delta_{13}}, q\rangle$, where q is the quadraric polynomial $q = (x_2 - x_1)(y_3 - y_1) - (y_2 - y_1)(x_3 - x_1)$. The quadric q is anti-invariant for $\mathfrak{S}(\{2, 3\})$, hence $\mathcal{I}^{\mathfrak{S}(\{2,3\})}_{\Delta_3} = (\mathcal{I}_{\Delta_{12}}\mathcal{I}_{\Delta_{13}}\mathcal{I}_{\Delta_{13}})^{\mathfrak{S}(\{2,3\})}$. Writing down all nine degree-3 generators of $\mathcal{I}_{\Delta_{12}}\mathcal{I}_{\Delta_{13}}\mathcal{I}_{\Delta_{13}}$ and taking $\mathfrak{S}(\{2, 3\})$-invariants, we get that $\mathcal{I}^{\mathfrak{S}(\{2,3\})}_{\Delta_3}$ is generated, up to elements of degree 4, by $q(x_3 - x_2)$ and $q(y_3 - y_2)$. Now it is easy to see that the image of d^1_Δ contains $(\Omega^1_A \otimes_{\mathbb{C}} A)(-3\Delta)$. Indeed, if $\alpha, \beta \in \mathbb{N}$, $\alpha + \beta > 1$, we have

$$\begin{aligned}
&d^1_\Delta\Big[[(x_2 - x_1)(x_3 - x_1)^\alpha(y_3 - y_1)^\beta - \\
&\qquad (x_3 - x_1)(x_2 - x_1)^\alpha(y_2 - y_1)^\beta](x_3 - x_2)\Big] = (x_3 - x_1)^{\alpha+1}(y_3 - y_1)^\beta dx \\
&d^1_\Delta\Big[[(x_2 - x_1)(x_3 - x_1)^\alpha(y_3 - y_1)^\beta - \\
&\qquad (x_3 - x_1)(x_2 - x_1)^\alpha(y_2 - y_1)^\beta](y_3 - y_2)\Big] = (x_3 - x_1)^{\alpha}(y_3 - y_1)^{\beta+1} dx \\
&d^1_\Delta\Big[[(y_2 - y_1)(x_3 - x_1)^\alpha(y_3 - y_1)^\beta - \\
&\qquad (y_3 - y_1)(x_2 - x_1)^\alpha(y_2 - y_1)^\beta](x_3 - x_2)\Big] = (x_3 - x_1)^{\alpha+1}(y_3 - y_1)^\beta dy \\
&d^1_\Delta\Big[[(y_2 - y_1)(x_3 - x_1)^\alpha(y_3 - y_1)^\beta - \\
&\qquad (y_3 - x_1)(x_2 - x_1)^\alpha(y_2 - y_1)^\beta](y_3 - y_2)\Big] = (x_3 - x_1)^{\alpha}(y_3 - y_1)^{\beta+1} dy
\end{aligned}$$

Consider now an element of the form

$$\tau = (adx + bdy)(x_3 - x_1)^2 + (cdx + edy)(x_3 - x_1)(y_3 - y_1) + (fdx + gdy)(y_3 - y_1)^2$$

in $(\Omega^1_A \otimes_{\mathbb{C}} A)(-2\Delta)$. The image of τ in $S^3\Omega^1_A$ is $adx^3 + (b + c)dx^2dy + (e + f)dxdy^2 + gdy^3$. If τ is in the kernel of the map $(\Omega^1_A \otimes_{\mathbb{C}} A)(-2\Delta) \longrightarrow S^3\Omega^1_A$, then $a = g = 0$ and $b = -c$, $e = -f$ and τ is of the form $\tau = b(x_3 - x_1)^2dy - b(x_3 - x_1)(y_3 - y_1)dy + e(x_3 - x_1)(y_3 - y_1)dy - e(y_3 - y_1)^2dx$. But is now easy to see that τ is a linear combination of $d^1_\Delta[q(x_3 - x_2)]$ and $d^1_\Delta[q(y_3 - y_2)]$ and hence in the image of d^1_Δ. These facts show that the sequence (4.6) is exact. □

[1] here we mean: if it holds over any open set V_j of an open cover $\{V_j\}_j$ of $(S^nX)_{\mathrm{an}}$, where each V_j is chosen to be sufficiently small

A Appendix: Determination of higher differentials in the spectral sequence of invariants

We will determine here explicitely the higher differentials in the spectral sequence of invariants $(E_1^{p,q})^{\mathfrak{S}_n}$ for $n = 3, 4$, appearing in theorem 3.14, 3.15 and 3.20.

Remark A.1. In order to prove that a certain higher differential $d_r^{\mathfrak{S}_n} : (E_r^{p,q})^{\mathfrak{S}_n} \longrightarrow (E_r^{p+r,q-r+1})^{\mathfrak{S}_n}$ has a certain expression, we define explicitely another morphism between the same coherent sheaves of invariants, say $D_r : (E_r^{p,q})^{\mathfrak{S}_n} \longrightarrow (E_r^{p+r,q-r+1})^{\mathfrak{S}_n}$, and then we prove that the two maps coincide. By GAGA principle (as done in proposition 4.17), or, alternatively, by localization an completion, in order to compare the two maps we can always reduce the problem to the case $X = \mathbb{C}^2$, where computations are much easier.

Remark A.2. Over $X^n = (\mathbb{C}^2)^n = \operatorname{Spec} \mathbb{C}[x_1, \dots, x_n, y_1, \dots, y_n]$ we resolve the sheaves $\mathcal{K}_I^1$, $I \subseteq \{1, \dots, n\}$ with Koszul complexes $K_I^\bullet(F_I, s_I)$, where F_I is the trivial rank 2 bundle, with global frame γ_I, δ_I, and with global section $s_I = x_I \gamma_I + y_I \delta_I$, where, if $I = \{i, j\}$, $i < j$, we denoted briefly x_I and y_I the differences $x_i - x_j$ and $y_i - y_j$, respectively. We then build a term by term free resolution $R_I^{\bullet,\bullet}$ of the complex $\mathcal{K}_I^\bullet$. The spectral sequence $E_1^{p,q}$ can be seen as the spectral sequence associated to the bicomplex $L^{\bullet,\bullet} := \bigotimes_{|I|=2} R_I^{\bullet,\bullet}$, where the tensor product is taken respecting the lexicographic order of the multi-indexes I: we see it as a bicomplex with respect to the sum of the first indexes and the sum of the second. We denote with ∂ and δ the (commuting) horizontal and vertical differentials, respectively. It is straightforward to see that, remembering the notation used in the proof of proposition 2.14,

$$\begin{aligned} L^{p,\bullet} &\simeq \bigoplus_{I_1,\dots,I_p \subseteq \{1,\dots,n\}} K^\bullet(F_{I_1}, s_{I_1}) \otimes \cdots \otimes K^\bullet(F_{I_p}, s_{I_p}) \\ &\simeq \bigoplus_{I_1,\dots,I_p \subseteq \{1,\dots,n\}} K^\bullet(F_{I_1} \oplus \cdots \oplus F_{I_p}, s_{I_1} \oplus \cdots \oplus s_{I_p}) \\ & \bigoplus_{\Gamma \in \mathcal{G}_{p,n}} K^\bullet(F_\Gamma, s_\Gamma) \end{aligned}$$

where the direct sums are over distinct $I_1, \dots, I_p \subseteq \{1, \dots, n\}$, in lexicographic order.

Remark A.3. Let $X = \mathbb{C}^2$. After the description of Q_Γ given in 2.12, it is practical to think of F_I as $\mathbb{C}^2 \otimes \rho_I$ with $\gamma_I = e_1 \otimes e_I$ and $\delta_I = e_2 \otimes e_I$. The quotient bundle Q_Γ can then be seen as

$$Q_\Gamma \simeq \mathcal{O}_{\Delta_\Gamma} \otimes (\mathbb{C}^2 \otimes q_\Gamma) .$$

The isomorphism $Q_\Gamma^* \simeq (\Omega_X^1 \otimes q_\Gamma)_\Gamma \simeq \mathcal{O}_{\Delta_\Gamma} \otimes (\Omega_{\mathbb{C}^2}^1 \otimes q_\Gamma)$ is given by identifying, over Δ_Γ, the vector $e_1 \otimes v$, for $v \in q_\Gamma$, with $dx \otimes v$, with and $e_2 \otimes v$ with $dy \otimes v$. Of course, since every bundle here is trivial and the representation q_Γ is autodual, $Q_\Gamma \simeq Q_\Gamma^*$.

Notation A.4. Suppose that the graph $\Gamma \in \mathcal{G}_{p,n}$ contains an oriented 3-cycle $K_3(H)$. We will identify the 3-cycle $K_3(H)$ with the sequence of its vertices written in order according to the orientation. Hence we write e_H for the corresponding vector in q_Γ. Moreover, we write $\gamma_H = e_1 \otimes e_H$ and $\delta_H = e_2 \otimes e_H$ the corresponding vectors in $\mathbb{C}^2 \otimes q_\Gamma$, which can be seen as elements of Q_Γ, by the previous remark

Determination of the map D

Notation A.5. For $J \subseteq \{1, \dots, n\}$ a cardinality 2 multi-index and for $i \in \mathbb{N}^*$, we denote with $d_{\Delta_J}^i$ the i-th order differential $d_{\Delta_J}^i : \mathcal{I}_{\Delta_J}^i \longrightarrow \mathcal{I}_{\Delta_J}^i / \mathcal{I}_{\Delta_J}^{i+1}$ and with r_J the restriction $r_J : \mathcal{O}_{X^n} \longrightarrow \mathcal{O}_{\Delta_J}$. Sometimes we see the operator $d_{\Delta_J}^i$ as taking values in $(S^i \Omega_X^1)_J$, via the isomorphism $\mathcal{I}_{\Delta_J}^i / \mathcal{I}_{\Delta_J}^{i+1} \simeq S^i N_{\Delta_J}^* \simeq (S^i \Omega_X^1)_J$.

Remark A.6. Let Y, Z be smooth subvarieties of a smooth variety M intersecting transversely in a smooth subvariety $Y \cap Z$. It is easy to show that $N_{Y \cap Z/Y} \simeq N_{Z/M}\big|_{Y \cap Z}$.

Remark A.7. Let I and J be two distinct cardinality 2 multi-indexes in $\{1, \dots, n\}$, such that $I \cap J \neq \emptyset$. The diagonal Δ_J is isomorphic to X^{n-1} and Δ_I defines a pairwise diagonal in $\Delta_J \simeq X^{n-1}$, that we still indicate with Δ_I. By remark A.6 we can define the composition

$$d_{\Delta_I} \circ r_J : \mathcal{I}_{\Delta_{I \cup J}} \xrightarrow{r_J} \mathcal{I}_{\Delta_{I \cup J}} / \mathcal{I}_{\Delta_J} \simeq \mathcal{I}_{\Delta_I / \Delta_J} \xrightarrow{d_{\Delta_I}} N_{\Delta_I / \Delta_J}^* \simeq N_{\Delta_I / X^n}^*\big|_{\Delta_{I \cup J}} .$$

Remark A.8. Recall that $E_2^{3,-1} \simeq \oplus_{H\subseteq\{1,\dots,n\},|H|=3} Q^*_{K_3(H)} \otimes \cap_{J\not\subseteq H} \mathcal{I}_{\Delta_J}$. Now each of the sheaves $Q^*_{K_3(H)} \otimes \cap_{J\not\subseteq H} \mathcal{I}_{\Delta_J}$ can be indentified with $(\Omega^1_X \boxtimes \mathcal{I}_{\Delta_{n-2}})_{K_3(H)}$ (see notations 3.6 and 2.7). Hence elements in $Q^*_{K_3(H)} \cap_{J\not\subseteq H} \mathcal{I}_{\Delta_J}$ can be identified with differential forms in $\Omega^1_X \boxtimes \mathcal{O}_{X^{n-3}}$ over the product $X \times X^{n-3}$ vanishing on the diagonal Δ_{n-2} in $\Delta_H \simeq X \times X^{n-3}$. For brevity's sake, we will denote the sheaf $(\Omega^1_X \boxtimes \mathcal{I}_{\Delta_{n-2}})_{K_3(H)}$ more briefly with $(\Omega^1_X \boxtimes \mathcal{I}_{\Delta_{n-2}})_H$.

Recalling notation $(\oplus_{|I|=2}\mathcal{O}_{\Delta_I})_0$ for the kernel of $d_1 : \oplus_{|I|=2}\mathcal{O}_{\Delta_I} \longrightarrow \oplus_{I\neq J,|I|=|J|=2}\, \mathcal{O}_{\Delta_{I\cup J}}$, we have the following

Lemma A.9. *The morphism of coherent sheaves $\tilde{D} : (\oplus_{|I|=2}\mathcal{O}_{\Delta_I})_0 \longrightarrow \oplus_{|H|=3}\, (\Omega^1_X \boxtimes \mathcal{I}_{\Delta_{n-2}})_H$, given by*

$$\tilde{D}((f_L)_L)_H = d_{\Delta_I} r_J(\tilde{f}_J - \tilde{f}_I) - d_{\Delta_I} r_K(\tilde{f}_K - \tilde{f}_I)$$

— where $E_{K_3(H)} = \{I, J, K\}$, $I < J < K$, and where $\tilde{f}_L$ are liftings to $\mathcal{O}_{X^n}$ of functions f_L in $\mathcal{O}_{\Delta_L}$ — descends to a morphism of coherent sheaves $\tilde{D} : E_2^{1,0} \longrightarrow E_2^{3,-1}$, which coincides with the differential d_2 of the spectral sequence $E_1^{p,q}$.

Proof. It is easy to prove that the formula for $\tilde{D}$ well defines a morphism of sheaves $(\oplus_{|I|=2}\mathcal{O}_{\Delta_I})_0 \longrightarrow \oplus_{|H|=3}\, (\Omega^1_X \boxtimes \mathcal{I}_{\Delta_{n-2}})_H$ as in the statement, which is clearly zero on $d_1(E_1^{0,0})$ and hence induces a morphism of sheaves $\tilde{D} : E_2^{1,0} \longrightarrow E_2^{3,-1}$, by lemma 3.7.

We now prove that $\tilde{D} = d_2$: we put ourselves in the situation explained in remarks A.1, A.2, A.3. Let f_I, $|I| = 2$, functions in $\mathcal{O}_{\Delta_I}$ such that $(f_I)_I$ is in $E_1^{1,0}$. For all I, let $\tilde{f}_I$ be regular functions in $\mathcal{O}_{X^n}$ restricting to f_I. The element $(\tilde{f}_I)_I$ is in $L^{1,0}$ and its image $\partial((\tilde{f}_I)_I) \in L^{2,0}$ is zero when projected to $E_1^{2,0}$. In other word we have that

$$\partial((\tilde{f}_I)_I)_{I_1\cup I_2} = \varepsilon_{I_1,I_1\cup I_2}\tilde{f}_{I_1} + \varepsilon_{I_2,I_1\cup I_2}\tilde{f}_{I_2} \in \mathcal{I}_{\Delta_{I_1\cup I_2}}$$

for pairs of cardinality 2-multi-indexes I_1, I_2 with $I_1 \neq I_2$, where we indicated graphs with two edges just as a union of these. Therefore the element $\partial((f_I)_I)_{I_1\cup I_2}$ can be lifted to $L^{2,-1}$ as

$$\varepsilon_{I_1,I_1\cup I_2}\tilde{f}_{I_1} + \varepsilon_{I_2,I_1\cup I_2}\tilde{f}_{I_2} = a_{I_1,I_2}x_{I_1} + b_{I_1,I_2}y_{I_1} + c_{I_1,I_2}x_{I_2} + d_{I_1,I_2}y_{I_2} = \delta(w_{I_1,I_2})$$

for elements $w_{I_1,I_2} = a_{I_1,I_2}\gamma^*_{I_1} + b_{I_1,I_2}\delta^*_{I_1} + c_{I_1,I_2}\gamma^*_{I_2} + d_{I_1,I_2}\delta^*_{I_2}$, where, according to remark A.2, we indicated with $\gamma^*_{I_i}, \delta^*_{I_i}$ a frame of $F^*_{I_i}$. The image in $L^{3,-1}$ via ∂ of liftings w_{I_1,I_2} will represent the image of d_2. The complex $L^{3,\bullet}$ is now a direct sum of Koszul complexes $K^\bullet(F_\Gamma, s_\Gamma)$, where Γ is a simple graph with 3 edges and without isolated vertices. But if Γ is acyclic then the correspondent Koszul complex is acyclic in negative degree, and the Γ-component of the image via ∂ will be zero in vertical cohomology and hence in $E_2^{3,-1}$. Hence we are just interested in components of the second differential d_2 indexed by 3-cycles, determined by cardinality 3 multi-indexes. Suppose now H is such a multi-index and that $\{I, J, K\}$ are the edges of the corresponding 3-cycles, with $I < J < K$. From $(\partial \circ \partial((\tilde{f}_I)_I))_{K_3(H)} = 0$ we deduce

$$\partial((\tilde{f}_I)_I)_{I\cup J} - \partial((\tilde{f}_I)_I)_{I\cup K} + \partial((\tilde{f}_I)_I)_{J\cup K} = 0\ ,$$

which implies that

$$(a_{I,J} - a_{I,K})x_I + (c_{I,J} + a_{J,K})x_J + (-c_{I,K} + c_{J,K})x_K = 0$$
$$(b_{I,J} - b_{I,K})y_I + (d_{I,J} + b_{J,K})y_J + (-d_{I,K} + d_{J,K})y_K = 0$$

and hence, since $x_I - x_J + x_K = 0$ and $y_I - y_J + y_K = 0$, that

$$(a_{I,J} - a_{I,K}) = (-c_{I,K} + c_{J,K}) = -(c_{I,J} + a_{J,K}) \tag{A.1a}$$
$$(b_{I,J} - b_{I,K}) = (-d_{I,K} + d_{J,K}) = -(d_{I,J} + b_{J,K}) \tag{A.1b}$$

Finally the image of d_2 is represented in $L^{3,-1}$ by $w_{I\cup J} - w_{I\cup K} + w_{J\cup K}$, which is equal to

$$(a_{I,J} - a_{I,K})\gamma^*_I + (c_{I,J} + a_{J,K})\gamma^*_J + (-c_{I,K} + c_{J,K})\gamma^*_K + \\ + (b_{I,J} - b_{I,K})\delta^*_I + (d_{I,J} + b_{J,K})\delta^*_J + (-d_{I,K} + d_{J,K})\delta^*_K$$

and which, using relations (A.1), can be rewritten as

$$(a_{I,J}-a_{I,K})\gamma_H^* + (b_{I,J}-b_{I,K})\delta_H^* \ ,$$

which, by notation A.4 and in the identifications explained in remark A.3, is precisely the formula in the statement. □

Over an affine open set of the form U^n or S^nU, with $U=\mathrm{Spec}(A)$, we will identify sheaves with their modules of global sections. In particular, we will denote with $\Omega^1_A(-\Delta_{n-2}) = (\Omega^1_A\otimes A^{n-3})\otimes \mathcal{I}_{\Delta_{n-2}}$ the module of sections of the sheaf $(\Omega^1_U\boxtimes\mathcal{O}_{U^{n-3}})\otimes\mathcal{I}_{\Delta_{n-2}}$ over U^n, where $\mathcal{I}_{\Delta_{n-2}}$ is the ideal of the big diagonal in $U^{\otimes n-2}$ and $\Omega^1_A\otimes A^{n-3}$ is seen as a $A^{\otimes n-2}$-module (see notation 3.6) and hence a $A^{\otimes n}$-module, via the contraction of the first three factors $A^{\otimes n}\longrightarrow A\otimes A^{n-3}$. We denote with $\oplus_{|I|=2}(A\otimes A^{n-2})_0$ the module of global sections of the sheaf $(\oplus_{|I|=2}\mathcal{O}_{\Delta_I})_0$ over U^n and with $(A\otimes S^{n-2}A)_0$ its $\mathfrak{S}_n$-invariants over S^nU.

Notation A.10. If $w_1\otimes\cdots\otimes w_l$ is an element of $A^{\otimes l}$, and $1\le i\le l$, we indicate with $\widehat{w_i}$ the element $w_1\otimes\cdots\otimes w_{i-1}\otimes w_{i+1}\otimes\cdots w_l\in A^{\otimes l-1}$. We use an analogous notation for an element $w_1.\cdots.w_l\in S^lA$.

Corollary A.11. *Over an affine open set $U^n=\mathrm{Spec}\,A^{\otimes n}$, the differential $d_2: E_2^{1,0}\longrightarrow E_2^{3,-1}$ is induced by the map $\tilde{D}:\big(\oplus_{|I|=2}A\otimes A^{\otimes n-2}\big)_0\longrightarrow \oplus_{H\subseteq\{1,\dots,n\},|H|=3}\Omega^1_A(-\Delta_{n-2})$, determined by*

$$\tilde{D}((f_L)_L)_H = adu_{k-2}\otimes\widehat{u_{k-2}} + bdv_{j-1}\otimes\widehat{v_{j-1}} - w_i dc\otimes\widehat{w_i}$$

where $H=\{i,j,k\}$, $i<j<k$, $I=\{i,j\}, J=\{i,k\}, K=\{j,k\}$, and where $f_I=a\otimes u_1\otimes\cdots\otimes u_{n-2}$, $f_J=b\otimes v_1\otimes\cdots\otimes v_{n-2}$, $f_K=c\otimes w_1\otimes\cdots\otimes w_{n-2}$.

Proposition A.12. *The invariant differential $d_2:(E_2^{1,0})^{\mathfrak{S}_n}\longrightarrow (E_2^{3,-1})^{\mathfrak{S}_n}$ is induced locally, over an affine open set of the form S^nU, by the map $D:(A\otimes S^{n-2}A)_0\longrightarrow \Omega^1_A(-\Delta_{n-2})^{\mathfrak{S}_{n-3}}$ determined by*

$$D(a\otimes b_1.\dots.b_{n-2}) = \sum_{i=1}^{n-2}(2adb_i - b_i da)\otimes\widehat{b_i}\ .$$

Here the group $\mathfrak{S}_{n-3}$ acts on the factor A^{n-3} of the tensor product $A\otimes A^{\otimes n-3}$.

Determination of the map A for $n=4$. In what follows we use the second convention of notation 3.5. Any graph of the kind $C_4\cup L$ has a distinguished edge L (the only edge whose vertices are of degree 3) and therefore it can be decomposed uniquely as a union of two 3-cycles H and K, determined by two cardinality 3 multi-indexes H and K such that $H\cap K=L$. In what follows we will write such a graph just as $H\cup K$, instead of $K_3(H)\cup K_3(K)$.

Suppose K is a 3-cycle $K=\{i_1,i_2,i_3\}$ with $i_1<i_2<i_3$. We say that a simple path in K is positively oriented if, in the orientation of the path, the vertex following i_1 is i_2, negatively oriented if it is not positively oriented. When writing the coefficient $\eta_{I,K}$ for I and edge of K (see notation 2.9), we will alwasy assume that K is positively oriented.

We introduce a general sign $\varepsilon_{\Gamma,\Gamma'}$ for a couple (Γ,Γ') where Γ is a subgraph of Γ'. If $E_{\Gamma'}\setminus E_\Gamma = \{I_1,\dots,I_l\}$, in lexicographic order, then $\varepsilon_{\Gamma,\Gamma'}=\prod_{j=0}^{l-1}\varepsilon_{\Gamma\cup I_1\cup\cdots I_j,\Gamma\cup I_1\cup\cdots I_{j+1}}$.

A final notation before describing the map A. Let H be a 3-cycle and $I\subseteq\{1,\dots,4\}$ such that $I\not\subseteq H$. Surely $I\cap H=\{a\}$, for some $a\in\{1,\dots,4\}$. We denote the sign $\delta_{I,H}$ as $+1$ if $a=\min I$, and $\delta_{I,H}=-1$ otherwise. According to these facts and notations, we have the first

Lemma A.13. *Consider the map $\tilde{A}:\oplus_{|H|=3}(\Omega^1_X\boxtimes\mathcal{I}_{\Delta_2})_H\longrightarrow\oplus_{\Gamma\in\mathcal{G}_{5,4}}\Lambda^2(\Omega^1_X\otimes q_\Gamma)_\Gamma$, whose component $\tilde{A}^\Gamma_H$ is zero if H is not a subgraph of Γ and is defined by the formula*

$$\tilde{A}^\Gamma_H(\omega\otimes f) = \varepsilon_{H,H\cup K}\delta_{I,H}\eta_{I,K}(\omega\otimes e_H)\wedge(d_\Delta f\otimes e_K)$$

if $\Gamma=H\cup K$, for some 3-cycle K, with $I=\min E_{H\cup K}\setminus E_H$, where $\omega\in\Omega^1_X$, $f\in\mathcal{I}_{\Delta_2}$ and where e_H, e_K are base elements in $q_{H\cup K}$. Then the image of $\tilde{A}$ is in $\ker d_1$; hence it induces a map $\tilde{A}:E_2^{3,-1}\longrightarrow E_2^{5,-2}$, which coincides with the second differential d_2.

Proof. It is clear that the map $\tilde{A}$ is well defined and it is easy to see that its image lies in $\ker d_1$: this yields the good definition of the map $\tilde{A} : E_2^{3,-1} \longrightarrow E_2^{5,-2}$. We just have to prove that it coincides with d_2. We put ourselves in the situation of remarks $A.1, A.2, A.3$. Consider a differential form $\omega \otimes f$ in $(\Omega_X^1 \boxtimes \mathcal{I}_{\Delta_2})_H$, where $\omega \in \Omega_X^1$ and $f \in \mathcal{I}_{\Delta_2}$. In what follows we identify the element $\omega \otimes f \in (\Omega_X^1 \boxtimes \mathcal{I}_{\Delta_2})_H$ with the element in $\oplus_{|L|=3}(\Omega_X^1 \boxtimes \mathcal{I}_{\Delta_2})_L$ whose H-component is $\omega \otimes f$ and whose every other L-component, with $L \neq H$ is zero.

To prove the statement, it is sufficient to compute the component $d_2(\omega \otimes f)_{H\cup K}$ of the second differential, where $K = \{I, J, L\}$, where it will always be assumed that $I < J$ in the lexicographic order. The form $\omega \otimes f$ can be rewritten as

$$\omega \otimes f = h(dx \otimes 1) + g(dy \otimes 1)$$

over $X \times X \simeq \Delta_H$, where $h, g \in \mathcal{I}_{\Delta_2}$. Now $\mathcal{I}_{\Delta_2}$ is generated over $X \times X$ by (classes in $\mathcal{O}_{\Delta_H}$ of) regular functions x_I, y_I in $\mathcal{O}_{X^4}$: hence we can lift h and g to regular functions (which we will still call h and g) $h = ax_I + by_I$, $g = cx_I + dy_I$ in $\mathcal{O}_{X^4}$. By remark A.3, the differential form $\omega \otimes f$ can be represented, in $L^{3,-1}$ as

$$\omega \otimes f = h\gamma_H^* + g\delta_H^* = (ax_I + by_I)\gamma_H^* + (cx_I + dy_I)\delta_H^* \; .$$

Since the element $\omega \otimes f$ was chosen in $E_2^{3,-1} = \ker(\partial : E_1^{3,-1} \longrightarrow E_1^{4,-1})$, this means that the vertical cohomology class of $\partial(\omega \otimes f)$ in $E_1^{4,-1} = H_\delta^{-1}(L^{4,\bullet})$ is zero. This is equivalent to saying that $\partial(\omega \otimes f)_{H\cup I}$ and $\partial(\omega \otimes f)_{H\cup J}$ come from elements in $L^{4,-2}$; for the first we can we can write

$$\partial(\omega \otimes f)_{H\cup I} = \varepsilon_{H,H\cup I}(\omega \otimes f) = \delta\Big(\varepsilon_{H,H\cup I}\big((a\gamma_I^* + b\delta_I^*) \wedge \gamma_H^* + (c\gamma_I^* + d\delta_I^*) \wedge \delta_H^*\big)\Big) \tag{A.2}$$

For the second we have, taking into account that, by definitions of the signs $\eta_{I,K}$ and omitting for brevity's sake the index K, we can write $\eta_I x_I + \eta_J x_J + \eta_L x_L = 0$:

$$\begin{aligned}\partial(\omega \otimes f)_{H\cup J} = \varepsilon_{H,H\cup J}(\omega \otimes f) = \delta\Big(&\varepsilon_{H,H\cup J}\big((a(-\eta_I\eta_J\gamma_J^* - \eta_I\eta_L\gamma_L^*)+\\ &b(-\eta_I\eta_J\delta_J^* - \eta_I\eta_L\delta_L^*)) \wedge \gamma_H^* + (c(-\eta_I\eta_J\gamma_J^* - \eta_I\eta_L\gamma_L^*)+\\ &+ d(-\eta_I\eta_J\delta_J^* - \eta_I\eta_L\delta_L^*)) \wedge \delta_H^*\big)\Big) \; .\end{aligned} \tag{A.3}$$

Now the element between parenthesis in (A.2) has image

$$\varepsilon_{H\cup I,H\cup K}\varepsilon_{H,H\cup I}\big((a\gamma_I^* + b\delta_I^*) \wedge \gamma_H^* + (c\gamma_I^* + d\delta_I^*) \wedge \delta_H^*\big)$$

via ∂ in $L^{5,-2}$, while the element between parenthesis in (A.3) has image

$$\begin{aligned}\varepsilon_{H\cup J,H\cup K}\varepsilon_{H,H\cup J}\big((a(-\eta_I\eta_J\gamma_J^* - \eta_I\eta_L\gamma_L^*) + b(-\eta_I\eta_J\delta_J^* - \eta_I\eta_L\delta_L^*)) \wedge \gamma_H^*+\\ (c(-\eta_I\eta_J\gamma_J^* - \eta_I\eta_L\gamma_L^*) + d(-\eta_I\eta_J\delta_J^* - \eta_I\eta_L\delta_L^*)) \wedge \delta_H^*\big)\end{aligned}$$

via ∂ in $L^{5,-2}$. The sum of the two terms is given by

$$\begin{aligned}\varepsilon_{H\cup I,H\cup K}\varepsilon_{H,H\cup I}\Big((a(\gamma_I^* - \eta_I\eta_J\gamma_J^* - \eta_I\eta_L\gamma_L^*) + b(\delta_I^* - \eta_I\eta_J\delta_J^* - \eta_I\eta_L\delta_L^*)) \wedge \gamma_H^*+\\ (c(\gamma_I^* - \eta_I\eta_J\gamma_J^* - \eta_I\eta_L\gamma_L^*) + d(\delta_I^* - \eta_I\eta_J\delta_J^* - \eta_I\eta_L\delta_L^*)) \wedge \delta_H^*\Big)\end{aligned}$$

since we can easily see that $\varepsilon_{H\cup J,H\cup K}\varepsilon_{H,H\cup I} = -\varepsilon_{H\cup I,H\cup K}\varepsilon_{H,H\cup J}$ because $\varepsilon_{H,H\cup I} = \varepsilon_{H\cup J,H\cup K}$ and $\varepsilon_{H,H\cup J} = -\varepsilon_{H\cup I,H\cup K}$. Note now that $\gamma_K^* = \eta_I\gamma_I^* + \eta_J\gamma_J^* + \eta_L\gamma_L^*$, $\delta_K^* = \eta_I\delta_I^* + \eta_J\delta_J^* + \eta_L\delta_L^*$. By lemma [Sca15a, Lemma A.3], we have that $d_2(\omega \otimes f)_{H\cup K}$ is represented by the vertical cohomology class of

$$\varepsilon_{H,H\cup K}\eta_{I,K}\big((a\gamma_K^* + b\delta_K^*) \wedge \gamma_H^* + (c\gamma_K^* + d\delta_K^*) \wedge \delta_H^*\big) \; .$$

Since we identified the classes of γ_H^*, γ_K^*, δ_H^*, δ_K^* with $dx \otimes e_H$, $dx \otimes e_K$, $dy \otimes e_H$, $dy \otimes e_K$ in $\Omega_X^1 \otimes q_{H\cup K}$, respectively, we obtain the formula in the statement. □

Remark A.14. For future use the following computation wil turn out handy. Let $X = \mathbb{C}^2$. Consider the differential form $\omega \otimes f$, as above, but now lift it to the element $(ax_M + by_M)\gamma_H^* + (cx_M + dy_M)\delta_H^* \in L^{3,-1}$, for functions $a, b, c, d \in \mathcal{O}_{X^4}$ and let K a 3-cycle as above with edges $\{I, J, L\}$, such that the edge M satisfies $M \notin H$, $M \notin K$. The stairway process in order to compute the component $d_2(\omega \otimes f)_{H\cup K}$ provides the element

$$\delta_{M,H}\varepsilon_{H,H\cup K}\eta_{L,K}\big(\gamma_H^* \wedge (a\gamma_K^* + b\delta_K^*) + \delta_H^* \wedge (c\gamma_K^* + d\delta_K^*)\big) = \delta_{M,H}\varepsilon_{H,H\cup K}\eta_{L,K}(\omega \otimes e_H) \wedge (d_\Delta f \otimes e_K) \in L^{5,-2} ,$$

up to elements coming from $L^{4,-2}$.

As an immediate corollary, taking $\mathfrak{S}_4$-invariants, and using Danila's lemma for morphisms, we have

Corollary A.15. *The $\mathfrak{S}_4$-invariant higher differential $A = d_2^{\mathfrak{S}_4} : (E_2^{3,-1})^{\mathfrak{S}_4} \longrightarrow (E_2^{5,-2})^{\mathfrak{S}_4}$ takes an element $\omega \otimes f$ in $w_{3*}(\Omega_X^1 \boxtimes \mathcal{I}_{\Delta_2})$ to the element $-\omega \wedge d_\Delta f$ in $w_{4*}(K_X)$.*

Proof. Indeed it is sufficient to take $H = \{1, 2, 3\}$ and $K = \{1, 3, 4\}$. Then $L = \{1, 3\}$, $I = \{1, 4\}$, $J = \{3, 4\}$. In order to compute the invariant differential $d_2^{\mathfrak{S}_4}(\omega \otimes f)$ of an element $\omega \otimes f$, by [Sca09, Lemma A.1], we just have to compute $d_2(\omega \otimes f + (24)_*\omega \otimes f) = -(\omega \otimes e_H) \wedge (d_\Delta f \otimes e_K) - (\omega \otimes e_K) \wedge (d_\Delta f \otimes e_H)$ — where we omit writing the push-forward π_* — but this can be identified with $-\omega \otimes d_\Delta f + d_\Delta f \otimes \omega = -\omega \wedge d_\Delta f$ in $\Omega_X^1 \otimes \Omega_X^1 \subseteq \Lambda^2(\Omega_X^1 \otimes q_{H\cup K})$. □

Determination of the map C.

Remark A.16. Recall notation 3.19. Let $X = \mathbb{C}^2$. The differential forms $x_{14}dx$, $y_{14}dy$ are in the image of $D : w_{2*}(\mathcal{O}_X \boxtimes \mathcal{O}_{S^2X})_0 \longrightarrow w_{3*}(\Omega_X^1 \boxtimes \mathcal{I}_{\Delta_2})_0$. Indeed, by corollary A.11, it is clear that $A(x_{14}dx) = -dx \wedge dx = 0 = -dy \wedge dy = A(y_{14}dy)$. So both differential forms belong to $w_{3*}(\Omega_X^1 \boxtimes \mathcal{I}_{\Delta_2})_0$. On the other hand we have that $x \otimes x.1 \in w_{2*}(\mathcal{O}_X \boxtimes \mathcal{O}_{S^2X})_0$, since $d_1^{\mathfrak{S}_4}(x \otimes x.1) = 2x \otimes x - 2x \otimes x = 0$ and, by proposition A.12, $D(x \otimes x.1) = (2xdx - xdx) \otimes 1 - dx \otimes x = (dx \otimes 1)(x \otimes 1 - 1 \otimes x)$, which can be identified with $x_{14}dx$. Similarly $y_{14}dy = D(y \otimes y.1)$, and $y \otimes y.1 \in w_{2*}(\mathcal{O}_X \boxtimes \mathcal{O}_{S^2X})_0$.

Remark A.17. Since we have surjective maps $E_1^{6,-3} \twoheadrightarrow E_2^{6,-3}$ and $E_2^{6,-3} \twoheadrightarrow E_3^{6,-3}$, and since $E_1^{6,-3} \simeq \Lambda^3 Q_{K_4}^*$, we can see $E_3^{6,-3}$ as a quotient of the bundle $\Lambda^3 Q_{K_4}^*$ over the small diagonal Δ_{1234}. If a is an element of $\Lambda^3 Q_{K_4}^*$, we denote with $[a]$ the class of its image in $E_3^{6,-3}$.

Remark A.18. The natural composition $w_{3*}(\Omega_X^1 \boxtimes \mathcal{I}_{\Delta_2}^2) \longrightarrow w_{3*}(\Omega_X^1 \boxtimes \mathcal{I}_{\Delta_2})_0 \twoheadrightarrow (E_3^{3,-1})^{\mathfrak{S}_4}$ is surjective.

Proof. By GAGA principle, it is sufficient to prove the statement for $X = \mathbb{C}^2$. But in this case it follows by remark A.16 and by corollary A.15; indeed a differential form $\omega \otimes f \in w_{3*}(\Omega_X^1 \boxtimes \mathcal{I}_{\Delta_2})$ is in the kernel of $d_2^{\mathfrak{S}_4}$ if and only if it is of the form $\omega \otimes f = ax_{14}dx + by_{14}dy + \omega_1 \otimes g$, where g is in $\mathcal{I}_{\Delta_2}^2$. But now the term $ax_{14}dx + by_{14}dy$ is zero in $(E_3^{3,-1})^{\mathfrak{S}_4}$, because of remark A.16. □

By the previous remark, we can represent any element in $(E_3^{3,-1})^{\mathfrak{S}_4}$ by an element in $w_{3*}(\Omega_X^1 \boxtimes \mathcal{I}_{\Delta_2}^2)$. In the proof of the next proposition we will use the following notation: if I is a cardinality 2-multi-index in $\{1, \ldots, 4\}$, we will indicate with $\Gamma(\widehat{I})$ the graph obtained by the complete graph K_4 removing the edge I, that is $V_{\Gamma(\widehat{I})} = \{1, \ldots, 4\}$, $E_{\Gamma(\widehat{I})} = E_{K_4} \setminus \{I\}$.

Proposition A.19. *Consider the map $C : w_{3*}(\Omega_X^1 \boxtimes \mathcal{I}_{\Delta_2}^2) \longrightarrow w_{4*}(S^3\Omega_X^1)$ defined by the formula*

$$C(\omega \otimes f) = -\operatorname{sym}(\omega \otimes d_\Delta^2 f) ,$$

where $\omega \in \Omega_X^1$ and $f \in \mathcal{I}_{\Delta_2}^2$. It descends to a map $C : E_3^{3,-1} \longrightarrow E_3^{6,-3}$, which coincides, up to a constant, with $d_3^{\mathfrak{S}_4}$.

Proof. It is clear that the formula induces a well defined map $C : E_3^{3,-1} \longrightarrow E_3^{6,-3}$. It is sufficient to prove that this map coincides up to constants with the invariant differential $d_3^{\mathfrak{S}_4}$. We put ourselves in the situation explained in remarks A.1, A.2, A.3. For brevity's sake, we indicate with H_0 the cardinality 3-multi-index $\{1, 2, 3\}$ and the associated 3-cycle. We identify $(E_2^{3,-1})^{\mathfrak{S}_4}$ with $w_{3*}(\Omega_X^1 \boxtimes \mathcal{I}_{\Delta_2}) \simeq \pi_*((\Omega_X^1)_{H_0} \otimes \mathcal{I}_{\Delta_{14}})$; hence

$w_{3*}(\Omega^1_X \boxtimes \mathcal{I}^2_{\Delta_2})$ can be identified with $\pi_*((\Omega^1_X)_{H_0} \otimes \mathcal{I}^2_{\Delta_{14}}) \subseteq \ker d_2^{\mathfrak{S}_4}$. By Danila's lemma for morphisms, if $\omega \otimes f \in \pi_*((\Omega^1_X)_{H_0} \otimes \mathcal{I}^2_{\Delta_{14}})$, then

$$d_3^{\mathfrak{S}_4}(\omega \otimes f) = d_3\big(\sum_{[\tau] \in \mathfrak{S}_4/\mathfrak{S}_3} \tau_*(\omega \otimes f)\big) = \sum_{[\tau] \in \mathfrak{S}_4/\mathfrak{S}_3} \tau_* d_3(\omega \otimes f) \ ,$$

where, when writing $d_3(\omega \otimes f)$ we think of $\omega \otimes f$ as an element in $\ker d_2 \subseteq \oplus_H (\Omega^1_X \boxtimes \mathcal{I}_{\Delta_2})_H$.

Hence we just need to compute $d_3(\omega \otimes f)$ in $E_3^{6,-3}$. The element $\omega \otimes f \in \pi_*((\Omega^1_X)_{H_0} \otimes \mathcal{I}^2_{\Delta_{14}})$ can be written as $\omega \otimes f = h(dx \otimes 1) + g(dy \otimes 1)$, where $h, g \in \mathcal{I}_{\Delta_{14}}$. Writing $h = ax_{14}^2 + bx_{14}y_{14} + cy_{14}^2$ and $g = a'x_{14}^2 + b'x_{14}y_{14} + c'y_{14}^2$, we see that is sufficient to compute the image for d_3 of differential forms of the kind $\alpha x_{14}^2 dx$, $\alpha x_{14}y_{14}dx$, $\alpha y_{14}^2 dx$, $\alpha x_{14}^2 dy$, $\alpha x_{14}y_{14}dy$, $\alpha y_{14}^2 dy$, for an arbitrary function $\alpha \in \mathcal{O}_{X \times X}$, of course, thinking of these differential forms as elements in $\ker d_2$. Let's begin with $\alpha x_{14}^2 dx$. We have that $d_2(\alpha x_{14}^2 dx) = 0$ in $E_2^{5,-2}$: this means that its components are zero:

$$d_2(\alpha x_{14}^2 dx)_{\Gamma(\widehat{14})} = 0 \ , \qquad d_2(\alpha x_{14}^2 dx)_{\Gamma(\widehat{24})} = 0 \ , \qquad d_2(\alpha x_{14}^2 dx)_{\Gamma(\widehat{34})} = 0 \ .$$

By lemma A.13, we have that, in terms of representants in $\Lambda^2(Q^*_{\Gamma(\widehat{24})})$ and $\Lambda^2(Q^*_{\Gamma(\widehat{34})})$

$$d_2(\alpha x_{14}^2 dx)_{\Gamma(\widehat{24})} = [-\alpha x_{14} \gamma^*_{123} \wedge \gamma^*_{134}] \ , \qquad d_2(\alpha x_{14}^2 dx)_{\Gamma(\widehat{34})} = [-\alpha x_{14} \gamma^*_{123} \wedge \gamma^*_{124}] \ .$$

In order to compute the representant of $d_2(\alpha x_{14}^2 dx)_{\Gamma(\widehat{14})}$ in $L^{5,-2}$, we invoke remark A.14 and we find, in terms of representants in $\Lambda^2(Q^*_{\Gamma(\widehat{14})})$:

$$d_2(\alpha x_{14}^2 dx)_{\Gamma(\widehat{14})} = [\alpha x_{14} \gamma^*_{123} \wedge \gamma^*_{234}] \ .$$

We now lift the elements we found to $L^{5,-3}$. We get

$$\begin{aligned} -\alpha x_{14} \gamma^*_{123} \wedge \gamma^*_{134} &= \delta(-\alpha \gamma^*_{14} \wedge \gamma^*_{123} \wedge \gamma^*_{134}) \in \Lambda^3(\mathbb{C}^2 \otimes W_{\Gamma(\widehat{24})}) \\ -\alpha x_{14} \gamma^*_{123} \wedge \gamma^*_{124} &= \delta(-\alpha \gamma^*_{14} \wedge \gamma^*_{123} \wedge \gamma^*_{124}) \in \Lambda^3(\mathbb{C}^2 \otimes W_{\Gamma(\widehat{34})}) \\ \alpha x_{14} \gamma^*_{123} \wedge \gamma^*_{234} &= \delta(\alpha(\gamma^*_{12} + \gamma^*_{24}) \wedge \gamma^*_{123} \wedge \gamma^*_{234}) \in \Lambda^3(\mathbb{C}^2 \otimes W_{\Gamma(\widehat{14})}) \ . \end{aligned}$$

By [Sca15a, Lemma A.3], the term $d_3(\alpha x_{14}^2 dx)$ in $E_3^{6,-3}$ is represented by the sum of the images of the preceding liftings via the horizontal differential ∂: hence:

$$d_3(\alpha x_{14}^2 dx) = [-\alpha \gamma^*_{14} \wedge \gamma^*_{123} \wedge (\gamma^*_{134} - \gamma^*_{124}) - \alpha(-\gamma^*_{12} - \gamma^*_{24}) \wedge \gamma^*_{123} \wedge \gamma^*_{234})]$$

Note now that $\gamma^*_{234} = \gamma^*_{123} + \gamma^*_{134} - \gamma^*_{124}$ and that the term $(-\gamma^*_{12} - \gamma^*_{24}) \wedge \gamma^*_{123} \wedge \gamma^*_{123}$ is zero in $E_3^{6,-3}$ since it comes from something in $L_3^{5,-3}$. Hence we get that

$$d_3(\alpha x_{14}^2 dx) = [\alpha(\gamma^*_{12} - \gamma^*_{14} + \gamma^*_{24}) \wedge \gamma^*_{123} \wedge (\gamma^*_{134} - \gamma^*_{124})] = [-\alpha \gamma^*_{123} \wedge \gamma^*_{124} \wedge \gamma^*_{134}]$$

where again we simplified terms coming from $L^{5,-3}$. The term $-\alpha \gamma^*_{123} \wedge \gamma^*_{124} \wedge \gamma^*_{134}$ belongs to $\Lambda^3(Q^*_{K_4}) \simeq \Lambda^3(\mathbb{C}^2 \otimes q_{K_4})$ and can be identified with $-\alpha(dx \otimes e_{123}) \wedge (dx \otimes e_{124}) \wedge (dx \otimes e_{134}) = -\alpha(dx)^3 \otimes (e_{123} \wedge e_{124} \wedge e_{134}) \in S^3\Omega^1_X \otimes \Lambda^3 q_{K_4} \subseteq \Lambda^3(\Omega^1_X \otimes q_{K_4})$.

When computing $d_3(\alpha x_{14}y_{14}dx)$, we have, analogously to the previous case that $d_2(\alpha x_{14}y_{14}dx)_{\Gamma(\widehat{24})}$ is represented in $L^{5,-2}$ by $-\alpha x_{14}\gamma^*_{123} \wedge \delta^*_{134}$ and hence we have the lifting $-\alpha x_{14}\gamma^*_{123} \wedge \delta^*_{134} = \delta(-\alpha \gamma^*_{14} \wedge \gamma^*_{123} \wedge \delta^*_{134})$; moreover $d_2(\alpha x_{14}y_{14}dx)_{\Gamma(\widehat{34})}$ is represented by $-\alpha x_{14}\gamma^*_{123} \wedge \delta^*_{134}$ and hence can be lifted to $-\alpha\gamma^*_{14} \wedge \gamma^*_{123} \wedge \delta^*_{134})$ in $L^{5,-2}$; finally, by remark A.14,

$$d_2(\alpha x_{14}y_{14}dx)_{\Gamma(\widehat{14})} = [\alpha x_{14}(\delta^*_{123} - \delta^*_{234}) \wedge \gamma^*_{123}] = [-\alpha x_{14}\gamma^*_{123}(\delta^*_{123} - \delta^*_{234})]$$

and we have the lifting $-\alpha x_{14}\gamma^*_{123}(\delta^*_{123} - \delta^*_{234}) = -\alpha(\gamma^*_{12} + \gamma^*_{24}) \wedge \gamma^*_{123} \wedge (\delta^*_{123} - \delta^*_{234})$. Then, using that $\delta^*_{234} = \delta^*_{123} + \delta^*_{134} - \delta^*_{124}$, we get that $d_3(\alpha x_{14}y_{14}dx)$ is given by the class

$$d_3(\alpha x_{14}y_{14}dx) = [-\alpha\gamma^*_{123}(\delta^*_{134} - \delta^*_{124}) \wedge (\gamma^*_{14} - \gamma^*_{12} - \gamma^*_{24})] = [-\alpha\gamma^*_{123} \wedge \gamma^*_{124} \wedge \delta^*_{134}]$$

were we simplified elements coming from $L^{5,-3}$. Analogously

$$d_3(\alpha y_{14}^2 dx) = [-\alpha\gamma_{123}^* \wedge \delta_{124}^* \wedge \delta_{134}^*] \, .$$

The computation of all other elements is done by symmetry. We than finally have that, for a differential form $\omega \otimes f$ in $\pi_*((\Omega_X^1)_{H_0} \otimes \mathcal{I}_{\Delta_{14}})$ as in the beginning, the differential $d_3(\omega \otimes f)$ is represented by the class of $-\omega \otimes d_\Delta^2 f$ in $(\Omega_X^1 \otimes S^2\Omega_X^1)_{K_4} \subseteq \Lambda^3(\Omega_X^1 \otimes q_{K_4})_{K_4}$.

To finish the compution we remark the following two facts. Firstly, the terms $\gamma_{123}^* \wedge \gamma_{124}^* \wedge \delta_{134}^*$, satifies the equality $[\gamma_{123}^* \wedge \gamma_{124}^* \wedge \delta_{134}^*] = [\gamma_{123}^* \wedge \delta_{124}^* \wedge \gamma_{134}^*]$. Indeed, simplifying at each step elements coming from $L^{5,-3}$,

$$\begin{aligned}
[\gamma_{123}^* \wedge \gamma_{124}^* \wedge \delta_{134}^*] &= [\gamma_{123}^* \wedge (\gamma_{123}^* + \gamma_{134}^* - \gamma_{234}^*) \wedge \delta_{134}^*] = -[\gamma_{123}^* \wedge \gamma_{234}^* \wedge \delta_{134}^*] \\
&= -[\gamma_{123}^* \wedge \gamma_{234}^* \wedge (\delta_{234}^* - \delta_{123}^* + \delta_{124}^*)] = -[\gamma_{123}^* \wedge \gamma_{234}^* \wedge \delta_{124}^*] \\
&= -[\gamma_{123}^* \wedge (\gamma_{123}^* + \gamma_{134}^* - \gamma_{124}^*) \wedge \delta_{124}^*)] \\
&= -[\gamma_{123}^* \wedge \gamma_{134}^* \wedge \delta_{124}^*] = [\gamma_{123}^* \wedge \delta_{124}^* \wedge \gamma_{134}^*]
\end{aligned}$$

The same is true for any nontrivial triple wedge products of vectors associated to 3-cycles γ_H^*, δ_K^*. Secondly, the class in $E_3^{6,-3}$ of any such nontrivial triple wedge product is $\mathfrak{S}_4$-invariant. The proof of this fact is similar to that of the previous fact. Hence, up to positive constants

$$d_3^{\mathfrak{S}_4}(\omega \otimes f) = -\operatorname{sym}(\omega \otimes d_\Delta^2 f)$$

in $w_{4*}(S^3\Omega_X^1 \otimes \Lambda^3 q_{K_4})^{\mathfrak{S}_4} \simeq w_{4*}(S^3\Omega_X^1)$. □

References

[BKR01] Tom Bridgeland, Alastair King, and Miles Reid. The McKay correspondence as an equivalence of derived categories. *J. Amer. Math. Soc.*, 14(3):535–554 (electronic), 2001.

[BS91] M Beltrametti and Andrew J Sommese. Zero cycles and k-th order embeddings of smooth projective surfaces. In *Problems in the theory of surfaces and their classification, Symposia Math*, volume 32, pages 33–48, 1991.

[CG90] Fabrizio Catanese and Lothar Gœttsche. d-very-ample line bundles and embeddings of Hilbert schemes of 0-cycles. *Manuscripta Mathematica*, 68(1):337–341, 1990.

[Dan01] Gentiana Danila. Sur la cohomologie d'un fibré tautologique sur le schéma de Hilbert d'une surface. *J. Algebraic Geom.*, 10(2):247–280, 2001.

[Die10] Reinhard Diestel. *Graph theory*, volume 173 of *Graduate Texts in Mathematics*. Springer, Heidelberg, fourth edition, 2010.

[DN89] J.-M. Drezet and M. S. Narasimhan. Groupe de Picard des variétés de modules de fibrés semi-stables sur les courbes algébriques. *Invent. Math.*, 97(1):53–94, 1989.

[FH91] William Fulton and Joe Harris. *Representation theory*, volume 129 of *Graduate Texts in Mathematics*. Springer-Verlag, New York, 1991. A first course, Readings in Mathematics.

[Hai99] Mark Haiman. Macdonald polynomials and geometry. In *New perspectives in algebraic combinatorics (Berkeley, CA, 1996–97)*, volume 38 of *Math. Sci. Res. Inst. Publ.*, pages 207–254. Cambridge Univ. Press, Cambridge, 1999.

[Hai01] Mark Haiman. Hilbert schemes, polygraphs and the Macdonald positivity conjecture. *J. Amer. Math. Soc.*, 14(4):941–1006 (electronic), 2001.

[Hai02] Mark Haiman. Vanishing theorems and character formulas for the Hilbert scheme of points in the plane. *Invent. Math.*, 149(2):371–407, 2002.

[Leh99] Manfred Lehn. Chern classes of tautological sheaves on Hilbert schemes of points on surfaces. *Invent. Math.*, 136(1):157–207, 1999.

[Sca09] Luca Scala. Cohomology of the Hilbert scheme of points on a surface with values in representations of tautological bundles. *Duke Math. J.*, 150(2):211–267, 2009.

[Sca15a] Luca Scala. Higher symmetric powers of tautological bundles on Hilbert schemes of points on a surface. arXiv: 1502.07595v1, 2015.

[Sca15b] Luca Scala. Singularities of the Isospectral Hilbert Scheme. arXiv: 1510.03071v1, 2015.

[Ser77] Jean-Pierre Serre. *Linear representations of finite groups*. Springer-Verlag, New York-Heidelberg, 1977. Translated from the second French edition by Leonard L. Scott, Graduate Texts in Mathematics, Vol. 42.

Departamento de Matemática, Puc-Rio, Rua Marquês São Vicente 225, 22451-900 Gávea, Rio de Janeiro, RJ, Brazil
Email address: `lucascala@mat.puc-rio.br`