4 Entanglement in Phase Space

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4.1 Introduction

The realization that quantum mechanics admits entangled states goes back
to Schrödinger in 1926 [1]. He went on to coin the term *entanglement* in
1935 [2], but the dramatic example of a biological cat coupled to a decaying
nucleus was never meant to be operational. Einstein, Rosen and Podolsky
(EPR) [3] discussed an example of a simple entangled bipartite state in the
same year. Their concern was the compatibility between Heisenberg's in-
determinacy principle and the generation of strong correlations through a
measurement on a member of a pair of particles, even when they could no
longer be interacting. It was the formulation of Bell inequalities, starting in
1964 [4] that provided a *litmus test* for nonlocal correlations in quantum me-
chanics. The initial concern was centred on hidden variable theories and the
possibility of their emulating quantum correlations even for particles that
have ceased to interact. Such violations of local causality, detected by Bell
inequalities, could not have developed within any kind of classically evolved
ensemble, irrespective of whether the variables are explicit or hidden.

Quantum information theory [6] has given a new boom to the study of
the qualitative distinctions between classical and quantum mechanics and to
establishing their quantitative measures. There is nothing so dramatic about
the development of nonclassical correlations between particles that are still
undergoing an interaction, but this question has acquired promising appli-
cations in future quantum computations. Necessarily, these deal with finite-
dimensional (Hilbert) state spaces, for which the appropriate entanglement
measures are now well established.

One of the difficulties in applying semiclassical methods to the study of
entanglement is that the former have been developed for infinite-dimensional
Hilbert spaces. Not only are these an extrapolation from the few qubits that
have been usually considered in quantum information theory, but entangle-
ment is most clearly exhibited through the correlations in elementary *either-
or* experiments. This seems to privilege simple state spaces of a single qubit,
such as spin-1/2 systems. For this reason, Bohm's version of EPR [7] has
become much more popular than the original full phase space version. A way
around this difficulty is to consider the measurement of special observables
that have only a pair of eigenvalues, even though they operate on states

within an infinite space. It turns out that one of the most renowned phase space representations in quantum mechanics, the Wigner–Weyl representation, is based on such operators. Usually, this representation is viewed as a way of eliciting classical features in a quantum state, but it will be used here mainly as a probe into nonclassical correlations.

The development of semiclassical theory throughout the last century allows us to trace the classical skeleton underlying many features of quantum evolution. These classical structures are the core of approximations that improve asymptotically in the limit of large classical actions, or, more formally, as Planck's constant, $\hbar \to 0$. In the case of a finite-dimensional Hilbert space, this becomes the limit of large dimensions. Even though entanglement is a subtle phenomenon, it leads to gross violation of inequalities and to quantitative measures that are not beyond the accuracy of semiclassical approximations. Therefore, it is appropriate to enquire into the manner in which classical structures can be implicated in such a very nonclassical feature of quantum mechanics.

Traditionally, semiclassical theory is concerned with the unitary quantum evolution of closed systems, which are thus described classically by Hamiltonian dynamical systems. Each point in phase space accounts completely for the state of the classical system that evolves along a trajectory. A bipartite or multipartite system is accommodated in this correspondence by a higher dimensional phase space. Each point still evolves as a single-dimensional (1-D) trajectory, but its projections onto the subspaces, which describe the succession of possible states of each component of the system, are also 1-D trajectories in their own right, as shown in Fig. 4.1.
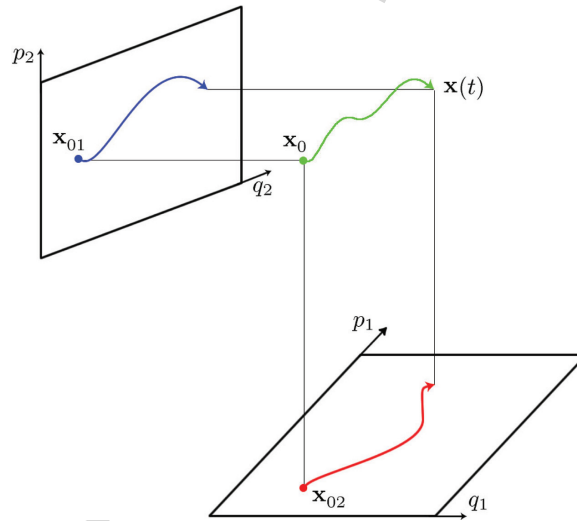

Fig. 4.1. The classical trajectories $\mathbf{x}_1(t)$ and $\mathbf{x}_2(t)$, for each component in its own phase space, are projections of the full trajectory for the entire system

Part of the power of Hamiltonian dynamics lies in the freedom to transform between different sets of phase space coordinates. This *canonical invariance* emphasizes the importance of the unified evolution of the full system over that of the component trajectories, which are seen to depend on the particular choice of coordinates. In contrast, it is in the separation into components that the phenomenon of quantum entanglement emerges. The particular nature of quantum measurement lies behind this difference, as is discussed in Sect. 4.2: It is only when this is combined to the preceding unitary evolution, that the unclassical correlations between the components become manifest.

Therefore, the study of the properties related to entanglement should be viewed as an objective that is imposed externally on semiclassical physics, which perhaps explains the low priority received by this goal so far. In these lectures, we will only be concerned with the most elementary kind of entanglement, i.e., that of pure bipartite states, for which the measures of entanglement are well established. Even so, it will be seen that this simple case requires the introduction of theoretical instruments of semiclassical theory that are far from elementary. Not only do we need to cope with a higher dimensional phase space for the description of a bipartite system, but it will be shown how the simplest semiclassical description of operators is achieved in a phase space with double the dimension of the one corresponding to the states on which they act. Conversely, some of the most relevant structures for entanglement, such as partial traces and probability densities, can be interpreted as projections of the Wigner function, or sections of its Fourier transform.

The following section reviews the different ways in which features of quantum mechanics, interference and entanglement are *nonclassical*. Simple examples introduce the reflection symmetries, quite familiar for classical waves, that will play a major role in the Wigner function formalism. Prior to this though, it is useful to consider classical–quantum correspondence in a more simple-minded way. This is the subject of Sect. 4.3, which introduces product spaces for both quantum states and classical probability distributions in phase space. In either case, the factorizability is broken by an interaction Hamiltonian, leading to correlations for measurements on the different components. In the classical case, these correlations are constrained by general Bell inequalities. This section also introduces the Schmidt decomposition of quantum states.

Section 4.4 reviews standard semiclassical theory for quantum states. Special emphasis is given to products and factorization of both the phase spaces themselves and the internal *Lagrangian* surfaces that support the quantum states. This product structure is then generalized in Sect. 4.5 to the representation of operators. Dyadic operators of position or momentum eigenstates form a complete linear basis for all quantum operators, which correspond to planes in double phase space. Linear canonical transformations take these into the phase space coordinates for the Weyl representation and the chord

representation, its Fourier transform. These bases are associated respectively to phase space reflections and translations and to the corresponding quantum operators. In the case of the density operator, we thus obtain the Wigner function and the chord function, both presented in Sect. 4.6. Though the Wigner function cannot be interpreted as a probability distribution, because it may be negative, it coincides with the difference for probabilities of measuring either the positive or the negative eigenvalue of the reflection operator. Section 4.7 is dedicated to projections of the Wigner function and sections of the chord function, which represent the reduced density operators. The loss of purity of the latter, obtained as integrals of the square of either the reduced Wigner function or the reduced chord function, indicates that the overall state is entangled.

It may be guessed that an initially *classical* pure state, the product of Gaussian Wigner functions, would not be entangled by a simple rotation of positions and momenta. After all, this class of states, including the original EPR states, could stand in for a classical phase space distribution. However, this is not so, as shown in Sect. 4.8: The reflection correlations for such states violate Bell inequalities, even though measurements of positions and momenta can only correlate classically. The transformation to centre of mass coordinates for any number of particles, studied in Sect. 4.9, has similar features. By invoking the Central Limit Theorem for Wigner functions, we obtain features of the nonunitary evolution of the centre of mass in agreement with Markovian theory, i.e., the exact solution of the Lindblad equation for the density operator.

The final section relates double phase space geometry to the semiclassical Wigner and chord functions. These are not known in detail for eigenstates of chaotic Hamiltonians, but it has been proved that *ergodic eigenstates* are supported by the entire energy shell. In this case, the unitary transformation that factorizes the state can have no classical correspondence.

A lot of the experimental work related to entanglement has been carried out in quantum optics. Rarely is the full generality of semiclassical states employed there and one can rely mainly on states derived from the eigenstates of the harmonic oscillator, even when phase space is invoked [8]. For this reason, the initial examples of phase space structures are here chosen among states of this type. The reader who wishes to avoid the more subtle aspects of semiclassical theory can mostly skip Sects. 4.4, 4.10 and parts of 4.5.

4.2 Entanglement and Classical Physics

Entanglement is considered to be a quintessential quantum property which defies all attempts at a classical correspondence. For this reason, its description in terms of the classical concept of phase space might appear foolhardy. It could be that the semiclassical program of uncovering meaningful relationships between XIX'th and XX'th century mechanics would be overstretched.

Perhaps, though, such an endeavour would make more sense if it were recalled that the usual validity of a classical description of macroscopic phenomena can be attributed to the effect of decoherence. In its turn, this results from the entanglement of a given system with an uncontrolled environment, caused by interactions that can be minimized, but never entirely eliminated. Thus, in spite of the fact that the common working languages employed in classical and quantum mechanics are quite alien to each other, it is hard to fully comprehend why the outcome of decoherence should be the emerging appropriateness of a classical description for quantum systems, unless we can detect its traces even within entanglement itself. A simplified version of this program will be sketched in Sect. 4.10.

Before attempting to establish a bridge between some features of quantum entanglement and classical mechanics, it is worthwhile to consider the more obvious way in which interference already separates these theories. In contrast, the analogy of quantum mechanics with classical waves is much smoother: The latter may be superposed linearly and they interfere in the same way as matter waves. In a simple two-slit experiment, the initial quantum state is prepared as a coherent superposition of momentum eigenstates, with eigenvalues that can be *classically* measured: The probability for each momentum direction is the same as for a uniform ensemble of *classical* states. The evolution through a pair of slits generates classical interference, equally observable in water waves, or sound waves. Quantum strangeness only emerges if the intensity of the resulting interference pattern for the conjugate variable, the position, is interpreted as the probability for the position measurement of a single particle, moving according to classical mechanics. Even so, the particular nature of quantum measurement itself does not play a prominent role in the phenomenon of quantum interference. The subsequent quantum state is certainly redefined by the result of the measurement, but this is not a crucial feature of quantum interference, no matter how unclassical its interpretation for a single particle.

The success of the semiclassical treatment of interference phenomena is no real surprise. If we start from Feynman's path integral formalism [9, 10], quantum evolution is described by a continuum of interfering paths. Semiclassical theory merely groups these around a few particular classical trajectories with their Feynman phase. The amplitude of each of these discrete interfering terms is then given by a local integration over the continuum of paths. Classical mechanics takes its part in the theory, but there is no limitation to classical phenomena and interference is well described. Indeed, the role of classical mechanics is the same as ray optics in classical wave theory.

In contrast to interference, the unintuitive nature of entanglement is derived from that of quantum measurement itself. In no way does this tally with the common sense description of the macroscopic world. Nothing in our everyday experience prepares us for the collapse of a state that is measured into one of several possibilities. The common sense presupposition would be that the effect of the measurement on the system could and should be made

negligible. Entanglement highlights this phenomenon in a specially subtle way because it involves pairs of measurements on systems with at least two degrees of freedom, or components.

If we consider classical waves, or particles, it would be indeed strange to imagine that such a collapse could result from the measurement of a subsystem, thus constraining the possible states of the complementary subsystem: It is well known that playing a note on a piano, i.e., exciting a finite string, will provoke a response on the next octave string. Here we have two nearly independent systems, stretched strings, weakly coupled by the surrounding air. Perhaps, it is better to consider the same note on two nearby pianos, so that we consider the interaction of identical systems. The wave form assumed instantaneously by the pair of strings may be used to describe the state of the whole system, or else, we may prefer the Fourier representation, in terms of the eigenstates for the discrete set of allowed frequencies of each string. These classical strings are completely analogous to the textbook example in quantum mechanics of particles moving in 1-D, each in its own box. But there is no way in which a photograph of one of the piano strings will affect the sound produced by the other string, no matter how entangled the quantum analogues happen to be! Likewise, the measurement of the frequency spectrum of the vibrations of one of the strings does not oblige it to choose among the various overtones and we would be even more surprised if this led to a correlated jump in the other string.

Yet this is just what we would expect for the analogous quantum system composed of two particles in their 1-D boxes, coupled by the same Hamiltonian that may account for the atmospheric interaction. Such a measurement would single out a discrete energy, or equivalently a discrete momentum modulus. Furthermore, in the quantum system, we could also measure the position of the particle, with a probability density that is specified by the wave intensity. No equivalent interpretation can be imputed to the classical wave, so that such a *position measurement* would then be devoid of meaning.[1]

Just as there are measurements on a quantum system that are meaningless for a classical wave, there are others that make no sense for a classical particle. Consider the excitation of a piano string by the same note, but played on a clarinet. This has only even harmonics because it is equivalent to a string that is free on one side. Then only the even harmonics will be excited in the string, which will be symmetric about its midpoint. Such an even (or odd) parity, i.e., the symmetry (or antisymmetry) of the classical stationary wave, is certainly a measurable property of the analogous quantum state. Indeed, even a classical wave, a string that is free on one side, could in principle be used as a probe to measure directly the even component of the wave, instead of exciting it. But what would it mean to measure the parity of the corresponding classical particle in a box? Generalizations of such *parity*

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[1] It should be remembered that the classical particle analogy here is not related to the phonons that are generated by second quantization within each mode.

measurements, distinguishing the eigenvalues of *non-mechanical observables*, will play a major role in the following discussions of entanglement.

Measurement theory lies outside the scope of a semiclassical treatment. However, such experimental outcomes will be preceded by (unitary) quantum evolution, which is not so adverse to a classical description. Indeed the process by which subsystems become entangled is a *preparation* that precedes any quantum measurement. It is only in the probabilistic interpretation of the subsequent measurement on the system that the quantum and the classical viewpoints fundamentally diverge. As it happens, standard measures of entanglement require that the components of the system be completely defined, but do not pre-specify the measurements to be performed. Thus, the presence of entanglement only indicates the possibility that some subset of measurements will have nonclassical correlations. It is precisely this lack of definition with respect to future quantum measurements that allows space for a semiclassical treatment.

The study of classical waves displays many of the properties of a simple quantum system. Indeed, Rayleigh's *The Theory of Sound* [11] anticipates some results later rediscovered in semiclassical theory. However, each piano string is a system with an infinite number of degrees of freedom. Though it is not forbidden to consider coupled fields,[2] the following lectures will concern mainly systems with a finite number of degrees of freedom. In most cases, two degrees of freedom already suffice to discuss the relation between the concept of entanglement and classical mechanics. So we start with a review of classical–quantum correspondence.

4.3 Classical–Quantum Correspondence

The simplest quantum systems with a classical correspondence have a single degree of freedom, e.g., a particle constrained to move on a straight line. The classical state of the system is described by its position, q, and its momentum, p. Together they define a point in phase space, $x = (p, q)$, which is a 2-D plane. Perhaps, classical state space would be a more appropriate term because each point specifies all future motion of a classical system, once the Hamiltonian, $H(x)$, is specified, through Hamilton's equations:

$$\dot{p} = -\frac{\partial H}{\partial q}, \quad \dot{q} = \frac{\partial H}{\partial p}. \tag{4.1}$$

These equations may be compactified into the form

$$\dot{x} = \mathbf{J}\frac{\partial H}{\partial x}, \tag{4.2}$$

with the definition of the (2×2)-dimensional matrix

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[2] Perhaps, quantum superstring theory will tackle entanglement someday.

$$\mathbf{J} = \left[\begin{array}{c|c} 0 & -1 \\ \hline 1 & 0 \end{array} \right] , \tag{4.3}$$

acting on the phase space points, $x = (p, q)$. Unless $H(x)$ is quadratic, this motion is nonlinear.

Corresponding to this 2-D plane, quantum mechanics matches the states $|\psi\rangle$ of an infinite-dimensional Hilbert space, $\mathbf{H}$, on which act the operators, $\widehat{q}$ and $\widehat{p}$. Each eigenstate of $\widehat{q}$, labeled by the eigenvalue q_0, corresponds to the vertical line, $q = q_0$, whereas the horizontal phase space lines are matched by eigenstates of $\widehat{p}$. These operators do not commute, $[\widehat{p}, \widehat{q}] = i\hbar$, but if we appropriately symmetrize the order in which p and q appear in $H(x)$, then the motion of the states $|\psi\rangle$ is also determined by the quantum Hamiltonian $H(\widehat{x})$, through the linear equation

$$i\hbar \frac{\partial}{\partial t} |\psi\rangle = H(\widehat{x}) |\psi\rangle, \tag{4.4}$$

i.e., Schrödinger's equation.

The uncertainty principle excludes the existence of a quantum state that corresponds precisely to a phase space point. However, the unavoidable dispersion in measurements of position or momentum allows to seek an approximate correspondence with probability distributions of phase space points. This is unsatisfactory as far as interpretation is concerned because probabilities are associated to the square of a state rather than the state itself. Nonetheless, a certain intuition can be obtained through this analogy. Given a phase space probability density, $f(x)$, the expectation value of any *classical observable* $O(x)$ is given by

$$E(O) = \int \mathrm{d}x \, O(x) \, f(x) . \tag{4.5}$$

Hence, the dispersions in position and momentum are $\delta q^2 = E\left((q - E(q))^2\right)$ and $\delta p^2 = E\left((p - E(p))^2\right)$. The uncertainty principle then imposes that only phase space distributions for which $\Delta' = \delta q \delta p \geq \hbar$ should be considered.

However, this quantity is not a classical invariant. The flow, $x(0) \to x(t)$, generated by the Hamiltonian is a *canonical transformation*, so that [12], for all t,

$$\oint_{\gamma_0} p(0) \cdot \mathrm{d}q(0) = \oint_{\gamma_t} p(t) \cdot \mathrm{d}q(t) , \tag{4.6}$$

where γ_0 is any circuit and $\gamma_0 \to \gamma_t$. General Hamiltonian evolution will stretch and bend any closed curve that is initially compact, so that a probability distribution that is unity inside γ_0 and zero outside will not have constant Δ'. Linear canonical transformations, that is, *symplectic transformations*, are well known to be specially favourable for classical–quantum correspondence, as will be further discussed. It will be shown in Sect. 4.6 that, Δ, the determinant of the *covariance matrix*,

$$\mathbf{K} = \left[\begin{array}{c|c} \delta p & \delta pq \\ \hline \delta pq & \delta q \end{array} \right] , \tag{4.7}$$

where $(\delta pq)^2 = E\left(pq - E(p)E(q)\right)$, is invariant under symplectic transformations.

To discuss entanglement, we need more than one degree of freedom. Quantum states can then be decomposed into a basis of product states,

$$|\psi\rangle = |\psi_1\rangle \otimes \ldots |\psi_l\rangle \otimes \ldots |\psi_L\rangle , \tag{4.8}$$

which span the full Hilbert space, $\mathbf{H} = \boldsymbol{H}_1 \otimes \ldots \boldsymbol{H}_l \otimes \ldots \boldsymbol{H}_L$, i.e., the tensor product of the factor Hilbert spaces that describe each degree of freedom. Likewise, the full phase space is now a Cartesian product of 2-D conjugate planes, each the phase space for a particular degree of freedom,

$$x = x_1 \times \ldots x_l \times \ldots x_L \tag{4.9}$$

and thus has $2L$ dimensions. However, we must be wary of the difference between the classical and quantum geometries: a phase space strip, δq, corresponds to this range of eigenvalues for the operator $\widehat{q}$. This set of eigenstates spans an infinite-dimensional subspace of the product Hilbert space, whatever the number of degrees of freedom. On the other hand, each of these position eigenstates corresponds to one of the parallel L-D q-planes within the $2L$-D phase space strip.

The classical or quantum motion for systems with more than one degree of freedom is still defined by a Hamiltonian, $H(x)$, or $H(\widehat{x})$, but now $\partial H/\partial x$ is a $2L$-dimensional vector and $\mathbf{J}$ is a block matrix. We shall also use the *skew product*,

$$x \wedge x' = \sum_{n=1}^{L} (p_l q_l' - q_l p_l') = \mathbf{J}\, x \cdot x' . \tag{4.10}$$

This *symplectic area* of the parallelogram formed by the vectors x and x' is invariant with respect to symplectic transformations. Again, these are linear canonical transformations, with (4.6) interpreted as a line integral in the $2L$-D phase space. For higher dimensional systems, all even dimensional volumes, from 2 to $2L$, are preserved by canonical transformations [12].

If the degrees of freedom are completely decoupled, each with its own probability distribution, $f_l(x_l)$, the full probability distribution will be just the product,

$$f(x) = f_1(x_1) \ldots f_l(x_l) \ldots f_L(x_L) . \tag{4.11}$$

In this case, the probability distribution for a single degree of freedom is reobtained by *tracing over* the other variables:

$$f_1(x_1) = \int f(x)\, \mathrm{d}x_2 \ldots \mathrm{d}x_L . \tag{4.12}$$

If the full probability distribution cannot be factored into a product, the above equation then defines the *marginal distribution*. This process foreshadows that of partial tracing over the density operator, to be studied in Sect. 4.7, which is central to the study of entanglement.

The product nature of the classical distribution will be retained throughout the evolution if the full classical Hamiltonian is purely additive,

$$H(x) = H_1(x_1) + \ldots H_l(x_l) + \ldots H_L(x_L), \qquad (4.13)$$

i.e., if there is no coupling between the motions of the several degrees of freedom. This follows from the decoupling of Hamilton's equations into

$$\dot{x}_l = \mathbf{J}_l \frac{\partial H_l}{\partial x_l}, \qquad (4.14)$$

for each degree of freedom. In other words, if $k \neq l$, then $x_l(t; x_{l0})$ does not depend on x_k (nor on the initial value, x_{k0}). Furthermore, we then have

$$f_l(t; x_l) = f_l(t; x_l(-t; x_l)) \text{ and } f(t; x) = f_1(t; x_1) \ldots f_L(t; x_L), \qquad (4.15)$$

where $x_l(-t; x_l)$ specifies the past location of x_l. Likewise, the volumes in each subspace will be preserved, and the conservation of the $2L$-dimensional volume is just that resulting from the conservation of the factor volumes.

For a classical system, the transition from product probabilities to general probabilities can only be generated by coupling terms in the driving Hamiltonian, containing cross products, which are at least bilinear in the different variables. A general classical observable will be a function of all the phase space variables, and its expectation is accordingly given by (4.5). For instance, this might be the *either-or observable*, $O_1 = \pm 1$ for detecting some physical properties associated with one particle, or the detection of $O_2 = \pm 1$ for a second particle. For classical particles that have been allowed to drift sufficiently far from each other after interacting, the result of the O_1 measurement will not affect the O_2 measurement and vice versa. Therefore, the correlation must be represented in the form

$$E(O_1; O_2) = E(O_1 \, O_2) = \int dx_1 dx_2 \, O_1(x_1) \, O_2(x_2) \, f(x_1, x_2). \qquad (4.16)$$

This equation has the same form as correlations postulated for *local hidden variable theories* [4, 13]. Indeed, one of the reasons for this choice is that (4.16) must hold for any evolution of $f(x)$ governed by classical mechanics. This form for the correlation between different components of the system is then taken as a prerequisite for theories that in all other respects should give the same results as quantum mechanics. Since this is certainly not one of the objectives of classical mechanics, such conjectures then necessarily demand extra, unknown and hence *hidden* variables.

It is due to the seminal work of Bell [4] that we are able to compare, through inequalities, the correlations predicted by quantum mechanics with

a very wide range of possible local correlations. The point is that any measurement affects the entire quantum state, i.e., both its components, unless the state happens to be an eigenstate of the measured observable. So quantum measurements are not local in the sense that led to (4.16). In case of the general CHSH inequality [6, 14], involving either-or observables, O_{1a}, O_{1b}, O_{2a} and O_{2b}, (4.16) implies that

$$|E(O_{1a}; O_{2a}) + E(O_{1a}; O_{2b}) + E(O_{1b}; O_{2a}) - E(O_{1b}; O_{2b})| \leq 2 . \qquad (4.17)$$

As well as constraining possible hidden variable theories, this inequality can be used as a detector of nonclassical correlations in quantum mechanics. This kind of nonclassicality, *entanglement*, is much more subtle than quantum interference effects, as will be discussed in the later sections. A dip into *Bertlmann's socks and the nature of reality* [15] by Bell provides a delightful discussion of all the main points concerning classical locality versus quantum correlations. The book by Peres [13] is also recommended.

It is worthwhile to discuss some specific examples of systems with more than one degree of freedom. An obvious possibility is a collection of particles, each moving in one dimension. Another is a single particle moving in two, or three dimensions. Classical and quantum mechanics make no distinction between these alternative interpretations of the dynamical variables. All that is demanded is that the variables pertaining to different degrees of freedom commute, $[\widehat{p}_k, \widehat{q}_j] = i\hbar\delta_{kj}$, or, correspondingly, that the classical Poisson bracket $\{p_k, q_j\} = \delta_{kj}$ (see e.g., [16]). We can also use angular momentum and their conjugate angles. But are other variables, obtained through classical canonical transformations, allowed?

For example, consider our piano string, now modeled as L masses connected by harmonic springs. We can switch to the L normal modes of vibration. This is a linear canonical transformation, which substitutes the original L conjugate planes, $x_l = (p_l, q_l)$, by new conjugate planes, $x'_l = (p'_l, q'_l)$, that now describe collective motions of the L masses. This is also a proper phase space to be quantized, $x'_l \rightarrow \widehat{x}'_l$. Another important example of a quantizable canonical transformation follows from the description of a collection of particles in terms of the centre of mass together with internal coordinates.

Whatever the physical realization, symplectic transformations correspond exactly to unitary quantum transformations and hence to equivalent quantum systems [17]. These transformations generally redefine the components of the full system and may take an entangled state into a product state, or vice versa. Any measure of entanglement is affected by such a general transformation, so one requires only that the measure be invariant with respect to local unitary transformations, lying within each separate component. As for nonlinear canonical transformations, these are not exactly matched by quantum unitary transformations [17] and, hence, cannot be directly quantized. It might still be useful sometimes to push this correspondence through, but it must be remembered that the result is only a semiclassical approximation.

Taking again the continuum limit, $L \to \infty$, each normal mode of the finite chain converges onto one of the lower modes of the continuous string. There is no essential difference between the interaction and hence the entanglement among these modes of the continuum and that of finite modes (caused by residual non-quadratic terms in the Hamiltonian). In each case, there corresponds a plane in the phase space, which is of infinite-dimension in the case of a field. The entanglement between modes of the electromagnetic field within a finite cavity also has a similar interpretation in terms of a classical field. The unperturbed motion is now that of a quantized harmonic oscillator, corresponding to a classical oscillation in each phase plane .

Another example is that of a particle with *internal structure*. The latter may be described by an angular momentum, coupled to the translational degrees of freedom by an external field. The Stern–Gerlach experiment is just such a system, in which the magnetic moment, tied to the spin angular momentum of the electron is coupled to its position by an inhomogeneous magnetic field. The spin is an intrinsically quantum mechanical two-level system and the interest in quantum information theory tends to emphasise such simple quantum systems. But, in principle, there is no difference between this case and a Rydberg atom, prepared in a state with a large electric dipole moment, coupled to position through an inhomogeneous electric field. Such a system can be described more naturally in classical terms. Cavity quantum optics deals with the coupling and hence the entanglement of the internal states of individual Rydberg atoms with a specific mode of the electromagnetic field.

For all these systems, coupling terms in the overall Hamiltonian will destroy the product form of an initially decoupled quantum state, or classical distribution. We should bear in mind three basic differences between classical and quantum systems: (i) the nature of the initial state; (ii) the nature of the evolution and (iii) the effect of experiments. As we have seen, the last is the most radical difference, which, indeed, gives rise to the concept of entanglement. Our objective here is to cast the quantum mechanical description of (i) and (ii) in the most classical possible, so as to highlight the truly innovative elements of the quantum theory when (iii) is considered.

A fundamental difference between the quantum and classical descriptions should be discussed before proceeding: The analogy between the evolution of classical probability distributions and quantum states is somewhat deceptive in as much as the latter determine only probability amplitudes that can be complex and interfere with each other. To arrive at a closer analogue of probabilities, we should, in some sense, square the quantum states. The correct procedure is to define density operators, or their phase space representation, Wigner functions, to be studied in Sect. 4.6. However, their evolution is non-classical, unless the Hamiltonian is quadratic.

It will be only in the context of the density operator that it becomes meaningful to distinguish between pure states and mixed states. Taking an average over a set of probability distributions defines a new probability distribution.

Likewise, if we superpose the corresponding quantum states, $|\psi_j\rangle$, we obtain a new quantum pure state. But if we average over the corresponding *pure state density operators*, $\rho_j = |\psi_j\rangle\langle\psi_j|$, there results a mixed state. The latter will be discussed in Sect. 4.6.

Now it is important to bring out a special form of state superposition. This is the *Schmidt decomposition*,

$$|\psi\rangle = \sum_j \lambda_j \, |\psi_1\rangle_j \otimes |\psi_2\rangle_j, \qquad (4.18)$$

which exists for any bipartite state (see Sect. 2.3.1 or e.g., [6]). It must be emphasised that both factor states in the above tensor products may themselves correspond to several degrees of freedom, but the result is only proved if there are only two of them. The product states form a particular orthonormal basis in which to describe the state, $|\psi\rangle$, so that the real, non-negative coefficients, λ_j, satisfy $\sum_j \lambda_j{}^2 = 1$. The state is entangled, unless $\lambda_j = \delta_{1,j}$. The Schmidt decomposition is often employed for the description of entangled states in finite Hilbert spaces. In this case, the number of nonzero eigenvalues, λ_j, is a relevant quantifier of entanglement, known as the *Schmidt number*. For infinite-dimensional Hilbert spaces, there may be an infinite number of nonzero Schmidt coefficients.

4.4 Semiclassical Quantum States

Consider a momentum eigenstate $|p'\rangle$ for $L = 1$. In the momentum representation, this is just

$$\langle p|p'\rangle = \delta(p' - p), \qquad (4.19)$$

which is not in a good form for semiclassical extrapolation. For this purpose, it is better to use the complementary representation,

$$\langle q|p'\rangle = \exp\left(\frac{iqp'}{\hbar}\right) = \exp\left(i\frac{S_{p'}(q)}{\hbar}\right). \qquad (4.20)$$

The phase in this expression can be interpreted as the area between the classical curve (the straight line $p' = p$) and the q-axis. There is also an arbitrary constant phase, which is established by the choice of the initial point for the integral,

$$S(q) = \int_{q_0}^{q} p(q) \, dq. \qquad (4.21)$$

Consider now a general observable, $K(\hat{p}, \hat{q})$. Its eigenstates correspond classically to curves, γ, in phase space: $K(p, q) = k$. These may be viewed locally as (possibly multivalued) functions, $p_j(q)$. Then the simplest semiclassical approximation is

$$\langle q|k \rangle = \sum_j A_j(q) \exp\left[i\left(\frac{S_j(q)}{\hbar} + \nu_j\right)\right],\tag{4.22}$$

see e.g., [18]. The phases, $S_j(q)$, are again obtained from (4.21). The extra constant phases, ν_j, are known as *Maslov indices* [18, 19], but they will not be discussed here. The amplitudes, $A_j(q)$, are defined purely in terms of the classical structure. They are finite wherever the vertical line, $q = constant$, intersects the classical curve transversely. Where this vertical line is tangent to the classical curve, such as q_c in Fig. 4.2, the amplitude diverges. These points where the semiclassical approximation breaks down are known as *caustics*. The different branches of the function $p_j(q)$ are connected at caustic points.

In the case of bound eigenstates of $\widehat{K}$, the curves γ are closed. Then the eigenvalues are approximately obtained by the Bohr–Sommerfeld quantization condition,

$$\oint_\gamma p\, dq = (n + \tfrac{1}{2})\hbar.\tag{4.23}$$

The quality of the semiclassical approximation for both the states themselves and their eigenvalues improves for large quantum numbers n. Ground states, including that of the harmonic oscillator, are badly described by these approximations.

Even for large n, a closed curve, γ, must inevitably have at least a pair of caustics. The way around this is to switch to the p-representation. Then the vertical tangent at the caustic position, q_c, shown in Fig. 4.2, would correspond to the state $\langle p|q_c \rangle$, which is in a nice semiclassical form. This means that the local branch of the multivalued function $q(p)$ gives rise to a semiclassical approximation that is a superposition of terms of the form
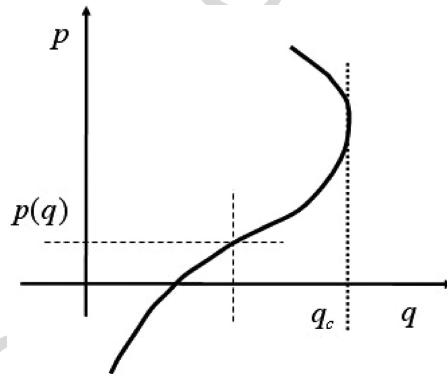

Fig. 4.2. The caustic of the semiclassical approximation to $\langle q|k \rangle$ lies in the neighbourhood of the point q_c, where the tangent to the *classical curve* is vertical. The projection of this region onto the p-axis is nonsingular, leading to a good semiclassical approximation for $\langle p|k \rangle$

$$\langle p|k\rangle = B(p)\exp\left[i\left(\frac{S(p)}{\hbar}+\mu\right)\right].\qquad(4.24)$$

This allows us to define the correct semiclassical approximation in the q-representation through the caustic region by the Fourier transform

$$\langle q|k\rangle = \frac{1}{(2\pi\hbar)^{1/2}}\int\mathrm{d}p\,\langle p|k\rangle\exp\left(\frac{iqp}{\hbar}\right),\qquad(4.25)$$

which leads to a more refined approximation in terms of Airy functions instead of exponentials. This is usually referred to as the Maslov method of dealing with caustics [20] (also discussed in [18]).

Let us now consider a product state for $L>1$. Then,

$$\langle q|p'\rangle = \exp\left(\frac{iq_1p'_1}{\hbar}\right)\ldots\exp\left(\frac{iq_Lp'_L}{\hbar}\right) = \exp\left(\frac{iq\cdot p'}{\hbar}\right),\qquad(4.26)$$

and we can generalize the definition of action,

$$S(q) = \int_{q_0}^{q}p(q)\cdot\mathrm{d}q.\qquad(4.27)$$

This does not depend on the choice of path between q_0 and q because $p'(q)$ is a constant in this simple case. Hence, this function defines a *Lagrangian surface*, i.e., a surface such that

$$\oint p\cdot\mathrm{d}q = 0,\qquad(4.28)$$

for any (reducible) circuit [12].

In general, the product state will involve arbitrary eigenstates of L observables, $\widehat{K}=\widehat{K}_1\widehat{K}_2\ldots\widehat{K}_L$, each in its own Hilbert space:

$$\langle q|k\rangle = \langle q_1|k_1\rangle\ldots\langle q_L|k_L\rangle.\qquad(4.29)$$

The wave function will be a superposition of terms with the form

$$\langle q|k\rangle = \prod_l A_l(q_l)\exp\left[\frac{i}{\hbar}\left(S_1(q_1)+\ldots+S_L(q_L)\right)\right],\qquad(4.30)$$

one term for each branch of the functions, $p_l(q_l)$.

Defining again $S(q)$ as the above phase, it is seen to be independent of the order in which we progress along each segment (q_{0l},q_l), while keeping the other integration variables constant: The definition (4.21), now reinterpreted as a path integral, is independent of the path on the surface. Therefore, this more general surface, $K(p,q)=k$, is also Lagrangian.

If the surface is the product of L-quantized *circles* (closed curves), it will be an L-torus, τ. Each of the L irreducible circuits, γ_l, must then satisfy the Bohr–Sommerfeld conditions,

$$\oint_{\gamma_l} p_l \cdot \mathrm{d}q_l = (n_l + \frac{1}{2})\hbar, \tag{4.31}$$

or some suitable generalization (see e.g., [18]). Notice that the line integral here used is not restricted to plane sections of τ because all topologically equivalent circuits on a Lagrangian surface must have the same action.

Let us now evolve the product state semiclassically. The basic result, due to van Vleck [21], can be reinterpreted as the statement that classical and quantum evolutions *commute*. In other words, we can evolve classically each curve, γ_l, if there are no cross terms in the Hamiltonian, so that the different degrees of freedom are decoupled. Each evolved observable then corresponds to $K_l(x_l, t) = K_l(x_l(x_{l0}, t), 0)$ and we approximately reconstruct the classically evolved state from the evolved torus, $\tau_l(t)$, which is the product of the $\gamma_l(t) : K_l(x_l, t) = k_l$.

Notice that this classical evolution of products of curves fits into the general view concerning the evolution of product probability distributions in the previous section, by merely choosing $f_l(t; x_l) = \delta(K_l(x_l, t) = k_l)$ and running time backwards. The important distinction between classical and semiclassical evolution is that the latter contains interferences between the different branches of the evolving classical curve. Each representation exhibits these interferences in a different way.

Just as cross terms containing products of the different variables in the Hamiltonian destroy the product form of a classical probability distribution, the classically evolved L-D surface corresponding to an original product state also ceases to be a product. However, the smoothness of the evolution implies that the topology of the surface must be preserved (be it plane, torus, or, in between cylindrical). Furthermore, the classical evolution, $x_0 \to x_t$, is a canonical transformation, and hence all reducible circuits on the evolved surface have zero action, i.e., τ_t still has the Lagrangian property, which allows to define the path-independent action $S(q)$, and the irreducible circuits of τ_t still satisfy the same Bohr–Sommerfeld conditions to first order in $\hbar$.

Let us investigate further the case of two degrees of freedom. The separable torus, $\tau = \gamma_1 \otimes \gamma_2$, can be pictured through the separate γ_1 and γ_2 curves. These coincide with sections of the 2-D torus by alternative 3-D planes (the normal case for *Poincare sections*, see e.g., [18]). The γ_2 curve does not depend on the choice of the $q_1 = constant$ section. The separable torus projects as a rectangle onto position space (q_1, q_2), as shown in Fig. 4.3. Within this rectangle, there are four different branches of the torus, which project onto each position, q, corresponding to the combinations of the two branches of each circle. The caustics at the side of the rectangle are *double fold lines*.

After a general canonical evolution, the sections of τ are no longer equal for different choices of $q_1 = constant$ (or $q_2 = constant$), though all the sections have the same area, S_1 (or S_2). In some cases (to do with time invariance of the Hamiltonian), the projection onto the q-plane will merely distort the rectangle, which will still have finite-angled corners connecting
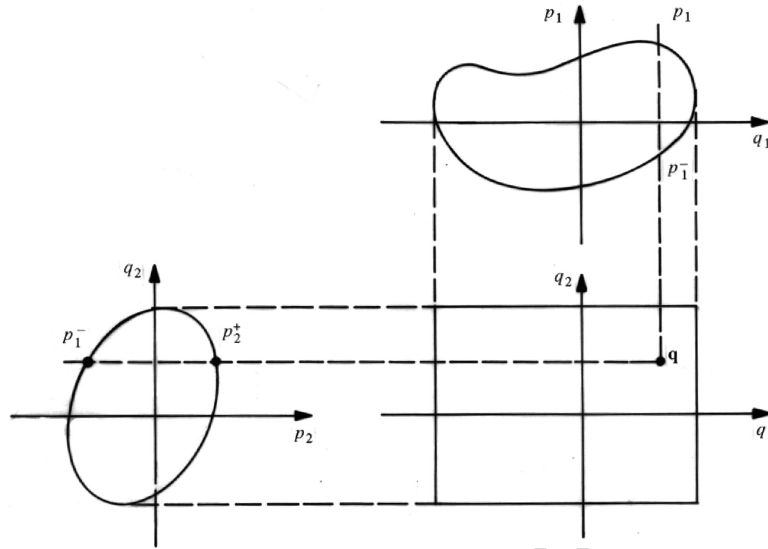
Fig. 4.3. Each point within the rectangular caustic of a two-dimensional product torus is the image of four phase space points under p-projection

double-fold lines. But in general, these corners, *hyperbolic umbilic points*, will unfold in the generic form specified by catastrophe theory, as shown in Fig. 4.4. There are four possibilities for the topology of the unfolding of the rectangle, shown in Fig. 4.5. For $L > 2$, the projection of the L-torus onto the L-D q-plane will be a solid hypercube that will be distorted, or unfolded by the motion generated by a coupling Hamiltonian. (These geometries are reviewed in [18], but are more thoroughly discussed in [22].)

The representations of quantum states in terms of orthogonal position, or, alternatively, momentum eigenstates are the best that we can do because of Heisenberg's uncertainty principle. Semiclassically, this corresponds to viewing a Lagrangian surface through a set of Lagrangian planes that *foliate*
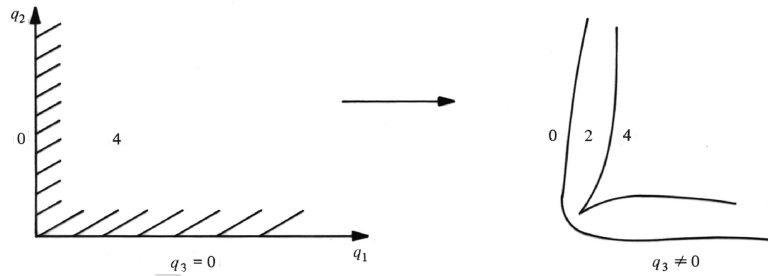

Fig. 4.4. Catastrophe theory establishes the generic form for the unfolding of the double-fold caustic at each corner of the projection of a product torus as it evolves
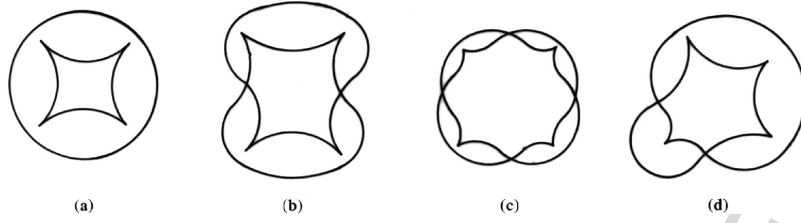
Fig. 4.5. The full topology of the full fold lines is not determined by catastrophe theory: Each of the above forms corresponds to a different symplectic evolution from an initial product torus

phase space. We switch from the q-representation to the p-representation by means of a Fourier transform of $\langle q|\psi\rangle$. This corresponds classically to taking the *Legendre transform* of $S(q)$ [12]. For $L > 1$, we may take the Fourier transform for a subset of the degrees of freedom. This corresponds to using a classical description in terms of the alternative Lagrangian planes $(p_1, \ldots, p_l, q_{l+1}, \ldots, q_l)$.

One way to achieve a full phase space description is to use the basis of *coherent states* [8, 23–26], labeled by the phase space vector, $\eta = (\eta_p, \eta_q)$,

$$\langle q|\eta\rangle = \left(\frac{\omega}{\pi\hbar}\right)^{1/4} \exp\left[-\frac{\omega}{2\hbar}(q - \eta_q)^2 + i\frac{\eta_p}{\hbar}(q - \frac{\eta_q}{2})\right] . \qquad (4.32)$$

Even though the coherent state basis is overcomplete, the exact decomposition,

$$|\psi\rangle = \frac{1}{\pi} \int \mathrm{d}\eta |\eta\rangle\langle\eta|\psi\rangle , \qquad (4.33)$$

is unique. The coherent states are phase space translations of the ground state of the harmonic oscillator (with unit mass):

$$\langle q|0\rangle = \left(\frac{\omega}{\pi\hbar}\right)^{1/4} \exp\left(-\frac{\omega}{2\hbar}q^2\right) . \qquad (4.34)$$

These result from the action of the *translation operator*:

$$\hat{T}_\eta = \exp\left[\frac{i}{\hbar}(\eta_p \cdot \hat{q} - \eta_q \cdot \hat{p})\right] = \exp\left(\frac{i}{\hbar}\eta \wedge \hat{x}\right), \qquad (4.35)$$

using the skew product (4.10). If either $\eta_p = 0$ or $\eta_q = 0$, we obtain the usual translation operators for momenta, or positions, respectively. The arbitrary phase due to noncommutation of $\hat{p}$ and $\hat{q}$ is here chosen in the most symmetric way, using the Baker–Hausdorff relation [27].

In quantum optics, it is customary to switch to the basis of creation and annihilation operators $(\hat{q} \pm i\hat{p})/\sqrt{2\hbar}$. In this context, the translation operator

(4.35) depends on the complex chords $(\xi_p \pm i\xi_q)/\sqrt{2\hbar}$ and is called the *displacement operator* [23]. The semiclassical limit for a complex phase space is not as transparent as the real theory treated here. However, it is quite feasible to effect phase space translations in an experimental optical context [28].

The coherent state representation is not orthogonal and is overcomplete. The alternative, to be explored in the next section, is to work directly with operators: We represent operators in *orthogonal* operator bases in analogy to the way that quantum states are commonly decomposed. This allows us to work directly with the translation operators, without having to apply them to the ground state of the harmonic oscillator.

4.5 Operator Representations and Double Phase Space

The linear operators, $\widehat{A}$, that act on the quantum Hilbert space form a vector space of their own: $|A\rangle\!\rangle$. Defining the *Hilbert–Schmidt product*,

$$\langle\!\langle A|B\rangle\!\rangle = \mathrm{tr}\,\widehat{A}^\dagger\widehat{B}\,, \tag{4.36}$$

we find that the dyadic operators $|Q\rangle\!\rangle = |q_-\rangle\langle q_+|$ form a complete basis, i.e.,

$$\langle\!\langle Q|A\rangle\!\rangle = \langle q_+|\widehat{A}|q_-\rangle = \mathrm{tr}\,|q_-\rangle\langle q_+|\widehat{A}\,, \tag{4.37}$$

provides a complete representation of the operator $\widehat{A}$. Here, $\widehat{A}^\dagger$ is the adjoint of $\widehat{A}$. One should note the similarity between this dyadic basis, $|q_-\rangle\langle q_+|$, in the case of $L = 1$ with the basis of product states, $|q_1\rangle\otimes|q_2\rangle$. The substitution of a bra by a ket in the former will in most cases imply no more than complex conjugation.

Thus, we may relate the vector space of quantum operators to a *double Hilbert space* with respect to that of quantum states. Since we have explored the correspondence of the state-Hilbert space with classical phase space, it is now natural to relate the double Hilbert space to a double phase space : $X = x_- \times x_+$ (see e.g., [29]). The operator $|Q\rangle\!\rangle$ should then correspond to the Lagrangian plane $Q = constant$ in the double phase space. This does hold, within a minor adaptation, analogous to the use of the adjoint operator in the definition of the Hilbert–Schmidt product. That is, we should define $Q = (q_-, q_+)$, but $P = (-p_-, p_+)$ as coordinates of the double phase space $X = (P, Q)$.

A good reason for this is that then we include among the set of Lagrangian surfaces in double phase space all the canonical transformations in single phase space, $x_- \to x_+ = \mathbb{C}(x_-)$. This also transports closed curves, $\gamma_- \to \gamma_+$, so that we may rewrite the definition of a canonical transformation as

$$\oint_\Gamma P \cdot \mathrm{d}Q = 0\,, \tag{4.38}$$

where $\Gamma = (\gamma_-, \gamma_+)$. Thus, we may consider $\gamma_\pm$ as projections of the curve Γ defined on the $(2L)$-dimensional surface, $\Lambda_{\mathbb{C}}$, which specifies the canonical transformation, within the $(4L)$-dimensional double phase space, $X = (P, Q)$.

It is worthwhile to consider the richness of structures in double phase space. On the one hand, a canonical transformation defines a Lagrangian surface as $x_+(x_-)$, a one-to-one function. On the other hand, the product of a Lagrangian surface, λ_- in x_- with another surface λ_+ in x_+, $\Lambda = \lambda_- \otimes \lambda_+$, is also Lagrangian in double phase space, but projects singularly onto either of the factor spaces. In the case that both surfaces are tori, we obtain a double phase space torus, $\tau = \tau_- \otimes \tau_+$, as if we had doubled the number of degrees of freedom. (All Lagrangian surfaces will hereon be labeled τ, even when they are not necessarily a torus; in the case that $L = 1$, τ is just a closed curve, γ.) If $L = 1$, it will be a 2-D product torus, with the only difference that $p_- \to -p_-$ in the present construction. If each Lagrangian surface corresponds to a state, i.e., $|\psi_-\rangle$ and $|\psi_+\rangle$, then we represent $|\Psi\rangle\rangle = |\psi_+\rangle\langle\psi_-|$ in the $|Q\rangle\rangle$ representation as

$$\langle\langle Q|\Psi\rangle\rangle = \langle q_+|\psi_+\rangle\langle\psi_-|q_-\rangle. \tag{4.39}$$

Therefore, the semiclassical approximation is just a superposition of terms of the form

$$\langle\langle Q|\Psi\rangle\rangle = A_J(Q)\exp[iS_J(Q)/\hbar] \tag{4.40}$$

with

$$A_J(Q) = A_{j-}(q_-)^* \, A_{j+}(q_+) \tag{4.41}$$

and

$$S_J(Q) = \oint_0^Q P_J(Q') \cdot \mathrm{d}Q'. \tag{4.42}$$

Again this is in strict analogy to the construction of semiclassical product states of higher degrees of freedom. Note that the projection of the double Lagrangian torus onto P or Q is just the rectangle discussed previously for product states, whereas the projections onto the planes, x_- and x_+, are specially singular.

The semiclassical approximation for a unitary operator, $\widehat{U}$, that corresponds to a canonical transformation, $\mathbb{C} : x_- \to x_+$, has exactly the same form, i.e., a superposition

$$\langle\langle Q|U\rangle\rangle = \langle q_+|\widehat{U}|q_-\rangle = U_J(Q)\exp[iS_J(Q)/\hbar], \tag{4.43}$$

for each branch of the function $P_J(Q)$ defined by the Lagrangian surface in double phase space. Note that the situation with respect to projection singularities is now reversed, as compared to $|\Psi\rangle\rangle$. The fact that the projections of the Lagrangian surface, $\Lambda_{\mathbb{C}}$, onto either x_-, or x_+ are both nonsingular in no way guarantees that the projections onto the P or the Q Lagrangian planes will be likewise free of caustics.

Conversely, any function, $S(Q)$, is, at least locally, the generating function of a canonical transformation through the implicit equations:

$$\frac{\partial S}{\partial Q} = P(Q)\,, \quad \text{or} \quad \frac{\partial S}{\partial q_+} = p_+\,, \quad \frac{\partial S}{\partial q-} = -p_-\,. \tag{4.44}$$

Here we recognize the standard generating functions $S(q_-, q_+)$ in Goldstein [16]. If $S(Q)$ is quadratic, then these implicit equations will be linear, so that the explicit transformation will result from a matrix inversion (if it is nonsingular). There will be a single branch in $S(Q)$ for such a symplectic transformation, and it turns out that the semiclassical approximation is exact in this case.

The well-known alternatives to these generating functions are usually obtained by Legendre transforms. However, we can consider the $\pi/2$ rotation, $q_+ \to p_+, p_+ \to -q_+$, times the identity in x_-, as an example of canonical transformation in double phase space: $X \to X'$. Then $Q' = (q_-, p_+)$ is also a good Lagrangian plane that can be used as the new coordinate plane for the description of $\Lambda_{\mathbb{C}}$. In the new coordinates, the implicit equations for the canonical transformation are just

$$\frac{\partial S'}{\partial Q'} = P'(Q')\,, \quad \text{or} \quad \frac{\partial S'}{\partial p_+} = -q_+\,, \quad \frac{\partial S'}{\partial q-} = -p_-\,. \tag{4.45}$$

The correspondence with a semiclassical state,

$$\langle\!\langle Q'|\Psi\rangle\!\rangle = A'_J(Q') \exp[iS'_J(Q')/\hbar]\,, \tag{4.46}$$

will be exact in the case of a symplectic transformation. Note that $|Q'\rangle\!\rangle$ is a first example of an operator basis that corresponds to a set of parallel Lagrangian planes in double phase space, which, nonetheless, have internal coordinates that can be identified with a phase space on its own.

The crucial step is now to explore other kinds of canonical transformations in double phase space [30]. In particular,

$$Q' = \mathbf{x} = \frac{x_+ + x_-}{2}, \quad P' = \mathbf{y} = \mathbf{J}(x_+ - x_-) = \mathbf{J}\xi\,. \tag{4.47}$$

Here, the $\mathbf{J}$ symplectic matrix in single phase space is essential to *canonize* what would be just a $\pi/4$ rotation. It accounts for the change of sign in the p_- coordinate. We will here have to bare the discomfortable situation that the canonical coordinate in double phase space is $\mathbf{y}$, but the geometrically meaningful variable in single phase space is ξ, the trajectory *chord*, in the case of continuous evolution. The coordinate $\mathbf{x}$ will be referred to as the *centre*.

If we consider the *horizontal* Lagrangian planes $\mathbf{y} = constant$, each is identified with a uniform classical *translation*. Thus, we have departed from coordinate planes corresponding to dyadic operators to those planes in double phase that describe canonical transformations and hence correspond to unitary transformations. In this case, $x_- \to x_+ = x_- + \xi$ are the group of phase space translations, which include the the identity, i.e., the *identity plane* is defined as $\xi = 0$.

On the other hand, the *vertical* plane, $\mathbf{x} = 0$, defines the canonical *reflection* through the origin, $x_- \to x_+ = -x_-$ (or inversion), since all the chords for this transformation are centred on the origin. Other vertical planes specify reflections through other points, $x_- \to x_+ = -(x_- - 2\mathbf{x})$. The reflections do not form a group on their own (no identity), but together with the translations they form the *affine group* of geometry [31].

Since there is an exact correspondence between linear canonical transformations and unitary transformations, each plane $\mathbf{y} = constant$ corresponds precisely to the translation operator, $\widehat{T}_\xi$, previously defined as . Notice that this was written with a phase that is a skew product involving ξ, but we could also use $\widehat{T}_\xi = \exp(i\mathbf{y} \cdot \widehat{x}/\hbar)$. In terms of the previous dyadic $|Q\rangle\rangle$ basis, this is expressed as

$$\hat{T}_\xi = \int \mathrm{d}q \left| \mathbf{q} + \frac{\xi_q}{2} \right\rangle \left\langle \mathbf{q} - \frac{\xi_q}{2} \right| e^{i\xi_p \cdot \mathbf{q}/\hbar}, \qquad (4.48)$$

a symmetrized Fourier transform (see e.g., [32]).

Just as a $\pi/2$ rotation in single phase space, $q \to p$ and $p \to -q$ correspond to a Fourier transform, so the transformation between *horizontal* and *vertical* planes in double phase space is also achieved by a full Fourier transform (except for an annoying factor of 2^L):

$$2^L \widehat{R}_{\mathbf{x}} = \int \frac{\mathrm{d}\xi}{(2\pi\hbar)^L} \, \widehat{T}_\xi \exp(\frac{i}{\hbar}\mathbf{x} \wedge \xi). \qquad (4.49)$$

In terms of the dyadic $|Q\rangle\rangle$ basis, we have

$$2^L \hat{R}_{\mathbf{x}} = \int \mathrm{d}\xi_q \left| \mathbf{q} + \frac{\xi_q}{2} \right\rangle \left\langle \mathbf{q} - \frac{\xi_q}{2} \right| e^{i\mathbf{p} \cdot \xi_q/\hbar}, \qquad (4.50)$$

the complementary symmetrized Fourier transform to .

We are now free to switch from the usual (position) dyadic basis to the unitary operator basis, $|\mathbf{y}\rangle\rangle = \widehat{T}_\xi$:

$$\langle\langle \mathbf{y}|A\rangle\rangle = \mathrm{tr}\, \widehat{T}_{-\xi}\widehat{A} = A(\xi), \qquad (4.51)$$

where we use $\widehat{T}_{-\xi} = \widehat{T}_\xi^\dagger$. $A(\xi)$ is the *chord representation* of the operator $\widehat{A}$ (also referred to as the *chord symbol*). Notice that the chord basis includes the identity operator, $\widehat{I} = |I\rangle\rangle = |\mathbf{y} = 0\rangle\rangle$. To verify that (4.51) is indeed the expansion coefficient for an arbitrary operator in the basis of translation chords, we use

$$\mathrm{tr}\, \widehat{T}_\xi = (2\pi\hbar)^L \delta(\xi) = \langle\langle \mathbf{y}|I\rangle\rangle \qquad (4.52)$$

(note the double phase space analogy with $\langle p'|(p=0)\rangle = \delta(p')$), as well as the quantum version of the group of translations:

$$\widehat{T}_{\xi_2}\widehat{T}_{\xi_1} = \widehat{T}_{\xi_1+\xi_2} \exp\left[\frac{-i}{2\hbar}\xi_1 \wedge \xi_2\right] \qquad (4.53)$$

(see e.g., [33]). Then, the expansion,

$$\widehat{A} = \int \frac{\mathrm{d}\xi}{(2\pi\hbar)^L} \, A(\xi) \, \widehat{T}_\xi \,, \tag{4.54}$$

leads to

$$
\begin{aligned}
\mathrm{tr}(\widehat{T}_{-\xi}\widehat{A}) &= \mathrm{tr} \int \frac{\mathrm{d}'}{\xi}(2\pi\hbar)^L \, A(\xi')\widehat{T}_{-\xi}\widehat{T}_{\xi'} \\
&= \int \frac{\mathrm{d}\xi'}{(2\pi\hbar)^L} \, A(\xi')\exp\Big[\frac{i}{2\hbar}\,\xi'\wedge\xi\Big]\,\mathrm{tr}\,\widehat{T}_{\xi'-\xi} \\
&= A(\xi)\,.
\end{aligned}
\tag{4.55}
$$

The chord representation is thus a second example of a representation of operators in terms of an operator basis that can be identified uniquely to a phase space. Indeed, each chord corresponds to a Lagrangian surface in double phase space and hence a particular uniform translation in single phase space.

The next representation will be based on phase space reflections, $\widehat{R}_\mathbf{x}$. But first, it is worthwhile to examine some characteristics of these operators. Unlike the translations, they do not form a group on their own, though they combine with the latter to form the affine group. The products are [33]

$$\widehat{R}_\mathbf{x}\widehat{T}_\xi = \exp[-\frac{i}{\hbar}\mathbf{x}\wedge\xi] \, \widehat{R}_{\mathbf{x}-\xi/2}\,, \tag{4.56}$$

$$\widehat{T}_\xi\widehat{R}_\mathbf{x} = \exp[-\frac{i}{\hbar}\mathbf{x}\wedge\xi] \, \widehat{R}_{\mathbf{x}+\xi/2} \tag{4.57}$$

and

$$\widehat{R}_{\mathbf{x}_2}\widehat{R}_{\mathbf{x}_1} = \exp[\frac{2i}{\hbar}\mathbf{x}_1\wedge\mathbf{x}_2] \, \widehat{T}_{2(\mathbf{x}_2-\mathbf{x}_1)}\,. \tag{4.58}$$

Except for the phases, these are just the classical relations. The last one is specially interesting. Note that $\widehat{R}_\mathbf{x}^2 = \widehat{I}$, the identity, hence the (degenerate) eigenvalues of $\widehat{R}_\mathbf{x}$ must be either $+1$, or -1. Therefore, these operators are Hermitian, as well as unitary.

Are they true observables? Consider the effect of $\widehat{R}_0$ on the eigenstates of the harmonic oscillator. Taking $q \to -q$ and $p \to -p$, leads to a change of sign for all the odd states, while preserving the even states. In other words, the latter are just the $(+1)$-eigenstates, while the odd states are (-1)-eigenstates. Though it is hard to imagine measuring the parity of a particle, we saw in Sect. 4.2 that the parity decomposition of even a classical wave can certainly be effected. Measurements of the eigenvalues of this *non-mechanical observable* are currently performed for single photons in optical cavities [34]. It is true that these measurements are performed on a mode of the electromagnetic field rather than a particle, but it only makes sense to discuss the parity *within a specific mode* if it is quantized.

Reflection operators are very strange observables as far as phase space correspondence is concerned. It was discussed in Sect. 4.4 that usual observables correspond to smooth phase space functions and their eigenvalues correspond to level curves if $L = 1$. This is just not the case of reflection operators with their infinitely degenerate ± 1 eigenvalues. In their dual role as both unitary and Hermitian (observable) operators, reflections are almost schizophrenic: They are perfectly ordinary unitary operators, corresponding to Lagrangian planes in double phase space, but they do not correspond to any smooth classical function in phase space, as expected of a mechanical observable.

This should furnish sufficient motivation to investigate the representation of arbitrary operators in terms of reflection centres. The assumption that

$$\widehat{A} = \int d\mathbf{x}\, A(\mathbf{x})\, 2^L \widehat{R}_\mathbf{x} \tag{4.59}$$

leads to

$$\langle\!\langle \mathbf{x}|A \rangle\!\rangle = \text{tr}\,(2^L \widehat{R}_\mathbf{x})\widehat{A} = \text{tr} \int \frac{d\mathbf{x}'}{(2\pi\hbar)^L} A(\mathbf{x}')(2^L \widehat{R}_\mathbf{x})(2^L \widehat{R}_{\mathbf{x}'}) = A(\mathbf{x})\,. \tag{4.60}$$

This is the *Weyl representation* of the operator $\widehat{A}$ (also known as the Weyl symbol). Once again we use half the coordinates of double phase space, within a Lagrangian plane that is a phase space on its own, to describe a quantum operator. This perception that we are really dealing with different phase spaces for each operator representation was clearly stated in the excellent review by Balazs and Jennings [35]. What was lacking was merely the identification of each of these different phase spaces with a specific foliation of Lagrangian planes in double phase space.

As far as unitary operators, $\widehat{U}$, are concerned, the semiclassical limit of the representations, either in terms of centres or chords, has exactly the same form as for any other Lagrangian basis. For instance, the Weyl symbol will be a superposition of terms, such as

$$U(\mathbf{x}) = A(\mathbf{x}) \exp[iS(\mathbf{x})/\hbar]\,, \tag{4.61}$$

in terms of the centre action, defined as

$$S(\mathbf{x}) = \int_0^\mathbf{x} \mathbf{y}(\mathbf{x}') \cdot d\mathbf{x}' = \int_0^\mathbf{x} \xi(\mathbf{x}') \wedge d\mathbf{x}'\,. \tag{4.62}$$

For symplectic transformations, the Lagrangian surface is a plane, and so there is only a single branch of the action function $S(\mathbf{x})$, which is quadratic. Then (4.61) is an exact representation of the corresponding quantum *metaplectic transformation*. However, in the general nonlinear case, there may be caustics in the projection of the Lagrangian $\mathbf{y}(\mathbf{x}')$ surface onto the $\mathbf{x}$-plane. Recall that this is just the plane that defines the identity operator, $\widehat{I}$ (corresponding to $\mathbf{y} = 0$, or $S = 0$).

For the canonical transformation generated by a Hamiltonian, $H(x)$, it turns out that the generating function has the limit [33]

$$S_\epsilon(\mathbf{x}, t = \epsilon) \to -\epsilon H(\mathbf{x}) + \mathcal{O}(\epsilon^3)\,. \qquad (4.63)$$

There are no caustics for small times in the centre representation, since the corresponding Lagrangian surface is nearly horizontal.

The smooth real Hamiltonian itself can be equated to the Weyl symbol for the corresponding operator, $\widehat{H}$, within semiclassically small ordering terms. This is the case of the Weyl representation for any observable that corresponds classically to a smooth classical function of the points in phase space [17]. Since we can always consider classical observables as infinitesimal generators of motion through Hamilton's equations, it is appropriate to picture them as functions on the $\mathbf{y} = 0$ plane, so that the Hamiltonian vectors form a field on this plane that indicates which way it will evolve. In contrast, the chord symbol for these smooth mechanical observables is not at all smooth. This is because the chord and centre symbols are related to each other through the Fourier transform,

$$A(\xi) = \frac{1}{(2\pi\hbar)^L} \int d\mathbf{x} \ \exp\left(-\frac{i}{\hbar}\xi \wedge \mathbf{x}\right) A(\mathbf{x})\,, \qquad (4.64)$$

just as the translation and reflection operators themselves in . This Fourier transform takes the symbol for the identity, $I(\mathbf{x}) = 1$, into $I(\xi) = \delta(\xi)$ and a Taylor series in $\mathbf{x}$ into a series of derivatives of δ-functions. However, we shall see in the next section that the chord representation of density operators have very useful properties.

It is fitting to consider here another feature which distinguishes the reflection operators from mechanical observables. Far from being represented by a smooth phase space function, their centre representation is just

$$R_\mathbf{x}(\mathbf{x}') = 2^{-L}\delta(\mathbf{x}' - \mathbf{x})\,. \qquad (4.65)$$

These singular functions cannot be interpreted as corresponding to classical states (i.e., individual phase space points) because the $\widehat{R}_\mathbf{x}$ have the eigenvalue -1, so they are not density operators.

Probably the first to remark on the general structure of translations and reflections underlying the Weyl and the chord representations were Grossmann and Huguenin [36]. There exists an exact correspondence, between these operators of the affine quantum group, together with the unitary operators of the metaplectic group with the classical transformations of the inhomogeneous symplectic group [17]. In other words, all linear canonical transformations, including reflections and translations, are exactly matched by quantum unitary transformations. Thus, the unitary transformation, $\widehat{U}_\mathbf{C}$, corresponding to $x \to x' = \mathbf{C}x$, where $\mathbf{C}$ is a symplectic matrix, takes

$$\widehat{R}_\mathbf{x} \to \widehat{R}_\mathbf{x}' = \widehat{U}_\mathbf{C}^\dagger \widehat{R}_\mathbf{x} \widehat{U}_\mathbf{C} = \widehat{R}_{\mathbf{x}'} \ \text{ and } \ \widehat{T}_\xi \to \widehat{T}_\xi' = \widehat{U}_\mathbf{C}^\dagger \widehat{T}_\xi \widehat{U}_\mathbf{C} = \widehat{T}_{\xi'}\,. \qquad (4.66)$$

This has the consequence that both the centre and the chord representations are invariant with respect to metaplectic transformations because the transformed operator $\widehat{A} \to \widehat{A}'$ is represented by

$$A'(\mathbf{x}) = \operatorname{tr} \widehat{U}_{\mathbf{C}} \widehat{A} \, \widehat{U}_{\mathbf{C}}^{\dagger} \widehat{R}_{\mathbf{x}} = \operatorname{tr} \widehat{A} \, \widehat{U}_{\mathbf{C}}^{\dagger} \widehat{R}_{\mathbf{x}} \widehat{U}_{\mathbf{C}} = \operatorname{tr} \widehat{A} \, \widehat{R}'_{\mathbf{x}} = A(\mathbf{x}') \qquad (4.67)$$

and, likewise, $A'(\xi) = A(\xi')$.

This section is concluded with some general formulae concerning these representations. For the trace of an operator, we have the alternative forms:

$$\operatorname{tr} \widehat{A} = \operatorname{tr} \widehat{I} \, \widehat{A} = \langle\!\langle \widehat{T}_{\xi=0} | A \rangle\!\rangle = A(\xi = 0) = \frac{1}{(2\pi\hbar)^{L}} \int \mathrm{d}\mathbf{x} \, A(\mathbf{x}) \,. \qquad (4.68)$$

The adjoint operator, $\widehat{A}^{\dagger}$, is represented by

$$A^{\dagger}(\mathbf{x}) = [A(\mathbf{x})]^{*}, \quad \text{or} \quad A^{\dagger}(\xi) = [A(-\xi)]^{*}, \qquad (4.69)$$

where * denotes complex conjugation. Thus, if $\widehat{A}$ is Hermitian, $A(\mathbf{x})$ is real, though $A(\xi)$ may well be complex. The Weyl or chord symbols for products of operators are not at all obvious (see e.g., [33]), but

$$\operatorname{tr} \widehat{A}_{2} \widehat{A}_{1} = \int \frac{\mathrm{d}\xi}{(2\pi\hbar)^{L}} A_{2}(\xi) A_{1}(-\xi) = \int \frac{\mathrm{d}x}{(2\pi\hbar)^{L}} A_{2}(\mathbf{x}) A_{1}(\mathbf{x}) \,. \qquad (4.70)$$

4.6 The Wigner Function and the Chord Function

It is customary to alter the normalization of the centre and the chord symbols for the density operator, $\widehat{\rho}$, so as to define

$$W(\mathbf{x}) = \frac{\rho(\mathbf{x})}{(2\pi\hbar)^{L}} \quad \text{and} \quad \chi(\xi) = \frac{\rho(\xi)}{(2\pi\hbar)^{L}} \,, \qquad (4.71)$$

respectively the *Wigner function* and the *chord function*. Combining with the general definition of the Weyl representation and the expression for the reflection operator, we obtain the original definition of $W(\mathbf{x})$, proposed by Wigner [37]. In both cases of , the representation of the trace of a product leads to the expectation of any observable, $\widehat{A}$, as

$$\langle \widehat{A} \rangle = \int \mathrm{d}\mathbf{x} \, W(\mathbf{x}) \, A(\mathbf{x}) = \int \mathrm{d}\xi \, \chi(-\xi) \, A(\xi) \,. \qquad (4.72)$$

The first integral is more interesting because $A(\mathbf{x})$ is at least semiclassically close to the classical variable, which tempts us to identify the Wigner function with a *nearly classical* probability distribution. However, we will see below that $W(\mathbf{x})$, though real and normalized so

$$\int \mathrm{d}\mathbf{x}\, W(\mathbf{x}) = 1\,, \tag{4.73}$$

may well take on negative values.

The chord function behaves like a classical characteristic function, in as much as the *moments* are

$$\langle q^n \rangle = \mathrm{tr}\, \widehat{q}^n\, \widehat{\rho} = (i\hbar)^n\, \frac{\partial^n}{\partial \xi_p^n} (2\pi\hbar)^L\, \chi(\xi)\Big|_{\xi=0} \tag{4.74}$$

and

$$\langle p^n \rangle = \mathrm{tr}\, \widehat{p}^n\, \widehat{\rho} = (-i\hbar)^n\, \frac{\partial^n}{\partial \xi_q^n} (2\pi\hbar)^L\, \chi(\xi)\Big|_{\xi=0}\,. \tag{4.75}$$

Taking the zeroth moment, we obtain the normalization,

$$1 = (2\pi\hbar)^L\, \chi(0)\,, \tag{4.76}$$

because $\mathrm{tr}\, \widehat{\rho} = \rho(\xi = 0) = 1$.

Shifting the phase space origin to $\langle x \rangle = (\langle p \rangle, \langle q \rangle)$, we can define the *Schrödinger covariance matrix* [38] just as its classical counterpart (4.7), with $\delta p^2 = \langle \widehat{p}^2 \rangle$, $\delta q^2 = \langle \widehat{q}^2 \rangle$ and $(\delta pq)^2 = \langle (\widehat{p}\widehat{q} + \widehat{q}\widehat{p})/2 \rangle$. It is then obvious that the expansion of the chord function at the origin is given by a quadratic form

$$\chi(\xi) = (2\pi\hbar)^{-L} - \xi\, \mathbf{K}\, \xi + \ldots, \tag{4.77}$$

and we can interpret the *uncertainty*,

$$\Delta_\mathbf{K} = \sqrt{\det \mathbf{K}}\,, \tag{4.78}$$

as proportional to the volume of the ellipsoid: $\xi\, \mathbf{K}\, \xi = 1$. Evidently, this volume is invariant with respect to symplectic transformations, so that $\Delta_\mathbf{K}$ is a symplectically invariant measure of the uncertainty of the state.

The projection of the Wigner function

$$\int \mathrm{d}\mathbf{p}\, W(\mathbf{p}, \mathbf{q}) = \mathrm{Pr}(q) \tag{4.79}$$

is a true probability for position measurements [37]. Furthermore, the invariance of the chord and the centre representations with respect to symplectic transformations then guarantees that the projection of the Wigner function along any set of Lagrangian planes $\mathbf{p}'$ supplies the probability distribution for the conjugate variable $\mathbf{q}'$. In particular, the probability $\mathrm{Pr}(p)$ results from the projection of $W(\mathbf{p}, \mathbf{q})$ with respect to $\mathbf{q}$. All these planes are Lagrangian, so it follows that the projection of the Wigner function onto any Lagrangian plane in phase space is a probability distribution for the corresponding variable. It may appear somewhat contrived, as far as measurement is concerned, to consider general linear combinations of position and momentum. However, it should be recalled that these observables will evolve from an initial position

for the motion driven by any quadratic Hamiltonian, even including free motion through a laboratory. The reconstruction of the Wigner function from a suitable set of these marginal distributions is known as quantum tomography. This is achieved through the *Radon transform* (see e.g., [39]).

It is equally remarkable, but less well known, that the characteristic function corresponding to the marginal probability distribution for positions is obtained by merely taking a section of the chord function:

$$\int d\mathbf{q} \ \Pr(\mathbf{q}) \ \exp\left(-\frac{i}{\hbar}\eta_q \cdot \mathbf{q}\right) = (2\pi\hbar)^L \ \chi(0, \eta_q). \tag{4.80}$$

Since the chord function is also symplectically invariant, it follows that the characteristic functions for all the probability distributions, which result from Wigner projections onto Lagrangian planes, are equal to the corresponding sections of the chord function.

So far, we have emphasised the seemingly classical aspects of the Wigner function. However, it must be remembered that the Weyl representation is defined in terms of a very anomalous observable, as far as classical correspondence is concerned. In order to reveal the full quantum nature of the Wigner function, let us divide the Hilbert space of quantum states into even and odd subspaces for a given reflection operator, $\widehat{R}_\mathbf{x}$. This is achieved through the projection operator introduced by Grossmann [40] and Royer [41],

$$\widehat{P}_\pm^\mathbf{x} = \frac{1}{2}\left(1 \pm \widehat{R}_\mathbf{x}\right), \tag{4.81}$$

so that, in its turn, we can express each reflection operator as the superposition of this pair of projections onto the even and the odd subspaces:

$$\widehat{R}_\mathbf{x} = \widehat{P}_+^\mathbf{x} - \widehat{P}_-^\mathbf{x}. \tag{4.82}$$

But

$$\operatorname{tr} \widehat{\rho} \, \widehat{P}_\pm^\mathbf{x} = \Pr_\pm^\mathbf{x} \tag{4.83}$$

is just the probability of measuring $\widehat{R}_\mathbf{x}$ to have the eigenvalue ± 1, so it follows that [41]

$$W(\mathbf{x}) = \frac{1}{(\pi\hbar)^L}[\Pr_+^\mathbf{x} - \Pr_-^\mathbf{x}] = \frac{1}{(\pi\hbar)^L}[2\Pr_+^\mathbf{x} - 1]. \tag{4.84}$$

We thus find that the Wigner function does not admit the interpretation as a probability distribution in phase space because it can certainly be negative. Even so, it is a simple linear function of a distribution of probabilities of positive eigenvalues for all possible reflection measurements. Its maximum possible value $(\pi\hbar)^{-L}$ is attained for any point, $\mathbf{x}$, such that $\widehat{P}_+^\mathbf{x} \, \widehat{\rho} = \widehat{\rho}$, whereas the commutation of the density operator with $\widehat{P}_-^\mathbf{x}$ specifies a phase space point where $W(\mathbf{x}) = -(\pi\hbar)^{-L}$.

Let us now investigate the effect of reflections and translations on a density operator. Evidently, the centre and chord representations are specially

suitable for this purpose. In the case of a phase space translation by the vector, η, i.e., $\widehat{\rho}_\eta = \widehat{T}_\eta \, \widehat{\rho} \, \widehat{T}_{-\eta}$, the respective Wigner and chord functions become

$$W_\eta(\mathbf{x}) = W(\mathbf{x} - \eta) \ \text{ and } \ \chi_\eta(\xi) = \mathrm{e}^{\mathrm{i}\eta \wedge \xi/\hbar} \, \chi(\xi) \,, \tag{4.85}$$

which shows that, unlike the Wigner function, the chord function is not generally real. The sensitivity of a state to translations is described by the phase space correlations of a given density operator, defined as [32]

$$\begin{aligned} C(\xi) &= \mathrm{tr} \, \widehat{\rho} \, \widehat{T}_\xi \, \widehat{\rho} \, \widehat{T}_\xi^\dagger \\ &= (2\pi\hbar)^L \int \mathrm{d}\mathbf{x} \, W(\mathbf{x}) \, W(\mathbf{x} - \xi) \\ &= (2\pi\hbar)^L \int \mathrm{d}\eta \, e^{\mathrm{i}\eta \wedge \xi/\hbar} \, |\chi(\eta)|^2 \,. \end{aligned} \tag{4.86}$$

From the reciprocal relation that supplies the intensity of the chord function as the Fourier transform of these correlations and the normalization condition , we see that

$$\int \mathrm{d}\xi \, C(\xi) = (2\pi\hbar)^{3L} |\chi(\eta = 0)|^2 = (2\pi\hbar)^L \,. \tag{4.87}$$

So, even though these correlations are defined in terms of classical translations in phase space, they are purely quantum and disappear in the classical limit. However, if we fix $\hbar$ and adopt this constant as our phase space scale, then we can picture $C(\xi)$ as a classical-like phase space distribution for which the characteristic function is just $|\chi(\xi)|^2$.

Specializing to the case of a pure state, $\widehat{\rho} = |\psi\rangle\langle\psi|$, we find that

$$\langle\psi|\widehat{T}_\xi|\psi\rangle \ = (2\pi\hbar)^L \chi(-\xi) \,, \tag{4.88}$$

so that the phase space correlations take the form [32]

$$C(\xi) = |\langle\psi|\widehat{T}_\xi|\psi\rangle|^2 = (2\pi\hbar)^{2L} |\chi(\xi)|^2 \,. \tag{4.89}$$

Thus, for instance, in the case that $\xi = (0, \xi_q)$, the phase space correlations, $C(\xi)$, are just the usual spacial correlations inferred from neutron scattering experiments. Nonetheless, we must be careful to distinguish between phase space correlations and the correlations between the quantum measurements of observables defined on the different components of a bipartite system, such as the CHSH inequality. For a pure state, (4.89) is the square modulus of the expectation for a translation, which is not a quantum observable. However, (4.86) defines the phase space correlation in the same way as for a classical distribution.

The chord function always assumes its maximum value $1/(2\pi\hbar)^L$ at the origin. But also an average of overlaps cannot exceed one, so $\chi(0)$ is the maximum even for mixed states. As for the correlations, we always have

$$\operatorname{tr} \widehat{\rho}^2 = (2\pi\hbar)^L \int d\mathbf{x} \, [W(\mathbf{x})]^2 = (2\pi\hbar)^L \int d\xi \, |\chi(\xi)|^2 = C(0) \,, \qquad (4.90)$$

being that $\operatorname{tr} \widehat{\rho}^2 = 1$ for pure states. But consider a mixture of orthogonal states,

$$\widehat{\rho} = \sum_j \Pr(n) \, |n\rangle\langle n| \,, \qquad (4.91)$$

then the *purity*

$$C(0) = \sum_j \Pr(n)^2 \le 1 \,. \qquad (4.92)$$

Another form in which this quantity appears is the *linear entropy* : $1 - \operatorname{tr} \widehat{\rho}^2$. This may be considered as a first-order expansion of the *von Neumann entropy*:

$$-\operatorname{tr} \widehat{\rho} \, \ln \widehat{\rho} = - \int d\mathbf{x} \, W(\mathbf{x}) \, \ln[(2\pi\hbar)^L \, W(\mathbf{x})] \,, \qquad (4.93)$$

a quantum version of the classical Shannon entropy. In [32], the correlations were normalized by the purity so as to be always unity at the origin, but it is convenient to include this quantity as a special case of the correlations.

General invariance with respect to Fourier transformation characterizes the correlation in the case of pure states. Indeed, inserting the above expression in the definition of the phase space correlation , we obtain [32]

$$C(\xi) = \int \frac{d\eta}{(2\pi\hbar)^L} \, e^{i\eta\wedge\xi/\hbar} \, C(\eta) \,. \qquad (4.94)$$

This is a remarkable property of all pure states and is in no way restricted by special symmetry properties that will be shown to relate certain Wigner functions to their respective chord functions. An immediate consequence is that oscillations of the phase space correlation of a pure state involving a large displacement, ξ, are necessarily bound to small ripples on the scale, $|\xi|^{-1}$, in the direction, $\mathbf{J}\xi$. Of course, these small-scale oscillations of the phase space correlations, which have been attractively described as *subplanckian* [42], show up in the pure state Wigner function because of (4.86).

The Fourier invariance condition (4.94) includes as a special case the more familiar one obtained by tracing over the full pure state condition $\hat{\rho}^2 = \widehat{\rho}$. It follows that the difference of both sides of (4.94) for each chord ξ generalizes (4.92) as a measure of the degree of *purity* of a state. All the same, the loss of the phase information in $C(\xi)$, but contained in the chord function, would seem to imply that these are necessary conditions, whereas the full sufficient condition of purity is $\hat{\rho}^2 = \widehat{\rho}$, which is expressed in the chord representation as [33]

$$\int d\eta \, \chi(\eta) \, \chi(\xi - \eta) \, e^{i\xi\wedge\eta/2\hbar} = \int d\eta \, \chi_{\xi/2}(\eta) \, \chi(\xi - \eta) = \chi(\xi) \qquad (4.95)$$

with $\chi_{\xi/2}(\eta)$ defined by (4.85).[3] However, the particular condition $C(0) = 1$ is indeed a sufficient condition, because, for any mixture of pure states, $\widehat{\rho} = \sum_n \Pr(n)|\psi_n\rangle\langle\psi_n|$, we obtain

$$\operatorname{tr}\widehat{\rho}^2 = 1 - \sum_{n \neq n'} \Pr(n)\Pr(n')[1 - |\langle\psi_n|\psi_{n'}\rangle|^2]. \tag{4.96}$$

A single phase space point does not correspond to any pure state in Hilbert space. The only pure states that are *classical-like*, i.e., have positive Wigner functions, are either coherent states, or their image by a symplectic transformation [43, 44].

Let us now consider the effect of measuring a general phase space reflection, $\widehat{R}_{\mathbf{x}}$. The density operator, $\widehat{\rho}$, will be projected by $\widehat{P}_{\pm}^{\mathbf{x}}$, defined by , onto either the even or odd subspace for this particular reflection:

$$\widetilde{\rho}_{\pm}^{\mathbf{x}} = \frac{\widehat{P}_{\pm}^{\mathbf{x}} \widehat{\rho} \widehat{P}_{\pm}^{\mathbf{x}}}{\operatorname{tr}\widehat{\rho}\,\widehat{P}_{\pm}^{\mathbf{x}}}. \tag{4.97}$$

The Weyl symbol for $\widehat{\rho}\,\widehat{P}_{\pm}^{\mathbf{x}}$ defines the symmetric Wigner function, $W_{\mathbf{x}}^{\pm}(\mathbf{x}')$, within a normalization factor, so that, using the group relations (4.56), (4.57) and (4.58), we obtain [45]:

$$\begin{aligned} W_{\mathbf{x}}^{\pm}(\mathbf{x}') &= (\pi\hbar)^{-L}\operatorname{tr}\widehat{R}_{\mathbf{x}'}\,\widetilde{\rho}_{\pm}^{\mathbf{x}} \\ &= \frac{W(\mathbf{x}') + W(2\mathbf{x} - \mathbf{x}') \pm \Re\, 2^{(L+1)}\,e^{2i\,\mathbf{x}'\wedge\mathbf{x}/\hbar}\chi(2(\mathbf{x}' - \mathbf{x}))}{2\,[1 \pm (\pi\hbar)^L\,W(\mathbf{x})]}, \end{aligned} \tag{4.98}$$

where $\Re$ denotes the real part of a number. It follows that the Wigner function and the chord function for a reflection symmetric density operator are trivially related. Shifting the origin of phase space to the symmetry point leads to [32]

$$W_0^{\pm}(\mathbf{x}) = \pm\, 2^L \chi_0^{\pm}(-2\mathbf{x}). \tag{4.99}$$

Thus, all Wigner functions for density operators that commute with a reflection symmetry attain the largest amplitude at the symmetry point, but this will be negative in the case of odd symmetry.

Let us consider some standard examples of Wigner and chord functions. All the following cases are related to eigenstates of a harmonic oscillator with one degree of freedom and unit mass.

(i) Coherent states: the Wigner function is just a Gaussian centred on η,

$$W_\eta(\mathbf{x}) = \frac{1}{\pi\hbar}\exp\left[-\frac{\omega}{\hbar}\,(\mathbf{q} - \eta_q)^2 - \frac{1}{\hbar\omega}\,(\mathbf{p} - \eta_p)^2\right] \xrightarrow{\omega=1} \frac{1}{\pi\hbar}e^{-(\mathbf{x}-\eta)^2/\hbar}, \tag{4.100}$$

---

[3] For distributions $\Pr(n)$ over eigenstates $|n\rangle$ of an observable with discrete spectrum, the condition $\Pr(n)^2 = \Pr(n)$ also singles out a *pure state*, $\Pr(n) = \delta_{n,m}$, but this condition is not generalizable to a continuous spectrum.

whereas

$$\chi_\eta(\xi) = \frac{1}{2\pi\hbar} \exp\left(\frac{i\eta \wedge \xi}{\hbar}\right) \exp\left[-\frac{\omega}{\hbar}\left(\frac{\xi_q}{2}\right)^2 - \frac{1}{\hbar\omega}\left(\frac{\xi_p}{2}\right)^2\right]$$

$$\xrightarrow{\omega=1} \frac{1}{2\pi\hbar} e^{i\eta\wedge\xi/\hbar} e^{-\xi^2/4\hbar} . \tag{4.101}$$

So, any translation of the coherent state merely alters the phase of the Gaussian chord function that sits on the origin. The coherent states, or more generally all equivalent Gaussian states obtained from them by symplectic transformations, are the only examples of pure states for which the Wigner function is nowhere negative [43]. This is one of the reasons why these are sometimes considered to be the most classical of pure quantum states. Since the projection of a Gaussian is also a Gaussian, the measurement of position, or any other Lagrangian phase space coordinate, does not display interference fringes. The fact that the uncertainty, $\Delta = \delta p \delta q = \hbar$, is minimal allows us to interpret them as *quantum phase space points*.

(ii) A superposition of a pair of coherent states $|\eta\rangle \pm |-\eta\rangle$ is sometimes known as a *Schrödinger cat state*. Its Wigner function is[4]

$$W_\pm(\mathbf{x}) = \frac{1}{2\pi\hbar\left(1 \pm e^{-\eta^2/\hbar}\right)}$$

$$\times \left[e^{-(\mathbf{x}-\eta)^2/\hbar} + e^{-(\mathbf{x}+\eta)^2/\hbar} \pm 2e^{-\mathbf{x}^2/\hbar} \cos\left(\frac{2}{\hbar}\mathbf{x} \wedge \eta\right)\right] . \tag{4.102}$$

It consists of two *classical* Gaussians centred on $\pm\eta$ and an interference pattern with a Gaussian envelope centred on their midpoint. The frequency of this oscillation increases with the separation $|2\eta|$. In Fig. 4.6, the displacements $\pm\eta$ have been chosen as $(\pm 3, \pm 3)$. The phase of the pair of coherent states merely shifts the phase of the interference fringes, so that the midpoint is an absolute maximum for $W_+(\mathbf{x})$ and an absolute minimum for $W_-(\mathbf{x})$. It might be supposed that for small $\eta \to 0$, we would have $W_+(\mathbf{x}) > 0$ for all $\mathbf{x}$, but it is easy to verify that there are very shallow negative regions far removed from the classical superposed Gaussians, in agreement with [43, 44]. The interference pattern of the Wigner function does not survive the projection orthogonal to η: In this direction, the interference disappears to produce a purely classical pattern. Conversely, the projection along η is marked by interference fringes.

For the chord function,

$$\chi_\pm(\xi) = \frac{1}{4\pi\hbar\left(1 \pm e^{-\eta^2/\hbar}\right)}$$

$$\times \left[e^{-(\xi/2-\eta)^2/\hbar} + e^{-(\xi/2+\eta)^2/\hbar} \pm 2e^{-\xi^2/4\hbar} \cos\left(\frac{1}{\hbar}\xi \wedge \eta\right)\right] , \tag{4.103}$$

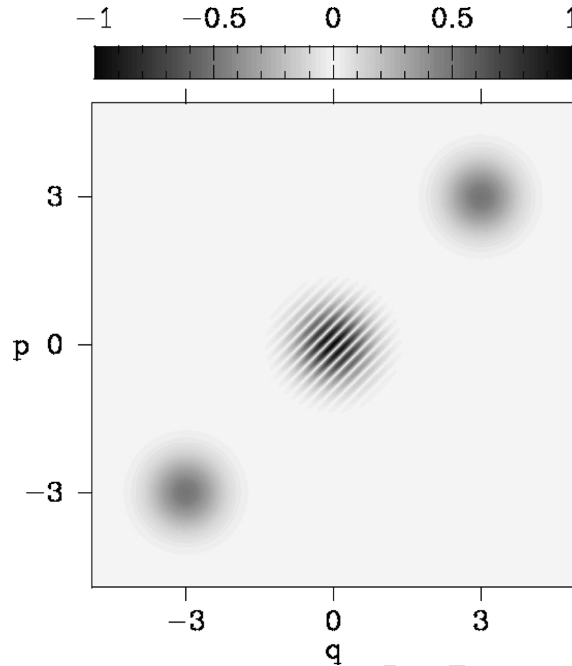[4] Here and below we set $\omega = 1$.

Fig. 4.6. The Wigner function for the Schrödinger cat state displays a pair of *classical* Gaussians, one for each coherent state, and a third Gaussian modulated by interference fringes halfway between them. The chord function is a mere rescaling of the Wigner function if the midpoint lies on the origin

this same configuration has to be reinterpreted. Now the local phase space correlations of the individual coherent states, as in (i), are placed in the neighbourhood of the origin, where they interfere, while their cross-correlation generates new Gaussians centred on the separation vectors $\pm 2\eta$. The general case of coherent states $|\eta_1\rangle$ and $|\eta_2\rangle$ merely leads to Gaussians centred on $\pm(\eta_1 - \eta_2)$ with addition of the phase factor $\exp[i(\eta_1 + \eta_2) \wedge \xi/2\hbar]$.

Recalling that the phase space correlations of a pure state are just the square modulus of the chord function, we can immediately verify the general relation between large- and small-scale structures in the case of Schrödinger cat states. Indeed, the spacial frequency of the oscillations of the chord function increases directly with the separation of the pair of coherent states.

The particular superpositions of coherent states, $|+\rangle$ and $|-\rangle$, are respectively even and odd eigenstates of the parity operator $\widehat{R}_0$, i.e., reflection about the origin. Therefore, they are the two possible states that could be produced by a parity measurement effected on the single coherent state $|\eta\rangle$. Thus, the parity measurement would generate a sizable probability of finding a particle near $x = -\eta$, even though this was most unlikely before the measurement.

The states $|\pm\rangle$ are orthogonal, even though the coherent states, $|\eta\rangle$ and $|-\eta\rangle$, are not. It is true that such a pair of coherent states will be nearly orthogonal if η is large enough and thus considered to form a qubit. Within this approximation, the symmetrical states would then be a mere unitary transformation of a single qubit. However, no approximation is needed in this process of carving a qubit from an infinite-dimensional system, if we use $|\pm\rangle$ as the original basis states. We would then consider a *common garden* coherent state to be the superposition of a symmetrical pair of Schrödinger cats. (Is there some approximation involved?) Indeed, this generation of a qubit by a reflection is not limited to coherent states, but could in principle be realized for any unsymmetrical initial state.

(iii) Fock states, $|n\rangle$, i.e., the excited states of the harmonic oscillator, also have reflection symmetry with respect to the origin. Thus, from the exact Wigner function, first derived by Grönewold [46],

$$W_n(\mathbf{x}) = \frac{(-1)^n}{\pi\hbar} e^{-\mathbf{x}^2/\hbar} L_n\left(\frac{2\mathbf{x}^2}{\hbar}\right), \tag{4.104}$$

where L_n is a Laguerre polynomial, we obtain the chord function

$$\chi_n(\xi) = \frac{e^{-\xi^2/4\hbar}}{2\pi\hbar} L_n\left(\frac{\xi^2}{2\hbar}\right). \tag{4.105}$$

It is interesting to note that the symmetry centre, which produces the maximum amplitude of the Wigner function, is nowhere near the classical manifold with energy $E_n = \left(n + \frac{1}{2}\right)\hbar\omega$. However, this point lies in a region of narrow oscillations, so that it does not affect the average of smooth observables. Figure 4.7 shows the Wigner function for the Fock state with $n = 2$; the origin is a maximum because of the positive parity. The unfolding of this peak for nonsymmetric Wigner functions is discussed in Sect. 4.10.

The Wigner function exhibits the interference fringes for the measurement of any variable $ap + bq$. In the case of the Fock state, these are always present. A simple way to see this is that any direction for the projection will be somewhere tangent to each of the continuous curves that form the Wigner function fringes. These regions dominate the projection. This example thus illustrates the *necessity* for the Wigner function to have negative regions: This is the only way that interference can result from a mere projection in phase space.

The Fock states are an example of a complete parity basis, which is even or odd according to the state label, n. Hence, if a pure state, $|\psi\rangle$, is specified in this basis, then

$$W_\psi(0) = \frac{1}{\pi\hbar} \sum_n \left[|\langle 2n|\psi\rangle|^2 - |\langle(2n+1)|\psi\rangle|^2\right]. \tag{4.106}$$

Such a decomposition can in principle be achieved for general arguments of the Wigner function, but then it is necessary to translate the whole Fock state
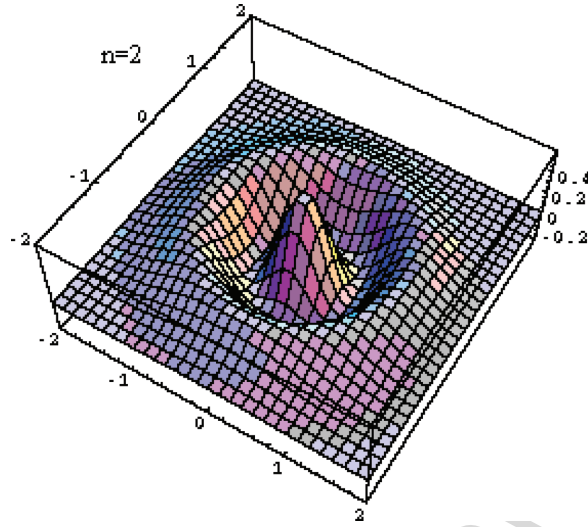
Fig. 4.7. The Wigner function for the $n = 2$ Fock state. The classical Bohr-quantized circle lies just outside the maximum of the outer fringe. The chord function is just a rescaling of the Wigner function

basis instead of just the ground state, as in the definition of coherent states. If we similarly translate the Hamiltonian, it will commute with $\widehat{R}_{\mathbf{x}}$ instead of commuting with $\widehat{R}_0$. The eigenstates of all such Hamiltonians will form a good odd–even basis. The difficulty with defining a semiclassical correspondence for both these classes of eigenstates is that the odd and even Bohr-quantized curves approach each other without a limit as $\hbar \to 0$.

All the above examples are singled out by some point of reflection symmetry, which needs to be chosen as the origin for the chord function to be real. The chord function must assume its maximum value $1/(2\pi\hbar)^L$ at the origin, whatever the symmetry, because of normalization. The Wigner amplitude, $|W(\mathbf{x})|$, need not have such a prominent peak in general. However we shall see in Sect. 4.10 that the large-scale features of the semiclassical forms of the Wigner function and the chord function maintain a mutual correspondence, even in the absence of a reflection symmetry.

It is important to note that the commutation, $[\widehat{\rho}, \widehat{R}_0] = 0$, guarantees that $W(\mathbf{x})$ is a symmetric function with respect to (classical) reflection at the origin. This is a consequence of the fact that if $\widehat{\rho}_R = \widehat{R}\,\widehat{\rho}\,\widehat{R}$, then $W_R(\mathbf{x}) = W(R(\mathbf{x}))$, the classical reflection of the argument. However, it is the maximum (or minimum) value at the origin which guarantees that the density operator is pure with respect to parity, i.e., it is either $\widehat{\rho}_+$, or $\widehat{\rho}_-$. Indeed, even though a mixture of an even density and an odd density (i.e., $\widehat{\rho} = c_+\widehat{\rho}_+ + c_-\widehat{\rho}_-$) will trivially satisfy $[\widehat{\rho}, \widehat{R}_0] = 0$, we see that $W(0) = (c_+ - c_-)/(\pi\hbar)^L$ will not be maximal.

Perhaps the converse property is of even more interest: If a Wigner function is symmetric about the origin, but $|W(0)| < (\pi\hbar)^{-L}$, then the state must be a mixture. After all, if it is a mixture of parities, it cannot be a pure state. It is only if the mixture is restricted to states of the same parity that it will not be detected by $W(0)$. There are many measures of degrees of mixedness, or impurity, but it is specially nice to be able to spot this property by a mere glance at the Wigner function. Furthermore, the Wigner function, i.e., the parity decomposition, is a measurable property [28, 34, 47].

The Wigner function may be considered as a field of probabilities for parity decompositions in phase space. Each reflection separates the infinite-dimensional Hilbert space into a pair of orthogonal components. If we just consider a single reflection, this goes a long way to reducing the Hilbert space to that of a single qubit, a two state system. No matter how classical the appearance of the Wigner function (i.e., it may be smooth and positive), it is always fully quantum as far as parity measurements are concerned. The situation is quite different, for instance, for position measurements. Then there is an important difference between the Wigner function for a pure Schrödinger cat state and a mixture of cats with different phases. This is revealed by the fine interference fringes between the two classical regions, but it is even more clearly displayed by the pair of correlation peaks far from the origin of the chord function. The relation between these features and entanglement is discussed in Sect. 4.8.

There is a vast literature concerning the Wigner function. Only a few topics have been mentioned here, and it has been necessary to leave out even as relevant a topic as quantum tomography. The adaptation of the Wigner function for finite Hilbert spaces is of special relevance for quantum computing and quantum information theory. Then the rule that each quantum state corresponds to a volume of $(2\pi\hbar)^L$ in classical phase space restricts the overall phase space volume. Thus, one must first face a choice of the topology in which to compactify phase space. It turns out that the simplest choice is a torus, though single qubits are more naturally displayed on a *Bloch sphere*. In spite of the intrinsic interest in many of the aspects of finite space Wigner functions [48, 49], there is no overall agreement on the choice of Wigner function properties to emphasise. Not all formalisms lead to a corresponding natural definition of a conjugate chord function as in [50], nor is there an overall preoccupation with invariance with respect to those symplectic transformations which preserve the torus topology of phase space [51]. A final difficulty concerns the appearance of ghost images and dimensionally dependent features [52].

So far nothing has been said of an alternative phase space representation, the *Husimi function* [53, 54]. Defined in terms of coherent states $|\eta\rangle$ as

$$\rho_H(\eta) = \langle\eta|\rho|\eta\rangle = \mathrm{tr}\,\rho\,|\eta\rangle\langle\eta|, \tag{4.107}$$

it can be interpreted as a smoothed Wigner function,

$$\rho_H(\eta) = \int d\mathbf{x}\, W_\eta(\mathbf{x})\, W(\mathbf{x}) \tag{4.108}$$

because of (4.70). The lack of purity of a state can be described in terms of the *Wehrl entropy*,

$$S_W = -(2\pi\hbar)^L \int d\eta\, \rho_H(\eta) \ln \rho_H(\eta). \tag{4.109}$$

According to Wehrl's inequality [55, 56] (see also [57]), the Wehrl entropy is always bounded from below by the von Neumann entropy (4.93). For more recent developments concerning the Wehrl entropy, see e.g., [58].

The Husimi function is most appropriate for the study of *quantum chaos*, because it highlights the classical region. But such a downplay of the quantum interferences, achieved by coarse graining the Wigner function, is not what one would ordinarily seek in quantum information theory. In a way, this is just the opposite of the chord function, which squashes all classical structure to the neighbourhood of the origin, so as to display the purely quantum coherences. It is remarkable that both these antithetical representations are intimately related to the translation operators, since the Husimi function for a pure state, $\widehat{\rho} = |\psi\rangle\langle\psi|$, can be rewritten as

$$\rho_H(\eta) = |\langle\psi|\eta\rangle|^2 = |\langle\psi|\widehat{T}_\eta|(\eta = 0)\rangle|^2. \tag{4.110}$$

Hence, the basic difference with respect to C_η in is the exchange of $|0\rangle$, the Gaussian ground state of the harmonic oscillator, for $|\psi\rangle$ itself.

A further comment is that the quantum interferences are displayed by the isolated zeroes of Husimi functions [59], in the case that $L = 1$. A uniform distribution of zeroes has been used to characterize the eigenstates of classically chaotic systems. Even though this is of great theoretical interest, these zeroes are usually located in regions where the Husimi function is already tiny, so that they may be very hard to compute. For instance, in the case of the cat state (4.102) with small η, they are found in the shallow negative regions where $|W_+(\mathbf{x})|$ is exponentially small.

4.7 The Partial Trace: Sections and Projections

Recall that the representation of operators, $\widehat{A} = |A\rangle\rangle$, in a given basis, such as $\langle\langle Q|A\rangle\rangle$, corresponds to the foliation of the double phase space, $X = (P, Q)$, by a set of Lagrangian planes, $Q = constant$. Performing linear canonical transformations in double phase space, we are free to choose the alternative coordinate planes, $Q = (q_-, q_+)$, or $Q = \mathbf{x}$, or $Q = \mathbf{y} = J\xi$ among others. In all cases, it is the fact that

$$\langle\langle Q'|Q\rangle\rangle = \delta(Q' - Q), \tag{4.111}$$

which permits us to identify the expansion coefficient in

$$\widehat{A} = |A\rangle\!\rangle = \int \mathrm{d}Q \; A(Q) \, |Q\rangle\!\rangle \qquad (4.112)$$

with $\langle\!\langle Q|A\rangle\!\rangle$.

Let us now assume that the (single) phase space is itself a product of a pair of phase spaces, $X = X_1 \otimes X_2$, each with $2L_j$ dimensions, and that these correspond to Hilbert spaces, $\boldsymbol{H}_j$, so that $\boldsymbol{H} = \boldsymbol{H}_1 \otimes \boldsymbol{H}_2$. Then we can always decompose the Lagrangian planes chosen as a basis for double phase space as the product $Q = Q_1 \otimes Q_2$, corresponding to operators $|Q\rangle\!\rangle = |Q_1\rangle\!\rangle \otimes |Q_2\rangle\!\rangle$. Thus the complete $|Q\rangle\!\rangle$ representation becomes

$$\widehat{A} = |A\rangle\!\rangle = \int \mathrm{d}Q_1 \mathrm{d}Q_2 \; A(Q_1, Q_2) \, |Q_1\rangle\!\rangle \otimes |Q_2\rangle\!\rangle . \qquad (4.113)$$

The definition of the partial trace is then

$$\mathrm{tr}_2 \, \widehat{A} = \mathrm{tr}_2 \, \widehat{I}_2 \, \widehat{A} = \int \mathrm{d}Q_1 \mathrm{d}Q_2 \; A(Q_1, Q_2) \, |Q_1\rangle\!\rangle \, \langle\!\langle I_2|Q_2\rangle\!\rangle , \qquad (4.114)$$

so that

$$A_1(Q_1) = \int \mathrm{d}Q_2 \; A(Q_1, Q_2) \langle\!\langle I_2|Q_2\rangle\!\rangle \qquad (4.115)$$

defines the $|Q_1\rangle\!\rangle$ representation of a reduced operator $\widehat{A}_1$, which acts on the Hilbert space $\boldsymbol{H}_1$. It is well known that in the case of the density operator $\widehat{\rho}$, the reduced operator $\widehat{\rho}_1$ describes the same probability as the full density operator for all measurements concerning the subsystem-1.[5]

The different forms of the partial trace depend essentially on the Hilbert–Schmidt product of each basis with the identity. In the case of the position basis, we have

$$\langle\!\langle I|Q\rangle\!\rangle = \mathrm{tr}\,\widehat{I} \; |q_-\rangle\langle q_+| = \delta(q_- - q_+) , \qquad (4.116)$$

so that

$$\begin{aligned}
A_1(Q_1) &= \int \mathrm{d}q_{2-}\mathrm{d}q_{2+} \; A(Q_1, Q_2 = (q_{2-}, q_{2+})) \; \delta(q_{2-} - q_{2+}) \\
&= \int \mathrm{d}q_2 \; A(Q_1, (q_2, q_2)) .
\end{aligned} \qquad (4.117)$$

Here, we should recall that,

$$A(Q_1, (q_2, q_2)) = \langle q_{1-}, q_2| \, \widehat{A} \, |q_{1+}, q_2\rangle , \qquad (4.118)$$

in matrix notation.

---

[5] A measurement on subsystem-2 only affects $\widehat{\rho}_1$ if the information on the outcome of the measurement is made available [6].

In the centre representation, we have simply

$$\langle\!\langle I|\mathbf{x}\rangle\!\rangle = \operatorname{tr}\widehat{I}\ (2^L\widehat{R}_{\mathbf{x}}) = 1\,, \tag{4.119}$$

leading to the phase space projection:

$$A_1(\mathbf{x}_1) = \int d\mathbf{x}_2\ A(\mathbf{x}_1,\mathbf{x}_2)\,. \tag{4.120}$$

In the case of the density operator, the corresponding reduced Wigner function, $W_1(\mathbf{x}_1)$, is thus obtained from $W(\mathbf{x})$ in the same way as a marginal probability distribution is projected out of the full distribution .

The simplest choice turns out to be the chord representation. Then, $|I\rangle\!\rangle = \widehat{T}_{\xi=0}$ is an element of the operator basis, so that

$$\langle\!\langle I|\mathbf{y}\rangle\!\rangle = \delta(\mathbf{y}) = \delta(\xi)\,. \tag{4.121}$$

Thus in this case, instead of projecting, we obtain the reduced operator merely by slicing through the chord symbol:

$$A_1(\xi_1) = A(\xi_1,\xi_2\!=\!0)\,. \tag{4.122}$$

Of course, the reduced operator $\widehat{A}_1$ itself is insensitive to the procedure used to obtain it within the various representations, but the ease of calculating the reduction is a special bonus of the chord representation.

It should be recalled that the partial trace is invariant with respect to unitary transformations performed internally within the factor Hilbert space $\boldsymbol{H}_2$: $\widehat{U} = \widehat{U}_2\otimes\widehat{I}_1$ (see, e.g., [6]). In the example where the subsystems are particles that have separated by a large distance, then these are truly *local transformations*. In other words, if $\widehat{A}' = \widehat{U}\widehat{A}\widehat{U}^{-1}$, then $\operatorname{tr}_2\widehat{A}' = \operatorname{tr}_2\widehat{A}$. This invariance corresponds semiclassically to the freedom of performing canonical transformations that leave invariant the $\mathbf{x}_1$ variables: $(x_1,x_2) \to (x_1,x_2')$. This also implies that only the double phase space corresponding to x_2 changes: $(X_1,X_2) \to (X_1,X_2')$. If the canonical transformation is linear in the single phase space, then both the centres, $\mathbf{x}$, and the chords, ξ, are propagated in the arguments of their respective functions by this same transformation.

Another point that is worth discussing concerns the completeness of the operator representations. Notice that the restricted translation operators

$$|\mathbf{y}_1\rangle\!\rangle_1 = \widehat{T}'_{\xi_1} = \widehat{T}_{\xi_1}\otimes\widehat{I}_2 \tag{4.123}$$

are a subset of the translation operators used in the chord basis for the full Hilbert space, $\boldsymbol{H}_1\otimes\boldsymbol{H}_2$. It follows that a representation in terms of the restricted translation operators, $\widehat{T}'_{\xi_1}$, would not be complete. Likewise, we may define the restricted unitary reflection operators,

$$|\mathbf{x}_1\rangle\!\rangle_1 = 2^{L_1}\widehat{R}'_{\mathbf{x}_1} = \widehat{R}_{\mathbf{x}_1}\otimes\widehat{I}_2\,, \tag{4.124}$$

but these do not belong to the centre basis for $\boldsymbol{H}_1 \otimes \boldsymbol{H}_2$. Even so, we may also define directly the reduced operator $\widehat{A}_1$ as

$$\widehat{A}_1 = \int d\mathbf{x}_1 \, A_1(\mathbf{x}_1) \, |\mathbf{x}_1\rangle\!\rangle_1 \,, \tag{4.125}$$

with

$$A_1(\mathbf{x}_1) = \operatorname{tr} \widehat{A} \left(2^{L_1} \widehat{R}'_{\mathbf{x}_1}\right). \tag{4.126}$$

Let us now specialize to density operators. In the case of the chord function, we must take care of the normalization, which depends on the number of degrees of freedom. Hence, the validity of between the chord representation of density operators $\rho(\xi)$ and $\rho_1(\xi_1)$ implies that the reduced chord function is

$$\chi_1(\xi_1) = (2\pi\hbar)^{L_2} \, \chi(\xi_1, \xi_2 = 0) \,. \tag{4.127}$$

Clearly, $\chi_1(\xi_1)$ is the Fourier transform of $W_1(\mathbf{x}_1)$. Since the definition of phase space correlations is valid for the reduced system, we obtain the reduced correlations as a projection of the correlations of the entangled pure state:

$$\begin{aligned}
C_1(\xi_1) &= \operatorname{tr} \widehat{\rho}_1 \, \widehat{T}_{(\xi_1,0)} \, \widehat{\rho}_1 \, \widehat{T}^\dagger_{(\xi_1,0)} \\
&= (2\pi\hbar)^{L_1} \int d\eta_1 \, e^{i\eta_1 \wedge \xi_1/\hbar} \left|\chi_1(\eta_1)\right|^2 \\
&= (2\pi\hbar)^{L_1} \int d\eta_1 d\eta_2 \, e^{i\eta_1 \wedge \xi_1/\hbar} \delta(\eta_2) \, \left|\chi(\eta)\right|^2 \\
&= \int \frac{d\xi_2}{(2\pi\hbar)^{L_2}} C(\xi) \,.
\end{aligned} \tag{4.128}$$

It should be recalled that the relation between the Wigner function and the chord function mimics that between a classical probability distribution and its characteristic function. The definition of correlations and the classical marginal distributions also goes through as above. Therefore, the property that the correlation of the reduced state for a given displacement, ξ_1, is just the integral over all correlations in the larger space over displacements that share this component also holds for classical probability distributions. This relation does not depend on the full density operator being a pure state.

All the representations that we have been discussing will factor in the case that $\widehat{\rho} = (|\psi_1\rangle \otimes |\psi_2\rangle)(\langle\psi_2| \otimes \langle\psi_1|)$ is a product pure state. Thus we obtain product Wigner functions, $W(\mathbf{x}) = W_1(\mathbf{x}_1)W_2(\mathbf{x}_2)$, and product chord functions, $\chi(\xi) = \chi_1(\xi_1) \chi_2(\xi_2)$. These relations may be interpreted in terms of average values of the basis operators, i.e., $\langle\widehat{R}_\mathbf{x}\rangle = \langle\widehat{R}'_{\mathbf{x}_1}\rangle \langle\widehat{R}'_{\mathbf{x}_2}\rangle$ and $\langle\widehat{T}_\xi\rangle = \langle\widehat{T}'_{\xi_1}\rangle\langle\widehat{T}'_{\xi_2}\rangle$. Thus, a sufficient criterion for the existence of entanglement would be that either of these equalities not hold for some centre $\mathbf{x}$ or some chord ξ.

Curiously, it is not the generation of cross-correlations that is usually taken as a measure of entanglement, but instead the loss of correlations of the

reduced density operator. Its von Neumann entropy (4.93) is often referred to as *the entanglement*. Expanding this to first order, results in the linear entropy of a partial trace of the full density operator,

$$1 - \operatorname{tr} \widehat{\rho}_1^2 = 1 - C_1(0) \,, \tag{4.129}$$

recalling (4.86). The square of this last expression is the *concurrence*, another widely used entanglement measure (see, e.g., Sect. 2.4.2 or [60]). At first sight, these are not obvious measures of overall entanglement, because we should obtain the same measure by singling out instead the reduced density operator for subsystem-2. But, it is a simple consequence of , the invariance of the quantum correlations with respect to Fourier transforms for a pure state, that

$$
\begin{aligned}
C_1(0) &= \int \frac{\mathrm{d}\xi_2}{(2\pi\hbar)^{L_2}} C(0, \xi_2) \\
&= \int \frac{\mathrm{d}\xi_2}{(2\pi\hbar)^{L_2}} \int \frac{\mathrm{d}\eta}{(2\pi\hbar)^L} \, C(\eta) \, e^{i\eta_2 \wedge \xi_2} \\
&= \int \frac{\mathrm{d}\eta_1}{(2\pi\hbar)^{L_1}} C(\eta_1, 0) = C_2(0) \,.
\end{aligned} \tag{4.130}
$$

Reinterpreted in terms of Wigner functions,

$$\int \mathrm{d}\mathbf{x}_1 \, [W_1(\mathbf{x}_1)]^2 = \int \mathrm{d}\mathbf{x}_2 \, [W_2(\mathbf{x}_2)]^2 \,, \tag{4.131}$$

this is another remarkable property of pure quantum states, for it is highly unusual for the second moment of a pair of marginal probability distributions to display a similar equality. Indeed, it is not even generally true for product distributions.

The focus on properties of the reduced density matrix makes sense when it is recalled that the concept of entanglement involves separate measurement on each of the components. The invariance of the partial traces with respect to local transformations carries over to the above measures of entanglement. Even better, it has been shown that it is possible to concentrate the entanglement within a few elements of an ensemble of identical states, by performing local measurements [61].

In terms of Husimi functions (4.108), it is natural to describe entanglement in terms of the Wehrl entropy (4.109) for the reduced density operator. Another way of describing entanglement is through the Schmidt decomposition . The corresponding Wigner and chord functions are then

$$W(\mathbf{x}) = (\pi\hbar)^{-L} \sum_{i,j} \lambda_i \lambda_j \, \langle \psi^i{}_1 | \widehat{R}'_{\mathbf{x}_1} | \psi^j{}_1 \rangle \langle \psi^i{}_2 | \widehat{R}'_{\mathbf{x}_2} | \psi^j{}_2 \rangle \tag{4.132}$$

and

$$\chi(\xi) = (2\pi\hbar)^{-L} \sum_{i,j} \lambda_i \lambda_j \, \langle \psi^i{}_1 | \widehat{T}'_{-\xi_1} | \psi^j{}_1 \rangle \langle \psi^i{}_2 | \widehat{T}'_{-\xi_2} | \psi^j{}_2 \rangle , \qquad (4.133)$$

recalling the definitions of the restricted reflection operators and the restricted translation operators . In both cases, the partial trace over subsystem-2 substitutes the second Dirac bracket by $\delta_{i,j}$, so that

$$W_1(\mathbf{x}_1) = (\pi\hbar)^{-L_1} \sum_i \lambda_i{}^2 \, \langle \psi^i{}_1 | \widehat{R}'_{\mathbf{x}_1} | \psi^i{}_1 \rangle = \sum_i \lambda_i{}^2 \, W_i(\mathbf{x}_1) \qquad (4.134)$$

and

$$\chi_1(\xi_1) = (2\pi\hbar)^{-L_1} \sum_i \lambda_i{}^2 \, \langle \psi^i{}_1 | \widehat{T}'_{-\xi_1} | \psi^i{}_1 \rangle = \sum_i \lambda_i{}^2 \, \chi_i(\xi_1) . \qquad (4.135)$$

Therefore, the reduced density operator is just a mixture of the factor states in the Schmidt decomposition for subsystem-1, with probabilities specified by the square of the Schmidt coefficients. The square of the concurrence is then given by

$$1 - \mathrm{tr} \, (\widehat{\rho}_1)^2 = 1 - \sum_i \lambda_i{}^4 , \qquad (4.136)$$

in terms of the second moment of the weighing factors for the mixed state. Note that, contrary to the Schmidt number, this is a well-defined entanglement measure for systems with infinite Hilbert spaces, if the above sum converges. Clearly, the purity of subsystem-2 involves the same sum over Schmidt coefficients, in agreement with our previous calculation (4.130).

Consider now the case that a subsystem can again be split up into a pair of components. If the full original state was entangled, the reduced density operator is not pure. Hence, it is an average over pure states. Obviously, this cannot be a product state overall, but if all of the pure states are products, the mixed state is not characterized as entangled, rather it is a *separable state*. The problem with mixed states is that the decomposition into pure states is not unique, so a state is considered separable if there exists any decomposition where it is separated (see Sect. 2.2.2).

Let us now define a *classical pure state* as a δ-function in phase space. Then all pure states, $f(x) = \delta(x)$, in a higher dimensional phase space will be product states, because the higher dimensional δ-functions factor as $\delta(x) = \delta(x_1)\,\delta(x_2)$. In this sense, the expression

$$f(x) = \int f(x') \, \delta(x - x') \mathrm{d}x' \qquad (4.137)$$

can be reinterpreted as a *classical separable state*: Any probability distribution in phase space can be considered as a linear combination of products of classical pure states. Thus, we can never consider a classical phase space distribution to be entangled, no matter how strong the correlations may be between variables pertaining to different subsystems.

What if a mixed Wigner function for a bipartite state is everywhere positive? Can we mimic the above reasoning to conclude that there is no entanglement? In general this is not so, because the function $\delta(\mathbf{x})$ does not represent a density operator in the Weyl representation. It represents instead the reflection operator, which has an infinitely degenerate negative eigenvalue, as discussed in Sect. 4.5. The closest that is possible is the coherent state (4.100), which approaches a δ-function as $\hbar \to 0$, but imposes an extra smoothing on the Wigner function for any combination of these pure states. Indeed, a general superposition of coherent states is defined by a weight function, known as the Glauber–Sudarshan *P-function* in quantum optics [8, 62, 63]. So, it is the positivity of a P-function that guarantees a separable state, rather than that of the Wigner function, because each coherent state can be factored.

To close this section, let us now study another kind of projection of the Wigner function. Whereas, by projecting onto a component subspace we generate a reduced Wigner function, a projection onto a Lagrangian plane (4.79) results in a probability density. All the coordinates of such a plane correspond to commuting operators. In the case of a bipartite system, we can define this Lagrangian plane by choosing some linear combination of the variables for each subsystem, $q_1' = \alpha_1 p_1 + \beta_1 q_1$ and $q_2' = \alpha_2 p_2 + \beta_2 q_2$, so that each coordinate, q_j', pertains to a different subsystem.

Consider now pairs of either–or measurements on both these variables, i.e., we can define observables $\widehat{O}_{1a}$, $\widehat{O}_{1b}$, $\widehat{O}_{2a}$ and $\widehat{O}_{2b}$ which take the value $+1$, for q_j' in the interval ja, and -1 outside. In terms of projection operators $\widehat{P}_{ja}$, we have $\widehat{O}_{ja} = 2\widehat{P}_{ja} - \widehat{I}$ and $\widehat{O}_{1a}\widehat{O}_{2a} = 4\widehat{P}_{1a}\widehat{P}_{2a} - 2\widehat{P}_{1a} - 2\widehat{P}_{2a} + \widehat{I}$, with similar formulae for the other products of commuting operators. Combining the expectation values for these products in the form of the CHSH inequality (4.17),

$$\langle\widehat{O}_{1a}\widehat{O}_{2a}\rangle + \langle\widehat{O}_{1a}\widehat{O}_{2b}\rangle + \langle\widehat{O}_{1b}\widehat{O}_{2a}\rangle - \langle\widehat{O}_{1b}\widehat{O}_{2b}\rangle$$
$$= 4\left(\langle\widehat{P}_{1a}\widehat{P}_{2a}\rangle + \langle\widehat{P}_{1a}\widehat{P}_{2b}\rangle + \langle\widehat{P}_{1b}\widehat{P}_{2a}\rangle - \langle\widehat{P}_{1b}\widehat{P}_{2b}\rangle\right) - 4\left(\langle\widehat{P}_{1a}\rangle + \langle\widehat{P}_{2a}\rangle\right) + 2\,,$$
$$(4.138)$$

we can now evaluate each expectation value on the right-hand side as a definite integral of the probability density over some region of the (q_1', q_2') plane. This is a purely classical set-up, so that by regrouping,

$$2 - 4\left(\widehat{P}_{1a} - \langle\widehat{P}_{1a}\widehat{P}_{2b}\rangle\right) - 4\left(\widehat{P}_{2a} - \langle\widehat{P}_{1b}\widehat{P}_{2a}\rangle - \langle\widehat{P}_{1a}\widehat{P}_{2a}\rangle\right) - 4\langle\widehat{P}_{1b}\widehat{P}_{2b}\rangle \leq 2\,,$$
$$(4.139)$$

we rederive the CHSH inequality, because none of the terms with the factor -4 can be positive.

We thus verify that the correlations measured among commuting pairs of either–or observables of each subsystem lie within strictly classical bounds, irrespective of any possible entanglement of their combined state. It makes no difference whether, or not, the Wigner function has negative regions. The

point is that we need to deal only with a single positive projection, which is a true probability distribution. To obtain a violation of the CHSH inequality, we must choose pairs of observables for each component which do not commute. The correlation for a given choice of observables from each pair may still be computed from the probabilities in the respective Lagrangian plane, but we must use different planes in each of the four correlations. Then, if the overall Wigner function that generates all these densities has negative regions, the CHSH inequality may be violated, as discussed in the following section.

Apparently, there has not been much effort to relate the intuitively appealing picture of entanglement as the source of nonclassical correlations in Bell inequalities to the technical entanglement measures appropriate to quantum information theory. However, a recent paper by Cirone [64] bridges this gap for finite-dimensional systems. The main point is that measurements are restricted to projection operators for the factor states in the Schmidt basis. It is then shown that the same concurrence, which was introduced in terms of the partial trace, can be expressed as a sum over conditional probabilities for measurements on either component.

4.8 Generating a *Classical* Entanglement: The EPR State

We have seen how symplectic transformations correspond exactly to unitary transformations in Hilbert space. Let us now examine how these can produce entangled states, given that the initial state, $\widehat{\rho}$, is a product of states, each represented by its Wigner function, $W_j(\mathbf{x}_j)$, or its chord function, $\chi_j(\xi_j)$, so that $W(\mathbf{x}) = W_1(\mathbf{x}_1)W_2(\mathbf{x}_2)$ and $\chi(\xi) = \chi_1(\xi_1)\chi_2(\xi_2)$. For the canonical transformation to be linear, the classical interaction Hamiltonian $H(x_1, x_2)$ can only be bilinear in the phase space variables. A convenient choice is $H = p_1 q_2 - p_2 q_1$, which may be interpreted as angular momentum, L_3, if the degrees of freedom refer to Cartesian coordinates in a plane. This Hamiltonian merely rotates both p and q coordinates in the argument of $W(\mathbf{x})$ and $\chi(\xi)$. Then, after a rotation by $\pi/4$, the density operator becomes $\widehat{\rho}'$, represented by

$$\chi'(\xi) = \chi_1\big(\frac{\xi_{p_1} + \xi_{p_2}}{\sqrt{2}}, \frac{\xi_{q_1} + \xi_{q_2}}{\sqrt{2}}\big)\ \chi_2\big(\frac{\xi_{p_1} - \xi_{p_2}}{\sqrt{2}}, \frac{\xi_{q_1} - \xi_{q_2}}{\sqrt{2}}\big). \qquad (4.140)$$

Since the partial trace is specified by (4.127), a section of the chord function, the reduced density for the first component becomes

$$\chi'_1(\xi_1) = (2\pi\hbar)\,\chi_1\big(\frac{\xi_{p_1}}{\sqrt{2}}, \frac{\xi_{q_1}}{\sqrt{2}}\big)\ \chi_2\big(\frac{\xi_{p_1}}{\sqrt{2}}, \frac{\xi_{q_1}}{\sqrt{2}}\big), \qquad (4.141)$$

in the chord representation.

So as to emphasise how *classical* an entanglement can be, let us choose for example initial Gaussian states, the product of harmonic oscillator ground states, described by

$$W_j(\mathbf{x}_j) = \frac{1}{\pi\hbar} \exp\left(-\frac{\omega_j}{\hbar}\, \mathbf{q}_j^2 - \frac{1}{\hbar\omega_j}\, \mathbf{p}_j^2 \right) \qquad (4.142)$$

or

$$\chi_j(\xi_j) = \frac{1}{2\pi\hbar} \exp\left(-\frac{\omega_j}{\hbar}\left(\frac{\xi_{q_j}}{2}\right)^2 - \frac{1}{\hbar\omega_j}\left(\frac{\xi_{p_j}}{2}\right)^2 \right). \qquad (4.143)$$

Thus, the probability distribution for positions,

$$f(\mathbf{q}) = \int \mathrm{d}\mathbf{p}\, W(\mathbf{x}), \qquad (4.144)$$

is also a Gaussian with elliptic level curves that are also rotated if $\omega_1 \neq \omega_2$. In this case, the effect of rotation, followed by the partial trace, is just a narrowing of the Gaussians in the chord representation:

$$\chi_1'(\xi_1) = \frac{1}{2\pi\hbar} \exp\left[-\frac{\omega_1 + \omega_2}{2\hbar}\left(\frac{\xi_{q_1}}{2}\right)^2 - \frac{1}{2\hbar}\left(\frac{1}{\omega_1} + \frac{1}{\omega_2}\right)\left(\frac{\xi_{p_1}}{2}\right)^2 \right]. \quad (4.145)$$

Notice that normalization is maintained, because we still have $\chi_1'(\xi_1) = (2\pi\hbar)^{-1}$ at the chord origin, but now the widths of the position Gaussian and of the momentum Gaussian are obtained through different kinds of average. The overall narrowing indicates that this is no longer a pure state.

The Wigner function presents a more intuitive picture of a mixed state. Taking the Fourier transform:

$$W_1'(\mathbf{x}_1) = \frac{1}{\pi\Delta}\, \exp\left[-\frac{2\omega_1\omega_2\mathbf{q}_1^2}{\hbar(\omega_1 + \omega_2)} - \frac{2\mathbf{p}_1^2}{\hbar(\omega_1 + \omega_2)} \right]. \qquad (4.146)$$

This still integrates to one, as demanded by normalization, but the Gaussian is now broader, with the uncertainty $\Delta = (\omega_1 + \omega_2)/2\sqrt{\omega_1\omega_2} > \hbar$, if $\omega_1 \neq \omega_2$. Therefore, this is not a pure state. The way that this example relates entanglement to initial states and evolution, which may both be considered *classical*, is even more extreme than those discussed in [65], which relies on projections of the Husimi function, in the approximate role of phase space probability density.

Another confirmation that this is not a pure state is that

$$\mathrm{tr}\,(\widehat{\rho}_1')^2 = 2\pi\hbar \int \mathrm{d}\mathbf{x}_1\, [\mathrm{W}_1'(\mathbf{x}_1)]^2 = 2\pi\hbar \int \mathrm{d}\xi_1\, |\chi_1'(\xi_1)|^2 < 1, \qquad (4.147)$$

and yet it might seem that this is just a freak result. After all, our state has remained a smooth *classical-like* Gaussian throughout. There are none of the quantum oscillations which are supposed to be the fingerprint of nonclassicality: For a start, nothing would prevent us from identifying the original Wigner function with a classical probability distribution. We then perform a simple rotation with perfect classical correspondence and obtain a new Gaussian, which pretends to be a quintessentially quantum entangled state! Have we been led astray?

Let us go back to the full Wigner function, resulting from choice (4.142) of Gaussians for the initial product state. After the $\pi/4$ rotation, this is just

$$W'(\mathbf{x}) = \left(\frac{1}{\pi\hbar}\right)^2 \exp\left[-\frac{\omega_1}{2\hbar}\,(q_1+q_2)^2 - \frac{1}{2\hbar\omega_1}\,(p_1+p_2)^2\right]$$
$$\times \exp\left[-\frac{\omega_2}{2\hbar}\,(q_1-q_2)^2 - \frac{1}{2\hbar\omega_2}\,(p_1-p_2)^2\right]. \quad (4.148)$$

In the extreme limit where $\omega_1 \to 0$ and $\omega_2 \to \infty$, we would obtain a normalized version of

$$W'(x) = \delta(q_1 - q_2)\,\delta(p_1 + p_2), \qquad (4.149)$$

which is just the Wigner function derived by Bell [66] for the original EPR wave function [3], namely $\langle q|\psi\rangle = \delta(q_1 - q_2)$. It so happens that the rotation that transformed the coordinates of our initial state, i.e., the ground state of an anisotropic plane harmonic oscillator, is essentially the same as the transformation from the individual coordinates for a pair of particles into a centre of mass, together with a relative *internal* coordinate. (These transformations differ only by local unitary transformations.) The EPR state is a δ-function both in the relative position and in the total momentum, which is the conjugate variable to the centre of mass.

Thus, the entanglement verified in our initial example implies that the centre of mass is likewise entangled with the relative coordinate in the EPR state. Perhaps, it is then even more surprising that the example that was picked is in some sense *classical*, if we consider that the discussion of the nonlocal and hence nonclassical nature of quantum mechanics started off with the historic EPR paper [3]. The fact that the full Wigner function is positive, not only allows us to interpret it as a classical probability distribution, but it also ensures that there is a wide range of measurements that can be performed on either component which may be considered as classical and hence local. We already found in the previous section that any measurement of pairs of either–or variables, $\widehat{O}_{1a}$, $\widehat{O}_{1b}$, $\widehat{O}_{2a}$ and $\widehat{O}_{2b}$ which take the value $+1$, for general phase space coordinates, q'_j, in the interval ja, and -1 outside, have correlations that satisfy the CHSH inequality, even if the Wigner function has negative regions. That was the case where the quantum observables which are measured commute. The statement for positive Wigner functions, due to Bell [66], is stronger: The inequality is then satisfied even if we choose different

variables for each measurement, $q'_{1a} \neq q'_{1b}$ and $q'_{2a} \neq q'_{2b}$, corresponding to different Lagrangian planes in phase space and, hence, quantum operators that do not commute. The argument is essentially the same as in the last section, except that now we can obtain all the expectation values from the full Wigner function, acting as a global probability distribution, instead of dealing with different probability distributions, each restricted to the Lagrangian plane specific to a given pair of variables.

Let us now reexamine our classically entangled states from the point of view of the reduced reflection operators, $\widehat{R}'_{\mathbf{x}j}$, defined as (4.126), that act on each component and, in particular, the parity operators, $\widehat{R}'_{0j}$. Such observables do not correspond to smooth phase space functions in classical mechanics, indeed, the Weyl representation of these operators (4.65) is singular. Nonetheless, parity, or reflection measurements can also be carried out on classical waves, as discussed in Sect. 4.2, and the question now concerns the possible correlations between measurements for different reflections carried out on both components. The fact that the full Wigner function (4.148) is symmetric with respect to the origin implies that the density operator commutes with the full reflection operator, $\widehat{R}_0$. However, $W'_1(0) < \pi\hbar$, so it does not have pure parity, i.e., $\widehat{\rho}'_1$ does not commute with $\widehat{R}'_{01}$. Hence, according to the discussion in Sect. 4.6, there is a finite probability to obtain negative (odd) parity, if such a measurement is performed on subsystem-1.

Perhaps this would not be so obvious a priori: The original state, represented by $W_0(\mathbf{x})$, is a pure state with pure positive (even) parity and this is also a property of the rotated state. This property can be verified directly, or it may be noticed that the driving Hamiltonian commutes with $\widehat{R}_0$, so that $H(\mathbf{x}) = H(R_0(\mathbf{x}))$. But now we find that a measurement of the parity of subsystem-1 has a finite probability to be negative. How is that?

Notice that the same also holds for subsystem-2: The derivation of the reduced density operator, $\widehat{\rho}'_2$, represented by $W'_2(\mathbf{x}_2)$ and $\chi'_2(\xi_2)$, goes through exactly as above. Therefore, there is also a finite probability of measuring negative parity in subsystem-2. As was shown in Sect. 4.6, the fact that, in both cases, the Wigner function is symmetric about the origin implies that all the pure states, into which the mixed reduced density operator can be decomposed, must have pure parity, but they are not all even. For this reason, the Wigner function (4.146) had to be obtained as a Fourier transform of the chord function; not a mere rescaling.

The crucial point is that the rotated state, $\widehat{\rho}'$, does not commute with either of the restricted reflections defined by , i.e., $\widehat{R}'_{01}$ or $\widehat{R}'_{02}$, even though it commutes with their product: $\widehat{R}_0 = \widehat{R}'_{01}\widehat{R}'_{02} = \widehat{R}'_{02}\widehat{R}'_{01}$. It should be recalled that the reduced Wigner functions are entirely determined by and in terms of the restricted reflections. Thus, to understand the results of measurements of either $\widehat{R}'_{0j}$, we need a common basis for all these operators. This is just the product of an even–odd basis for subsystem-1 and subsystem-2, for which we obtain the table:

$$even \otimes even \rightarrow even,$$
$$even \otimes odd \rightarrow odd,$$
$$odd \otimes even \rightarrow odd,$$
$$odd \otimes odd \rightarrow even.$$

(4.150)

Since $\widehat{\rho}'$ is even, it must be a superposition of the subset of basis states: *even* $\otimes$ *even* or *odd* $\otimes$ *odd*. Furthermore, we now find that the evolved state has a full parity correlation: If the measurement of $\widehat{R}'_{01}$ specifies even parity, then this must be the outcome of a measurement on $\widehat{R}'_{02}$. Conversely, if one of the subsystems has odd parity, then we know this to be the parity of the other subsystem.

An initial product state of an even Schrödinger cat state with a coherent state, which is rotated by $\pi/4$, is also susceptible to the foregoing analysis. However, an odd symmetry Schrödinger cat would have perfectly anticorrelated odd–even, or even–odd subsystems. In the case of the rotated cat, the evidence for entanglement is much more obvious. The pair of Gaussians is not centred on either of the planes in the chord phase space pertaining to the pair of subsystems. The partial trace that generates the reduced chord functions is a section of the full chord function, so that it does not capture these local maxima. Therefore, there is a deficit of phase space correlations in the reduced density operators.

Returning to the original rotated squeezed state, or, equivalently, the original EPR state, we must conclude that this is truly quantum and correctly described as entangled, i.e., just as nonclassical as the spin states in the Bohm version of EPR [7] that are commonly used to exemplify entanglement. The secret lies in choosing the property to be measured: A position measurement on one of the subsystems would not distinguish between this pure quantum state and a classical distribution. However, a measurement of reflection eigenvalues evokes a *spin-like duality* of this apparent classical state.

The violation of the CHSH inequality for reflection measurements of the smoothed EPR state completes the evidence of its nonclassicality. Banaszek and Wodkiewicz [67] first pointed out that the full pure state Wigner function of a bipartite state is proportional to the correlation for relection measurements on each subsystem: $\langle \widehat{R}'_{\mathbf{x}1} \widehat{R}'_{\mathbf{x}2} \rangle = (\pi\hbar)^2 W(\mathbf{x})$. This leads to a violation of the CHSH inequality for reflection measurements of the EPR state. They also proposed a realistic experiment for this in quantum optics [68]. We have already verified the complete correlation for parity measurements about the origin, which is in agreement with the maximal value that the full Wigner function (4.148) attains there. Its decay for large $\mathbf{x}_1$ or $\mathbf{x}_2$ signifies that $\langle \widehat{R}'_{\mathbf{x}1} \widehat{R}'_{\mathbf{x}2} \rangle \rightarrow 0$, so that

$$C_{CHSH} = \langle \widehat{R}'_{01} \widehat{R}'_{02} \rangle + \langle \widehat{R}'_{01} \widehat{R}'_{\mathbf{x}2} \rangle + \langle \widehat{R}'_{\mathbf{x}1} \widehat{R}'_{02} \rangle - \langle \widehat{R}'_{\mathbf{x}1} \widehat{R}'_{\mathbf{x}2} \rangle \qquad (4.151)$$

sinks from 2, its maximal classical value at the origin, to the limiting value 1. However, the origin is not the maximum of $C_{CHSH}(\mathbf{x}_1, \mathbf{x}_2)$, because the lowest order expansion of

$$(\pi\hbar)^2 W'(\mathbf{x}) = 1 - \frac{\omega_1}{2\hbar} (q_1 + q_2)^2 - \frac{1}{2\hbar\omega_1} (p_1 + p_2)^2$$
$$- \frac{\omega_2}{2\hbar} (q_1 - q_2)^2 - \frac{1}{2\hbar\omega_2} (p_1 - p_2)^2 + \ldots \quad (4.152)$$

leads to

$$C_{CHSH}(\mathbf{x}_1, \mathbf{x}_2) = 2 + (\omega_1 - \omega_2)q_1 q_2 + \left(\frac{1}{\omega_1} - \frac{1}{\omega_2} \right) p_1 p_2 + \ldots \quad (4.153)$$

Hence, the origin is a saddle point of $C_{CHSH}(\mathbf{x}_1, \mathbf{x}_2)$, which *increases* from its maximal classical value along the directions $q_2 = -q_1$ and $p_2 = p_1$, if one chooses the EPR conditions, $\omega_2 >> \omega_1$, i.e., if reflections are chosen in the directions where the Wigner function decays rapidly.

So we find that nonlocal correlations between two subsystems can arise even if the Wigner function for the full system is non-negative everywhere. It would thus appear that there is no relation between fringes in the Wigner function, where it attains negative values, and entanglement. The former project as interference fringes for possible measurements, but this is quite a different kind of *nonclassicality* than the delicate nonclassical correlations resulting from entanglement. But even here, one must be wary! If the measurements on the different components concern mechanical observables, natural for classical particles, then there is at least one case where negativity of the Wigner function has been shown to produce nonclassical correlations. Indeed, Bell [66] constructed an example where the CHSH inequality is violated for measurements on pairs of different variables,[6] $q'_j = q_j + t_j p_j$ and $q'_j = q_j + \tau_j p_j$. The state for which this is proved is a variation of our rotated state, where one of the factor Gaussians is substituted by the second excited state of the harmonic oscillator.

It should always be remembered that entanglement is not an intrinsic property, but only acquires its meaning within a specified basis, *the computational basis*, or the basis where measurements are made. In this respect, it resembles semiclassical caustics, which depend on our choice of representation. If the physical realization of the foregoing example were the ground state of a 2-D harmonic oscillator, then the rotation, which was found to produce entanglement, could be dismissed as merely an inconvenient coordinate transformation: Unless all measurements were to be restricted to the original coordinate axes, it would not be relevant, though true, to say that the rotated system became entangled, while the original system was a mere product. In contrast, for the alternative physical interpretation of one of the new coordinates as the centre of mass for a pair of particles, its entanglement with the internal coordinate can be important.

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[6] Even though Bell refers to the transformation parameters as *times*, these should be understood as specifications of the variables and hence of the planes onto which the Wigner function is projected.

4.9 Entanglement and Decoherence

The process of decoherence also results from the interaction of a pair of systems: the (small) *open system* and a (large) system, which we call the *environment* (see, e.g., Chap. 5). In contrast to the previous example, the component over which we trace, so as to obtain the reduced density operator, is on a scale which defies anything but a statistical description. The usual picture is that the environment lies *somewhere outside*, but it may just as well consist of the internal degrees of freedom for the centre of mass (CM) of a large system of particles. Exchanges between the large-scale motion and the internal variables lead to macroscopic energy dissipation as well as decoherence of the quantum state for the CM.

Let us consider the simplest possible example of the decoherence of the CM, because of its entanglement with internal variables. The CM for a system of L identical particles, assumed to be distinguishable is $Q = (q_1 + \ldots + q_L)/L$. The conjugate variable to Q is the total momentum, $P = p_1 + \ldots + p_L$. Let us further imagine that they are each in the same single particle state, $\widehat{\rho}$, and that these are independent, i.e., both the Wigner and the chord function are products over those of the individual states. This may seem too restrictive, because we should allow for different values of each average position $\langle q_j \rangle$, but we can redefine this as the origin for each j, so that we then measure Q from $\langle Q \rangle$.

In the case of $L = 2$, $X = (P, Q)$ is obtained from the rotated coordinate in the previous section by a mere canonical rescaling of $2^{\pm 1/2}$. It has been repeatedly emphasised that all such symplectic transformations on the argument of the Wigner or the chord function correspond exactly to unitary quantum transformations. So let us now reverse this transformation in the case of general L: We define $Q' = L^{1/2}Q$ and $P' = L^{-1/2}P$. Then, if the individual Wigner functions, $W(\mathbf{x}_j)$, were classical probability distributions, the *central limit theorem* would imply that the distribution for $X' = (P', Q')$ converges to

$$W_L(\mathbf{X}') \rightarrow [\pi \boldsymbol{\Delta_K}]^{-1} \exp[-\mathbf{X}'\mathbf{K}^{-1}\mathbf{X}'/\mathbf{2}], \qquad (4.154)$$

as $L \rightarrow \infty$, where we recall the definition of the Schrödinger covariance matrix, $\mathbf{K}$, in (4.7) and its determinant $\Delta_{\mathbf{K}}{}^2$. It is remarkable that positivity is not a necessary ingredient for the proof of the central limit theorem: In the case of identical square-integrable pure state Wigner functions, it is shown by Tegmark and Shapiro [69] that convergence onto a Gaussian again results. If the state for the individual particles is not represented by a pure state Gaussian, then the moments for this state will be such that $\Delta_{\mathbf{K}} > \hbar$. Therefore, the centre of mass Wigner function, $W_L(\mathbf{X})$, is a broader Gaussian than is permissible for a pure state and hence it must be a mixture. So, the CM of independent particles with identical Wigner functions is generally entangled with the internal phase space variables (which it has not been necessary to describe explicitly). Curiously, the potential entanglement resulting from the central limit theorem was overlooked in [69].

How does this entanglement with the internal coordinates evolve in time? It is easy to verify that free motion, generated by the Hamiltonian $H = p_1{}^2 + \ldots + p_L{}^2$, will not alter $\Delta_{\mathbf{K}}$. Thus the entanglement of the centre of mass with the environment is invariant in this simple case. Let us suppose instead that, though the particles do not interact, there is an external nonlinear field. Furthermore, the particles are sufficiently separated and the field is smooth enough so that it is legitimate to linearize the field locally around each $\langle q_j \rangle$. Then the Wigner function for each particle will evolve classically in different ways. The restrictive form of the central limit theorem in [69] cannot be applied in this case, but one can readily adapt Levy's proof [70] to allow for different Wigner functions, as long as the moments are finite and their average values converge [71].

The averages of the moments resulting from the different evolutions of many Wigner functions lead to a progressive loss of purity for the CM. Just as in (4.146) for the simple example of the last section, the uncertainty, $\Delta_{\mathbf{K}}$, increases. On top of that, the central limit theorem supplies the statistical ingredient for the decoherence process. It might appear strange to obtain decoherence even for a system of noninteracting particles, but it should be recalled that the CM momentum P, or P', appears linearly in each of the terms, $p_j{}^2$, in the kinetic energy, which accounts for the coupling to the internal momenta.

So as to make contact with the theory of Markovian open systems, we can now reinterpret this evolution of the reduced density matrix as a convolution of the original (Gaussian) Wigner function for the CM with a broadening Gaussian. For its Fourier transform, the chord function, this evolution is merely the product of an initial Gaussian with another Gaussian that narrows in time,[7] This is exactly the result for *quantum Markovian evolution* of an open system, in the case of quadratic internal Hamiltonian and linear coupling to the environment [72].

The deduction of the *canonical Lindblad equation* (see Sect. 5.3.2 or [73]),

$$\frac{\partial \widehat{\rho}}{\partial t} = -\frac{i}{\hbar}\big[\widehat{H}, \widehat{\rho}\big] - \frac{1}{2\hbar} \sum_j \big[\widehat{L}_j, \big[\widehat{L}_j, \widehat{\rho}\big]\big], \qquad (4.155)$$

that governs the evolution of the density operator in the quantum Markovian theory does not proceed by tracing out a larger system. All the same, the mere fact that the evolution is entirely determined by a differential equation precludes any delayed participation of previous motion. The *Lindblad operators* $\widehat{L}_j$, account for the nonunitarity of the evolution, that is, they take the part of the coupling to the environment. The Markovian approximation can in principle include arbitrary (non-quadratic) internal Hamiltonians for the system.

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[7] It must be recalled that the average CM evolution $\langle X(t) \rangle$ has been hidden by a time-dependent coordinate transformation.

The derivation of the Markovian approximation in the context of quantum optics (the damped harmonic oscillator) was carried out originally by Agarwal [74], but this is all in the language of complex phase space. The exact solution of (4.155) in [74] and that of Diosy and Kiefer [75], for the free *open* particle, are special cases of the general result in [72]: The chord representation of (4.155) is particularly simple if the Lindblad operators are linear functions of positions and momenta, $\widehat{L}_j = l_j \cdot \widehat{x}$, and if the Hamiltonian is quadratic [72]:

$$\frac{\partial \chi}{\partial t}(\xi, t) = \{H(\xi), \chi(\xi, t)\} - \frac{1}{2\hbar} \sum_j (\mathbf{l}_j \cdot \xi)^2 \ \chi(\xi, t). \qquad (4.156)$$

Here, the first term is the classical Poisson bracket. The exact solution of this equation factors into the unitary evolution of the chord function, undisturbed by the Lindblad operators, and a narrowing Gaussian factor. In the Wigner representation this becomes a Gaussian smudging of the unitarily evolving Wigner function. It is remarkable that the Wigner function becomes positive after a time that depends only on the parameters of the Lindblad equation, regardless of the initial pure state [72, 75].

In our simple example of the evolution of the CM, the Lindblad operator for its one-dimensional motion should be chosen as the total momentum $\widehat{P}$, because this is the variable that couples to the internal motion, which is hidden within the Markovian approximation. Even though the central limit theorem supplied the Gaussian factor of the evolving chord function, the overall Gaussian form for the evolving CM does not reflect the richness of other possibilities for Markovian evolution. However, by considering the entanglement of a *small system* with the CM of a *large system* and following the treatment of the example in the preceding section, we obtain qualitatively the general Markovian picture. This allows an interpretation of the Gaussian smoothing as originating in the multiple small contributions contemplated in the central limit theorem.

The standard way of going beyond the Markovian approximation, so as to include memory kept by the environment of the previous motion of the system, is to use the Feynman–Vernon functional [76], in the manner exploited by Caldeira and Leggett [77, 78]. Some standard references for dissipation, noise and decoherence from a quantum optics point of view are the books by Louisell [79], Gardiner [80] and Weiss [81] (see also Chap. 5).

4.10 A Semiclassical Picture of Entanglement

A full semiclassical theory of entanglement is still a program for the future, fascinating but difficult. Even so, several of the main elements for this construction can be sketched in this concluding section.

For a start, one should note that it is feasible to fit semiclassical torus states with Gaussian coherent states placed along the classical torus in a very

satisfactory way [82]. The number of Gaussians required increases with a fractional power of $\hbar^{-1}$. The important qualitative feature is that the interference fringes of the Wigner function, near the midpoint of the pair of Gaussians composing a Schrödinger cat, have the same wavevector as the similar fringes at the centre of a geometrical chord of the classical torus. Therefore, in both cases we can describe very fine interference fringes related to long chords. It also follows that our preliminary study of entanglement and decoherence of cat states is not at all irrelevant for understanding the evolution of product semiclassical states. Refinements of the fitting procedure allow even the description of the diffraction effects near caustics [83].

Before analysing product states and their partial trace, recall that dyadic operators, $|\psi\rangle\langle\phi|$, live in a kind of squared Hilbert space, which corresponds to a double phase space. These operators were shown in Sect. 4.5 to correspond to a product Lagrangian surface in double phase space, $\tau_\psi \otimes \tau_\phi$, if each of these states corresponds to a Lagrangian surface on its own right. Thus, the projection operator, or pure state density operator, $\widehat{\rho} = |\psi\rangle\langle\psi|$, is just a particular case of this general rule. If the state, $|\psi\rangle$, corresponds to a Bohr–Sommerfeld quantized torus of L dimensions in $2L$-D phase space, then the full density operator must correspond to a $2L$-D product torus in $4L$-D double phase space. This is in exact analogy to the way that a product torus describing the state for several particles (4.30) is obtained from lower dimensional tori. Recalling that we can describe double phase space in terms of the centre coordinates, $\mathbf{x}$, and the conjugate variables, $\mathbf{y} = \mathbf{J}\xi$ (4.47), the semiclassical Wigner function, $W(\mathbf{x})$, is then a superposition of complex exponentials, such that each phase is obtained by integrating $\mathbf{y}(\mathbf{x})$ along one of the different branches of the torus. Even though this approximation breaks down along caustics, the latter provide ready indication of regions where the Wigner function has a large intensity.

The problem is then to relate the semiclassical Wigner function, defined on the centre plane, to classical structures that are also portrayed in this same single phase space. Let us consider first the semiclassical Wigner and chord functions in the simplest case where $L = 1$. The Fock states (4.104) are good examples of semiclassical torus states when the quantum number n is large. Introducing the asymptotic expression for Laguerre polynomials,

$$\lim_{n\to\infty} L_n\left(\frac{z^2}{2n}\right) = J_0\left(\sqrt{2}z\right) , \tag{4.157}$$

together with the large argument expansion,

$$J_0(y) \approx \frac{2}{\sqrt{\pi y}} \cos\left(y - \frac{\pi}{4}\right) , \tag{4.158}$$

brings the Wigner function (4.7) for these states into a semiclassical form. To understand this, we must investigate the geometry of the double torus from the point of view of the simpler quantized curve, which is just a circle in this case.

Every point on the double torus represents a pair of points on the quantized curve and vice versa. A given pair of points on the quantized curve, $x_\pm$, defines a geometric chord: $\xi = x_+ - x_-$. Hence, $\mathbf{y} = \mathbf{J}\xi$ is the chord coordinate on the double torus, which has the centre coordinate, $\mathbf{x} = (x_+ + x_-)/2$. Obviously, the exchange of x_+ with x_- produces a new chord of the quantized curve with the same centre, $\mathbf{x}$. Viewed in double phase space, there must always be pairs of chords of the double torus projecting onto each centre, $\mathbf{x}$. The symmetry of this surface with respect to the identity plane, $\mathbf{y} = 0$, leads to complex conjugate phase contributions, in line with the above cosine for the Fock state. Actually this is a general feature: Because the Wigner function is real, the chord pairs will always produce semiclassical contributions adding up to cosines.

To obtain the phase of the cosine contribution to the semiclassical Wigner function for each pair of chords, the best course is to use a result which was put in its most general form by Littlejohn [29]. This concerns the general overlap, $\langle \psi | \phi \rangle$, of quantum states associated semiclassically to curves τ_ψ and τ_ϕ: The semiclassical contributions arise from the intersections of these classical curves and the phase difference between a pair of contributions is just the area sandwiched between the corresponding pair of intersections, divided by Planck's constant, as shown in Fig. 4.8.

We can immediately apply this principle to the Wigner and chord functions for pure states, by recalling that $W_\psi(\mathbf{x}) = (\pi\hbar)^{-1}\langle\psi|(\widehat{R}_\mathbf{x}|\psi\rangle)$ and $\chi_\psi(\xi) = (2\pi\hbar)^{-1}\langle\psi|(\widehat{T}_\xi|\psi\rangle)$. The semiclassical state $\widehat{R}_\mathbf{x}|\psi\rangle$ is merely the state constructed from the reflected curve, $R_\mathbf{x}(\tau_\psi)$, whereas $\widehat{T}_\xi|\psi\rangle$ corresponds to
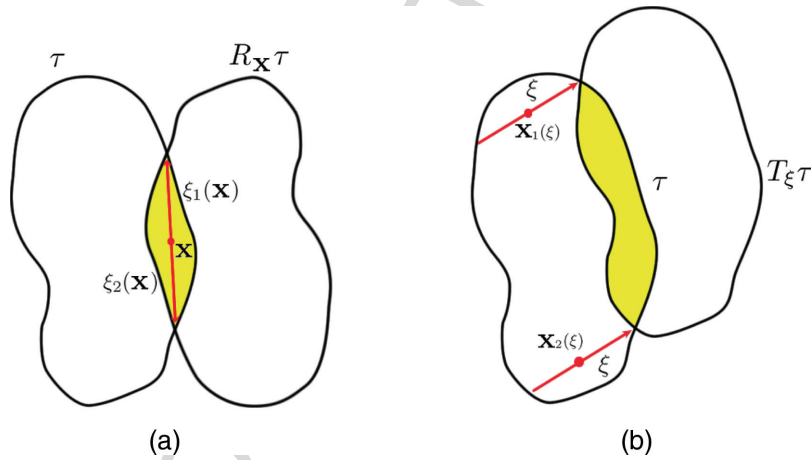

Fig. 4.8. (a) The semiclassical Wigner function is constructed by reflecting the torus τ around the point $\mathbf{x}$: The chords $\xi(\mathbf{x})$ are defined by the intersections with $R_\mathbf{x}\tau$. (b) The semiclassical chord function is constructed by translating τ by the vector ξ: The centres $\mathbf{x}(\xi)$ lie halfway back along ξ from each intersection of τ with $T_\xi\tau$

the translated curve $T_\xi(\tau_\psi)$. Therefore, in the case of the Wigner function, we obtain the phase of the cosine as half of the area sandwiched between the torus and its reflection at the centre $\mathbf{x}$ [22], which coincides with the area between the torus and the chord [84, 85] (see also [86]). Furthermore, this construction supplies, at a glance, the tips of all the chords centred on $\mathbf{x}$ as the intersections between both curves, as seen in Fig. 4.8(a). Note that once the curve has been reflected around the origin, we need only translate $R_0(\tau_\psi)$ to obtain the reflections around all other centres because of the group property, $R_\mathbf{x} = T_\mathbf{x} \circ R_0 \circ T_{-\mathbf{x}}$.

The same geometrical method can be used to study the structure of centres for a pre-specified chord ξ on the curve τ_ψ. Each intersection of τ_ψ with the translated curve $T_\xi(\tau_\psi)$ reveals one of the tips where the chord is to be placed and hence the centre of the chord, as shown in Fig. 4.8(b). In the case of open curves, the chord function may actually be simpler than the Wigner function because it is not necessary to have interference, as in the case of the parabola, for which there is only one intersection. In the case that τ_ψ has a centre of symmetry, as in the example of the Fock state, we thus find that the simple relation (4.99) between Wigner and chord functions is respected by the semiclassical approximation.

Viewed in single phase space, the caustics of the Wigner function arise from coalescing torus chords, as their centre, $\mathbf{x}$, is moved. This occurs at the tangencies of $R_\mathbf{x}(\tau_\psi)$ with the fixed curve, τ_ψ. Similarly, the caustics of the chord function are the loci of ξ such that $T_\xi(\tau_\psi)$ is tangent to τ_ψ. On the other hand, in double phase space, the Wigner caustics for a torus state are viewed as projection singularities of the double torus, $\tau_\psi \otimes \tau_\psi$, which lies *above* the area inside τ_ψ. The general geometric constructions underlying the semiclassical Wigner and chord functions are readily extended to phase space representations of dyadic operators, $|\psi\rangle\langle\phi|$, corresponding to double tori, $\tau_\psi \otimes \tau_\phi$. Their Weyl representation is known as *cross-Wigner functions* or *Moyal brackets* [87], whose semiclassical form is presented in [88].

There are many fascinating features of caustics in the phase space representations of pure states that have been studied and many more that must still be analyzed. For instance, the build up in the centre of the Wigner function for the Fock state is a caustic. Its semiclassical origin is the degeneracy of a continuum of chords conjugate to the same symmetry centre. However, this is a nongeneric feature of reflection-symmetric states. If the symmetry is broken, this supercaustic unfolds into a cusped triangle, first described by Berry [85]. The unfolding of higher dimensional caustics for rotated product tori, studied in [22], were also examined for the Wigner function. It turns out that the double fold surfaces of the Wigner caustic that meet along the torus do not unfold in the manner portrayed in Fig. 4.4 because of a symmetry constraint.

The limit of small chords is specially relevant for semiclassical theory. For the Wigner function, it singles out the classical torus itself as the Wigner caustic. The uniform approximation for the Wigner function throughout this
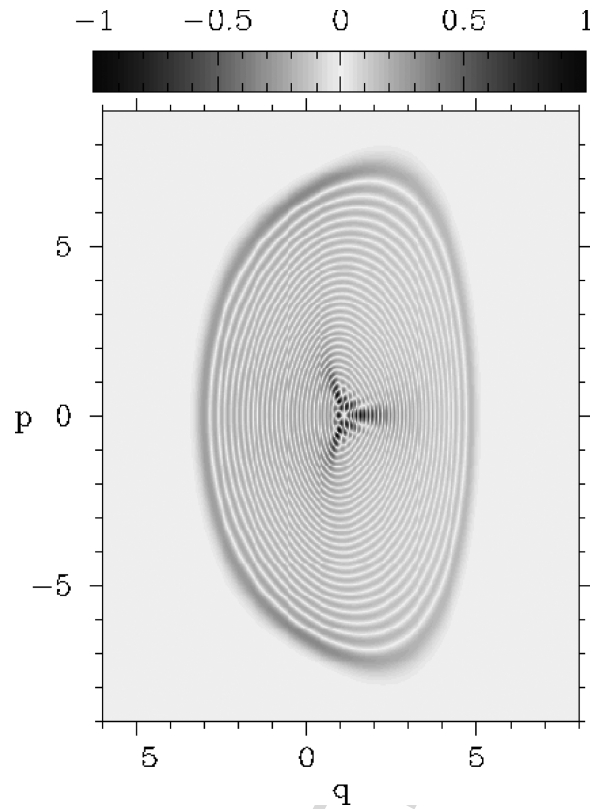
Fig. 4.9. Typical fringes for a semiclassical Wigner function. The classical curve follows closely the border of the fringes. The *triangular structure* of interfering fringes near the centre results from the unfolding of the maximum of the Fock states due to symmetry breaking

region is presented in [85], for the case of a curve and in [89], for a two-dimensional torus. A pair of sheets of the double torus are joined on the identity plane along this curve or torus. The large amplitude of the Wigner function oscillations near the quantized curve is due to this caustic. The corresponding caustic of the chord function collapses onto the origin, whatever the geometry of the classical region. The neighbourhood of this highly non-generic chord-caustic is discussed in [32]. Once again, we find that all regions where $C(\xi) = |\chi(\xi)|^2$ is large, outside of an $\hbar^{L/2}$-neighbourhood of the origin, point to phase space correlations that are truly quantum in nature.

So far, we have only discussed static properties of the density operator. In turning to dynamics, a preliminary point is that we should distinguish the *Weyl propagator*, $U(\mathbf{x})$, that is, the Weyl representation of the evolution operator, from the propagator for Wigner functions. The former is unitary and is hence supported by its own *static* Lagrangian surface in double phase

space, as discussed in Sect. 4.5, so that its semiclassical description is similar to that of the Wigner function itself. This can be deduced from a path integral in single phase space [33, 90].

In order to treat the unitary evolution of the density operator for a pure bound state, $|\psi\rangle\langle\psi|$, with L degrees of freedom, we need to consider the corresponding classical evolution of a $2L$-D Lagrangian torus, $\tau_\psi \otimes \tau_\psi$. Initially, this is separable within both single phase spaces, even though the product is not factored in the centre×chord coordinates. The classical motion must propagate the tips of each chord, x_- and x_+, in the same way. Taking account of the change of sign, $p_- \to -p_-$, in the definition of double phase space, we find that the double phase space Hamiltonian must be

$$\mathbb{H}(X) = H(x_+) - H(x_-) = H(\mathbf{x} - \mathbf{J}\mathbf{y}/2) - H(\mathbf{x} + \mathbf{J}\mathbf{y}/2). \qquad (4.159)$$

This classical Hamiltonian can be verified to preserve the product form of the geometric structures in each of the phase spaces $x_\pm$, but it will not preserve initial products within each of these in the general case that the single Hamiltonian $H(x)$ has coupling terms between different degrees of freedom. It propagates Lagrangian surfaces in double phase space that correspond either to density operators (according to the Liouville–von Neumann equation) or to unitary operators (the Heisenberg equation). The explicit formulae for the semiclassical evolution of the Wigner function are given in [91, 92], whereas the evolving action of the chord function is presented in [93]. The difficulty that cannot be avoided by changing the representation lies in the caustics of the initial state, which require more sophisticated semiclassical treatment.

A promising approach lies in the definition of integral propagators for the Wigner function, or for the chord function. The former may be defined in terms of the Weyl propagator as a kind of *second-order Wigner transform* (see e. g. [91])

$$\mathbb{U}(\mathbf{x}_t, \mathbf{x}) = \int d\mu\, U_{-t}\left(\frac{\mathbf{x} + \mathbf{x}_t}{2} - \mu\right) U_t\left(\frac{\mathbf{x} + \mathbf{x}_t}{2} + \mu\right) \exp\left(\frac{2i}{\hbar}\mu \wedge (\mathbf{x}_t - \mathbf{x})\right). \qquad (4.160)$$

Their explicit semiclassical form has been developed in [94], but these propagators also have their own intrinsic caustics. More recently, caustic-free propagators, from the Wigner to the chord function and vice versa, have been defined [93]. These are constructed either in terms of the propagation of the unitary reflection operators or the translation operators, instead of directly evolving the density operator itself.

Several of the geometrical structures underlying the semiclassical theory of the Wigner function for nonseparable tori in double phase space that evolve under the action of a general Hamiltonian were analyzed in [22] for the simplest case where $L = 2$. It will be necessary to push much further this analysis, while adapting it to the chord function. The reason is again that the partial trace of the density operator is obtained immediately by a section through the chord function, $\chi(\xi_1, \xi_2 = 0)$, which is defined semiclassically by

a nonseparable section of the double torus that evolved from an initial product $\tau_\psi \otimes \tau_\psi$. For instance, the quantum state and the corresponding double torus both loose their product form under the action of the simple Hamiltonian employed in Sect. 4.8. But slicing through a torus produces either a single or several lower dimensional tori.

This indicates that the semiclassical theory of reduced density operators that allow us to quantify entanglement, or to calculate the correlations on separate measurements effected on each component of the system, can still be associated to lower dimensional Lagrangian surfaces that are no longer products. The problem is to work out the actions and the amplitudes for this multidimensional geometry. This general picture agrees with initial results for nonunitary Markovian evolution of semiclassical Wigner functions [95]. It is notable that the same methods that have been used in [96] to show that the semiclassical approximation of the Wigner function satisfies the purity condition, tr $\widehat{\rho}^2 = \widehat{\rho}$, reveal the loss of purity with time due to decoherence.

A question that has deserved some attention concerns the effect of classically chaotic evolution on decoherence of an open system, or equivalently on the evolution of a system entangled to a larger system. There are indications that the reduced density matrix will loose its purity faster than in the case of regular internal motion [97], and this is verified exactly for quadratically hyperbolic Markovian systems [72], which capture a small element of chaotic behaviour. However, it must be stressed that there is no essential extra difficulty, in dealing semiclassically with the short-time evolution generated by chaotic hamiltonians, over regular motion.

In contrast, the evolution of initial pure states, which are eigenstates of chaotic Hamiltonians, is a much tougher problem. According to Shnirelman's theorem [98–100], in most cases these *ergodic states* are supported by the full energy shell, in the sense that averages of smooth observables are well approximated by classical averages over this surface. Often, such a state is thus described as having a *δ-function over the shell* for its Wigner function, but this is not true of its detailed structure: The intimacy between the Wigner function and phase space reflections implies that interference fringes generally exist halfway between classical regions of the Wigner function. According to the discussion of correlations of the Wigner function in Sect. 4.6, these are equal to their Fourier transform, for all pure states. Therefore, a Wigner function supported by an energy shell of large phase space dimensions must have oscillations in its interior of correspondingly small wavelength. These are readily derived semiclassically for mixtures of states within a narrow energy range [33, 101], but the fine features of pure states have so far eluded all theoretical efforts.

So far, our discussion has concerned a chaotic initial state for the relevant component, but nothing prevents us from defining an initial state that is ergodic for some chaotic Hamiltonian defined over the full product Hilbert space of the system with its environment. The situation is then radically different from the states that have evolved semiclassically from product

states: It is possible to *disentangle* each state that has lost its original product form by merely reversing time. This corresponds classically to running Hamilton's equations backwards in the double phase space; e.g., in the example of Sect. 4.8, the product of two harmonic oscillator ground states is recovered from the EPR state. This is always possible in quantum mechanics because one can always specify a unitary transformation on the entire Hilbert space, which transforms any given state into any other state and we can choose the latter to be a product state. However, the correspondence for this disentanglement cannot exist for ergodic eigenstates of a classically chaotic Hamiltonian defined on the full Hilbert space.

In the case where $L = 2$, the energy shell is 3-D and, because it has an extra dimension, there exists no classical canonical transformation, whether linear or nonlinear, that can transform it into the product of two closed curves. Therefore, ergodic eigenstates are *essentially entangled* from the point of view of classical correspondence. The study of traces of classical chaos in quantum mechanics is known as *quantum chaology* [86]. The characterization of ergodic states as those that are not classically disentangleable establishes a bridge between entanglement theory and quantum chaology. It remains to be seen whether this special type of entangled state has any application in quantum information theory.

Acknowledgement

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