NOISE AND DISSIPATION ON COADJOINT ORBITS

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Abstract. We derive and study stochastic dissipative dynamical systems on coadjoint orbits by incorporating noise and dissipation into mechanical systems arising from the theory of reduction by symmetry, including a semidirect-product extension. Random attractors are found for this general class of systems when the Lie algebra is semi-simple. We study two canonical examples, the free rigid body and the heavy top, whose stochastic integrable reductions are found and numerical simulations of their random attractors are shown.

Contents

1. Introduction

Geometric mechanics, introduced in Poincaré [Poi01], is a powerful formalism for understanding dynamical systems whose Lagrangian and Hamiltonian are invariant under the transformations of the configuration manifold M by a Lie group G. Examples of its applications range from the simple finite dimensional dynamics of the freely rotating rigid body, to the infinite dimensional dynamics of the ideal fluid equations. For a historical review and basic references, see, e.g., [HMR98]. See [MR99, Hol08, HSS09] for textbook introductions to geometric mechanics and background references. One of the main approaches of geometric mechanics is the method of reduction of the motion equations of a mechanical system by the Lie group symmetry G of either its Lagrangian formulation on the tangent space TM, or its Hamiltonian formulation on the cotangent space T^*M. This reduction by symmetry yields new equations for the momentum maps defined on the dual Lie algebra $\mathfrak{g}^*$ of the Lie symmetry group G.

This Lie group reduction procedure simplifies the motion equations of a mechanical system with symmetry by transforming them into new dynamical variables in

$\mathfrak{g}^*$ which are invariant under the same Lie group symmetries as the Lagrangian and Hamiltonian for the dynamics of the mechanical system. On the Lagrangian side, the new invariant variables under the Lie symmetries are obtained from Noether's theorem, via the tangent lift of the infinitesimal action of the Lie symmetry group on the configuration manifold. The unreduced Euler–Lagrange equations are replaced by equivalent Euler-Poincaré equations expressed in the new invariant variables in $\mathfrak{g}^*$, plus an auxiliary reconstruction equation, which restores the information in the tangent space of the configuration space lost in transforming to group invariant dynamical variables. On the Hamiltonian side, after a Legendre transformation, equivalent new invariant variables in $\mathfrak{g}^*$ are defined by a *momentum map* $J : T^*M \to \mathfrak{g}^*$ from the phase space T^*M of the original system on the configuration manifold M to the dual $\mathfrak{g}^*$ of the Lie symmetry algebra $\mathfrak{g} \simeq T_eG$, via the cotangent lift of the infinitesimal action of the Lie symmetry group on the configuration manifold. The cotangent lift momentum map is an equivariant Poisson map which reformulates the canonical Hamiltonian flow equations in phase space as noncanonical Lie-Poisson equations governing flow of the momentum map in an orbit of the coadjoint action of the Lie symmetry group on the dual of its Lie algebra $\mathfrak{g}^*$, plus an auxiliary reconstruction equation for lifting the Lie group reduced coadjoint motion back to phase space T^*M.

Thus, applying a Lie symmetry reduction to a dynamical system with symmetry yields coadjoint motion of the corresponding momentum map and, thus, reduces the dimension of the dynamical system by restricting its motion to certain subspaces of the original phase space, called coadjoint orbits; that is, orbits of the action of the group G on $\mathfrak{g}^*$, the dual space of its Lie algebra $\mathfrak{g} \simeq T_eG$. Coadjoint orbits lie on level sets of the distinguished smooth functions $C \in \mathcal{F} : \mathfrak{g}^* \to \mathbb{R}$ of the symmetry-reduced dual Lie algebra variables $\mu \in \mathfrak{g}^*$ called Casimir functions, which have null Lie-Poisson brackets $\{C, F\}(\mu) = 0$ with any other functions $F \in \mathcal{F}(\mathfrak{g}^*)$, including the reduced Hamiltonian $h(\mu)$. Level sets of the Casimirs, on which the coadjoint orbits lie, are symplectic manifolds which provide the framework on which geometric mechanics is constructed. These symplectic manifolds have many applications in physics, as well as in symplectic geometry, whenever Lie symmetries are present. In particular, coadjoint motion of the momentum map $J(t) = \mathrm{Ad}^*_{g(t)}J(0)$ for a solution curve $g(t) \in C(G)$ takes place on the intersections of level sets of the Casimirs with level sets of the Hamiltonian.

Given this framework Lie group reduction by symmetry for deterministic geometric mechanics, we shall seek strategies for adding stochasticity and dissipation in classical mechanical systems with symmetry which preserve the coadjoint motion structure of the unperturbed deterministic dynamics as much as possible. Specifically, we seek stochastic coadjoint motion equations whose solutions dissipate energy but also lie on the coadjoint orbit of the unperturbed equation. Consequently, our first goal in this paper will be to replace the deterministic equations for coadjoint motion by stochastic processes whose solutions lie on coadjoint orbits. However, simply inserting additive noise into the deterministic equations will not, in general, produce coadjoint motion on level sets of the Casimirs of a Lie-Poisson bracket. Instead, our approach in developing a systematic derivation of stochastic deformations that preserve coadjoint orbits will be to constrain the variations in Hamilton's principle to preserve the transport relations for infinitesimal transformations defined by the action of a stochastic vector field on the configuration manifold.

Having used the constrained Hamilton's principle to derive the stochastic coadjoint motion equation, the study of the associated Fokker-Planck equation and its invariant measure will follow naturally, and be well defined, at least provided one restricts to finite dimensional mechanical systems. The resulting Fokker-Planck equation defines a probability density for coadjoint motion on Casimir surfaces, since it takes the form of a Lie-Poisson equation for the transport part, and a double Lie-Poisson structure for the diffusion part, both of which generate motion along coadjoint orbits. As we will discover, this form of the Fokker-Planck equation in the absence of any additional energy dissipation will imply that the invariant measure (asymptotically in time) simply tends to a constant on Casimir surfaces.

Next, we shall include an additional energy dissipation mechanism, called double bracket dissipation, which preserves the coadjoint orbits while it decays the energy toward its minimum value, usually associated with an equilibrium state of the deterministic system. We refer to [BKMR96, GBH13, GBH14] and references therein for complete studies of double bracket dissipations. In a second step, we will include this double bracket dissipative term in our stochastic coadjoint motion equations and again study the associated Fokker-Planck equation and its invariant measure, which will no longer be a constant but instead will be an exponential function of the energy.

The procedure we shall follow will produce noisy dissipative dynamical systems on coadjoint orbits. The study of multiplicative noise and nonlinear dissipation in these systems is greatly facilitated by the symmetric structure of the equations for coadjoint motion. Indeed, a large part of standard dynamical system theory will still apply in our setting, In particular, the proof of existence of random attractors follows a standard approach. We will mainly focus on this particular feature of random attractors of our systems, as it is an important diagnostic and has recently been as active field of research. The main idea behind the random attractor is the decomposition of the invariant measure of the Fokker-Planck equation into random measures, called Sinai-Ruelle-Bowen, or SRB measures, whose expectation recovers the invariant measure of the Fokker-Planck equation. See, e.g., [You02] for a short insightful review. Besides these theoretical considerations, random attractors can help understanding the notion of reliability in complex dynamical systems, see for example [LSBY09]. The proof of existence of non-singular SRB measures requires some work, but it can be accomplished for our general class of mechanical systems written on semi-simple Lie algebras. Although geometric mechanics can also describe infinite dimensional systems such as fluid mechanics, [HMR98], we will only focus here on finite dimensional systems, and in particular on systems described by semi-simple Lie algebras. The natural non-degenerate and bi-invariant pairing admitted by semi-simple Lie algebras will facilitate the computations involved in proving our results, although some of the results may still apply more generally.

We will apply the theory of stochastic deformations that preserve coadjoint orbits for a particular class of semidirect product systems whose advected quantities live in the underlying vector space of the Lie algebra $\mathfrak{g}$. With this particular structure, which can be viewed as a generalised heavy top, we will be able to prove the existence of SRB measures. Although much of the present theory may also apply for more general systems than we treat here, as a first investigation we will show that simple mechanical systems in geometric mechanics exhibit interesting random attractors when both noise and a certain type of dissipation are included.

As illustrations, we will discuss in detail two canonical elementary examples in the science of stochastic dissipative geometric mechanics. These two examples are the rigid body and the heavy top, which are also the well known canonical examples for understanding symmetry reduction for deterministic geometric mechanics, [MR99, Hol08, HSS09]. Their extensions here to include stochasticity and dissipation which preserve coadjoint orbits may be regarded as natural counterparts for geometric mechanics of the standard nonlinear dissipative systems, such as the stochastic Lorenz systems, treated, e.g., in [KCG15].

Main contributions of this work. Section 2 uses the Clebsch approach of [Hol15] to introduce noise into the Euler-Poincaré equation for the momentum map, including its extension for semidirect product Lie symmetry groups. By construction, the momentum map for the stochastic vector field is the same as that for the deterministic vector field, so the stochastic and deterministic Euler-Poincaré equations for the momentum map may be compared directly. Section 3 introduces the selective decay mechanism for dissipation and studies the existence of random attractors. The first example of the Euler-Poincaré equation is treated in Section 4 with the free rigid body. Section 5 treats the Heavy Top as an example of the semidirect product extension. Finally, Section 6 briefly sketches the treatments of two other examples, the $SO(4)$ free rigid body and the spring pendulum.

2. Structure preserving stochastic mechanics

Stochastic Hamilton equations were introduced along parallel lines with the deterministic canonical theory in [Bis82]. These results were later extended to include reduction by symmetry in [LCO08]. Reduction by symmetry of expected-value stochastic variational principles for Euler-Poincaré equations was developed in [ACC14, CCR15]. Stochastic variational principles were also used in constructing stochastic variational integrators in [BRO09].

The present work is based on recent work of [Hol15], which used variational principles to introduce noise in fluid dynamics. This variational approach was developed further for fluids with advected quantities in [GBH16]. The inclusion of noise in fluid equations has a long history in the scientific literature. For reviews and recent advances in stochastic turbulence models, see [Kra94], [GK96]; and in the analysis of stochastic Navier-Stokes equations, see [MR01]. These studies of the stochastic Navier-Stokes equation are fundamental in the analysis of fluid turbulence. Expected-value stochastic variational principles leading to the derivation of the Navier-Stokes motion equation for incompressible viscous fluids have been investigated in [AC12]. For further references, we refer to [Hol15].

The present work incorporates stochasticity into finite dimensional mechanical systems admitting Lie group symmetry reduction, by using the standard Clebsch variational method for deriving cotangent lifted momentum maps. We review the standard approach to Lie group reduction by symmetry for finite dimensional systems in 2.1 and incorporate noise into this approach in section 2.2. Next, we describe the semidirect extension in 2.3 and study the associated Fokker-Planck equations and invariant measures in 2.4. The primary examples from classical mechanics with symmetry will be the free rigid body and the heavy top under constant gravity.

2.1. **Euler-Poincaré reduction.** Classical mechanical systems with symmetry can often be understood geometrically in the context of Lagrangian or Hamiltonian reduction, by lifting the motion $q(t)$ on the configuration manifold Q to a Lie symmetry group via the action of the symmetry group G on the configuration manifold, by setting $q(t) = g(t)q(0)$, where the multiplication has to be understood as the action of G to Q. This procedure lifts the solution of the motion equation from a curve $q(t) \in Q$ to a curve $g(t) \in G$, see [MR99, Hol08]. The simplest case is when $Q = G$. This case, called Euler-Poincaré reduction, will be described in the present section.

In the Lagrangian framework, reduction by symmetry may be implemented in Hamilton's principle via restricted variations in the reduced variational principle arising from variations on the corresponding Lie group. In the standard approach, for an arbitrary variation δg of a curve $g(t) \in G$ in a Lie group G, the left-invariant reduced variables are $g^{-1}\dot{g} \in \mathfrak{g}$ in the Lie algebra $\mathfrak{g} = T_eG$. Their variations arise from variations on the Lie group and are given by

$$\delta\xi = \dot{\eta} + \mathrm{ad}_\eta \xi \,,$$

for $\eta := g^{-1}\delta g$. Here, the operation $\mathrm{ad} : \mathfrak{g} \times \mathfrak{g} \to \mathfrak{g}$ represents the adjoint action of the Lie algebra on itself via the Lie bracket, denoted equivalently as $\mathrm{ad}_\xi \eta = [\xi, \eta]$, and we will freely use either notation throughout the text. If the Lagrangian $L(g, \dot{g})$ is left-invariant under the action of G, the restricted variations $\delta\xi$ of the reduced Lagrangian $L(e, g^{-1}\dot{g}) =: l(\xi)$ inherited from admissible variations of the solution curves on the group yield the Euler-Poincaré equation

$$\frac{d}{dt}\frac{\partial l(\xi)}{\partial \xi} + \mathrm{ad}^*_\xi \frac{\partial l(\xi)}{\partial \xi} = 0 \,. \tag{2.1}$$

In this equation, $\mathrm{ad}^* : \mathfrak{g} \times \mathfrak{g}^* \to \mathfrak{g}^*$ is the dual of the adjoint Lie algebra action, ad. That is, $\langle \mathrm{ad}^*_\xi \mu, \eta\rangle = \langle \mu, \mathrm{ad}_\xi \eta\rangle$ for $\mu \in \mathfrak{g}^*$ and $\xi, \eta \in \mathfrak{g}$, under the nondegenerate pairing $\langle \cdot\,, \cdot\rangle : \mathfrak{g} \times \mathfrak{g}^* \to \mathbb{R}$. Throughout this paper, we will restrict ourselves to semi-simple Lie algebras, so that the pairing is given by the Killing form, defined as

$$\kappa(\xi, \eta) := \mathrm{Tr}\left(\mathrm{ad}_\xi \mathrm{ad}_\eta\right). \tag{2.2}$$

In terms of the structure constants of the Lie algebra denoted as c^i_{jk} for a basis e_i, $i = 1, \ldots, \dim(\mathfrak{g})$, so that $[e_i, e_j] = c^i_{jk} e_i$, in which $\xi = \xi^i e_i$ and $\eta = \eta^j e_j$, the Killing form takes the explicit form

$$\mathrm{Tr}(\mathrm{ad}_\xi \mathrm{ad}_\eta) = c^n_{im} c^m_{jn} \xi^i \eta^j \,.$$

An important property of this pairing is its bi-invariance, written as

$$\kappa(\xi, \mathrm{ad}_\zeta \eta) = \kappa(\mathrm{ad}_\xi \zeta, \eta), \tag{2.3}$$

for every $\xi, \eta, \zeta \in \mathfrak{g}$. This pairing allows us to identify the Lie algebra with its dual, as the Killing form is non-degenerate. We will also use the property for compact Lie algebras, that the Killing form is negative definite and thus induces a norm on the Lie algebra, $\|\xi\|^2 := -\kappa(\xi, \xi)$.

Of course, the theory of semi-simple Lie algebra is very well-known and developed, see for example [Var84]. However, for the sake of clarity, we will express the abstract notations of adjoint and coadjoint actions with respect to the Killing form. We may then identify $\mathfrak{g}^* \cong \mathfrak{g}$ for each semi-simple Lie algebra we treat here.

We now turn to the equivalent Clebsch formulation of the Euler-Poincaré equations via a constrained Haamilton's principle, which we will use for implementing the noise in these systems. The Clebsch formulation of the Euler-Poincaré equation and its corresponding Lie-Poisson bracket on the Hamiltonian side has been explored extensively in ideal fluid dynamics [HK83, MW83] and more recently in optimal control problems [GBR11] and stochastic fluid dynamics [Hol15]. This earlier work should be consulted for detailed derivations of Clebsch formulations of Euler-Poincaré equations in the contexts of ideal fluids and optimal control problems. We will briefly sketch the Clebsch approach, as specialised to the applications treated here; since we will rely on it for the introduction of noise in finite-dimensional mechanical systems by following the approach of [Hol15] for stochastic fluid dynamics. We first introduce the Clebsch variables $q \in \mathfrak{g}$ and $p \in \mathfrak{g}^*$, where p will be a Lagrange multiplier which enforces the dynamical evolution of q given by the Lie algebra action of $\xi \in \mathfrak{g}$, as $\dot{q} + \mathrm{ad}_\xi q = 0$. Note the similarity of this equation with the constrained variations of the Lagrangian reduction theory. The Clebsch method in fluid dynamics (resp. optimal control) introduces auxiliary equations for advected quantities (resp. Lie algebra actions on state variables) as constraints in the Hamilton (resp. Hamilton-Pontryagin) variational principle $\delta S = 0$ with constrained action

$$S(\xi, q, p) = \int l(\xi) dt + \int \langle p, \dot{q} + \mathrm{ad}_\xi q \rangle dt \,. \tag{2.4}$$

Taking free variations of S with respect to ξ, q and p yields a set of equations for these three variables which can be shown to be equivalent to the Euler-Poincaré equation (2.1). The relation between the Lie algebra vector $\xi \in \mathfrak{g}$ and the phase-space variables $(q, p) \in T^*_e G$ is given by the variation of the action S with respect to the velocity ξ in (2.4). This variation yields the momentum map, $\mu : T^*_e G \to \mathfrak{g}^*$, given explicitly by

$$\mu := \frac{\partial l(\xi)}{\partial \xi} = \mathrm{ad}^*_q p. \tag{2.5}$$

Unless specified otherwise, we will always use the notation μ for the conjugate variable to ξ. This version of the Clebsch theory is especially simple, as the Clebsch variables are also in the Lie algebra $\mathfrak{g}$. In general, it is enough for them to be in the cotangent bundle of a manifold T^*M on which the group G acts by cotangent lifts. In this more general case, the adjoint and coadjoint actions must be replaced by their corresponding actions on T^*M but the method remains the same. Another generalisation, which will be useful for us later, allows the Lagrangian to depend on both ξ and q. In this case, the Euler-Poincaré equation will acquire additional terms depending on q and the Clebsch approach will be equivalent to semidirect product reduction [HMR98]. We will consider a simple case of this extension in Section 2.3 and in the treatment of the heavy top in Section 5.

2.2. **Structure preserving stochastic deformations.** We are now ready to deform the Euler-Poincaré equation (2.1) by introducing noise in the constrained Clebsch variational principle (2.4). In order to do this stochastic deformation, we introduce n independent Wiener processes W_t^i indexed by $i = 1, 2, \dots, n$, and their associated stochastic potential fields $\Phi_i(q,p) \in \mathbb{R}$ which are prescribed functions of the Clebsch phase-space variables, (q,p). The stochastic processes used here are standard Weiner processes, as discussed, e.g., in [CCR15, IW14]. The number of stochastic processes can be arbitrary, but usually we will take it as equal to the dimension of the Lie algebra, $n = \dim(\mathfrak{g})$. The constrained stochastic variational principle is then given by

$$S(\xi, q, p) = \int l(\xi) dt + \int \langle p, dq + \mathrm{ad}_\xi q\, dt\rangle + \int \sum_{i=1}^n \Phi_i(q,p) \circ dW_t^i. \tag{2.6}$$

In the stochastic action integral (2.6) and hereafter, the multiplication symbol $\circ$ denotes a stochastic integral in the Stratonovich sense. The Stratonovich formulation is the only choice of stochastic integral that admits the classical rules of calculus (e.g., integration by parts, the change of variables formula, etc.). Therefore, the Stratonovich formulation is indispensable in variational calculus and in optimal control. The free variations of the action functional (2.6) may now be taken, and they will yield stochastic processes for the three variables ξ, q and p.

For convenience in the next step of deriving a stochastic Euler-Poincaré equation, we will assume that the Lagrangian $l(\xi)$ in the action (2.6) is hyperregular, so that ξ may be obtained from the fibre derivative $\frac{\partial l(\xi)}{\partial \xi} = \mathrm{ad}_q^* p$. We will also specify that the stochastic potentials $\Phi_i(q,p)$ should depend only on the momentum map $\mu = \mathrm{ad}_q^* p$ so that the resulting stochastic equation will be independent of q and p. Following the detailed calculations in [Hol15], we then find the stochastic Euler-Poincaré equation

$$d\frac{\partial l(\xi)}{\partial \xi} + \mathrm{ad}_\xi^* \frac{\partial l(\xi)}{\partial \xi} dt - \sum_i \mathrm{ad}^*_{\frac{\partial \Phi_i(\mu)}{\partial \mu}} \frac{\partial l(\xi)}{\partial \xi} \circ dW_t^i = 0\,. \tag{2.7}$$

In terms of the stochastic process $dX = \xi dt - \sum_i \frac{\partial \Phi_i(\mu)}{\partial \mu} \circ dW_t^i$ with $\mu = \frac{\partial l(\xi)}{\partial \xi}$, the stochastic Euler-Poincaré equation (2.7) may be expressed in compact form, as

$$d\mu + \mathrm{ad}^*_{dX} \mu = 0\,. \tag{2.8}$$

The introduction of noise in the Clebsch-constrained variational principle rather than using reduction theory has simplified some of the technical difficulties linked with stochastic processes on Lie groups and constrained variations arising for such processes. See for example [ACC14] for a different approach to the derivation and analysis of deterministic expectation-value Euler-Poincaré equations using reduction by symmetry with conditional expectation.

As in the deterministic case, various generalisations of this theory are possible. For example, as mentioned earlier, the Clebsch phase-space variables can also be defined in T^*M, the Lagrangian can depend on q for systems of semidirect product types [GBH16]. Another generalisation is to let the stochastic potentials $\Phi_i(\mu)$ also depend separately on q in the semidirect product setting, as we will see later.

After having defined the Stratonovich stochastic process (2.7), one may compute its corresponding Itô form, which is readily given in terms of the ad* operation by

$$\begin{aligned} d\frac{\partial l(\xi)}{\partial \xi} + \mathrm{ad}^*_\xi \frac{\partial l(\xi)}{\partial \xi} dt + \sum_i \mathrm{ad}^*_{\sigma_i} \frac{\partial l(\xi)}{\partial \xi} dW^i_t + \\ - \frac{1}{2}\sum_i \mathrm{ad}^*_{\sigma_i}\left(\mathrm{ad}^*_{\sigma_i}\frac{\partial l(\xi)}{\partial \xi}\right) dt = 0, \end{aligned} \tag{2.9}$$

where $\sigma_i := -\frac{\partial \Phi_i(\mu)}{\partial \mu}$. Note that the indices for σ_i in the Itô sum in (2.9) are the same, and they be taken as a basis of the underlying vector space. In terms of $\mu := \frac{\partial l(\xi)}{\partial \xi}$ the Itô stochastic Euler-Poincaré equation (2.9) may be expressed equivalently as

$$d\mu + \mathrm{ad}^*_\xi \mu \; dt + \sum_i \mathrm{ad}^*_{\sigma_i}\mu \; dW^i_t - \frac{1}{2}\sum_i \mathrm{ad}^*_{\sigma_i}\left(\mathrm{ad}^*_{\sigma_i}\mu\right) dt = 0\,. \tag{2.10}$$

Another formulation of the stochastic Euler-Poincaré equation in (2.7) which will be used later in deriving the Fokker-Planck equation is the stochastic Lie-Poisson equation

$$df(\mu) = \left\langle \mu, \left[\frac{\partial f}{\partial \mu}, \frac{\partial h}{\partial \mu}\right]\right\rangle dt + \sum_i \left\langle \mu, \left[\frac{\partial f}{\partial \mu}, \frac{\partial \Phi_i}{\partial \mu}\right]\right\rangle \circ dW^i_t \tag{2.11}$$

$$=: \{f, h\} dt + \sum_i \{f, \Phi_i\} \circ dW_i\,, \tag{2.12}$$

where we have defined the Lie-Poisson bracket $\{\cdot,\cdot\}$ just as in the deterministic case, from the adjoint action and the pairing on the Lie algebra $\mathfrak{g}$.

2.3. **The extension to semidirect product systems.** As discussed in [HMR98], "It turns out that semidirect products occur under rather general circumstances when the symmetry in T^*G is broken". The geometric mechanism for this remarkable fact is that under reduction by symmetry, a semidirect product of groups emerges whenever a break of symmetry in the phase space occurs. The symmetry breaking produces new dynamical variables, living in the coset space formed from taking the quotient G/G_a of the original unbroken symmetry G by the remaining symmetry G_a under the isotropy subgroup of the new variables. These new dynamical variables form a vector space $G/G_a \simeq V$ on which the unbroken symmetry acts as a semidirect product, $G \circledS V$. In physics, elements of the vector space V corresponding to the new variables are called "order parameters". Typically, in physics, the original symmetry is broken by the introduction of potential energy depending on variables which reduce the symmetry to the isotropy subgroup of the new variables. Dynamics on semidirect products $G \circledS V$ results, and what had been only flow under the action of the unbroken symmetry before now becomes flow plus waves, or oscillations, produced by the exchange of energy between its kinetic and potential forms. The heavy top is the basic example, and it will be treated in Section 5. The semidirect product motion for the heavy top arises in the presence of gravity, when the support point of a freely rotating rigid body is shifted away from its centre of mass.

With this connection between symmetry breaking and semidirect products in mind, we now extend the stochastic Euler-Poincaré equations to include semidirect product systems. We refer to [HMR98] for a complete study of these systems. Although the deterministic equations of motion in [HMR98] are derived from reduction by symmetry, we will instead incorporate noise by simply extending the Clebsch-constrained variational principle used in the previous section.

The generalisation proceeds, as follows. We will begin by assuming that the Clebsch phase-space variables comprise the elements of T^*V for a given vector space V on which the Lie group G acts freely and properly. In fact we will have $(q,p) \in V \times V^*$. However, in this work, we will restrict ourselves to the case where V is the underlying vector space of $\mathfrak{g}$. Following the notation of [Rat81], we denote $\overline{\mathfrak{g}} = V$ in the sequel. Then, from the Killing form on $\mathfrak{g}$, denoted by $\kappa : \mathfrak{g} \times \mathfrak{g} \to \mathbb{R}$, there is a bi-invariant extension of the Killing form on $\mathfrak{g}Ⓢ\overline{\mathfrak{g}}$ defined as

$$\kappa_s\left((\xi_1,\xi_2),(\eta_1,\eta_2)\right) := \kappa(\xi_1,\eta_2) + \kappa(\xi_2,\eta_1). \tag{2.13}$$

Although this pairing is non-degenerate and bi-invariant, we will not use it for the definition of the dual of the semidirect algebra $\mathfrak{g}ⓈV$. Instead, we will use the sum of both Killing forms, namely

$$\kappa_0\left((\xi_1,\xi_2),(\eta_1,\eta_2)\right) := \kappa(\xi_1,\eta_1) + \kappa(\xi_2,\eta_2). \tag{2.14}$$

The group action is defined via the adjoint representation of G on $V = \overline{\mathfrak{g}}$, given by $(g_1,\eta_1)(g_2,\eta_2) = (g_1 g_2, \eta_1 + \mathrm{Ad}_{g_1}\eta_2)$. We then directly have the infinitesimal adjoint and coadjoint actions as

$$\begin{aligned}\mathrm{ad}_{(\xi_1,q_1)}(\xi_2,q_2) &= \left(\mathrm{ad}_{\xi_1}\xi_2, \mathrm{ad}_{\xi_1}q_2 + \mathrm{ad}_{q_1}\xi_2\right),\\ \mathrm{ad}^*_{(\xi,q)}(\mu,p) &= (\mathrm{ad}^*_\xi\mu + \mathrm{ad}^*_q p, \mathrm{ad}^*_\xi p),\end{aligned} \tag{2.15}$$

where the coadjoint action is taken with respect to κ_0 in (2.14).

The extended Killing form κ_s defined in (2.13), gives, apart from $\kappa(\eta,\eta)$ with $\eta \in \overline{\mathfrak{g}}$, a second invariant function on the coadjoint orbit

$$\kappa_s\left((\xi,\eta),(\xi,\eta)\right) = 2\kappa(\xi,\eta).$$

One then replaces the corresponding Lie algebra actions in the Clebsch-constrained variational principle (2.6), to obtain the stochastic process with semidirect product

$$d\,(\mu,q) + \mathrm{ad}^*_{(\xi,r)}\,(\mu,q)\,dt + \sum_i \mathrm{ad}^*_{\left(\frac{\partial\Phi_i(\mu,q)}{\partial\mu},\frac{\partial\Phi_i(\mu,q)}{\partial q}\right)}\,(\mu,q)\circ dW_t^i = 0, \tag{2.16}$$

where $l : \mathfrak{g}ⓈV \to \mathbb{R}$, $\Phi_i : \mathfrak{g}^*ⓈV^* \to \mathbb{R}$ and

$$\frac{\partial l(\xi,q)}{\partial\xi} =: \mu \qquad \text{and} \qquad \frac{\partial l(\xi,q)}{\partial q} =: r\,. \tag{2.17}$$

Consequently, after taking the Legendre transform of l, we have the Hamiltonian derivatives

$$\frac{\partial h(\mu,q)}{\partial\mu} =: \xi \qquad \text{and} \qquad \frac{\partial h(\mu,q)}{\partial q} =: -r\,, \tag{2.18}$$

for $h : \mathfrak{g}^* Ⓢ V^* \to \mathbb{R}$. By substituting into (2.16) the expressions in (2.15) for the coadjoint action, we obtain the system

$$
\begin{aligned}
&d\mu + \left(\mathrm{ad}^*_\xi \mu + \mathrm{ad}^*_r q\right) dt + \sum_i \left(\mathrm{ad}^*_{\frac{\partial \Phi_i(\mu,q)}{\partial \mu}} \mu + \mathrm{ad}^*_{\frac{\partial \Phi_i(\mu,q)}{\partial q}} q \right) \circ dW_t^i = 0\,, \\
&dq + \mathrm{ad}^*_\xi q dt + \sum_i \mathrm{ad}^*_{\frac{\partial \Phi_i(\mu,q)}{\partial \mu}} q \circ dW_t^i = 0\,.
\end{aligned}
\tag{2.19}
$$

Although the number of stochastic potentials Φ_i one may consider is arbitrary, but for our purposes we shall find it convenient to restrict to a maximum of $n = \dim(\mathfrak{g}) + \dim(V)$ such potentials. In fact, the potentials associated with V will not actually be fully treated here.

The semidirect product theory we have described here is the simplest instance of it, as we are using a particular vector space V. In general, the advected quantities can also be in a Lie algebra, or an arbitrary manifold, provided the action of the group G on it is free and proper [GBH16].

2.4. **The Fokker-Planck equation and invariant distributions.** We derive here a geometric version of the classical Fokker-Planck equation (or forward Kolmogorov equation) using our SDE (2.7). Recall that the Fokker-Planck equation describes the time evolution of the probability distribution $\mathbb{P}$ for the process driven by (2.7). We refer to the textbook [IW14] for a complete treatment of stochastic processes. Here, we will consider $\mathbb{P}$ as a function $C(\mathfrak{g}^*)$ with the additional property that $\int_{\mathfrak{g}^*} \mathbb{P} d\mu = 1$. First, the generator of the process (2.7) can be readily found from the Lie-Poisson form (2.9) of the stochastic process (2.12) to be

$$
Lf(\mu) = \left\langle \mathrm{ad}^*_\xi \mu, \frac{\partial f}{\partial \mu} \right\rangle - \sum_i \left\langle \mathrm{ad}^*_{\sigma_i} \mu, \frac{\partial}{\partial \mu} \left\langle \mathrm{ad}^*_{\sigma_i} \mu, \frac{\partial f}{\partial \mu} \right\rangle \right\rangle , \tag{2.20}
$$

where $\langle \cdot\,, \cdot \rangle$ still denotes the Killing form on the Lie algebra $\mathfrak{g}$ and $f \in C(\mathfrak{g}^*)$ is an arbitrary function of μ. Provided that the Φ_i's are linear functions of the momentum μ, the diffusion terms of the infinitesimal generator L will be self-adjoint with respect to the L^2 pairing $\int_{\mathfrak{g}^*} f(\mu)\mathbb{P}(\mu) d\mu$. If Φ is not linear, the advection terms of L^* will contain other terms but since we will restrict our considerations to the case of linear stochastic potentials, mainly for practical reasons, we will refer to (2.20) and its analogues as the Fokker-Planck operator L^*.

The Fokker-Planck equation describes the dynamics of the probability distribution $\mathbb{P}$ associated to the stochastic process for μ, in the standard advection diffusion form. Another step can be taken to highlight the underlying geometry of the Fokker-Planck equation (2.20), by rewriting it in terms of the Lie-Poisson bracket structure

$$
\frac{d}{dt}\mathbb{P} + \{h, \mathbb{P}\} - \sum_i \{\Phi_i, \{\Phi_i, \mathbb{P}\}\} = 0\,, \tag{2.21}
$$

where $h(\mu)$ is the Hamiltonian associated to $l(\xi)$ by the Legendre transform. In (2.21), we recover the Lie-Poisson formulation (2.12) of the Euler-Poincaré equation together with a dissipative term arising from the noise of the original SDE in a double Lie-Poisson bracket form.

This formulation gives the following theorem for invariant distributions of (2.20):

Theorem 2.1. *The invariant distribution* $\mathbb{P}_\infty$ *of the Fokker-Planck equation* (2.20), *i.e,* $L^*\mathbb{P}_\infty = 0$ *is* uniform *on the coadjoint orbits on which the SDE* (2.7) *evolves.*

Proof. By a standard result in functional analysis, see for example [Vil09], a linear differential operator of the form $L = B + \sum_i A_i^2$ has the property that $\ker(L) = \ker(A_i) \cap \ker(B)$ where here $A_i = \{\Phi_i, \cdot\}$ and $B = \{h, \cdot\}$. Consequently, for every smooth function f, the only functions g which satisfy $\{f, g\} = 0$ are the Casimirs, or invariant functions, on the coadjoint orbits. When restricted to a coadjoint orbit, these functions become constants. Hence, the invariant distribution $\mathbb{P}_\infty$ is a *constant* on the coadjoint orbit identified by the initial conditions of the system. □

Since the dynamics is restricted to the coadjoint orbits, for the probability distribution $\mathbb{P}$ to tend to a constant, yet remain normalisable, satisfying $\int_{\mathfrak{g}^*} \mathbb{P}(\mu) d\mu = 1$, the value of the density must tend to the inverse of the volume of the coadjoint orbit. Of course the compactness of the coadjoint orbit is equivalent to $\mathbb{P}_\infty > 0$. For non-compact orbits, Theorem 2.1 is still valid, and it will imply an asymptotically vanishing invariant distribution, in the same sense as for the invariant solution of the heat equation on the real line. In this case, a more detailed analysis of the invariant distribution can be performed by studying *marginals*, or projections onto a compact subspace of the coadjoint orbit.

Examples of non-compact coadjoint orbits arise in the semidirect product setting. First, the Fokker-Planck equation for the semidirect product stochastic process (2.16) is given by

$$\begin{aligned} Lf(\mu, q) = &\left\langle \mathrm{ad}^*_{(\xi,r)}(\mu, q), \left(\frac{\partial f(\mu, q)}{\partial \mu}, \frac{\partial f(\mu, q)}{\partial q} \right) \right\rangle - \\ &- \sum_i \left\langle \mathrm{ad}^*_{(\sigma_i,\eta_i)}(\mu, q), \left\{ \frac{\partial}{\partial \mu} \left\langle \mathrm{ad}^*_{(\sigma_i,\eta_i)}(\mu, q), \left(\frac{\partial f(\mu, q)}{\partial \mu}, \frac{\partial f(\mu, q)}{\partial q} \right) \right\rangle , \right.\right. \\ &\left.\left. , \frac{\partial}{\partial q} \left\langle \mathrm{ad}^*_{(\sigma_i,\eta_i)}(\mu, q), \left(\frac{\partial f(\mu, q)}{\partial \mu}, \frac{\partial f(\mu, q)}{\partial q} \right) \right\rangle \right\} \right\rangle, \end{aligned} \tag{2.22}$$

where $\sigma_i := -\partial\Phi_i/\partial\mu$ and $\eta_i := -\partial\Phi_i/\partial q$. The pairing used here is the sum of the pairings on $\mathfrak{g}$ and on V, given by κ_0 in (2.14). Note that for some values of index i, the vector fields σ_i or η_i may be absent. One can check that $L^* = L$; so that L generates the Lie-Poisson Fokker-Planck equation for the probability density $\mathbb{P}(\mu, q)$. As before, upon using the explicit form of the coadjoint actions, one finds

$$\begin{aligned} Lf(\mu, q) = &\left\langle \mathrm{ad}^*_\xi \mu + \mathrm{ad}^*_q r, \frac{\partial f}{\partial \mu} \right\rangle + \left\langle \mathrm{ad}^*_\xi q, \frac{\partial f}{\partial q} \right\rangle - \\ &- \sum_i \left\langle \mathrm{ad}^*_{\sigma_i} \mu + \mathrm{ad}^*_q \eta_i, \frac{\partial A_i}{\partial \mu} \right\rangle - \sum_i \left\langle \mathrm{ad}^*_{\sigma_i} q, \frac{\partial A_i}{\partial q} \right\rangle , \end{aligned} \tag{2.23}$$

$$\text{where} \qquad A_i := \left\langle \mathrm{ad}^*_{\sigma_i} \mu + \mathrm{ad}^*_q \eta_i, \frac{\partial f}{\partial \mu} \right\rangle + \left\langle \mathrm{ad}^*_{\sigma_i} q, \frac{\partial f}{\partial q} \right\rangle .$$

The Fokker-Planck equation (2.22) provides a direct corollary of Theorem 2.1.

Corollary 2.1.1. *The invariant probability density $\mathbb{P}_\infty(\mu, q)$ of (2.22) is constant on the coadjoint orbit corresponding to the initial conditions of the stochastic process (2.16).*

As mentioned before, the coadjoint orbit of this system is not compact, even if it had been compact for the Lie algebra $\mathfrak{g}$. Nevertheless, we can study the marginal distributions

$$\mathbb{P}^1(\mu) := \int \mathbb{P}(\mu, q) dq \qquad \text{and} \tag{2.24}$$

$$\mathbb{P}^2(q) := \int \mathbb{P}(\mu, q) d\mu \,, \tag{2.25}$$

which of course extend to invariant marginal distributions $\mathbb{P}^1_\infty$ and $\mathbb{P}^2_\infty$. With these marginal distributions, we can get more information on the invariant distribution of the semidirect product Lie-Poisson Fokker-Planck equation (2.22), as summarized in the next theorem.

Theorem 2.2. *For a semi-simple Lie algebra $\mathfrak{g}$ and $V = \overline{\mathfrak{g}}$, the marginal invariant distributions defined in (2.24) and (2.25) of the Fokker-Planck equation (2.22), with $\eta_i = 0$, for all i, have the following forms.*

(1) The invariant distribution $\mathbb{P}^2_\infty(q)$ is constant on the q-dependent subspace of the coadjoint orbit. If the Lie algebra $\mathfrak{g}$ is non-compact, the constant is zero.
(2) The invariant distribution $\mathbb{P}^1_\infty(\mu)$ restricted to $\kappa(\mu, \mu)$ is constant.
(3) If $\mathfrak{g}$ is compact, the invariant distribution $\mathbb{P}^1_\infty(\mu)$ is linearly bounded in time in the direction perpendicular to $\kappa(\mu, \mu)$.

Proof. We will compute the invariant marginal distributions separately, but first recall that the invariant distribution $\mathbb{P}(\mu, q)$ is constant on the Casimir level sets given by the initial conditions.

(1) By integrating the Fokker-Planck equation (2.22) over μ, one obtains

$$L\mathbb{P}^2(q) = \int \left\langle \mathrm{ad}^*_\xi q, \frac{\partial \mathbb{P}(\mu, q)}{\partial q} \right\rangle d\mu - \left\langle \mathrm{ad}^*_{\sigma_i} q, \frac{\partial}{\partial q} \left\langle \mathrm{ad}^*_{\sigma_i} q, \frac{\partial \mathbb{P}^2(q)}{\partial q} \right\rangle \right\rangle , \tag{2.26}$$

upon using the property that the coadjoint action is divergence-free (because of the anti-symmetry of the adjoint action, when identified with the coadjoint action via the Killing form) and recalling that the Lie algebra is either compact, or $\mathbb{P}(\mu, q) = 0$ for the boundary conditions.

Only the advection term remains in (2.26), as $\xi = \frac{\partial h}{\partial \mu}$ depends on μ. Nevertheless, an argument similar to that for the proof of Theorem 2.1 may be applied here to give the result of constant marginal distribution on the q dependent part of the coadjoint orbits. Again, if the Lie algebra is non-compact, then the probability density $\mathbb{P}^2_\infty(q)$ must vanish because of the normalisation.

(2) We first integrate the Fokker-Planck equation (2.22) with respect to the q variable to find

$$L\mathbb{P}^1(\mu) = \left\langle \mathrm{ad}^*_\xi \mu, \frac{\partial \mathbb{P}^1}{\partial \mu} \right\rangle - \sum_i \left\langle \mathrm{ad}^*_{\sigma_i} \mu, \frac{\partial}{\partial \mu} \left\langle \mathrm{ad}^*_{\sigma_i} \mu, \frac{\partial \mathbb{P}^1}{\partial \mu} \right\rangle \right\rangle , \tag{2.27}$$

where we have again used that the coadjoint action is divergence free, the same boundary conditions and the fact that $\langle \mathrm{ad}_q\xi, \frac{\partial \mathbb{P}}{\partial \mu}\rangle = 0$, $\forall \xi$ since $\frac{\partial \mathbb{P}}{\partial \mu}$ is aligned with q. Indeed, $\mathbb{P}$ is a function of the Casimirs, and thus is a function of $\kappa_s((\mu, q), (\mu, q))$. This fact prevents us from directly invoking Theorem 2.1 as we would find that $\mathbb{P}^1$ is indeed constant on $\kappa(\mu, \mu)$, but μ does not have an invariant norm. Nevertheless, we can still use this theorem by restricting $\mathbb{P}^1$ to the sphere $\kappa(\mu, \mu)$, or equivalently just considering polar coordinates for μ and discarding the radial coordinate. In this case we can invoke Theorem 2.1 and obtain the result of a constant marginal distribution $\mathbb{P}^1_\infty$ projected on the coadjoint orbit of the Lie algebra $\mathfrak{g}$ alone.

(3) We compute the time derivative of the quantity $\|\mu\|^2 := -\,\kappa(\mu, \mu)$, which is positive definite and thus defines a norm, to get an upper estimate of the form

$$\frac{d}{dt}\frac{1}{2}\|\mu\|^2 = \langle \mu, \dot\mu\rangle = \langle \mathrm{ad}_r q, \mu\rangle \leq \|r\|\|q\|\|\mu\|.$$

Then, because $\|q\| = \sqrt{-\kappa(q, q)}$ is constant, and provided that r is bounded, we can integrate to get

$$\|\mu(t)\| \leq \|\mu(0)\| + \alpha t, \tag{2.28}$$

where α is a constant depending on the Lie algebra and the Hamiltonian. □

This section has reviewed the framework for the study of noise in dynamical systems defined on coadjoint orbits, and has illustrated how noise may be included in these systems, so as to preserve the deterministic coadjoint orbits. The systems we have considered are the Euler-Poincaré equations on semi-simple finite dimensional Lie groups and the semidirect product structures which appear when the advected quantities are introduced in the underlying vector space of the Lie algebra of the Lie group. These structures are not the most general. However, their study has allowed us to use the properties of the natural pairing given by the Killing form to prove a few illustrative results in a simple and transparent way. In particular, we showed that the invariant measure of the Fokker-Planck equation, written in Lie-Poisson form, is constant on the coadjoint orbits. In the semidirect product setting, a bit more care was needed to obtain similar results of marginal distributions, as the coadjoint orbits are not compact in this case. We will illustrate our approach with the two basic examples of the rigid body and heavy top in sections 4 and 5, where more will be said about these systems, and in particular about their integrability.

3. Dissipation and random attractors

In the previous section we described a structure preserving stochastic deformation of mechanical systems with symmetries. The preserved structure is the coadjoint orbit of the deterministic system. Namely, the stochastic process still belongs to one of these orbits, characterised by the initial conditions of the system. This preservation is reflected in the strict conservation of particular integrals of motion, called Casimirs. In general, these are the only conserved quantities of our stochastic processes. Indeed, the energy is not conserved, apart from very particular choices of the energy and the stochastic potentials as we will see for some examples. The energy is not strictly decaying either, but is subject to random fluctuations with its own stochastic process coupled to the stochastic process of μ. The complexity of the energy evolution hindered us from studying it in full generality in the previous sections. In the present section, however, we will investigate the energy behaviour for particular mechanical examples subject to dissipation and random fluctuations. The type of energy dissipation that we will introduce in Section 3.1 also preserves the coadjoint orbits. Consequently, the dissipation is compatible with our stochastic deformation. The main outcome after introducing this dissipation is the emergence of a balance between noise and dissipation which will make the invariant measure of the Fokker-Planck equation energy dependent, as we will see in Section 3.2 and in the proof of existence of random attractors in Section 3.3.

3.1. **Double bracket dissipation.** To augment the stochastic processes introduced in the previous section, we will add a type of dissipation for which the solutions of the stochastic process will still lie on the deterministic coadjoint orbit. For this purpose, we will use *double bracket dissipation*, which was studied in detail in [BKMR96] and was generalised recently in [GBH13, GBH14]. We will follow the latter works to include an energy dissipation which preserves the Casimir functions. We will not review this theory in detail here. Instead, we refer the reader to [GBH13] for a detailed discussion of Euler-Poincaré selective decay dissipation and [GBH14] for the semidirect product extension.

For the stochastic process (2.8), the dissipative stochastic Euler-Poincaré equation written in Hamiltonian form is

$$d\mu + \mathrm{ad}^*_{\frac{\partial h}{\partial \mu}}\mu\, dt + \theta\, \mathrm{ad}^*_{\frac{\partial C}{\partial \mu}}\left[\frac{\partial C}{\partial \mu}, \frac{\partial h}{\partial \mu}\right]^\flat dt + \sum_i \mathrm{ad}^*_{\sigma_i}\mu \circ dW_t^i = 0\,, \tag{3.1}$$

where $\theta > 0$ parametrises the rate of energy dissipation and C is a chosen Casimir of the coadjoint orbit. For convenience, we are using the isomorphism $\flat : \mathfrak{g} \to \mathfrak{g}^*$ defined via the Killing form of $\mathfrak{g}$. The converse isomorphism will be denoted $\sharp : \mathfrak{g}^* \to \mathfrak{g}$. The corresponding generalisation of selective decay for the semidirect product stochastic process (2.16), following [GBH14], is given by

$$\begin{aligned} d(\mu, q) + \mathrm{ad}^*_{(\xi, r)}(\mu, q)\, dt + \theta\, \mathrm{ad}^*_{\left(\frac{\partial C}{\partial \mu}, \frac{\partial C}{\partial q}\right)}\left[\left(\frac{\partial C}{\partial \mu}, \frac{\partial C}{\partial q}\right), (\xi, r)\right]^\flat dt \\ + \sum_i \mathrm{ad}^*_{(\sigma_i, \eta_i)}(\mu, q) \circ dW_t^i = 0, \end{aligned} \tag{3.2}$$

where $\xi = \frac{\partial h}{\partial \mu}$, and the quantities h and r are defined in equation (2.18). Equation (3.2) may be written equivalently as a system of equations, by using the actions given in (2.15). Namely,

$$\begin{aligned} d\mu + (\mathrm{ad}^*_\xi \mu + \mathrm{ad}^*_r q)\, dt + \theta\, \mathrm{ad}^*_{\frac{\partial C}{\partial \mu}}\left[\frac{\partial C}{\partial \mu}, \xi\right]^\flat dt + \theta\, \mathrm{ad}^*_{\frac{\partial C}{\partial q}}\left(\mathrm{ad}_{\frac{\partial C}{\partial \mu}} r + \mathrm{ad}_{\frac{\partial C}{\partial q}} \xi\right)^\flat dt \\ + \sum_i \left(\mathrm{ad}^*_{\sigma_i}\mu + \mathrm{ad}^*_{\eta_i} q\right) \circ dW_t^i = 0 \end{aligned} \tag{3.3}$$

$$dq + \mathrm{ad}^*_\xi q\, dt + \theta\, \mathrm{ad}^*_{\frac{\partial C}{\partial \mu}}\left(\mathrm{ad}_{\frac{\partial C}{\partial \mu}} r - \mathrm{ad}_{\frac{\partial C}{\partial q}} \xi\right)^\flat dt + \sum_i \mathrm{ad}^*_{\sigma_i} q \circ dW_t^i = 0\,.$$

Recall for the deterministic equations that the energy decays for $\theta > 0$ as

$$\frac{d}{dt}h(\mu, q) = -\theta \left\| \mathrm{ad}_{\frac{\partial C}{\partial \mu}} \xi \right\|^2 - \theta \left\| \mathrm{ad}_{\frac{\partial C}{\partial \mu}} r + \mathrm{ad}_{\frac{\partial C}{\partial q}} \xi \right\|^2, \tag{3.4}$$

where the second term is present only in the semidirect product setting [GBH14].

Remark 3.1. *The selective decay approach preserves the entire coadjoint orbit, and the speed of decay depends upon which invariant function C one uses in implementing it. Indeed, either the first or second term of (3.4) can vanish depending on the choice of Casimir. We refer to the heavy top example in Section 5 for more details.*

Asymptotically in time, $t \to \infty$, the deterministic equations with selective decay will tend toward a state which is compatible with the state of minimal energy, as shown in [GBH14]. However, the presence of noise will balance the dissipation due to selective decay and prevent the system from reaching this deterministic equilibrium position. This feature will imply a non-constant invariant distribution of the corresponding Fokker-Planck solution to be studied in the next section, as well as the existence of *random attractors*, for which we refer to [KCG15, SH98] for background information.

3.2. **The Fokker-Planck equation and invariant distributions.** In order to study the balance between multiplicative noise and nonlinear dissipation, we compute the Fokker-Planck equation associated to the process (3.1) or, equivalently, (3.3), and its invariant solutions.

The Fokker-Planck equation for the Euler-Poincaré stochastic process (3.1) is modified by the double bracket dissipative term, to read as,

$$\frac{d}{dt}\mathbb{P}(\mu) + \{h, \mathbb{P}\} + \theta \left\langle \left[\frac{\partial \mathbb{P}}{\partial \mu}, \frac{\partial C}{\partial \mu}\right], \left[\frac{\partial h}{\partial \mu}, \frac{\partial C}{\partial \mu}\right] \right\rangle - \frac{1}{2}\sum_i \{\Phi_i, \{\Phi_i, \mathbb{P}\}\} = 0. \tag{3.5}$$

The invariant distribution of this Fokker-Planck equation is no longer a constant on the coadjoint orbits. Instead, it now depends on the energy, as summarized in the following theorem.

Theorem 3.2. *Let the noise amplitude be of the form $\sigma_i = \sigma e_i$ for an arbitrary $\sigma \in \mathbb{R}$, where the e_i's span the underlying vector space of the dual Lie algebra $\mathfrak{g}^* \cong \mathfrak{g}$. The invariant distribution of the Fokker-Planck equation (3.5) associated to (3.1) with Casimir $C = \kappa(\mu, \mu)$ is given by*

$$\mathbb{P}_\infty(\mu) = Z^{-1} e^{-\frac{2\theta}{\sigma^2} h(\mu)}, \tag{3.6}$$

where Z is the normalisation constant that enforces $\int \mathbb{P}_\infty(\mu) d\mu = 1$.

Proof. The invariant distribution is given by solving $\frac{d}{dt}\mathbb{P}_\infty(\mu) = 0$. From the structure of the Fokker-Planck equation in double bracket form (3.5), the advection term must vanish independently of the other terms. (See the argument of Theorem 2.1.) We therefore use the Ansatz $\mathbb{P}_\infty(\mu) = f(h(\mu))$, where the function f is to be determined. Consequently, only the selective decay and the double bracket term still remain. The selective decay is first rewritten, using the bi-invariance property of the Killing form (2.3), as

$$\theta \left\langle \left[\frac{\partial \mathbb{P}}{\partial \mu}, \frac{\partial C}{\partial \mu} \right], \left[\frac{\partial h}{\partial \mu}, \frac{\partial C}{\partial \mu} \right] \right\rangle = \theta \left\langle \frac{\partial \mathbb{P}}{\partial \mu}, \mathrm{ad}_{\frac{\partial C}{\partial \mu}} \left[\frac{\partial C}{\partial \mu}, \frac{\partial h}{\partial \mu} \right] \right\rangle$$
$$= \theta\, \mathbf{d} \left(f(h) \mathrm{ad}_{\frac{\partial C}{\partial \mu}} \left[\frac{\partial C}{\partial \mu}, \frac{\partial h}{\partial \mu} \right] \right),$$

where we have used the property that the coadjoint action for semi-simple Lie algebras is divergence-free. (Notice that the exterior derivative $\mathbf{d}$ is a divergence operation here.) Since $\kappa(\mu,\mu)$ is a Casimir and $\mu^\sharp = \frac{\partial C}{\partial \mu}$, we can rewrite the double bracket due to the noise as

$$-\frac{1}{2}\sum_i \{\Phi_i, \{\Phi_i, \mathbb{P}\}\} = -\sigma^2 \frac{1}{2} \sum_i \left\langle \mathrm{ad}^*_{e_i} \frac{\partial C^\flat}{\partial \mu}, \frac{\partial}{\partial \mu} \left\langle \mathrm{ad}^*_{e_i} \frac{\partial C^\flat}{\partial \mu}, \frac{\partial \mathbb{P}}{\partial \mu} \right\rangle \right\rangle$$
$$\text{(From bi-invariance of } \kappa) \quad = \sigma^2 \frac{1}{2} \sum_i \mathbf{d} \left(f'(h) \mathrm{ad}_{e_i} \frac{\partial C}{\partial \mu} \left\langle \mathrm{ad}_{\frac{\partial h}{\partial \mu}} \frac{\partial C}{\partial \mu}, e_i \right\rangle \right)$$
$$= \sigma^2 \frac{1}{2} \mathbf{d} \left(f'(h) \mathrm{ad}_{\mathrm{ad}_{\frac{\partial h}{\partial \mu}} \frac{\partial C}{\partial \mu}} \frac{\partial C}{\partial \mu} \right).$$

We have used the bi-invariance of the pairing to enforce the relation $\mathrm{ad}^\dagger_\xi \eta := \mathrm{ad}^*_\xi \eta^\flat = -\mathrm{ad}_\xi \eta$. See for example [Var84] for more details. The result (3.6) for the equilibrium distribution then follows by comparing the selective decay term with the double bracket term and noticing that the two terms will cancel, provided $f(x) = e^{-2\theta\, x/\sigma^2}$. □

Remark 3.3. *This calculation only uses the bi-invariance of the Killing form, which holds in general for semi-simple Lie algebras. Therefore, the same conclusion applies for other Lie algebras which admit a bi-invariant pairing. In statistical mechanics, the invariant measure* (3.6) *is often called the Gibbs measure.*

The Fokker-Planck equation with dissipation in the semidirect-product setting directly gives

$$\frac{d}{dt}\mathbb{P}(\mu, q) + \{h, \mathbb{P}\} - \frac{1}{2}\sum_i \{\Phi_i, \{\Phi_i, \mathbb{P}\}\} +$$
$$+\, \theta \left\langle \left[\left(\frac{\partial \mathbb{P}}{\partial \mu}, \frac{\partial \mathbb{P}}{\partial q} \right), \left(\frac{\partial C}{\partial \mu}, \frac{\partial C}{\partial q} \right) \right], \left[\left(\frac{\partial h}{\partial \mu}, \frac{\partial h}{\partial q} \right), \left(\frac{\partial C}{\partial \mu}, \frac{\partial C}{\partial q} \right) \right] \right\rangle = 0\,. \tag{3.7}$$

Consequently, for semidirect products, we have the analogue of the previous theorem, but for the marginal invariant distribution on the advected quantities.

Theorem 3.4. *Provided the Hamiltonian is of the form $h(\mu, q) = K(\mu) + V(q)$ for two functions K and V, the invariant marginal distribution $\mathbb{P}^2_\infty(q)$ with the selective decay from the Casimir $\kappa(\mu, q)$ is given by*

$$\mathbb{P}^2_\infty(q) = Z^{-1} e^{-\frac{2\theta}{\sigma^2} V(q)}, \tag{3.8}$$

where Z is the normalisation constant.

Proof. The proof here is similar to the proof for Theorem 3.2. Thus, we only show the main calculations. First, the selective decay term is given explicitly, using the Casimir $\kappa(\mu, q)$, by

$$\theta \left\langle \mathrm{ad}_{\frac{\partial \mathbb{P}}{\partial \mu}} q, \mathrm{ad}_\xi q \right\rangle + \theta \left\langle \mathrm{ad}_{\frac{\partial \mathbb{P}}{\partial \mu}} \mu + \mathrm{ad}_{\frac{\partial \mathbb{P}}{\partial q}} q, \mathrm{ad}_\xi \mu + \mathrm{ad}_r q \right\rangle .$$

Integrating the selective decay term of (3.7) in μ and assuming $\mathbb{P}^2(q) = f(V(q))$, gives

$$\theta \left\langle \mathrm{ad}_{\frac{\partial \mathbb{P}^2}{\partial q}} q, \mathrm{ad}_r q \right\rangle = -\theta \left\langle \mathrm{ad}^*_{\mathrm{ad}_r q} q, \frac{\partial \mathbb{P}^2}{\partial q} \right\rangle = -\theta \mathbf{d} \left(\mathrm{ad}^*_{\mathrm{ad}_r q} q f \right) ,$$

where we have used the bi-invariance property of κ (2.3), as well as the divergence-free property of the coadjoint action. Then, after integration over μ, the double bracket term becomes

$$\begin{aligned} -\frac{1}{2}\sigma^2 \mathbf{d} \left(\mathrm{ad}^*_{e_i} q \left\langle \mathrm{ad}^*_{e_i} q, \frac{\partial \mathbb{P}^2}{\partial q} \right\rangle \right) &= \frac{1}{2}\sigma^2 \mathbf{d} \left(f' \mathrm{ad}^*_{e_i} q \left\langle \mathrm{ad}_q r, e_i \right\rangle \right) \\ &= -\frac{1}{2}\sigma^2 \mathbf{d} \left(f' \mathrm{ad}^*_{\mathrm{ad}_r q} q \right) , \end{aligned}$$

upon again using bi-invariance. Thus, the result follows, as f must satisfy $\theta f = \frac{1}{2}\sigma^2 f'$. □

In the Euler-Poincaré setting, the invariant distribution was concentrated around the positions of minimum energy, and here the advected quantity q is concentrated around the position of minimal *potential* energy. We conjecture that the complete invariant distribution is concentrated around the minimal energy region, as in the Euler-Poincaré setting. However, we will not investigate this conjecture here, as we will be mainly interested in the dynamics of the advected quantities.

3.3. **Random attractors and Sinai-Ruelle-Bowen measures.** We now turn to the study of the existence of random attractors (RAs) in our stochastic dissipative systems, and its connection with the theory of random dynamical systems (RDS). The classic approach in studying the effect of stochastic forcing of nonlinear dynamical systems is by integrating the system forward in time and performing averages and then study the Fokker-Planck equation, as we have done up to now. A second point of view studies random dynamical systems via the so-called *pull-back approach*. We will not fully explore the theory of random dynamical systems and pull-back attractors here, but only invoke the main results from it and refer the interested reader to [Arn95, BDV06, KR11] for an in depth account of these subjects. In a nutshell, the realisation of the noise is fixed for each stochastic process and the average is taken over the initial conditions, but not over the realisation of the noise. The noise makes the system time-dependent; so the notion of attractor must be defined in the pull-back sense, such that for large times the attractive set does not depend on time. The pull-back attractor is thus defined by pulling back the same initial conditions from $t = 0$ to $t \to -\infty$ and letting the system evolve to $t = 0$. In the limit, the set obtained at $t = 0$ is the pull-back attractor. In random dynamical systems theory, it is usually called a random attractor, and if it is not singular, it may admit a particular measure, the Sinai-Ruelle-Bowen measure (SRB), which is also called a physical measure, see [You02]. We will denote the physical measure by $\mathbb{P}_\omega(\mu)$ for a given realisation of the noise ω and relate it to the invariant measure of the Fokker-Planck equation $\mathbb{P}_\infty(\mu)$ by the formula

$$\int_\Omega \mathbb{P}_\omega(\mu) d\omega = \mathbb{P}_\infty(\mu), \tag{3.9}$$

for the probability space Ω. Here we are informally referring to probability densities, and the SRB measure can be seen as the invariant measure most compatible with volume, although volume in phase space is not preserved, because of dissipation.

Remark 3.5. *We are considering the interaction of noise and dissipation. However, if the noise were replaced by a simpler deterministic forcing, similar results would emerge. In particular, periodic forcing or kicking of dissipative dynamical systems has been studied in great detail in numerous works, e.g., in* [LY10, LWY13]*. In our present setting, periodic kicking could also be implemented instead of noise. This might help for the understanding of basic mechanisms of the random attractors such as stretching and folding, as some solutions could perhaps be found explicitly. See the examples in Sections 4 and 5 where different types of random attractors are found. We have left such deeper studies for future investigations and will only treat the general case of noise here.*

We first determine that the stochastic processes (3.1) and (3.3) do indeed admit random attractors. See [KCG15, SH98] and references therein for more details about this type of approach. Then, we will give the region of the parameter space (θ, σ) where non-singular random attractors exist.

Theorem 3.6. *The stochastic process* (3.1) *admits a random attractor, for every Lie group G.*

Proof. We recast the SDE (3.1) into a random dynamical equation (RDE) using the Wiener processes z_i given by

$$dz_i = \sigma_i dW_t^i\,, \tag{3.10}$$

where z_i must be understood as a one dimensional process in the direction i (corresponding to σ_i). In the sequel, we will denote $z(t,\omega) = \sum_i z_i(t,\omega) \in \mathfrak{g}$. The process $z(t)$ thus defines a random path in the Lie algebra $\mathfrak{g}$ and, via the exponential map, a random path in the group G as $g(t,\omega) = e^{z(t,\omega)}$.

We then define a new variable $\widetilde{\mu}(t) = g(t)\mu(t) = \mathrm{Ad}^*_{g(t)}\mu(t)$ and we have, from (3.1) (see for example [MR99]),

$$d\widetilde{\mu}(t) = \mathrm{Ad}^*_{g(t)}\left(-\sum_i \mathrm{ad}^*_{\sigma_i}\mu \circ dW + d\mu\right) = \mathrm{Ad}^*_{g(t)}\left(F(\mathrm{Ad}^*_{g(t)^{-1}}\widetilde{\mu}(t))\right)dt,$$

where our stochastic process is generically written $d\mu = F(\mu)dt + G_i(\mu)\circ dW_i$ for convenience. From here, we have the RDE associated to (3.1) of the form

$$\frac{d}{dt}\widetilde{\mu}(t) = \widetilde{F}(\widetilde{\mu}(t), g(t)), \tag{3.11}$$

where $\widetilde{F}$ is defined from the previous calculation. Recall that from the theory of selective decay we have [GBH13]

$$\frac{d}{dt}h(\mu) = -\,\theta\left\|\left[\frac{\partial C}{\partial \mu},\frac{\partial h}{\partial \mu}\right]\right\|^2,$$

and $h(\mathrm{Ad}^*_g\mu) = h(\mu)$ so that this equality becomes for (3.11),

$$\frac{d}{dt}h(\widetilde{\mu}) = -\theta\left\|\left[\mathrm{Ad}_{g^{-1}}\frac{\partial C(\widetilde{\mu})}{\partial \widetilde{\mu}},\mathrm{Ad}_{g^{-1}}\frac{\partial h(\widetilde{\mu})}{\partial \widetilde{\mu}}\right]\right\|^2 \leq 0\,. \tag{3.12}$$

This inequality assures that the energy decays with a random strictly negative bound. The existence of the random attractor then follows from a standard argument, demonstrated, for example, in [SH98] or [KCG15], for another application in the linear case. □

The idea of the proof is to generalise the linear change of variables used to recast the original stochastic process into a random dynamical equation to allow a nonlinear group theoretical change of variable. The dissipative property is directly given by the selective decay theory, and the invariance of the dynamical equation under the group action. This theorem is very general and no specific assumptions on the Lie group need to be imposed. In particular, modulo difficulties in analysis, the theorem should also apply for the diffeomorphism group used in the description of compressible fluid equations. However, we have no intention of investigating the infinite dimensional theory here.

The same result persists in the semidirect-product theory, as developed earlier.

Corollary 3.6.1. *Theorem 3.6 applies to semidirect-product stochastic processes (3.3).*

Proof. The proof follows the same argument, upon using the action of the group G and the Lie algebra $\mathfrak{g}$ and the advected quantities in V to define the change of variables. The decay rate of the energy is given by using the deterministic selective decay formulae (3.4). □

3.4. **Existence of the SRB measure.** We now turn to the existence of the SRB measure. Theorem 3.7 below for the existence of SRB measures will invoke Hörmander's theorem about the smoothness of transition probabilities for a diffusion satisfying the so-called Hörmander (Lie) bracket conditions. The Lie bracket $[v,w](x)$ of two vector fields $v(x), w(x)$ in $\mathbb{R}^n$ is defined as

$$[v,w](x) = Dv(x)w(x) - Dw(x)v(x), \tag{3.13}$$

where we denote by Dv the derivative matrix given by $(Dv)_{ij} = \partial_j v_i = v_{i,j}$. Given an SDE of the form

$$dx = A_0(x) + \sum A_i(x)\circ dW^i_t, \tag{3.14}$$

the parabolic Hörmander condition states that if the following condition is satisfied

$$\cup_{k\geq 1} V_k(x) = \mathbb{R}^n, \quad \text{for all} \quad x\,, \tag{3.15}$$

where

$$\begin{aligned} &V_k(x) = V_{k-1}(x) \cup \mathrm{span}\{[v(x), A_j(x)] : v \in V_{k-1}, j \geq 0\} \quad \text{and} \\ &V_0(x) = \mathrm{span}\{A_j, j \geq 1\}, \end{aligned} \tag{3.16}$$

then the law of the solution to (3.14) has a smooth density with respect to Lebesgue measure. The distinction between V_0 and the remaining vectors is unnecessary: one can extend these vector fields to $\mathbb{R}^{n+1}$ and regard them all on equal footing.

We are now able to prove the following theorem for our stochastic dissipative systems.

Theorem 3.7. *If the largest Lyapunov exponent of* (3.1) *is positive, the random attractor is the support of a Sinai-Ruelle-Bowen (SRB) measure.*

Proof. The proof uses the corollary of Theorem B in [LY88], which assumes the existence of a random attractor. The only point left to show here is that the parabolic Hörmander condition (3.15) is fulfilled. Given the Stratonovich process (3.14) in $\mathfrak{g}^*$, we only need to check that the vector fields $A_1, \dots, A_N$ will span the tangent space to the coadjoint orbits as long as N is sufficiently large. Since $A_i(\mu) := \mathrm{ad}^*_{\sigma_i}\mu$, they are tangent to the coadjoint orbits. The minimal number of A_i needed cannot be found, in general, as it will depend on the Lie symmetry algebra and the form of the Hamiltonian. Nevertheless, in our case the $\langle \sigma_i \rangle$ span the vector space $\overline{\mathfrak{g}}$, and the Hörmander condition is fulfilled.

□

Corollary 3.7.1. *Theorem 3.7 also applies for the semidirect product case, even with $\eta_i = 0$.*

Proof. The same argument works here, even if $\eta_i = 0$, as the semidirect product structure will automatically span the whole space, provided $\mathfrak{g}$ is already spanned and h is not too degenerate on V. □

3.5. **Lyapunov exponents.** We now determine the region of the parameter space (σ, θ), for a given Lie algebra and Hamiltonian, where we are guaranteed that at least one of the Lyapunov exponents is positive. Having the positivity of the top Lyapunov exponent allows us to use the previous Theorem 3.7 to prove the existence of a non-singular random attractor with a SRB measure and positive entropy.

We will restrict ourselves here to the estimation of the lower bound of the sum of the Lyapunov exponents using the multiplicative ergodic theorem (MET), see [Arn95] for more details.

The stochastic systems which we can consider here are written on compact semi-simple Lie algebras, such that $c^2 = \|\mu\|^2$ is constant and defines a bounded set. The energy functional $h(\mu)$ is also a generic quadratic kinetic energy term, with a given inertia tensor $\mathbb{I}^{-1}$, corresponding to the Hessian matrix of $h(\mu)$.

Theorem 3.8. *Provided the Lie algebra is compact and semi-simple, the sum of the Lyapunov exponents is estimated from below by*

$$\sum_i \lambda_i \geq \frac{1}{2}|\epsilon| n\sigma^2 - \theta c^2 \mathbb{I}^{-1}_{\min} + \theta|\epsilon|\mathbb{E}_\infty h(\mu). \tag{3.17}$$

where $c = \|\mu\|^2$, ϵ is the Killing form constant, and n is the number of $\sigma_i = \sigma e_i$ spanning the Lie algebra. Thus, the dimension of the Lie algebra, $n = \dim(\mathfrak{g})$. The quantity $\mathbb{I}_{\rm min}^{-1}$ is the largest eigenvalue of the Hessian of the Hamiltonian. The expectation $\mathbb{E}_\infty$ is taken with respect to the invariant measure $\mathbb{P}_\infty$. An estimation from above is also valid, using $\mathbb{I}_{\rm max}^{-1}$, the minimal eigenvalue, instead.

Proof. Let us denote the stochastic process (3.1) in Itô form by

$$d\mu = F(\mu)dt + \sum_i G_i(\mu)dW_t^i.$$

The Multiplicative Ergodic Theorem (MET) states that [Arn95]

$$\sum_i \lambda_i = \lim_{t\to\infty} \frac{1}{t}\log\det D\varphi(t,\omega,x)\,, \tag{3.18}$$

where φ is the flow of the stochastic process and $D\varphi$ stands for its derivative (the Jacobian matrix). We can then use Jacobi's formula to rewrite (3.18) as

$$\lim_{t\to\infty} \frac{1}{t}\log\det D\varphi(t,\omega,x) = \lim_{t\to\infty}\frac{1}{t}\mathrm{Tr}\int^t DF(\varphi(t,x,\omega))ds\,. \tag{3.19}$$

Finally, ergodicity of this process gives

$$\sum_i \lambda_i = \int \mathrm{Tr}(DF(\mu))\mathbb{P}_\infty(\mu)d\mu\,, \tag{3.20}$$

where $\mathbb{P}_\infty$ is the invariant measure of the underlying stochastic process. The calculation of the trace simplifies in the case of a compact semi-simple Lie algebra with the Killing form $\mathrm{Tr}(\mathrm{ad}_A\mathrm{ad}_B) = \epsilon A\cdot B$, where $\epsilon < 0$ depends on the Lie algebra. Then, using the explicit form of F along with semi-simplicity for $\mathfrak{g}$, yields

$$F(\mu) = \mathrm{ad}_{\frac{\partial h}{\partial\mu}}\mu + \theta\,\mathrm{ad}_\mu\mathrm{ad}_\mu\frac{\partial h}{\partial\mu} - \frac{1}{2}\sum_i \mathrm{ad}_{\sigma_i}\mathrm{ad}_{\sigma_i}\mu\,.$$

Consequently, we arrive at

$$\begin{aligned}\mathrm{Tr}(DF(\mu)) &= \mathrm{Tr}\left(-\theta\,\mathrm{ad}_\mu\mathrm{ad}_{\frac{\partial h}{\partial\mu}} + \theta\,\mathrm{ad}_\mu\mathrm{ad}_\mu\frac{\partial^2 h}{\partial\mu^2} - \frac{1}{2}\sum_i\sigma^2\mathrm{ad}_{e_i}\mathrm{ad}_{e_i}\right)\\ &= |\epsilon|\theta h(\mu) + \theta A(\mu,\mu) + \frac{1}{2}|\epsilon|n\sigma^2,\end{aligned}$$

where n is the number of σ_i fields. The $A(\mu,\mu)$ term depends on the Lie algebra structure constants, and is difficult to obtain explicitly for every compact semi-simple Lie algebra. However, we can estimate it here using

$$\theta\kappa(\mu,\mu)\mathbb{I}_{\rm max}^{-1} \geq \theta\mathrm{Tr}\left(\theta\mathrm{ad}_\mu\mathrm{ad}_\mu\frac{\partial^2 h}{\partial\mu^2}\right) \geq \theta\kappa(\mu,\mu)\mathbb{I}_{\rm min}^{-1}. \tag{3.21}$$

Collecting terms then gives a lower and upper bound for the sum of the Lyapunov exponents (3.17). □

Notice that if c, the norm of the momentum map, increases in (3.17), the effect of the dissipation increases, which in turn decreases the sum of the Lyapunov exponents, since $\theta > 0$. Likewise, if the moment of inertia increases, the effect of the dissipation decreases. Consequently, for fixed moment of inertia and speed, the balance between θ and σ^2 gives the sign of the sum of Lyapunov exponents. The last term is the total energy, and gives a term in favour of the noise, because it is positive and if the damping increases, then this quantity decreases, as $\mathbb{P}_\infty$ concentrates in the low energy regions.

Theorem 3.8 only gives a lower bound for the sum of the Lyapunov exponents of an arbitrary compact semi-simple Lie algebra. A precise value can be computed explicitly for each Lie algebra, by using the structure constants to calculate the term $A(\mu,\mu)$. We will show this calculation in the case of $SO(3)$ in the free rigid body example in Section 4.

Remark 3.9. *This argument does not apply for non-compact semi-simple Lie algebras, as c is not bounded from above. It is possible that another argument exists for the top Lyapunov exponent, but so far we have not been able to find a useful estimate.*

We now turn to the semidirect product structure and also estimate the sum of the Lyapunov exponents in the following theorem.

Theorem 3.10. *The sum of the Lyapunov exponents for the semidirect stochastic process (3.3) with Casimir $C(\mu,q)=\frac{1}{\epsilon}\kappa(q,q)=c^2$ is given by*

$$\sum_i \lambda_i \geq |\epsilon|\, n\sigma^2 - \theta c^2 \mathbb{I}_{\min}^{-1}. \tag{3.22}$$

Proof. We follow closely the proof for the Euler-Poincaré case. Let us denote the stochastic process (3.3) in Itô form by

$$d(\mu,q) = [F^\mu(\mu,q)+F^q(\mu,q)]\,dt + \sum_i [G_i^\mu(\mu,q)+G_i^q(\mu,q)]\,dW_t^i,$$

where we denoted F^μ (resp. F^q) the μ (resp. q) component of F. The MET theorem, Jacobi's formula and ergodicity of this process gives

$$\sum_i \lambda_i = \int \mathrm{Tr}\left[D_\mu F^\mu(\mu,q) + D_q F^q(\mu,q)\right]\mathbb{P}_\infty(\mu,q)d(\mu,q), \tag{3.23}$$

where $\mathbb{P}_\infty(\mu,q)$ is the invariant measure of the underlying stochastic process, and D_μ and D_q denotes the Jacobian matrices taken with respect to μ or q respectively. We obtain

$$\mathrm{Tr}(D_\mu F^\mu + D_q F^q) = \mathrm{Tr}\left(-\theta \mathrm{ad}_q \mathrm{ad}_q \mathbb{I}^{-1} + \sigma^2\sum_i \mathrm{ad}_{e_i}\mathrm{ad}_{e_i}\right),$$

and, using again (3.21), we have the result (3.22).

□

Remark 3.11. *Note that because the energy is always bounded, reversing the sign of θ will not affect the results on the existence of the random attractor. However, the system will be attracted to the maximum energy position rather than the minimum energy.*

This section has been devoted to the study of the interaction of multiplicative noise and nonlinear dissipation on coadjoint orbits. In this section, we added a double bracket dissipation mechanism to the previously derived stochastic process in order to, again, preserve the coadjoint orbit structure where the solution of the stochastic process are supported. In the case of semi-simple Lie algebras we obtained the invariant measure of the Fokker-Planck equation and found the associated Gibbs measure on the coadjoint orbits. In the semidirect product case, this result was shown to hold for the marginal distribution on the advected quantity only, where the Gibbs distribution depends only on the potential energy. We then proved the existence of random attractors for a wide class of systems using the dissipative property of the double bracket dissipation, the Hörmander condition on the generating vector field and an ergodic theorem to prove the positivity of the sum of the Lyapunov exponents. In the next two sections we will study two concrete examples of stochastic deformations of the Euler-Poincaré dynamical equation, for the free rigid body and the heavy top, using both analytical and numerical tools.

4. Euler-Poincaré example: the stochastic free rigid body

This section treats the classic example of the Euler-Poincaré dynamical equation; namely, the equation for free rigid body motion. There has been active interest in stochastic models for stochastic rigid body models arising in different fields of application such as nanoparticles [STKH15, BBR+06], molecular biology [GHC09], polymer dynamics [Chi09][Section 13.7], filtering in aeronautics: guidance and tracking [Wil74]. We refer to [Chi12, Chi09] for more applications. A source of models for stochastic dynamics stems from the so-called rotational Brownian motion of molecules. Rotational Brownian motion comprises the random change in the orientation of a polar molecule due to collisions with other molecules and is an important element in the theory of dielectric materials. Perrin and Debye's non-inertial theories are the most well-known models, see for example [Chi09][section 16.3]. Rotational Brownian motions have also been observed in a laboratory setting and have been properly documented in [HAN+06]. Much of the current research in rotational Brownian motions is devoted to inertial models, non-spherical molecules and possibility of dipole-dipole interactions. Walter et al. [WGM10] took a step further in proposing an inertial, Langevin type of generalisation to the rigid body equations aiming at studying systems of rigid bodies as models for polymer dynamics. The coupling between linear and rotational dynamics was important in this case, to capture the motion features of long polymeric chains. Their models assume linearity in the noise for both linear and angular momentum variables, whereas the model used here is fully nonlinear with multiplicative noise and preserves strong geometrical features such as the coadjoint orbits.

Remark 4.1 (The LLG equation)**.** *We mention that the stochastic Landau-Lipschitz-Gilbert (LLG) equation studied for example in* [Gar97, BGJ12, KRVE05] *has the same structure as our stochastic dissipative rigid body equation. Indeed, we preserve the coadjoint orbit, thus the length of the momentum variable, which corresponds to the strength of magnetic moments in the LLG model. We will not study this link* *further here as the LLG equation is a PDE in two or three dimensions and requires different analytical methods than the rigid body equation.*

4.1. **The stochastic rigid body.** The canonical example for illustrating the Euler-Poincaré reduction by symmetry is the free rigid body described by the group of rotations $SO(3)$. For a complete treatment from the viewpoint of reduction we refer to [MR99], For simplicity here, we rely on the isomorphism $\mathfrak{so}(3) \cong \mathbb{R}^3$ which translates the commutator in the Lie algebra to the cross product of three-dimensional vectors, via $[A,B] \to \boldsymbol{A} \times \boldsymbol{B}$, where $\mathbb{R}^3$ vectors are denoted with bold font. This allows us to use a slightly different Killing form than the canonical one. Namely, we shall use the scalar product as our pairing, via the formula $\boldsymbol{A}\cdot\boldsymbol{B} = -\frac{1}{2}\kappa(A,B)$.

The reduced Lagrangian of the free rigid body is written in terms of the angular velocity $\Omega \in \mathfrak{so}(3)$ and a prescribed moment of inertia $\mathbb{I} \in \mathrm{Sym}(3)$ as

$$l(\boldsymbol{\Omega}) = \frac{1}{2}\boldsymbol{\Omega}\cdot\mathbb{I}\boldsymbol{\Omega} := \frac{1}{2}\boldsymbol{\Omega}\cdot\boldsymbol{\Pi}\,, \tag{4.1}$$

where the angular momentum $\boldsymbol{\Pi}$ is defined accordingly and the Legendre transform gives the reduced Hamiltonian $h(\boldsymbol{\Pi}) = \frac{1}{2}\boldsymbol{\Pi}\,\mathbb{I}^{-1}\boldsymbol{\Pi}$. We take the stochastic potential to be linear in the momentum variable $\boldsymbol{\Pi}$

$$\Phi_i(\boldsymbol{\Pi}) = \sum_{i=0}^{3}\boldsymbol{\sigma}_i\cdot\boldsymbol{\Pi}\,, \tag{4.2}$$

where the constants $\boldsymbol{\sigma}_i$ generically span $\mathbb{R}^3$ but can be chosen in various ways. The stochastic process for $\boldsymbol{\Pi}$ is then computed from (2.7) to be

$$d\boldsymbol{\Pi} + \boldsymbol{\Pi}\times\boldsymbol{\Omega}\,dt + \sum_i \boldsymbol{\Pi}\times\boldsymbol{\sigma}_i \circ dW_t^i = 0, \tag{4.3}$$

and the corresponding Itô process is

$$d\boldsymbol{\Pi} + \boldsymbol{\Pi}\times\boldsymbol{\Omega}\,dt + \frac{1}{2}\sum_i(\boldsymbol{\Pi}\times\boldsymbol{\sigma}_i)\times\boldsymbol{\sigma}_i\,dt + \sum_i \boldsymbol{\Pi}\times\boldsymbol{\sigma}_i\,dW_t^i = 0. \tag{4.4}$$

The coadjoint orbit defined by a level set of the quadratic Casimir $\|\boldsymbol{\Pi}\|^2 = \mathrm{c}^2$ is preserved in our geometrical construction, as may be checked by a direct computation in both the Stratonovich and the Itô stochastic representations. Although the Casmir is conserved, the energy $h(\boldsymbol{\Pi}) = l(\boldsymbol{\Omega})$ is not a conserved quantity in general. Indeed, since the moment of inertia $\mathbb{I}$ is a symmetric matrix, the stochastic process associated to h can be found to be

$$dh = \sum_i(\boldsymbol{\Pi}\times\boldsymbol{\sigma}_i)\cdot[\mathbb{I}^{-1}(\boldsymbol{\Pi}\times\boldsymbol{\sigma}_i) - (\boldsymbol{\Omega}\times\boldsymbol{\sigma}_i)]\,dt + 2\sum_i(\boldsymbol{\Pi}\times\boldsymbol{\sigma}_i)\cdot\boldsymbol{\Omega}\,dW_t^i. \tag{4.5}$$

In the general case, one only has bounds for the energy given by the two stable equilibrium positions of the rigid body, namely $E_{\min} = \frac{1}{2\mathrm{I}_3}|\Pi_3(0)|^2$ and $E_{\max} = \frac{1}{2\mathrm{I}_1}|\Pi_1(0)|^2$ if $I_1 \le I_2 \le I_3$. Thus, the energy may randomly fluctuate within these bounds.

Apart from the obvious case of $\mathbb{I} = Id$, one can check that the system with $\mathbb{I} = (I_1, I_1, I_3)$ and $\boldsymbol{\sigma} = (0,0,\sigma_3)$ conserves the energy for every values of I_1, I_3 and

σ_3 . In this case, the stochastic rigid body reduces to the Kubo oscillator of [KTH91]

$$d\Pi_1 = \Pi_2(a\Pi_3 dt + \chi_3 \circ dW), \quad d\Pi_2 = -\Pi_1(a\Pi_3 dt + \chi_3 \circ dW) \quad \text{and} \quad d\Pi_3 = 0,$$

where $a := \frac{I - I_3}{I\, I_3}$. This system is integrable by quadratures and a solution is

$$\Pi_1(t) = \Pi_1(0)\cos(\gamma t + \chi_3 W_t) - \Pi_2(0)\sin(\gamma t + \chi W_t),$$
$$\Pi_2(t) = \Pi_2(0)\cos(\gamma t + \chi_3 W_t) + \Pi_1(0)\sin(\gamma t + \chi W_t),$$

where $\gamma := a\Pi_3$. Although the deterministic free rigid body is integrable, the only known integrable stochastic rigid body is this particular case, which reduces to the Kubo oscillator.

4.2. **Fokker-Planck equation.** The Fokker-Planck equation of the process (4.3) is simply given for a probability density $\mathbb{P}$ by

$$\frac{d}{dt}\mathbb{P} + (\boldsymbol{\Pi} \times \boldsymbol{\Omega}) \cdot \nabla \mathbb{P} + \frac{1}{2}\sum_i (\boldsymbol{\Pi} \times \boldsymbol{\sigma}_i) \cdot \nabla[(\boldsymbol{\Pi} \times \boldsymbol{\sigma}_i) \cdot \nabla \mathbb{P}] = 0\,, \tag{4.6}$$

where $\nabla := \nabla_{\boldsymbol{\Pi}}$ is the gradient with respect to the independent variable $\boldsymbol{\Pi} \in \mathbb{R}^3$. According to Theorem 2.1, the invariant, or limiting distribution $\mathbb{P}_\infty$ is constant on coadjoint orbits, which are spheres.

Based on this result, more can be said about the energy evolution of the stochastic rigid body, without embarking on any deeper studies into the coupled stochastic processes 4.5 and (4.3). For example, by ergodicity of (4.3), the long time average of the stochastic rigid body motion follows the limiting distribution $\mathbb{P}_\infty$. In terms of energy, the distribution is not uniform, but will be proportional, at a given energy, to the length of the deterministic trajectory of the rigid body with this energy. This can be visualized with the Monte-Carlo simulations of Figure 1. The energy will thus randomly oscillate between two bounds, with a higher probability to be near the energy of the unstable equilibrium.

4.3. **Double bracket dissipation.** The double bracket dissipation for the rigid body involves the only Casimir $\|\boldsymbol{\Pi}\|^2$ and gives, with noise, the dissipative stochastic process

$$d\boldsymbol{\Pi} + \boldsymbol{\Pi} \times \boldsymbol{\Omega}\, dt + \theta\, \boldsymbol{\Pi} \times (\boldsymbol{\Pi} \times \boldsymbol{\Omega})\, dt + \sum_i \boldsymbol{\Pi} \times \boldsymbol{\sigma}_i \circ dW_t^i = 0. \tag{4.7}$$

The Itô formulation is similar to (4.4) and will not be written here. The corresponding the Fokker-Planck equation is

$$\frac{d}{dt}\mathbb{P} + (\boldsymbol{\Pi} \times \boldsymbol{\Omega}) \cdot (\nabla \mathbb{P} - \theta\, \boldsymbol{\Pi} \times \nabla \mathbb{P}) + \frac{1}{2}\sum_i (\boldsymbol{\Pi} \times \boldsymbol{\sigma}_i) \cdot \nabla[(\boldsymbol{\Pi} \times \boldsymbol{\sigma}_i) \cdot \nabla \mathbb{P}] = 0. \tag{4.8}$$

The Fokker-Planck equation for stochastic rigid body dynamics with selective decay may be found by specialising the general proof given for Theorem 3.2. Indeed, we can rewrite the Fokker-Planck equation as

$$\frac{d}{dt}\mathbb{P} + (\boldsymbol{\Pi} \times \boldsymbol{\Omega}) \cdot \nabla \mathbb{P} + \nabla \cdot \left(\theta\, \boldsymbol{\Pi} \times (\boldsymbol{\Pi} \times \boldsymbol{\Omega})\mathbb{P} - \frac{1}{2}\sigma^2\, \boldsymbol{\Pi} \times (\boldsymbol{\Pi} \times \nabla \mathbb{P})\right) = 0\,, \tag{4.9}$$

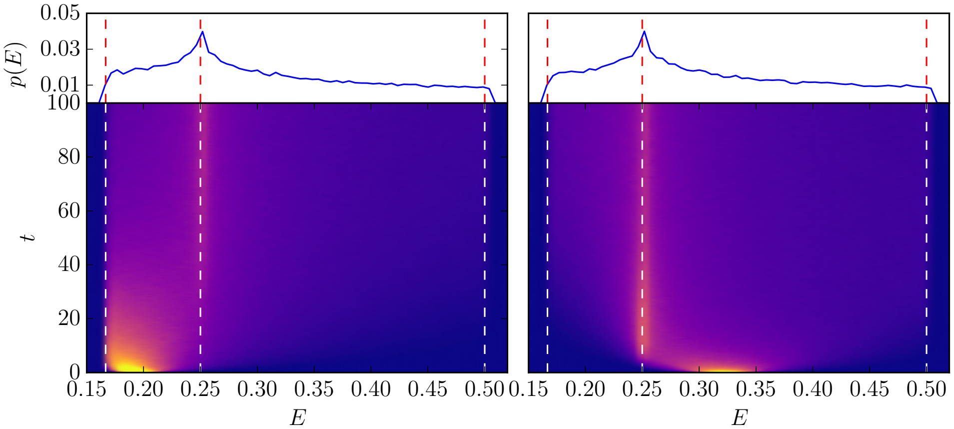


FIGURE 1. We display the time evolution of the probability distribution function of the stochastic process (4.5) driving the energy E. The yellow regions correspond to a value of $p = 0.1$. The vertical lines represent the energy of the three equilibrium points of the deterministic rigid body and the top panel displays the profile of the distribution function at $t = 100$. In both simulations we used $\mathbb{I} = (1, 2, 3)$, $\boldsymbol{\sigma}_i = \boldsymbol{e}_i$, $dt = 0.01$ and $5 \cdot 10^4$ runs of the stochastic rigid body. The left panel has the initial condition in spherical coordinates $\boldsymbol{\Pi} = (2, -1)$ and on the right $\boldsymbol{\Pi} = (0, 1)$. In both cases the distribution tends to the same limiting distribution. That the highest density occurs near the middle axis of the rigid body may be explained by noting that the orbits of this energy are longer than the others.

where we have used $\nabla \cdot (\boldsymbol{\Pi} \times (\boldsymbol{\Pi} \times \boldsymbol{\Omega})) = 0$. The last term in (4.8) simplifies as

$$\sum_i (\boldsymbol{\Pi} \times \boldsymbol{e}_i)[(\boldsymbol{\Pi} \times \boldsymbol{e}_i) \cdot \nabla \mathbb{P}] = \sum_i (\boldsymbol{\Pi} \times \boldsymbol{e}_i)[(\nabla \mathbb{P} \times \boldsymbol{\Pi}) \cdot \boldsymbol{e}_i] = \boldsymbol{\Pi} \times (\nabla \mathbb{P} \times \boldsymbol{\Pi}),$$

since the sum over i is simply the decomposition of the vector $(\nabla \mathbb{P} \times \boldsymbol{\Pi})$ into its $\boldsymbol{e}_i$ components. Consequently, the asymptotic equilibrium solution tends to

$$\mathbb{P}_\infty = Z^{-1} e^{-\frac{2\theta}{\sigma^2} h(\Pi)}\,, \tag{4.10}$$

in which the overall sign of the exponential argument is negative, since $\theta > 0$. We display in Figure 2 the limiting distribution of the Fokker-Planck equation (4.8) where the higher probabilities are around the stable equilibrium of minimal energy.

4.4. **Random attractor.** For $\mathfrak{so}(3)$, we can go beyond Theorem 3.8 to obtain the exact value of the sum of the Lyapunov exponents.

Proposition 4.2. *The sum of the Lyapunov exponents can be given exactly as*

$$\sum_i \lambda_i = 3\sigma^2 + \theta \left(c^2 \mathrm{Tr}\, \mathbb{I}^{-1} - 6 \mathbb{E}_\infty h \right), \tag{4.11}$$

where c is the value of the Casimir function, and $\theta > 0$.

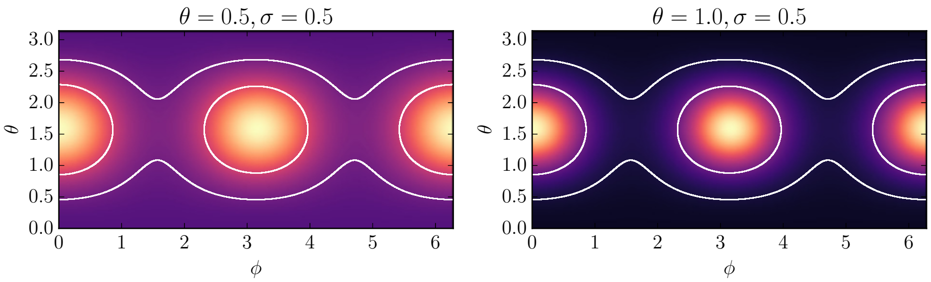


FIGURE 2. We display two invariant measures of the Fokker-Planck equation of the dissipative stochastic free rigid body equation (4.7) with different parameters $\theta = 0.5, 1$ for $\sigma = 0.5$. The density is higher (in yellow) in the lower energy regions and lower (in black) in the higher energy regions.

Proof. We can compute the term A of Theorem 3.8 exactly with the structure constants $c_k^{ij} = \epsilon_{ijk}$

$$
\begin{aligned}
-A(\boldsymbol{\Pi}, \boldsymbol{\Pi}) &= -\mathrm{Tr}(\mathrm{ad}_\Pi \mathrm{ad}_\Pi \mathbb{I}^{-1}) = -c_n^{im} c_m^{jn} \mathbb{I}^{-1} \Pi_i \Pi_j \\
&= \Pi_1^2(\mathbb{I}_2^{-1} + \mathbb{I}_3^{-1}) + \Pi_2^2(\mathbb{I}_1^{-1} + \mathbb{I}_3^{-1}) + \Pi_3^2(\mathbb{I}_1^{-1} + \mathbb{I}_2^{-1}),
\end{aligned}
$$

which, when combined with the Hamiltonian, yields the result in equation (4.11). □

We display in Figure 3 the condition of positivity of the sum of the Lyapunov exponents, computed with Monte-Carlo method for estimating the expectation of the energy on the momentum sphere.

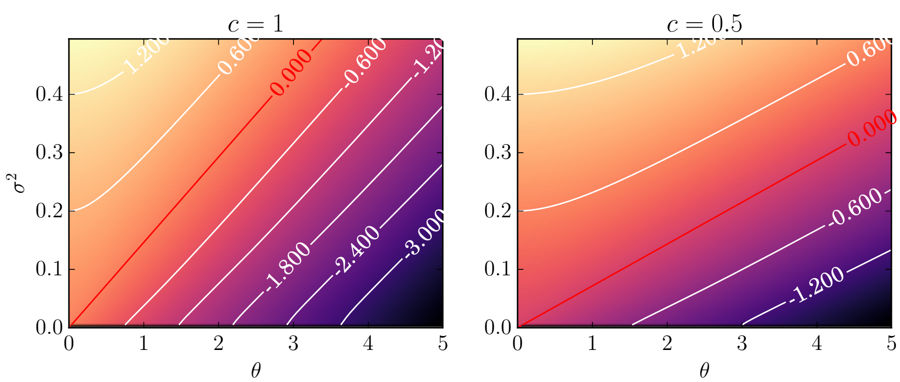


FIGURE 3. We display the value of the sum of the Lyapunov exponents for $\mathbb{I} = (1, 2, 3)$ and two values of $c = \|\Pi\|$ in the (θ, σ^2) plane. A positive value implies the existence of a SRB measure on a non singular random attractor.

This result, along with the theorem of existence of SRB measure, guaranteed by the fact that $\boldsymbol{\sigma}_i$ spans $\mathbb{R}^3$, and the dissipation of energy together give the condition required for the existence of a non-singular random attractor.

With the help of numerical simulations, we display in Figure 4 a realisation of a random attractor of the rigid body.[1]

[1]See http://wwwf.imperial.ac.uk/~aa10213/ for a video of this random attractor.

The plots in Figure 4 show the SRB measure, in log scale and exhibit the phenomena of stretching and folding, typical of strange attractors with a positive and negative Lyapunov exponent. The positive exponents produces the stretching mechanism and the negative ones produce the folding process. Asymptotically in time, these mechanisms may create a fractal, similar to Smale horseshoe structure, for the two dimensional random attractor. A more detailed study of this random attractor

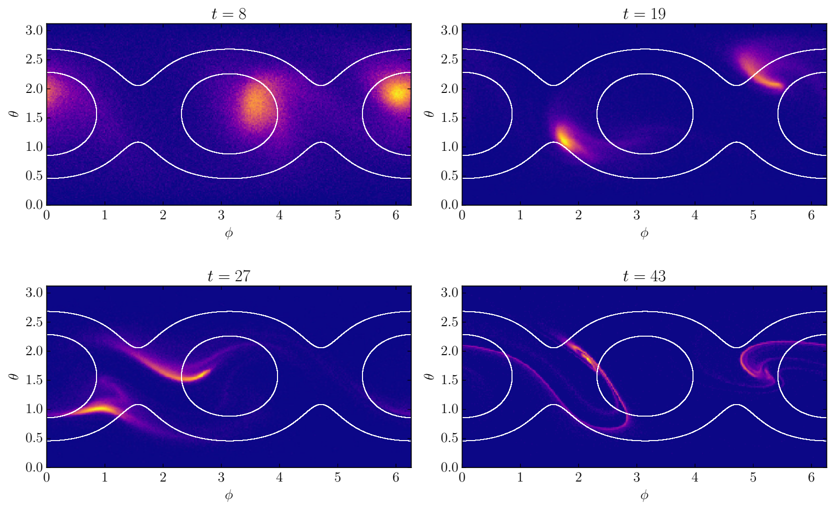


Figure 4. The four panels display snapshots of the same rigid body random attractor with $\mathbb{I} = \mathrm{diag}(1, 2, 3)$, $\theta = 0.5$ and $\sigma = 0.5$. The simulation started from a uniform distribution of rigid bodies on the momentum sphere and create finer and finder structures. The color is in log scale and we simulated 400 000 rigid body initial conditions with a split step scheme.

will certainly be interesting, but is out of the present scope of this work as it would require deeper dynamical systems analysis.

5. Semidirect product example: the stochastic heavy top

The basic example of the geometric mechanism for semidirect product motion is the heavy top, which arises in the presence of gravity, when the support point of a freely rotating rigid body is no longer at its centre of mass. The starting phase space for the heavy top is $T^*\mathrm{SO}(3)$, just as for the free rigid body. When the support point is shifted away from the centre of mass, gravity breaks the symmetry, and the system is no longer SO(3) invariant. Consequently, the motion can no longer be written entirely in terms of the body angular momentum $\mathbf{\Pi} \in \mathfrak{so}(3)^*$. One also needs to keep track of the unit vector $\mathbf{\Gamma}$, the "direction of gravity" as seen from the body ($\mathbf{\Gamma} = \mathbf{R}^{-1}\mathbf{k}$ where the unit vector $\mathbf{k}$ points upward in space and $\mathbf{R}$ is the element of SO(3) describing the current configuration of the body). The variable $\mathbf{\Gamma}$ may be identified with elements in the coset space $SO(3)/SO(2)$, where $SO(3)$ is the symmetry broken by introducing a special vertical direction for gravity, and $SO(2)$ is the remaining symmetry. This $SO(2)$ is the isotropy subgroup of $SO(3)$ corresponding to rotations around the unit vector $\mathbf{k}$ which leave the direction of gravity invariant.

5.1. **The stochastic heavy top.** The Lagrangian for the heavy top is the difference of the kinetic energy and the work against gravity, where the fixed vector $\boldsymbol{\chi}$ represents the position of the centre of mass of the body with respect to the fixed point. In body coordinates, the reduced Lagrangian is

$$l(\boldsymbol{\Omega}, \boldsymbol{\Gamma}) = \frac{1}{2}\boldsymbol{\Omega} \cdot \mathbb{I}\boldsymbol{\Omega} - mg\boldsymbol{\Gamma} \cdot \boldsymbol{\chi} \,. \tag{5.1}$$

We refer to see [HMR98, MR99] for a complete description of the semidirect product reduction for the heavy top that we will not explain here. The stochastic potential will be taken to be linear in both the $\boldsymbol{\Gamma}$ and $\boldsymbol{\Pi}$:

$$\Phi_i(\boldsymbol{\Gamma}, \boldsymbol{\Pi}) = \boldsymbol{\sigma}_i \cdot \boldsymbol{\Pi} + \boldsymbol{\eta}_i \cdot \boldsymbol{\Gamma} \,, \tag{5.2}$$

where $\boldsymbol{\sigma}_i$ and $\boldsymbol{\eta}_i$ need not span $\mathbb{R}^3$. The stochastic process describing the stochastic heavy top is then

$$\begin{aligned} &d\boldsymbol{\Pi} + (\boldsymbol{\Omega} dt + \sum_i \boldsymbol{\sigma}_i \circ dW_t^i) \times \boldsymbol{\Pi} + mg(\boldsymbol{\Gamma} \times \boldsymbol{\chi})dt + \sum_i mg(\boldsymbol{\Gamma} \times \boldsymbol{\eta}_i) \circ dW_t^i = 0 \,, \\ &d\boldsymbol{\Gamma} + (\boldsymbol{\Omega} dt + \sum_i \boldsymbol{\sigma}_i \circ dW_t^i) \times \boldsymbol{\Gamma} = 0 \,, \end{aligned} \tag{5.3}$$

and the corresponding Itô process is

$$\begin{aligned} d\boldsymbol{\Pi} &+ (\boldsymbol{\Omega} dt + \sum_i \boldsymbol{\sigma}_i dW_t^i) \times \boldsymbol{\Pi} + (\boldsymbol{\Gamma} \times mg\boldsymbol{\chi})dt \\ &\quad + \sum_i mg(\boldsymbol{\Gamma} \times \boldsymbol{\eta}_i) \circ dW_t^i - \frac{1}{2}\sum_i \boldsymbol{\sigma}_i \times (\boldsymbol{\sigma}_i \times \boldsymbol{\Pi})dt = 0 \,, \\ d\boldsymbol{\Gamma} &+ (\boldsymbol{\Omega} dt + \sum_i \boldsymbol{\sigma}_i dW_t^i) \times \boldsymbol{\Gamma} - \frac{1}{2}\sum_i \boldsymbol{\sigma}_i \times (\boldsymbol{\sigma}_i \times \boldsymbol{\Gamma})dt = 0 \,. \end{aligned} \tag{5.4}$$

The two Casimirs of the heavy top are conserved, $\|\boldsymbol{\Gamma}\|^2 = k$ and $\boldsymbol{\Pi} \cdot \boldsymbol{\Gamma} = c$. However, the energy is not conserved, as it satisfies the following stochastic process

$$\begin{aligned} \frac{d}{dt}E &= \frac{1}{4}\sum_i \left[\boldsymbol{\Omega} \cdot (\boldsymbol{\sigma}_i \times (\boldsymbol{\sigma}_i \times \boldsymbol{\Pi})) + \boldsymbol{\Pi} \cdot \mathbb{I}^{-1}(\boldsymbol{\sigma}_i \times (\boldsymbol{\sigma}_i \times \boldsymbol{\Pi}))\right] dt \\ &\quad + \frac{1}{2}\sum_i \left[(\boldsymbol{\Pi} \times \boldsymbol{\sigma}_i) \cdot \mathbb{I}^{-1}(\boldsymbol{\Pi} \times \boldsymbol{\sigma}_i) - mg(\boldsymbol{\sigma} \times \boldsymbol{\Gamma}) \cdot (\boldsymbol{\chi} \times \boldsymbol{\sigma}_i)\right] dt \\ &\quad + \frac{1}{2}\sum_i \left[\boldsymbol{\Omega} \cdot (\boldsymbol{\Pi} \times \boldsymbol{\sigma}_i) + \boldsymbol{\Pi} \cdot \mathbb{I}^{-1}(\boldsymbol{\Pi} \times \boldsymbol{\sigma}_i) + 2\boldsymbol{\chi} \cdot (\boldsymbol{\Gamma} \times \boldsymbol{\sigma}_i)\right] dW_t^i \,. \end{aligned} \tag{5.5}$$

The energy being only bounded from below, this stochastic process can lead to arbitrary large value for the energy, over a long enough time.

5.2. **The integrable stochastic Lagrange top.** When $\mathbb{I}$ is of the form $\mathbb{I} = \mathrm{diag}(I_1, I_1, I_3)$ and $\chi = (0, 0, \chi_3)$, the deterministic heavy top is called the Lagrange top and is integrable. The integrability comes from the extra conserved quantity

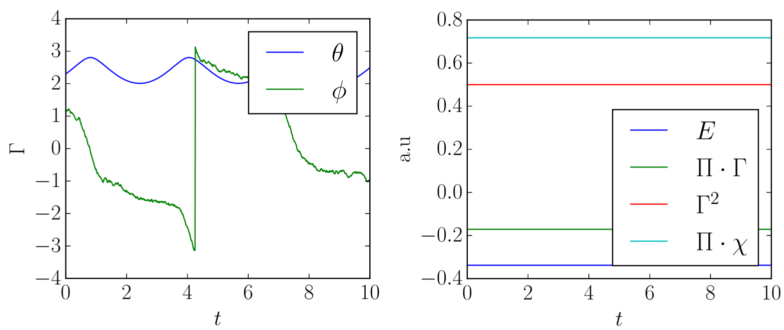
$\mathbf{\Pi}\cdot\boldsymbol{\chi}$, in this case. For noise, the stochastic process for this quantity is

$$\frac{d}{dt}(\mathbf{\Pi}\cdot\boldsymbol{\chi}) = -\frac{1}{2}\sum_i(\boldsymbol{\chi}\times\boldsymbol{\sigma}_i)\cdot(\boldsymbol{\sigma}_i\times\mathbf{\Pi})\,dt - \sum_i \boldsymbol{\chi}\cdot(\mathbf{\Pi}\times\boldsymbol{\sigma}_i)dW_t^i, \tag{5.6}$$

which is not a conserved quantity in general. However, the form of this equation implies that if one selects $\boldsymbol{\sigma}_i = \boldsymbol{\chi}$ then $\mathbf{\Pi}\cdot\boldsymbol{\chi}$ is a conserved quantity. It is remarkable that with this choice of noise, the energy is also a conserved quantity, as one can check from equation (5.5). We thus have a stochastic integrable Lagrange top, with a stochastic Lax pair given by

$$d(\lambda^2\boldsymbol{\chi} + \lambda\mathbf{\Pi} + \mathbf{\Gamma}) = ((\lambda\boldsymbol{\chi} + \mathbf{\Omega})dt + \boldsymbol{\chi}\circ dW)\times(\lambda^2\boldsymbol{\chi} + \lambda\mathbf{\Pi} + \mathbf{\Gamma}), \tag{5.7}$$

where λ is arbitrary and called a spectral parameter. We refer to [Rat81] for more details about the integrability of the Lagrange top. Following the framework of integrable hierarchies, further developed for infinite dimensional integrable hierarchies in [Arn15], there exists another integrable stochastic Lagrange top where the stochastic potential is the same as the Hamiltonian. The explanation for the integrability is straightforward, as the change of variable $t \to t + W_t$ maps the stochastic Lagrange top to the deterministic one; so we will not discuss it in more detail here.



(a) Γ motion (b) Conserved quantities

FIGURE 5. This figure displays a realisation of the motion of the integrable stochastic Lagrange top. The conserved quantities are displayed in the right panel.

We want to study this stochastic system further, as integrability means that an explicit solution can be found. Indeed, from the standard theory of the heavy top, see for example [Arn89, Aud96], the equation for Γ_3 can be found to be of the form $\dot\Gamma_3^2 = f(\Gamma_3)$, where f depends only on the constants of motion k and c. Then, a straightforward calculation with Euler angles gives

$$\begin{aligned}\dot\psi &= \frac{c - k\Gamma_3}{(1-\Gamma_3^2)I}\\ d\phi &= \left[\frac{c}{I_3\Gamma_3} - \frac{c-k\Gamma_3}{I_3\Gamma_3(1-\Gamma_3^2)I}((1-\Gamma_3^2)I - I_3\Gamma_3^2)\right]dt - \chi_3\circ dW\,,\end{aligned} \tag{5.8}$$

where $\cos(\theta) = \Gamma_3$ gives the third Euler angle. Surprisingly, only ϕ has a stochastic motion, while ψ and θ follow the deterministic Lagrange top motion. This is illustrated in Fig. 5 via a numerical integration of the stochastic Lagrange top equations.

The conservation of all the Lagrange top quantities is reproduced, as well as the fact that the noise only influences the ϕ component of the Euler angles.

5.3. The Fokker-Planck equation and invariant measures. We now analyse the associated Fokker-Planck equation for the stochastic heavy top, which is given by

$$
\begin{aligned}
\frac{d}{dt}\mathbb{P}(\boldsymbol{\Pi},\boldsymbol{\Gamma}) &= (\boldsymbol{\Pi}\times\boldsymbol{\Omega})\cdot\nabla_\Pi\mathbb{P} + (\boldsymbol{\Gamma}\times\boldsymbol{\Omega})\cdot\nabla_\Gamma\mathbb{P} \\
&\quad + \sum_i \frac{1}{2}(\boldsymbol{\Pi}\times\boldsymbol{\sigma}_i)\cdot\nabla_\Pi\left[(\boldsymbol{\Pi}\times\boldsymbol{\sigma}_i)\cdot\nabla_\Pi\mathbb{P}\right] \\
&\quad + \sum_i \frac{1}{2}(\boldsymbol{\Gamma}\times\boldsymbol{\sigma}_i)\cdot\nabla_\Gamma\left[(\boldsymbol{\Gamma}\times\boldsymbol{\sigma}_i)\cdot\nabla_\Gamma\mathbb{P}\right] \\
&\quad + \sum_i \frac{1}{2}(\boldsymbol{\Pi}\times\boldsymbol{\sigma}_i)\cdot\nabla_\Pi\left[(\boldsymbol{\Gamma}\times\boldsymbol{\sigma}_i)\cdot\nabla_\Gamma\mathbb{P}\right] \\
&\quad + \sum_i \frac{1}{2}(\boldsymbol{\Gamma}\times\boldsymbol{\sigma}_i)\cdot\nabla_\Gamma\left[(\boldsymbol{\Pi}\times\boldsymbol{\sigma}_i)\cdot\nabla_\Pi\mathbb{P}\right],
\end{aligned}
\tag{5.9}
$$

where in our notation ∇_Π denotes the gradient with respect to the $\boldsymbol{\Pi}$ variable only and similarly for ∇_Γ. By using the semidirect product Lie-Poisson structure of the heavy top

$$
\{H,G\}_{HT} := \begin{bmatrix}\nabla_\Pi H & \nabla_\Gamma H\end{bmatrix}\begin{bmatrix}\boldsymbol{\Pi}\times & \boldsymbol{\Gamma}\times \\ \boldsymbol{\Gamma}\times & 0\end{bmatrix}\begin{bmatrix}\nabla_\Pi G \\ \nabla_\Gamma G\end{bmatrix}, \tag{5.10}
$$

the Fokker-Planck equation (5.9) can be written in the double bracket form

$$
\frac{d}{dt}\mathbb{P} = \{h,\mathbb{P}\}_{HT} + \frac{1}{2}\{\Phi,\{\Phi,\mathbb{P}\}_{HT}\}_{HT}, \tag{5.11}
$$

where $h(\boldsymbol{\Pi},\boldsymbol{\Gamma})$ is the Legendre transform of (5.1).

Recall that the invariant marginal distribution on the Γ sphere is constant. We study here the distribution in the Π coordinate, following the general argument of Theorem 2.2, which gives the bound

$$
0 \le \|\Pi\|(t) \le \|\Pi_0\| + (mgc)t. \tag{5.12}
$$

This bound increases linearly with time and is unbounded only when $t\to\infty$. This effect is clearly illustrated in the Figure 6 where the probability distribution of $\|\Pi\|^2$ is plotted. The initial conditions are uniform distribution on the Γ sphere and a single position for all the momentum, with unit norm. Our system parameters are $m = g = c = 1$. Consequently, the linear bound is directly proportional to the time. According to Figure 6, the bound is reached almost immediately in the first stage of the diffusion, where the Γ and Π sphere are not yet uniformly covered. After this first short temporal regime, however, the diffusion rate slows considerably below this linear bound.

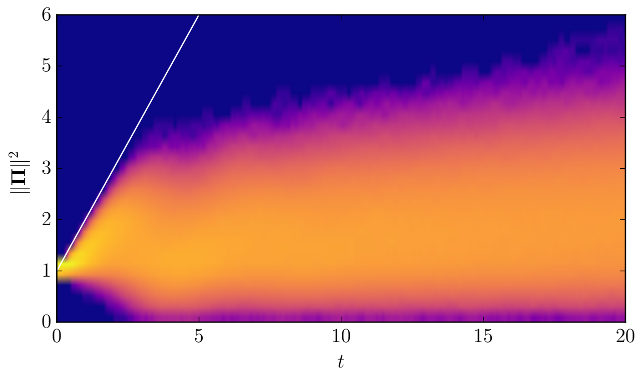


FIGURE 6. We display the probability distribution of the norm of the momentum of the heavy top, as a function of time. The distribution tends to 0 as time goes to ∞, but only linearly as shown by equation (5.12) and the white line in this Figure. The expansion is larger for small time, as the distribution is not yet uniform on the angles of the momentum but linearly bounded in time. After this rapid early expansion, the diffusion slows considerably.

5.4. **Random attractor.** The dissipative heavy top equations can be computed directly from the semidirect theory (see also [BKMR96]) and in Stratonovich form they read, when the Casimir $\mathbf{\Pi}\cdot\mathbf{\Gamma}$ is used,

$$\begin{aligned} d\mathbf{\Pi} &+ (\mathbf{\Omega} dt + \sum_i \boldsymbol{\sigma}_i \circ dW_t^i) \times \mathbf{\Pi} + mg(\mathbf{\Gamma}\times\boldsymbol{\chi})dt \\ &+ \theta\mathbf{\Gamma}\times(\mathbf{\Omega}\times\mathbf{\Gamma})dt + \theta\left[mg\mathbf{\Pi}\times(\boldsymbol{\chi}\times\mathbf{\Gamma}) - \mathbf{\Pi}\times(\mathbf{\Pi}\times\mathbf{\Omega})\right]dt = 0\,, \\ d\mathbf{\Gamma} &+ (\mathbf{\Omega} dt + \sum_i \boldsymbol{\sigma}_i \circ dW_t^i) \times \mathbf{\Gamma} + \theta\left[mg\mathbf{\Gamma}\times(\boldsymbol{\chi}\times\mathbf{\Gamma}) - \mathbf{\Gamma}\times(\mathbf{\Pi}\times\mathbf{\Omega})\right]dt = 0\,. \end{aligned} \tag{5.13}$$

Notice that the two Casimirs which define the coadjoint orbits are preserved by both the noise and the dissipation, as expected. Also recall the form of the deterministic energy decay

$$\frac{dh}{dt} = -\theta\,\|\mathbf{\Omega}\times\mathbf{\Gamma}\|^2 - \theta\,\|\mathbf{\Omega}\times\mathbf{\Pi} + mg\boldsymbol{\chi}\times\mathbf{\Gamma}\|^2\,, \tag{5.14}$$

which was used earlier to prove the existence of the random attractor after a nonlinear change of variables. The other Casimir $\|\mathbf{\Gamma}\|^2$ can also be used to derive dissipative equations, but energy dissipation will be slower, as only the first term in (5.14) and the first decay term of the $d\mathbf{\Pi}$ are left.

We can now compute the lower bound for the value of the sum of the Lyapunov exponents using Theorem 3.10 to find

$$\sum_i \lambda_i \geq 6\sigma^2 + \theta c^2 \mathrm{Tr}(\mathbb{I}^{-1})\,. \tag{5.15}$$

Figure 7 displays four snapshots of a random attractor for the stochastic heavy top.[2] on the Γ sphere, with parameters $\theta = 0.4$ and $\sigma = 0.8$. The random attractor lives

[2]See http://wwwf.imperial.ac.uk/~aa10213/ for a video of this random attractor.

in a four dimensional space, and clearly has different properties than the two dimensional rigid body attractor. In particular, it does not exhibit fractal structures created by the stretching and folding mechanism. Instead, it forms a rather complicated object in 4 dimensions. Further studies of this object will be undertaken elsewhere and will require more advanced tools from dynamical system theory.

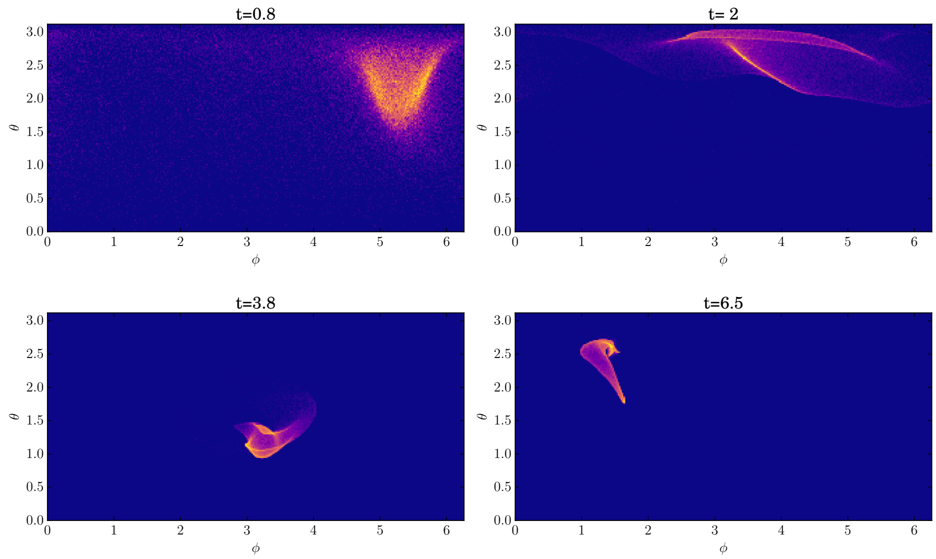


FIGURE 7. The figure displays four snapshots of a heavy top random attractors, projected onto the Γ sphere. The initial conditions are 25 000 heavy top uniformly distributed on the Γ sphere and all the same in the Π direction.

6. Two other examples

This section briefly sketches two other stochastic symmetry-reduced examples of the present theory which follow immediately from the examples of the $SO(3)$ rigid body and the heavy top, treated in the previous sections. These are the $SO(4)$ rigid body and the spring pendulum.

6.1. **The $SO(4)$ rigid body.** For a complete study of the rigid body motion on $SO(4)$ we refer to [BCRT12] and references therein. We use the generic elements

$$X = \begin{pmatrix} 0 & x_1 & x_2 & x_3 \\ -x_1 & 0 & x_4 & -x_5 \\ -x_2 & -x_4 & 0 & x_6 \\ -x_3 & x_5 & -x_6 & 0, \end{pmatrix}$$

or $X = (X_1, X_2) \in \mathbb{R}^6$.

In terms of vectors $(X_1, X_2) \in \mathbb{R}^6$ and $(X_1', X_2') \in \mathbb{R}^6$ we have

$$[(X_1, X_2), (X_1', X_2')] = (X_1 \times X_1' + X_2 \times X_2', X_1 \times X_2' + X_2 \times X_1')\,.$$

The coadjoint action is the same, under the trace-pairing.

The Casimir for $SO(4)$ are given by

$$C_1 = \mathrm{Tr}(X^2) = \sum_i x_i^2 = \|X_1\|^2 + \|X_2\|^2 \,,$$
$$C_2 = \sqrt{\det(X)} = x_1x_6 + x_2x_5 + x_3x_4 = X_1 \cdot X_2 \,.$$

The first Casimir is a 4-dimensional sphere and the second is the Pfaffian, or scalar product between two vectors.

The momentum-velocity relation is $\Pi = J\Omega + \Omega J$ where $J = \mathrm{diag}(\lambda_1, \ldots, \lambda_6)$ and the Hamiltonian $H(\Pi) = \frac{1}{2}(\Pi_1 \cdot \Omega_1 + \Pi_2 \cdot \Omega_2)$.

We thus have the following stochastic 4-dimensional rigid body equations

$$\begin{aligned} d(\Pi_1, \Pi_2) &= (\Pi_1 \times \Omega_1 + \Pi_2 \times \Omega_2, \Pi_1 \times \Omega_2 + \Pi_2 \times \Omega_1)\, dt \\ &\quad + \sum_i \left(\Pi_1 \times \sigma_1^i + \Pi_2 \times \sigma_2^i, \Pi_1 \times \sigma_2^i + \Pi_2 \times \sigma_1^i\right) \circ dW_i \,, \end{aligned} \tag{6.1}$$

which preserve the coadjoint orbit.

We now look at the selective decay term for the Casimir $C_2(\Pi) = \Pi_1 \cdot \Pi_2$. It reads, upon using semi-simplicity,

$$\begin{aligned} SD &= \mathrm{ad}_{(\Pi_2,\Pi_1)}\mathrm{ad}_{(\Pi_2,\Pi_1)}(\Omega_1, \Omega_2) \\ &= (\Pi_2 \times (\Pi_2 \times \Omega_1 + \Pi_1 \times \Omega_2) + \Pi_1 \times (\Pi_2 \times \Omega_2 + \Pi_1 \times \Omega_1), \\ &= (\Pi_2 \times \Pi_2 \times \Omega_1 + \Pi_2 \times \Pi_1 \times \Omega_2 + \Pi_1 \times \Pi_2 \times \Omega_2 + \Pi_1 \times \Pi_1 \times \Omega_1, \\ &\quad , \Pi_2 \times \Pi_2 \times \Omega_2 + \Pi_2 \times \Pi_1 \times \Omega_1 + \Pi_1 \times \Pi_2 \times \Omega_1 + \Pi_1 \times \Pi_1 \times \Omega_2) \,. \end{aligned}$$

One can directly check that the first Casimir C_1 is also preserved by this flow.

Proposition 6.1. *This stochastic dissipative $SO(4)$ free rigid body admits a random attractor.*

Proof. This is a direct application of the theory developed in Section 3. □

The invariant distribution will be centred around the minimal energy position, associated to the direction of the maximal moment of inertia. We will not numerically investigate the random attractors for this system here. However, further theoretical studies are indeed possible, and especially the integrable case, with a particular choice of the noise, would be interesting to discuss, elsewhere.

6.2. **Spring pendulum.** From the heavy top equation one can derive the spherical pendulum by letting one of the components of the diagonal inertia tensor in body coordinates tend to zero, e.g., $I_3 \to 0$. This follows, because the spherical pendulum is infinitely thin and, hence, does not have any inertia for rotations around its axis. We shall choose $\mathbb{I} = \mathrm{diag}(I, I, \epsilon)$ in the heavy top equations and then take the limit $\epsilon \to 0$ so that the dynamics on Π_3 vanishes. The similarity of this system with the rigid body allows us to consider an extension of the spherical pendulum which is called the spring pendulum [Lyn02]. To include the dynamics of the length of the spring pendulum, we introduce a new variable $R(t) \in \mathbb{R} \setminus \{0\}$ and enforce its dynamical evolution in the variational principle by adding $P(\dot{R} - v)dt$ where v

denotes the velocity of the mass along the pendulum and P denotes its associated momentum. The Lagrangian is then found to be

$$l(\boldsymbol{\Omega}, \boldsymbol{\Gamma}, R, v) = \frac{m}{2} R^2 \boldsymbol{\Omega} \cdot \mathbb{I} \boldsymbol{\Omega} - mgR\boldsymbol{\Gamma} \cdot \boldsymbol{\chi} + \frac{1}{2} m v^2 - \frac{k}{2}(R-1)^2 \,, \tag{6.2}$$

where $\boldsymbol{\chi}$ represents the initial position of the pendulum which is taken to be $(0, 0, 1)$ in accordance with our choice of inertia tensor. In (6.2), we denote the spring constant by k and the mass of the pendulum bob by m.

We shall assume a general linear stochastic potential of the form,

$$\Phi(\boldsymbol{\Pi}, \boldsymbol{\Gamma}, R, P) := \boldsymbol{\sigma} \cdot \boldsymbol{\Pi} + \boldsymbol{\eta} \cdot \boldsymbol{\Gamma} + \alpha R + \beta P \,, \tag{6.3}$$

for constant vectors $\boldsymbol{\sigma}, \boldsymbol{\eta}$, and constant scalars α, β. Consequently, the stochastic spring pendulum equations are given by

$$\begin{aligned}
d\boldsymbol{\Pi} &= \boldsymbol{\Pi} \times \boldsymbol{\Omega} dt + mgR\boldsymbol{\Gamma} \times \boldsymbol{\chi} dt + \boldsymbol{\Pi} \times \boldsymbol{\sigma}_i \circ dW_t^i + \boldsymbol{\Gamma} \times \boldsymbol{\eta}_i \circ dW_t^i \,, \\
d\boldsymbol{\Gamma} &= \boldsymbol{\Gamma} \times \boldsymbol{\Omega} dt + \boldsymbol{\Gamma} \times \boldsymbol{\sigma}_i \circ dW_t^i \,, \\
dR &= \frac{P}{m|\boldsymbol{\chi}|^2} dt + \beta dW_t \,, \\
dP &= -mg\boldsymbol{\Gamma} \cdot \boldsymbol{\chi} dt - k(R-1)|\boldsymbol{\chi}|^2 dt + \frac{1}{mR^3} \boldsymbol{\Pi} \cdot \mathbb{I}^{-1} \boldsymbol{\Pi} - \alpha dW_t \,.
\end{aligned} \tag{6.4}$$

The analysis above is valid, provided $\epsilon > 0$ in the inertia tensor. In the limit $\epsilon \to 0$, we may set $\boldsymbol{\Omega}_3 = 0$ and thereby recover the stochastic elastic spherical pendulum equations.

The equation set in (6.4) consists of two parts: the stochastic heavy top equations, coupled to a pair of stochastic canonical Hamilton equations for the (R, P) variables. The coupling between the two subsets of equations occurs through the dependence on R together with $\boldsymbol{\Omega}$ and $\boldsymbol{\Gamma}$ in the Lagrangian (6.2).

The Fokker-Planck equation is then easily derived and reads

$$\begin{aligned}
\frac{d}{dt}\mathbb{P} = \{H, \mathbb{P}\}_{HT} + \{H, \mathbb{P}\}_{\rm can} &+ \frac{1}{2}\{\Phi, \{\Phi, \mathbb{P}\}_{HT}\}_{HT} \\
&+ \{\Phi, \{\Phi, \mathbb{P}\}_{HT}\}_{can} + \frac{1}{2}\{\Phi, \{\Phi, \mathbb{P}\}_{can}\}_{can} \,,
\end{aligned} \tag{6.5}$$

where $\{\,\cdot\,,\,\cdot\,\}_{\rm can}$ is the canonical Poisson bracket with respect to the (R, P) variables. The coupling between the elastic and pendulum motions is too complicated to extract any information from the Fokker-Planck equation. Indeed, inspection of the motion on (R, P) shows that the advection equation for (R, P) depends on the other variables. This inextricable complex dependence precludes finding the limiting distribution explicitly, despite the simple Laplacian form of the diffusion operator.

As pointed out by [Lyn02], the deterministic elastic spherical pendulum system is a toy model for the lowest modes of atmosphere dynamics. For this application, the motion of the spring oscillations encoded in R is considerably faster than the pendulum motion and smaller in amplitude. Averaging the deterministic Lagrangian over the relatively rapid oscillations of the spring yields a nonlinear resonance between the modes of a type which also appears in the atmosphere. The noise can be included in either of the two types of dynamics and each will influence the other through the nonlinear coupling. Also, for small oscillations around the equilibrium, the deterministic nonlinear coupling produces star shaped orbits [HL02, Lyn02], which can be perturbed or even entirely destroyed by the introduction of the noise, depending on its amplitude.

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